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THE ASYMPTOTIC DUAL SMALE CONSTANT EQUALS 1, AND OTHER MISCELLANEA

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NOTE: This is an expanded version of the more focused article [8]: https://doi.org/10.5281/zenodo.24644423

Abstract. For a complex polynomial P of degree $d \geq 2$ and a regular point $a \in \mathbb{C}$ with $P'(a) \neq 0$, let

$$D(P,a) := \max_{P'(c)=0} \left| \frac{P(c)-P(a)}{(c-a)\,P'(a)} \right|, \qquad D_{\min}(d) := \inf_{\deg P = d,\ P'(a)\neq 0} D(P,a).$$

The dual Smale mean value conjecture, posed independently by T. W. Ng (2007) and by V. N. Dubinin and T. Sugawa (2009), asserts that $D(P,a) \geq 1/d$, with equality only when all critical points of P coincide. Previously the sharp bound $1/d$ was known only for $d \leq 7$, and the best unconditional bound for arbitrary degree was of order $1/d^2$.

In the focused core of this work (Sections 2–6), we prove by purely analytic, computer-free methods that the asymptotic dual Smale constant equals 1 ($L^* := \lim_{d\to\infty} d\, D_{\min}(d) = 1$) together with structural rigidity of near-extremal polynomials (locally uniform exponential convergence and critical-point concentration), that the sharp conjecture $D(P,a) \geq 1/d$ holds for every sufficiently large degree $d \geq d_0$ (strictly unless all critical points coincide), and that the unconditional all-degree lower bound $D(P,a) > 2/(5d)$ holds for every $d \geq 2$. In this expanded version (Sections 7–9), we include three further self-contained analytic theorems. First, we prove the sharp bound $D(P,a) \geq 1/d$ whenever all critical points lie on a circle centered at a (for every $d \geq 2$) or inside an annulus centered at a of aspect ratio at most 1.01 (for $d \geq 8$) or at most any fixed $M < \sqrt{2}$ (for sufficiently large d). Second, we prove a sharp ramified Blaschke derivative inequality yielding $D(P,a) > 1/(2d)$ for every $d \geq 2$ whenever every critical point of P has multiplicity at least 4 in P' (and more generally under a local multiplicity condition on the first-value component). Third, without compactness, we improve the linear constant $2/5$ effectively in large degree: $D(P,a) > 11/(20d)$ for every $d \geq 10^{75}$ and $D(P,a) > 2\sqrt{2}/(5d)$ for every $d \geq 10^{300\,000}$. These follow from an explicit family of effective bounds yielding $d\, D_{\min}(d) > \sqrt{\frac{8}{25}\cdot\frac{217}{216}} - 350\frac{\log\log d}{\log d}$ for large d. The all-degree bound $D(P,a) > 2/(5d)$ (Theorem 1.3), the sharp asymptotic constant (Theorem 1.2), and the empirical-law concentration along weakly convergent subsequences (Theorem 1.2(b)) are verified in Lean.[1]

1. Introduction

Let P be a complex polynomial of degree $d \geq 2$, and let $a \in \mathbb{C}$ be a *regular point* of P, meaning $P'(a) \neq 0$. Comparing the secant slope $(P(c)-P(a))/(c-a)$ to each critical point c (where $P'(c)=0$) against the derivative $P'(a)$ produces the affine-invariant quantity

$$D(P,a) := \max_{P'(c)=0} \left| \frac{P(c)-P(a)}{(c-a)\,P'(a)} \right|.$$

Replacing $P(z)$ by $(P(a+z)-P(a))/P'(a)$ reduces to the standard normalization $P(0)=0$ and $P'(0)=1$, where we write $D(P) := D(P,0) = \max_{P'(c)=0}|P(c)/c|$. We set $D_{\min}(d) := \inf\{D(P,a) : \deg P = d,\ P'(a)\neq 0\}$, write $L^* := \lim_{d\to\infty} d\, D_{\min}(d)$ for the *asymptotic dual Smale constant*

[1]These results are proved in Lean 4 from its three standard axioms alone and checked with `lake comparator` using `nanoda` 0.4.32: https://github.com/david44hello/paper_dual-mean-value-conjecture/tree/v1.10/lean

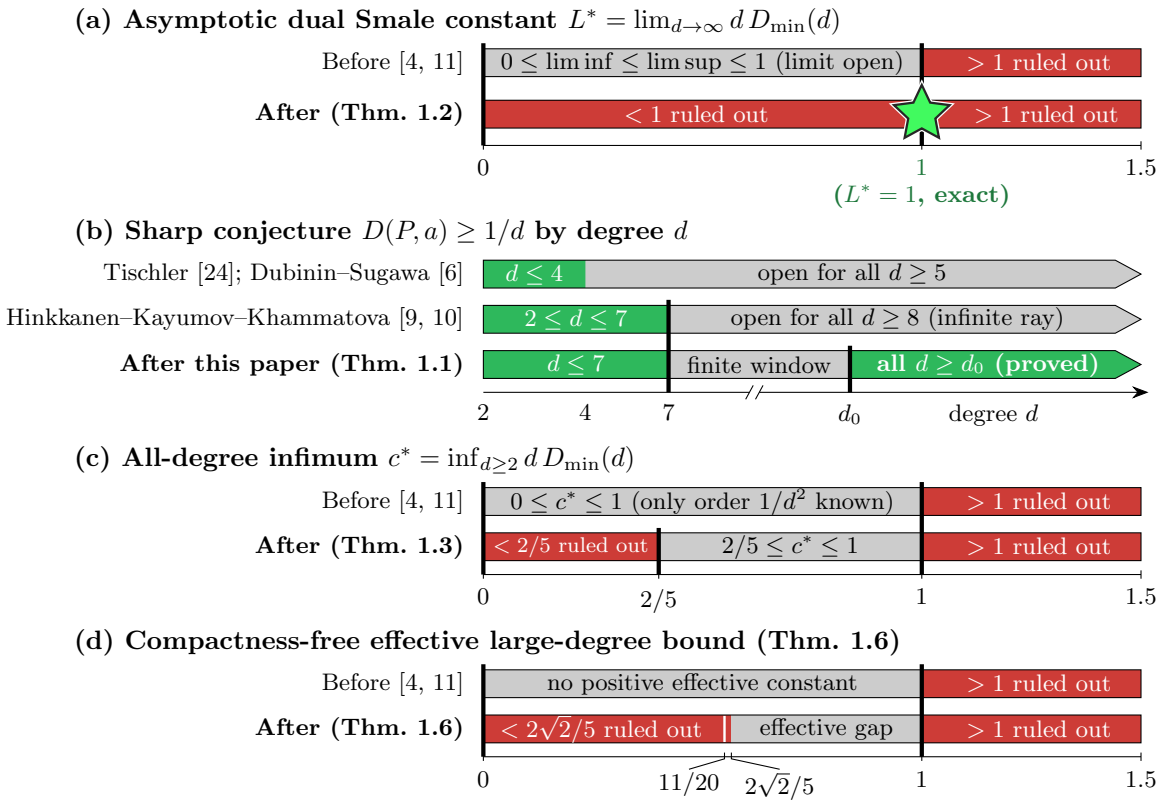

FIGURE 1. Historical survey before and after this expanded version. In (a) (to scale), red marks ruled-out values and gray marks uncertainty for the asymptotic constant $L^* = \lim_{d\to\infty} d\, D_{\min}(d)$: Theorem 1.2 pins L^* down to the exact value 1 (green star), leaving no gray. In (b) (not to scale across the axis break, as d_0 is ineffective), green marks proved degrees and gray marks open degrees: Theorem 1.1 reduces the infinite ray $d \ge 8$ to a finite window $8 \le d < d_0$ (and Theorem 1.4 settles centered-circle and 1.01-annulus polynomials inside it). In (c) (to scale), Theorem 1.3 excludes $[0, 2/5)$ for the all-degree infimum $c^* = \inf_{d\ge 2} d\, D_{\min}(d)$. In (d) (to scale), Theorem 1.6 establishes the compactness-free large-degree lower bounds $11/20$ ($d \ge 10^{75}$, white tick) and $2\sqrt{2}/5$ ($d \ge 10^{300\,000}$).

(proved to exist and equal 1 in Theorem 1.2), and write $c^* := \inf_{d\ge 2} d\, D_{\min}(d)$ for the all-degree infimum.

This problem is the direct dual to one of the most celebrated questions in the geometry of complex polynomials. In 1981, Stephen Smale [20] introduced the *mean value problem* while studying the complexity of Newton's method: replacing the maximum in $D(P)$ by the minimum, he asked how large $\min_{P'(c)=0} |P(c)/c|$ can be, proved an upper bound of 4 for every normalized polynomial, and conjectured that 4 could be replaced by $1 - 1/d$ (or 1). Smale included this question in the addenda to the second version of his landmark list *Mathematical problems for the next century* [21, p. 21] (see also [22]), and it has inspired decades of work in complex analysis and potential theory (see [18, 19]). In 2026, Smale's upper-bound conjecture at 1 was disproved in high degree [1, 2], subsequent work raised the lower bound on the universal constant further [12], and the true universal upper constant is shown to be 4 in [7].

Around 2007–2009, Tuen-Wai Ng [15] and, independently, Vladimir Dubinin and Toshiyuki Sugawa [6] posed the *dual mean value conjecture*: for every complex polynomial P of degree $d \ge 2$ and

every regular point a,

$$D(P,a) \geq \frac{1}{d}, \tag{1}$$

with equality if and only if all $d-1$ critical points coincide, i.e., $P(z) = A(z-c)^d + B$ ($A \neq 0$), for which the normalized representative $P(z) = ((1+z)^d - 1)/d$ has $c = -1$ and $|P(c)/c| = 1/d$.

Despite nearly twenty years of active work, two basic barriers remained. First, for arbitrary degree d, every known lower bound decayed strictly faster than $1/d$. Dubinin and Sugawa [6] proved $D(P) \geq 1/(d\,4^d)$ (improved to $> 4^{-d}$ by Ng and Zhang [16]), Dubinin [4] achieved the first polynomial bound $D(P) \geq 1/d^2$ (see also [5] for the outermost lemniscate), and Jatar and Ng [11] refined this to $(d-1/2)^{1/d}/d^2$, still missing the conjectured $1/d$ scaling by a full factor of d. Second, the sharp bound (1) was proved only degree by degree. The cases $d \leq 4$ and conservative polynomials already followed from Tischler's strong inequality [24] (as noted in [6]; Tischler's strong form fails for $d \geq 5$ [25]), while Hinkkanen, Kayumov, and Khammatova proved $d = 5, 6$ [9] and $d = 7$ [10], alongside special classes such as odd polynomials [23] and single-ray critical points [13]; every general degree $d \geq 8$ remained open (Figure 1 and Table 1).

Notation. Throughout, $\mathbb{D} = \{|z| < 1\}$, $\mathbb{T} = \{|z| = 1\}$, and $\mathbb{H} = \{\operatorname{Re} z > 0\}$. Critical points of a polynomial P of degree $d \geq 2$ are counted with multiplicity in P'; we refer to the $n := d-1$ roots of P' (with multiplicity) as *critical copies*. When $P(0) = 0$ and $P'(0) = 1$, the local inverse branch $f = P^{-1}$ with $f(0) = 0$ has a finite maximal radius of convergence $\rho > 0$, called the *first inverse radius*. By properness of P, every singular boundary point v of f on $\{|w| = \rho\}$ is a *first singular target*: the radial lift $t \mapsto f(tv)$ ($0 \leq t < 1$) converges to a critical point c_1 of P with $P(c_1) = v$, called a *first boundary critical point*. Rescaling $P(z) \mapsto \rho^{-1}P(\rho z)$ preserves $P(0) = 0$, $P'(0) = 1$, and $D(P)$, while setting $\rho = 1$; we call this *first inverse units*. We write $\log^+ x = \max(0, \log x)$ and $\log^- x = \max(0, -\log x)$.

Main theorems from the focused version. The first three theorems form the purely analytic, computer-free core of the focused companion [8] and are proved in full in Sections 2–6.

Theorem 1.1 (Large-degree resolution)**.** *There exists an integer d_0 such that for every degree $d \geq d_0$, every complex polynomial P of degree d, and every regular point $a \in \mathbb{C}$ with $P'(a) \neq 0$,*

$$D(P,a) \geq \frac{1}{d},$$

with equality if and only if $P(z) = A(z-c)^d + B$ for some $A \in \mathbb{C} \setminus \{0\}$ and $B, c \in \mathbb{C}$.

Theorem 1.2 (Sharp asymptotic constant and rigidity)**.** *Let $D_{\min}(d) := \inf\{D(P,a) : \deg P = d,\ P'(a) \neq 0\}$. Then*

$$\lim_{d\to\infty} d\, D_{\min}(d) = 1.$$

Moreover, let P_d be any sequence of polynomials of degree $d \to \infty$ with $P_d(0) = 0$, $P_d'(0) = 1$, first inverse radius $\rho_d = 1$, and $d\,D(P_d) \to 1$. Then:

(a) *for every choice of first singular targets $A_d \in \mathbb{T}$,*

$$P_d(z) - A_d\big(1 - e^{-z/A_d}\big) \to 0$$

locally uniformly on $\mathbb{C}$ with all derivatives (so every individual critical distance tends to ∞);

(b) *along every subsequence on which $A_d \to A \in \mathbb{T}$ and the empirical joint law of $(c/d, P_d(c))$ over the $d-1$ critical copies converges weakly to a probability measure μ, we have $\mu = \delta_{(u_0,A)}$ for some $u_0 \in \mathbb{T}$; in particular, along the full sequence, the empirical variance of c/d tends to 0 and the modulus of its centroid tends to 1.*

Indeed, rescaling the equality extremal $((1+z)^d - 1)/d$ to first inverse units ($\rho = 1$) produces $(1 + z/d)^d - 1 = -(1 - (1 + z/d)^d) \to -(1 - e^z)$, which is the case $A = -1$ of Theorem 1.2(a).

Theorem 1.3 (Unconditional all-degree linear bound)**.** *For every integer $d \geq 2$, every complex polynomial P of degree d, and every regular point $a \in \mathbb{C}$ with $P'(a) \neq 0$,*

$$D(P, a) > \frac{2}{5d}.$$

Additional theorems in this expanded version. Sections 7–9 establish three further self-contained, computer-free analytic theorems (summarized in Table 1 and Figure 1).

Theorem 1.4 (Critical circle and uniform annulus; Section 7)**.** *Let P be a complex polynomial of degree $d \geq 2$ with $P(0) = 0$ and $P'(0) = 1$, let $r := \min_{P'(c)=0} |c|$, let ρ be the first inverse radius at 0, and let $R_{\mathrm{hit}} := \min\{|c| : P'(c) = 0,\ |P(c)| \geq \rho\}$.*

(a) *If all critical points of P lie on a common circle $\{|z| = r\}$, then $D(P) \geq 1/d$ in every degree $d \geq 2$, witnessed by every first boundary critical point, with strict inequality whenever two critical points are distinct.*

(b) *If $d \geq 8$ and $R_{\mathrm{hit}}/r \leq 101/100$ (in particular, if $\max|c|/\min|c| \leq 1.01$), then $D(P) \geq 1/d$, with strict inequality whenever two critical points are distinct.*

(c) *For every fixed $1 \leq M < \sqrt{2}$, there exists $d_0(M)$ such that for all $d \geq d_0(M)$, $R_{\mathrm{hit}}/r \leq M$ implies $D(P) \geq 1/d$ (strictly unless all critical points coincide).*

Theorem 1.5 (Ramified multiplicity bound; Section 8)**.** *Let P be a complex polynomial of degree $d \geq 2$ with $P(0) = 0$, $P'(0) = 1$, and first singular target v ($\rho = |v|$, endpoint c_1). Let C_0 be the connected component of $\{|P - v| < \rho\}$ containing c_1 (the* first-value component*), and suppose every critical point of P in C_0 has multiplicity at least $m \geq 1$ in P'. Set $\lambda := 2m/(m+1)$ and*

$$A_m := \frac{4}{25} \cdot \frac{\lambda}{1 + \frac{49}{1250}(\lambda - 1)}.$$

Then, counting critical copies with multiplicity,

$$d \sum_{c \in C_0} \left| \frac{P(c)}{c} \right|^2 > A_m, \qquad \text{and hence} \qquad D(P) > \frac{\sqrt{A_m}}{d}.$$

In particular, if $m \geq 4$, then $A_m \geq 1600/6397 > 1/4$ and $D(P) > 1/(2d)$ in every degree $d \geq 2$.

Theorem 1.6 (Effective large-degree linear improvements; Section 9)**.** *For every complex polynomial P of degree $d \geq 2$ and every regular point $a \in \mathbb{C}$ with $P'(a) \neq 0$:*

(a) *If $d \geq 10^{75}$, then*

$$D(P, a) > \frac{11}{20d}.$$

(b) *For every $\varepsilon \in (0, 1/100]$ and every $d \geq (1/\varepsilon)^{100/\varepsilon}$,*

$$d\, D(P, a) > \sqrt{L(\varepsilon)}, \qquad L(\varepsilon) := \frac{8}{25} \cdot \frac{217}{216}(1 - 4\varepsilon)(1 - 6\varepsilon^2 + 4\varepsilon^3).$$

In particular, $D(P, a) > 2\sqrt{2}/(5d)$ for every $d \geq 10^{300\,000}$, and for all sufficiently large d,

$$d\, D_{\min}(d) > \sqrt{\frac{8}{25} \cdot \frac{217}{216}} - 350\, \frac{\log\log d}{\log d},$$

without any appeal to compactness.

Organization. Sections 2–6 reproduce the five-stage computer-free proof of Theorems 1.1–1.3 (Figure 2). Sections 7–9 present the three additional self-contained analytic theorems specific to this expanded version: the centered-circle and uniform-annulus theorem (Section 7), the ramified Blaschke and multiplicity-four theorem (Section 8), and the explicit effective large-degree linear improvements (Section 9).

Question on $D(P,a)$	Best earlier result	This expanded version
Asymptotic constant $L^* = \lim d\, D_{\min}(d)$	$\liminf \geq 0$ (open whether > 0)	$= 1$, with rigidity (Thm. 1.2); effective $\liminf \geq \sqrt{\frac{8}{25} \cdot \frac{217}{216}}$ (Thm. 1.6)
Sharp bound $D(P,a) \geq 1/d$	$d \leq 7$ [24, 6, 9, 10]; open for all $d \geq 8$	All $d \geq d_0$ (ineffective d_0, Thm. 1.1); centered circle ($d \geq 2$) and 1.01-annulus ($d \geq 8$) (Thm. 1.4)
All-degree lower bound ($d \geq 2$)	$(d - \frac{1}{2})^{1/d}/d^2$ [11] (after $1/d^2$ [4])	$> 2/(5d)$, i.e., $c^* \geq 2/5$ (Thm. 1.3); $> 1/(2d)$ for mult. ≥ 4 (Thm. 1.5)
Effective large-degree bound	$(d - \frac{1}{2})^{1/d}/d^2$ [11]	$> 11/(20d)$ for $d \geq 10^{75}$; $> 2\sqrt{2}/(5d)$ for $d \geq 10^{300\,000}$ (Thm. 1.6)

TABLE 1. Comparison with the best earlier results for the dual Smale mean value conjecture.

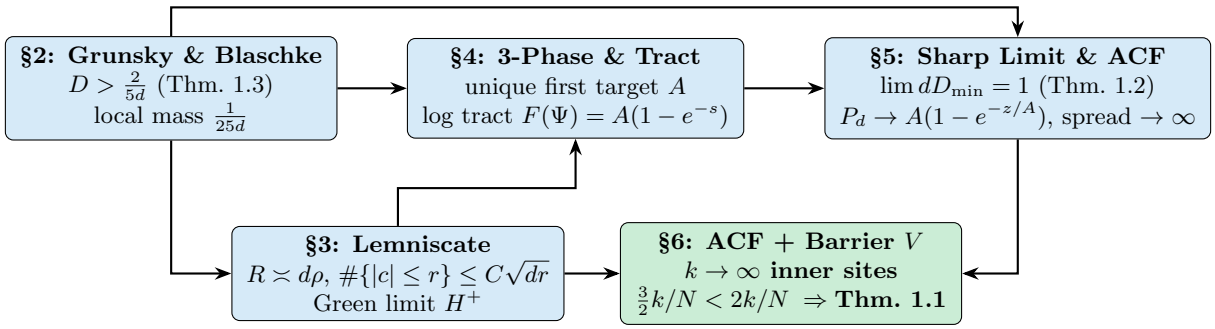

FIGURE 2. Logical architecture of the focused core (Sections 2–6). Sections 7–9 are self-contained additions establishing Theorems 1.4–1.6.

2. Two-sided Green estimate and the all-degree linear bound

2.1. **A two-sided Green estimate for disjoint conformal discs.** Let $\Omega \subset \mathbb{C}$ be the interior of a real-analytic Jordan curve Γ, and let Ω^* be its exterior on the Riemann sphere. For $\xi \in \Omega$, let $G_\Omega(\cdot, \xi)$ be the positive logarithmic Green function of Ω with pole at ξ ($G_\Omega(z,\xi) = \log|z-\xi|^{-1} + O(1)$ as $z \to \xi$), and let $G_{\Omega^*}(\cdot, \infty)$ be the exterior Green function with pole at ∞ ($G_{\Omega^*}(z,\infty) = \log|z| + O(1)$). At $b \in \Gamma$, write $P_i(b,\xi) := \partial_{n_i} G_\Omega(b,\xi)$ and $P_o(b,\infty) := \partial_{n_o} G_{\Omega^*}(b,\infty)$ for the magnitudes of the inward and outward normal derivatives (so on $\mathbb{D}$ with $\xi = 0$, $P_i = P_o = 1$).

Lemma 2.1 (Two-sided Green bound)**.** *For every real-analytic Jordan domain Ω, every $\xi \in \Omega$, and every $b \in \partial\Omega$,*

$$P_i(b,\xi)\, P_o(b,\infty) \leq \frac{1}{|b-\xi|^2}.$$

Proof. Let $\varphi : \mathbb{D} \to \Omega$ be conformal with $\varphi(0) = \xi$, and let $\psi : \mathbb{D} \to \Omega^*$ be conformal with $\psi(0) = \infty$. Fix $W \in \Omega^*$ and expand for $|z| < 1$:

$$-\log \frac{W - \varphi(z)}{W - \xi} = \sum_{n=1}^{\infty} p_n(W)\, \frac{z^n}{n},$$

taking the branch vanishing at $z = 0$. Each $p_n(W)$ is a polynomial in $(W-\xi)^{-1}$ vanishing at ∞. On $\mathbb{D}$, for $|z| < |\zeta| < 1$,

$$-\log \frac{\varphi(\zeta) - \varphi(z)}{\varphi(\zeta) - \xi} = -\log\Big(1 - \frac{z}{\zeta}\Big) - \log \frac{(\varphi(\zeta) - \varphi(z))/(\zeta - z)}{(\varphi(\zeta) - \xi)/\zeta},$$

where the second logarithm is holomorphic on $\mathbb{D}\times\mathbb{D}$; hence $p_n(\varphi(\zeta)) = \zeta^{-n} + h_n(\zeta)$ with h_n holomorphic on $\mathbb{D}$.

Fix $z \in \mathbb{D}$ and put $S_N(W) := \sum_{n=1}^{N} p_n(W)\, z^n/n$. On $|\zeta| = t < 1$, the Laurent expansion of $S_N \circ \varphi$ has negative-power coefficients $c_{-n} = z^n/n$ $(1 \le n \le N)$ and nonnegative-power coefficients c_l. Since S_N is holomorphic outside $\varphi(\{|\zeta| \le t\})$ and vanishes at ∞, Green's formula (integrating clockwise around the boundary) gives

$$\frac{1}{\pi}\int_{\mathbb{C}\setminus\varphi(\{|\zeta|\le t\})} |S_N'(W)|^2\, dA(W) = \sum_{n=1}^{N} n|c_{-n}|^2 t^{-2n} - \sum_{l=1}^{\infty} l|c_l|^2 t^{2l} \le \sum_{n=1}^{N} \frac{|z|^{2n}}{n}\, t^{-2n}.$$

Since $\Omega^* \subset \mathbb{C}\setminus\varphi(\{|\zeta|\le t\})$, letting $t\uparrow 1$ and using conformal invariance of the Dirichlet integral yields $\frac{1}{\pi}\int_{\mathbb{D}} |(S_N\circ\psi)'|^2\, dA \le \sum_{n=1}^{N} |z|^{2n}/n$. Because $(S_N\circ\psi)(0) = 0$, Cauchy–Schwarz on its Taylor series at $v \in \mathbb{D}$ gives

$$|S_N(\psi(v))| \le \sqrt{\Big(\sum_{n=1}^{N} \frac{|z|^{2n}}{n}\Big) \log\frac{1}{1-|v|^2}}.$$

Letting $N\to\infty$ gives the mixed Grunsky inequality

$$\left|\log\frac{\psi(v)-\varphi(z)}{\psi(v)-\xi}\right| \le \sqrt{\log\frac{1}{1-|z|^2}\,\log\frac{1}{1-|v|^2}} \le \frac{1}{2}\log\frac{1}{(1-|z|^2)(1-|v|^2)},$$

whence $|\psi(v)-\varphi(z)|/|\psi(v)-\xi| \ge \sqrt{(1-|z|^2)(1-|v|^2)}$. Both φ and ψ extend conformally across $\mathbb{T}$ because $\partial\Omega$ is real-analytic. Choose $e_i, e_o \in \mathbb{T}$ with $\varphi(e_i) = \psi(e_o) = b$, and set $z = e_i(1-\varepsilon/|\varphi'(e_i)|)$, $v = e_o(1-\varepsilon/|\psi'(e_o)|)$ as $\varepsilon\downarrow 0$. Then $|\psi(v)-\varphi(z)|/|\psi(v)-\xi| \le (2\varepsilon+o(\varepsilon))/|b-\xi|$, while the right-hand side is $2\varepsilon/\sqrt{|\varphi'(e_i)|\,|\psi'(e_o)|} + o(\varepsilon)$. Thus $|b-\xi|^2 \le |\varphi'(e_i)|\,|\psi'(e_o)|$. Since $G_\Omega(\varphi(z),\xi) = -\log|z|$ and $G_{\Omega^*}(\psi(v),\infty) = -\log|v|$, we have $P_i(b,\xi) = 1/|\varphi'(e_i)|$ and $P_o(b,\infty) = 1/|\psi'(e_o)|$, proving the lemma. □

2.2. **Tangent-disc heights and the sharp Blaschke critical-product bound.** Now normalize $P(0) = 0$, $P'(0) = 1$, and let v be any first singular target of P^{-1} with $\rho := |v|$ and critical endpoint c_1. For any $s > 1/2$, consider the open tangent disc $\Delta_s := \{w \in \mathbb{C} : |w - sv| < s\rho\}$ (which has $0 \in \partial\Delta_s$ and contains the segment $(0, v]$), and let U_s be the connected component of $P^{-1}(\Delta_s)$ having 0 on its boundary. Since P is a polynomial, U_s is simply connected (by the maximum principle) and $P : U_s \to \Delta_s$ is proper of some degree $m \in [2, d]$, containing c_1. The map

$$W(z) := \frac{sv}{P(z)} - \frac{1}{2}$$

maps U_s properly onto $\mathbb{H}$, with a simple pole at $z = 0$. Choose a conformal map $\xi : U_s \to \mathbb{H}$ sending 0 to ∞ with the normalization $W(z)/\xi(z) \to 1$ as $z \to 0$ (so $\xi(z) = sv/z + O(1)$ as $z \to 0$). For each $z \in U_s$, write $x(z) := \operatorname{Re}\xi(z) > 0$.

Corollary 2.2 (Kernel-to-secant bound)**.** *For every $s > 1/2$ and every $z \in U_s$,*

$$\frac{\rho^2}{|z|^2} \ge \frac{2\,x(z)}{d\,s^2}, \qquad \textit{and hence} \qquad \left|\frac{P(z)}{z}\right|^2 \ge \frac{2\,x(z)}{d}\left|\frac{P(z)}{s\rho}\right|^2.$$

Proof. Near $z = 0$, P is conformal on a small disc $B(0,\delta_0)$. For regular levels $r \uparrow s\rho$ of $|P - sv|$, let $b_r := P^{-1}((s\rho - r)v/\rho) \in B(0,\delta_0)$ (via the local inverse near 0), so that $|P(b_r) - sv| = r$ and $(P(b_r) - sv)/r = -v/\rho$, and let $U^{(r)}$ be the component of $\{|P - sv| < r\}$ adjoining b_r. Then $U^{(r)}$ is a real-analytic Jordan domain, $U^{(r)} \uparrow U_s$, and $b_r \to 0$ with $|P'(b_r)| \to 1$. Outside $U^{(r)}$, the subharmonic function $\log(|P - sv|/r) - d\,G_{(U^{(r)})^*}(\cdot,\infty)$ vanishes on $\partial U^{(r)}$ and is bounded above

at ∞, so it is ≤ 0 everywhere; comparing outward normal derivatives at b_r gives $P_o^{(r)}(b_r,\infty) \ge |P'(b_r)|/(dr)$. By Lemma 2.1,

$$P_i^{(r)}(b_r,z)\,\frac{|P'(b_r)|}{dr} \le \frac{1}{|b_r-z|^2}.$$

Fix a small half-disc $D_\varepsilon = \{\zeta \in \mathbb{D} : |\zeta + v/\rho| < \varepsilon\}$ whose image under $w_r(\zeta) := P^{-1}(sv + r\zeta)$ stays in a neighborhood $B(0,\delta_1) \subset B(0,\delta_0)$ disjoint from z for r near $s\rho$. Since $U^{(r)} \subset U_s$, domain monotonicity gives $0 \le g_r(\zeta) := G_{U^{(r)}}(w_r(\zeta),z) \le G_{U_s}(w_r(\zeta),z) \le \sup_{U_s\cap B(0,\delta_1)} G_{U_s}(\cdot,z) < \infty$ uniformly on D_ε, and $g_r = 0$ on $\partial D_\varepsilon \cap \mathbb{T}$. For each fixed $\zeta \in D_\varepsilon$, $w_r(\zeta) \to w_{s\rho}(\zeta) \in U_s \setminus \{z\}$ while $G_{U^{(r)}}(\cdot,z) \uparrow G_{U_s}(\cdot,z)$ locally uniformly on $U_s \setminus \{z\}$, so $g_r(\zeta) \to g_{s\rho}(\zeta)$. Extending g_r by odd Schwarz reflection across $\mathbb{T}$ yields uniformly bounded harmonic functions converging pointwise on $\{|\zeta + v/\rho| < \varepsilon\}$, hence with all derivatives on compact subsets; by the chain rule, $\frac{r}{|P'(b_r)|}P_i^{(r)}(b_r,z) = \partial_{n_\zeta} g_r(-v/\rho) \to \partial_{n_\zeta} g_{s\rho}(-v/\rho) = s\rho\, P_i(0,z)$, so $P_i^{(r)}(b_r,z) \to P_i(0,z)$. On $\mathbb{H}$, $G_{\mathbb{H}}(\xi,\xi(z)) = \log|(\xi + \overline{\xi(z)})/(\xi - \xi(z))| = 2x(z)\,\mathrm{Re}(1/\xi) + O(|\xi|^{-2})$ as $\xi \to \infty$; since the inward normal to U_s at 0 is v/ρ and $\xi(u) \sim sv/u$ as $u \to 0$, we obtain $P_i(0,z) = 2x(z)/(s\rho)$. Letting $r \uparrow s\rho$ yields $2x(z)/(ds^2\rho^2) \le 1/|z|^2$, proving both inequalities. □

Lemma 2.3 (Sharp Blaschke critical-product bound)**.** *Let $B : \mathbb{D} \to \mathbb{D}$ be a finite Blaschke product of degree ≥ 2, and let C be a finite Blaschke product whose zeros in $\mathbb{D}$ are precisely the critical points of B with multiplicity. Write $\beta(z) := (1-|z|^2)|B'(z)|/(1-|B(z)|^2)$. Then*

$$\beta(z) \ge \frac{2|C(z)|}{1+|C(z)|^2} \qquad (z \in \mathbb{D}). \tag{2}$$

In particular, if $B(0) = 0$ and the local inverse of B at 0 continues along $[0,\sigma)$ to a critical point over $\sigma \in (0,1)$, then the critical copies u_j of B in $\mathbb{D}$ satisfy

$$\prod_j |u_j|^2 \le \sigma. \tag{3}$$

Proof. Set $h := 2|C|/(1+|C|^2)$ and $u := \log\beta - \log h$. Because B'/C is holomorphic and non-vanishing on $\mathbb{D}$, u extends smoothly across the critical points of B. Writing $B(z) = \lambda\prod_{k=1}^m (z-a_k)/(1-\overline{a}_k z)$ ($|\lambda| = 1$, $|a_k| < 1$), for every $\zeta \in \mathbb{T}$ we have $|B'(\zeta)| = \sum_{k=1}^m (1-|a_k|^2)/|\zeta - a_k|^2 > 0$, and expanding $1-|B(z)|^2$ in elementary factors gives $(1-|B(z)|^2)/(1-|z|^2) \to |B'(\zeta)|$ as $z \to \zeta$; hence both β and h extend continuously to 1 on $\mathbb{T}$. Away from the zeros of C, direct computation of the Laplacian of $\log(1-|z|^2)$ and $\log(1-|B|^2)$ gives

$$\Delta\log\beta = \frac{4(\beta^2-1)}{(1-|z|^2)^2}, \qquad \Delta\log h = -\frac{4|C'|^2}{(1+|C|^2)^2} \ge \frac{4(h^2-1)}{(1-|z|^2)^2},$$

where the inequality uses Schwarz–Pick $|C'| \le (1-|C|^2)/(1-|z|^2)$ and $1-h^2 = (1-|C|^2)^2/(1+|C|^2)^2$. Subtracting yields $\Delta u \le 4(\beta^2-h^2)/(1-|z|^2)^2$, which extends continuously across the zeros of C. On $\{u < 0\}$ we have $\beta < h$, so u is superharmonic there with boundary limit 0; the minimum principle forces $u \ge 0$ everywhere, proving (2).

For (3), put $b := |B'(0)|$ and $r := \sqrt{\sigma}$. We claim $b \le 2r/(1+r^2)$. Indeed, $g(z) := B(z)/z$ maps $\mathbb{D}$ into $\overline{\mathbb{D}}$ with $|g(0)| = b$. If $b > 2r/(1+r^2)$, then Schwarz–Pick makes g zero-free on $\{|z| \le r\}$ with $|g(z)| \ge (b-r)/(1-br)$ on $\{|z| = r\}$, whence $|B(z)| \ge r(b-r)/(1-br) > \sigma$ on $\{|z| = r\}$; by Rouché's theorem, B takes every value in $\{|w| \le \sigma\}$ (and a slightly larger disc) once and simply in $\{|z| < r\}$, continuing the inverse past σ, a contradiction. By (2) at $z = 0$, $2p/(1+p^2) \le b \le 2r/(1+r^2)$ where $p = |C(0)| = \prod_j |u_j|$, so $p \le r = \sqrt{\sigma}$. □

Proposition 2.4 (Localized first-hit mass)**.** *Let $P(0) = 0$, $P'(0) = 1$, and let v be a first singular target with $\rho = |v|$ and critical endpoint c_1. For any $\delta \in (0,1)$, let V_δ be the component of $\{|P - v| <$*

$\delta\rho\}$ containing c_1. Counting critical copies with multiplicity in P',

$$\sum_{c\in V_\delta}\left|\frac{P(c)}{c}\right|^2 > \frac{4\delta(1-\delta)^2}{3(1+\delta)(4-\delta)\,d}. \tag{4}$$

In particular, at $\delta = 1/4$, $\sum_{c\in V_{1/4}} |P(c)/c|^2 > 1/(25d)$.

Proof. Set $s := (1+\delta)/2$ and $t := s - 1/2 = \delta/2$. Then $\{|w - v| < \delta\rho\} \subset \Delta_s$, so $V_\delta \subset U_s$. In the ξ-coordinate, the marked inverse $F := \xi \circ W^{-1}$ of $W \circ \xi^{-1}$ is defined on $\{\operatorname{Re} w > t\}$ (since then $P = sv/(w + 1/2)$ lies in $\{|P| < \rho\} \cap \Delta_s$) and maps it into $\mathbb{H}$ with $F(w) = w + O(1)$ as $w \to +\infty$. Applying Schwarz–Pick to $z \mapsto F(t + z)$ on $\mathbb{H}$ with a second point tending to $+\infty$ gives $\operatorname{Re} F(w) \geq \operatorname{Re} w - t$. For $\sigma = 1/4$, the marked preimage of $(1 - \sigma\delta)v$ has W-value $w_* = s/(1 - \sigma\delta) - 1/2$ and $\mathbb{H}$-coordinate $\xi_* := F(w_*)$, whose height $X := \operatorname{Re}\xi_*$ obeys $X \geq w_* - t = s\sigma\delta/(1-\sigma\delta) = s\delta/(4-\delta)$.

Since V_δ is simply connected and $P : V_\delta \to \{|w - v| < \delta\rho\}$ is proper, let $\phi : \mathbb{D} \to V_\delta$ be conformal with $\phi(0)$ equal to this marked preimage. Post-composing $(P \circ \phi/v - 1)/\delta$ with the automorphism $q \mapsto (q + \sigma)/(1 + \sigma q)$ gives a finite Blaschke product B with $B(0) = 0$ whose radial inverse hits a critical point at $\sigma = 1/4$. Let u_j be the critical copies of B in $\mathbb{D}$ and $x_j := x(\phi(u_j))$ their heights in U_s. By Lemma 2.3, $\prod_j |u_j| \leq \sqrt{\sigma} = 1/2$. On the other hand, Schwarz–Pick for $\xi \circ \phi : \mathbb{D} \to \mathbb{H}$ gives

$$|u_j| \geq \left|\frac{\xi(\phi(u_j)) - \xi_*}{\xi(\phi(u_j)) + \overline{\xi_*}}\right| \geq \frac{|X - x_j|}{X + x_j}.$$

Put $Y := \sum_j x_j/X$. If $Y < 1$, then $\prod_j (1 - x_j/X)/(1 + x_j/X) \geq (1 - Y)/(1 + Y)$ (by superadditivity of artanh on $[0, 1)$), so $(1 - Y)/(1 + Y) \leq \sqrt{\sigma}$ and hence

$$\sum_{c\in V_\delta} x_c \geq X\,\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}}. \tag{5}$$

For $Y \geq 1$, (5) is trivial. With $\sigma = 1/4$, $(1 - \sqrt{\sigma})/(1 + \sqrt{\sigma}) = 1/3$, so $\sum_{V_\delta} x_c \geq s\delta/[3(4 - \delta)]$. On V_δ we have $|P(c)| > (1 - \delta)\rho$, so summing Corollary 2.2 gives

$$\sum_{c\in V_\delta}\left|\frac{P(c)}{c}\right|^2 > \frac{2}{d}\cdot\frac{(1-\delta)^2}{s^2}\cdot\frac{s\delta}{3(4-\delta)} = \frac{4\delta(1-\delta)^2}{3(1+\delta)(4-\delta)\,d},$$

which equals $1/(25d)$ at $\delta = 1/4$. □

2.3. **Proof of Theorem 1.3.** To boost the constant from $1/5$ to $2/5$, we replace the inner disc by a half-disc in the W-plane, sharpen the lower bound for X with a rational comparison domain, and integrate over the modulus cutoff h (Figure 3). We isolate the elementary rational positivity check as a standalone lemma.

Lemma 2.5 (Explicit rational positivity)**.** *For $x \in (0, 1)$, set $A := 8 - 3x^2$, $s := A/8$, $t := s - 1/2$, $T := s/x^2 - 1/2$, $w := s/x - 1/2$,*

$$\sigma := \frac{w-t}{w+t}\cdot\frac{T^2+tw}{T^2-tw}, \qquad H := \frac{2}{sx(1+x)}\Big(1 + \frac{t}{w+1/3}\Big).$$

Then $0 < \sigma < 1$, $H > 1$, and

$$H\,\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}} > \frac{12}{25} \qquad (0 < x < 1). \tag{6}$$

Proof. Direct substitution gives $0 < t < w < T$, so $0 < \sigma < 1$, and $s \le 1$, $x(1+x) < 2$ give $H > 1$. Clearing fractions, $H = n_h/d_h$ and $\sigma = n_s/d_s$, where

$$\begin{aligned} n_h &= 16[3A(1+x) - 16x], & d_h &= Ax(1+x)(3A-4x), \\ b &= (A-4x^2)^2, & e &= (A-4)(A-4x)x^3, \\ n_s &= A(1-x)(b+e), & d_s &= [A(1+x)-8x](b-e), \end{aligned}$$

with $d_h, d_s > 0$ on $(0,1)$. Since $H > 1 > 12/25$, (6) is equivalent to $(25n_h - 12d_h)^2 d_s - n_s(25n_h + 12d_h)^2 > 0$. Factoring out $32\,xA(1-x)$ (strictly positive on $(0,1)$) yields $32\,xA(1-x)\,n(x)$, where $n(x)$ is a primitive degree-17 polynomial with positive constant term. Under $x = y/(1+y)$ ($y > 0$), the polynomial $(1+y)^{17} n(y/(1+y))$ has exact integer coefficients (from y^0 to y^{17}):

$$\begin{aligned} &7372800,\ 137625600,\ 1120735232,\ 5311864832,\ 16340546048,\ 34169743360,\\ &48985449152,\ 46608536192,\ 26046649808,\ 4353192768,\ -3506415528,\ -623233648,\\ &2320073300,\ 1944574760,\ 713222245,\ 138205862,\ 13259420,\ 441800. \end{aligned}$$

Only degrees 10 and 11 are negative. For $0 < y \le 1$, the degree-9 term dominates both negative terms: $4353192768\,y^9 > (3506415528 + 623233648)\,y^{10}$. For $y \ge 1$, degrees 12 and 13 dominate both: $2320073300\,y^{12} + 1944574760\,y^{13} > 3506415528\,y^{10} + 623233648\,y^{11}$. Hence $n(x) > 0$ on $(0,1)$, and therefore $(25n_h - 12d_h)^2 d_s - n_s(25n_h + 12d_h)^2 = 32\,xA(1-x)\,n(x) > 0$ throughout $(0,1)$, proving (6). □

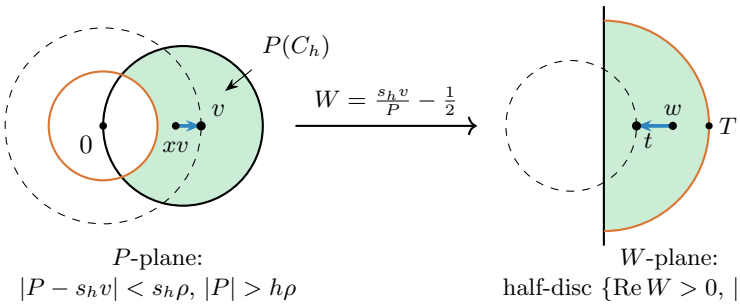

FIGURE 3. Geometry of Theorem 1.3 in the P- and W-planes. The green region $\{|P - s_h v| < s_h\rho,\ |P| > h\rho\}$ contains the preimage of the right half-disc $\{\mathrm{Re}\,W > 0,\ |W| < T\}$; the dashed circles mark $\{|P| = \rho\}$ (left) and its image $\{|W + \frac{1}{2}| = s_h\}$ (right), and the blue arrow shows the radial segment from xv to the first singular target v (mapped from w to t in W).

Proof of Theorem 1.3. Normalize $P(0) = 0$, $P'(0) = 1$, choose a first singular target v with $\rho = |v|$ and endpoint c_1, and for $0 \le h < 1$ set $s_h := 1 - 3h/8$. Let C_h be the component containing c_1 of $\{|P - s_h v| < s_h\rho,\ |P| > h\rho\}$. We prove the stronger layer bound

$$d \sum_{c \in C_h} \left| \frac{P(c)}{c} \right|^2 > \frac{4}{25}(1 - h^3) \qquad (0 \le h < 1). \tag{7}$$

At $h = 0$, C_0 contains $N \le d - 1$ critical copies, so (7) yields $D(P) > 2/(5\sqrt{dN}) > 2/(5d)$.

Fix $0 < h < 1$ and set $x := \sqrt{h} \in (0,1)$, $s := s_h$, $t := s - 1/2$, $T := s/h - 1/2$, and $w := s/x - 1/2$. In the right-half-plane coordinate ξ of U_s, $W = sv/P - 1/2$ maps U_s properly onto $\mathbb{H}$ with $W(\xi) = \xi + O(1)$ at ∞. Let $V \subset U_s$ be the preimage component of the half-disc $\{\mathrm{Re}\,W > 0,\ |W| < T\}$ containing the tip over $W = t$. On this half-disc $|W + 1/2| < T + 1/2 = s/h$, so $|P| > h\rho$ and hence

$V \subset C_h$. The radial lift from $P = xv$ to $P = v$ runs along $[t, w]$ in V, which is simply connected and maps properly onto the half-disc.

The conformal map $U(W) := 1/W - W/T^2$ maps the half-disc onto $\mathbb{H}$, sending $t < w$ to $U(t) > U(w) > 0$, whose pseudohyperbolic distance in $\mathbb{H}$ is the σ of Lemma 2.5. Uniformizing V from $\mathbb{D}$ with 0 at the marked preimage of w (whose height in U_s is X), Lemma 2.3 and the Schwarz–Pick argument of (5) give

$$\sum_{c \in V} x_c \geq X \, \frac{1 - \sqrt{\sigma}}{1 + \sqrt{\sigma}}. \tag{8}$$

To bound X from below, note that in the ξ-coordinate, the marked inverse $F := \xi \circ W^{-1}$ maps $\Omega_s := \{\operatorname{Re} z > 0,\ |z+1/2| > s\}$ into $\mathbb{H}$ with $F(z) = z + O(1)$ at $+\infty$. Set $J(z) := z - t(t+1/3)/(z+1/3)$. The quadratic inverse G of J selected by $G(u) = u + O(1/u)$ at $+\infty$ is holomorphic on $\mathbb{H}$ (its branch points lie on $\operatorname{Re} u = -1/3$). On the imaginary axis, $\operatorname{Re} J(iy) = -t(t+1/3)/(3(y^2+1/9)) < 0$; on the arc $|z + 1/2| = s$ with $z_1 = \operatorname{Re} z \in [0, t]$, we have $|z|^2 = t^2 + t - z_1$ and therefore

$$\left| z + \frac{1}{3} \right|^2 \operatorname{Re} J(z) = (z_1 - t)\Big(\frac{t - z_1}{3} + \frac{1}{9}\Big) \leq 0.$$

Hence $G(\mathbb{H}) \subset \Omega_s$ by connectedness (as $G(u) \in \Omega_s$ for large $u > 0$ and $J(\partial\Omega_s) \subset \{\operatorname{Re} u \leq 0\}$), so $F \circ G : \mathbb{H} \to \mathbb{H}$ has unit slope at $+\infty$. By Julia's lemma, $\operatorname{Re} F(G(u)) \geq \operatorname{Re} u$; evaluating at $u = J(w) > 0$ gives $X \geq J(w) = (w - t)(1 + t/(w + 1/3))$.

Combining Corollary 2.2, (8), and Lemma 2.5 (noting $2J(w)/[s^2(1-h)] = H$) shows that the cumulative reciprocal-square mass $M(h) := d \sum_{c \in C_h} \rho^2/|c|^2$ satisfies

$$M(h) > \frac{12}{25}(1 - h) \qquad (0 < h < 1). \tag{9}$$

The domains C_h are decreasingly nested as h increases (the tangent discs shrink about 0, the lower cutoff $h\rho$ rises, and all contain c_1). Therefore, for any $0 \leq b < 1$, each copy $c \in C_b$ contributes to $M(h)$ on a subinterval of $[b, |P(c)|/\rho)$, so

$$d \sum_{c \in C_b} \left| \frac{P(c)}{c} \right|^2 \geq b^2 M(b) + \int_b^1 2h \, M(h) \, dh > \frac{12}{25}\Big(b^2(1-b) + \int_b^1 2h(1-h)\, dh\Big) = \frac{4}{25}(1 - b^3),$$

proving (7) and Theorem 1.3. □

3. Macroscopic critical-point geometry and the filled lemniscate

Throughout this section we normalize $P(0) = 0$ and $P'(0) = 1$, and write $n := d - 1$, $R := \max_j |c_j|$, $M := \max_j |P(c_j)|$, $X := dD(P)$, and $G := (\prod_{j=1}^n |c_j|^{-1})^{1/n}$.

Proposition 3.1 (Lemniscate projection and farthest-radius scale)**.** *For every degree $d \geq 2$, with $K := 4(2X)^{1/n}$,*

$$X \geq \frac{(RG/4)^n}{2} \qquad \text{and} \qquad \frac{1}{4K^{25X^2}} \leq \frac{d\rho}{R} \leq X. \tag{10}$$

In particular, in first inverse units ($\rho = 1$) along any sequence with $X = dD$ bounded, $R \asymp d$.

Proof. For any $T > M$, every component of $\{|P| < T\}$ is bounded, simply connected, and maps properly onto $\{|w| < T\}$ with degree equal to its number of zeros and critical-copy count equal to that degree minus one. Summing over all components gives $n = d -$ (number of components), so $\{|P| < T\}$ is connected and contains a path from 0 to a farthest critical point $c_{\max}$ with $|c_{\max}| = R$. Rotate so that $c_{\max} = R > 0$, project all d zeros ζ_k of P onto $\mathbb{R}$, and set $Q(x) := \prod_{k=1}^d (x - \operatorname{Re} \zeta_k)$. For every $x \in [0, R]$ there is z on the path with $\operatorname{Re} z = x$, so $|Q(x)| \leq \prod |z - \zeta_k| = (d/G^n)|P(z)| < (d/G^n)T$ (since the leading coefficient of P has modulus G^n/d). By Chebyshev's minimal-deviation

theorem on $[0,R]$, $\max_{[0,R]}|Q| \geq 2(R/4)^d$. Letting $T \downarrow M$ and using $M \leq XR/d$ gives $X \geq (RG/4)^n/2$, so $RG \leq K$.

For the second inequality in (10), put $x_j := d\rho/|c_j|$ and $x_{\min} := d\rho/R$. Any first singular endpoint c_1 has $\rho = |P(c_1)| \leq D|c_1| \leq DR$, so $x_{\min} \leq X$. For the lower bound, (9) at $h = 1/2$ gives $d\sum_{c\in C_{1/2}} \rho^2/|c|^2 > 6/25$. Every $c \in C_{1/2}$ has $\rho/2 < |P(c)| \leq D|c|$, hence $x_c < 2X$. Let f be the fraction of all n copies with $1/4 < x_j < 2X$. Using $d\sum \rho^2/|c|^2 = d^{-1}\sum x_c^2$, we get $6/25 < (n/d)(1/16 + 4X^2 f)$, forcing $f > 1/(25X^2)$. If $x_{\min} < 1/4$, these copies contribute at least $f\log(1/(4x_{\min}))$ to $n^{-1}\sum \log(x_j/x_{\min}) = \log(RG) \leq \log K$, giving $x_{\min} \geq 1/(4K^{25X^2})$; if $x_{\min} \geq 1/4$, this holds since $K \geq 1$. □

Proposition 3.2 (Radial Lagrange interpolation and square-root counting)**.** *Write $y_j := |c_j|/R$. For every $s > 0$,*

$$\sum_{j=1}^{n} \log^+ \frac{s}{y_j} \leq \log \frac{X}{ds} + d\,\operatorname{arcosh}(1+4s) \leq \log \frac{X}{ds} + d\sqrt{8s}. \tag{11}$$

Consequently, for any continuum $K \subset \mathbb{C}$ with $\operatorname{diam} K > 0$ and any polynomial Q of degree d,

$$\max_{x\in K} |Q'(x)| \leq \frac{2\cosh(2)\, d^2}{\operatorname{diam} K} \max_{z\in K} |Q(z)|. \tag{12}$$

Moreover, if $X = dD$ stays bounded, then:

(a) *in first inverse units ($\rho = 1$), there is $C = C(X)$ such that $\#\{j : |c_j| \leq r\} \leq C\sqrt{dr}$ for every $r > 0$;*

(b) *(scale-invariantly, with no unit normalization required) the empirical logarithms $\log(|c_j|/R)$ are uniformly integrable, and if the empirical measures σ_d of c_j/R converge weakly to σ, then $\sigma(\{|z| \leq t\}) \leq 4\sqrt{t}/\log 2$ for all $t > 0$ and $n^{-1}\sum \log(|c_j|/R) \to \int \log|z|\, d\sigma(z)$.*

Proof. For $T > M$, pick a path in $\{|P| < T\}$ from 0 to a farthest critical point of modulus R, and let T_d denote the degree-d Chebyshev polynomial ($T_d(x) = \cosh(d\operatorname{arcosh} x)$ for $x \geq 1$). Choose nodes z_j on the path with $|z_j| = t_j := R(1 - \cos(j\pi/d))/2$ ($0 \leq j \leq d$). By Lagrange interpolation and the reverse triangle inequality, for $|z| \leq u$,

$$|P(z)| \leq T \sum_{j=0}^{d} \prod_{k\neq j} \frac{u + t_k}{|t_j - t_k|} = T\, T_d\Big(1 + \frac{2u}{R}\Big),$$

where the identity follows by evaluating the Lagrange basis of $T_d(1 - 2x/R)$ at $x = -u$ (its values at t_j are $(-1)^j$). Letting $T \downarrow M$ and applying Cauchy's estimate on the disc of radius sR centered at any z with $|z| = sR$ gives $|P'(z)| \leq MT_d(1+4s)/(sR) \leq XT_d(1+4s)/(ds)$. Averaging $\log|P'|$ over $\{|z| = sR\}$ yields $\sum_{j=1}^n \log^+(s/y_j)$ by Jensen's formula and $P'(0) = 1$. Since $T_d(x) \leq \exp(d\operatorname{arcosh} x)$ and $\operatorname{arcosh}(1+4s) \leq \sqrt{8s}$, this proves (11).

For (12), given $x \in K$, pick $y \in K$ with $|y - x| \geq (\operatorname{diam} K)/2$, translate x to 0, and by connectedness of K choose nodes $z_j \in K$ with $|z_j - x| = t_j$ (with $R_K := |y - x|$ in place of R) and repeat the same radial Lagrange interpolation to get $|Q(z)| \leq \|Q\|_K\, T_d(1 + 2|z - x|/|y - x|)$; Cauchy's estimate on $|z - x| = |y - x|/d^2$ gives $|Q'(x)| \leq \|Q\|_K\, T_d(1 + 2/d^2)\, d^2/|y - x| \leq 2\cosh(2)\, d^2\|Q\|_K/\operatorname{diam} K$ (using $\operatorname{arcosh}(1 + 2/d^2) \leq 2/d$).

For (a), put $s = 2r/R$ in (11): for $r \geq 1$, every copy with $|c_j| \leq r$ contributes at least $\log 2$, while $R \asymp d$ (Proposition 3.1) bounds $\log(X/(ds)) \leq C$ and $d\sqrt{8s} \leq C\sqrt{dr}$. By Koebe's quarter theorem applied to the normalized inverse $f : \mathbb{D} \to \mathbb{C}$, every critical point satisfies $|c_j| \geq 1/4$, so for $r < 1/4$ the count is 0, and for $1/4 \leq r < 1$ it is bounded by the $r = 1$ count $\leq C\sqrt{d} \leq 2C\sqrt{dr}$. For (b), applying (11) at $s = 2t$ gives $\limsup n^{-1}\#\{y_j \leq t\} \leq 4\sqrt{t}/\log 2$; hence $\sigma(\{|z| < t'\}) \leq \liminf \sigma_d(\{|z| < t'\}) \leq 4\sqrt{t'}/\log 2$ for every $t' > t$, and letting $t' \downarrow t$ yields $\sigma(\{|z| \leq t\}) \leq 4\sqrt{t}/\log 2$.

On $\{y_j \le t\}$, $n^{-1}\sum_{y_j\le t}(-\log y_j) = n^{-1}\sum \log^+(t/y_j) + \log(1/t)\, n^{-1}\#\{y_j \le t\}$, whose lim sup is at most $\sqrt{8t} + (4/\log 2)\sqrt{t}\log(1/t) \to 0$ as $t \downarrow 0$, proving uniform integrability and convergence of the log potential at 0. □

Proposition 3.3 (Filled lemniscate and exterior Green limit)**.** *Normalize $P_d(0) = 0$, $P_d'(0) = 1$, and $\max_c |c| = 1$. Suppose $d \to \infty$, $dD(P_d) \le Q$, and the empirical measure σ_d of the $d-1$ critical copies converges weakly to σ. Put $V(z) := \int \log|z-u|\, d\sigma(u)$, $\ell := V(0)$, and $H(z) := V(z) - \ell$. Then for every fixed $B > Q$ and $\tau_d := B/d$, the functions $g_d := d^{-1}\log^+(|P_d|/\tau_d)$ converge locally uniformly on $\mathbb{C}$ to H^+.*

Proof. Set $K_d := \{|P_d| \le \tau_d\}$; by Proposition 3.1 each K_d is connected and contains 0 and a point of modulus 1. Let $\ell_d := \int \log|u|\, d\sigma_d(u) \to \ell$ (Proposition 3.2(b)). The leading coefficient of P_d has modulus $e^{-(d-1)\ell_d}/d \ge 1/d$. Applying Chebyshev projection along a path between any two points of K_d at distance S gives $\tau_d \ge (2/d)(S/4)^d$, so $\operatorname{diam} K_d \le 4(B/2)^{1/d}$. For any $x \in K_d$, one of 0 or the unit-modulus point lies at distance $\ge 1/2$ from x; radial Lagrange interpolation from x yields $g_d(z) \le \sqrt{8\,\operatorname{dist}(z, K_d)}$ uniformly on $\mathbb{C}$.

Pass to a Hausdorff-convergent subsequence $K_d \to K$. Off K, g_d is positive harmonic and locally bounded, hence converges locally uniformly along a further subsequence to a continuous function g on $\mathbb{C}$ that vanishes on K and is harmonic off K (so $g = 0$ on every bounded complementary component of K). Since all zeros of P_d lie in K_d, $g(z) = \log|z| - \ell + O(1/|z|)$ as $z \to \infty$, so g is the Green function of the unbounded complementary component Ω_∞ of K. On compact subsets of Ω_∞, $2\partial g_d = P_d'/(dP_d) \to 2\partial g \ne 0$ (since all critical points lie in K_d, by Hurwitz and growth at ∞), whence $(d-1)^{-1}\log|P_d'| \to g$. Since $(d-1)^{-1}\log|P_d'(z)| = \int \log|z-u|\, d\sigma_d(u) - \ell_d$, weak convergence gives $g = H$ on Ω_∞. On the filled hull $\widehat{K} = \mathbb{C}\setminus\Omega_\infty$, averaging $\log|P_d'|$ over $|y-z| = s$ (via $\operatorname{avg}_{|y-z|=s}\log|y-u| = \log\max(s, |z-u|)$) and applying Cauchy's estimate on $B(y,s)$ (which gives $|P_d'(y)| \le (\tau_d/s)\exp(d\max_{|w-z|\le 2s} g_d(w))$) yields $\int \log\max(s,|z-u|)\, d\sigma(u) \le \ell + \max_{|w-z|\le 2s} g(w)$; letting $s \downarrow 0$ gives $H \le 0$ on $\widehat{K}$. Thus $g = H^+$ everywhere, independent of the subsequence, giving full-sequence convergence. □

4. Coalescence of first targets and the logarithmic tract

4.1. **Multiphase Dirichlet-energy monotonicity.** We collect the planar two-phase and three-phase monotonicity inequalities (in the spirit of Alt–Caffarelli–Friedman [3]) in one reusable lemma. For a nonnegative, continuous, locally Lipschitz subharmonic function w on a disc $B_r = B(z_0, r)$, write $I_w(r) := \int_{B_r} |\nabla w|^2\, dA$.

Lemma 4.1 (Planar multiphase monotonicity)**.** *Let $w_1, \dots, w_k \ge 0$ ($k \ge 2$) be continuous, locally Lipschitz, subharmonic functions on $B(z_0, R)$ with pairwise disjoint positive sets ($w_i w_j = 0$ for $i \ne j$).*

(a) Flux and cutoff bounds. *For a.e. $r \in (0, R)$, $I_{w_i}(r) \le \int_{\partial B_r} w_i\, \partial_r w_i\, ds = \frac{r}{2} m_i'(r)$ where $m_i(r) := \int_0^{2\pi} w_i(z_0 + re^{i\vartheta})^2\, d\vartheta$. For $r \le R/2$, $I_{w_i}(r) \le C \operatorname{avg}_{B_{2r}} w_i^2$ with an absolute constant C. Moreover, if $w_1(y) \ge a > 0$ for some $y \in B_{r/4}$ and w_2 is positive somewhere in $B_{r/4}$, then $I_{w_1}(r) \ge c\, a^2$ with an absolute constant $c > 0$.*

(b) Multiphase growth. *If $[r_1, r_2] \subset (0, R)$ and $I_{w_i}(r_1) > 0$ for every i, and $\theta_i(r)$ denotes the angular length of $\{w_i > 0\}$ on ∂B_r, then $r I_{w_i}'(r)/I_{w_i}(r) \ge 2\pi/\theta_i(r)$ for a.e. r, and consequently*

$$\frac{d}{dr}\log\prod_{i=1}^k I_{w_i}(r) \ge \frac{k^2}{r} \qquad \text{for a.e. } r \in [r_1, r_2], \qquad \prod_{i=1}^k \frac{I_{w_i}(r_2)}{I_{w_i}(r_1)} \ge \Big(\frac{r_2}{r_1}\Big)^{k^2}.$$

(c) ACF mean-square bound ($k = 2$). *If $k = 2$ and near z_0 we have $w_i = (h_i)_+$ where h_1, h_2 are harmonic with $h_i(z_0) = 0$ and opposite nonzero gradients $\nabla h_1(z_0)$, $\nabla h_2(z_0)$ supported on opposite sides of a smooth curve through z_0, then $I_{w_1}(r)I_{w_2}(r)/r^4$ is nondecreasing on $(0, R)$ with limit $(\pi/2)^2|\nabla h_1(z_0)|^2|\nabla h_2(z_0)|^2$ as $r \downarrow 0$, and for every $r \in (0, R)$ (and at $r = R$ if w_1, w_2 extend continuously to $\overline{B(z_0, R)}$),*

$$\int_0^{2\pi} \big(w_1^2 + w_2^2\big)(z_0 + re^{i\vartheta})\, d\vartheta \geq \pi r^2 |\nabla h_1(z_0)|\, |\nabla h_2(z_0)|.$$

Proof. (a) Since $w \geq 0$ is subharmonic and locally Lipschitz, $\Delta(w^2) \geq 2|\nabla w|^2$ distributionally, giving $I_w(r) \leq \int_{\partial B_r} w\partial_r w\, ds = \frac{r}{2}m'(r)$ by Green's formula and $I_w(r) \leq C \operatorname{avg}_{B_{2r}} w^2$ by testing against a smooth radial cutoff supported in B_{2r}. If $w_1(y) \geq a$ and w_2 is positive at some point in $B_{r/4}$, the submean property for w_1^2 gives $\operatorname{avg}_{B_{r/2}} w_1^2 \geq c_1 a^2$, so every circle mean of w_1^2 on $[r/2, r]$ is $\geq c_1 a^2$; by the maximum principle w_2 attains a positive value on every circle ∂B_s ($s \in [r/2, r]$), so w_1 vanishes at some point on ∂B_s, and the angular Poincaré inequality $\int_{\partial B_s} w_1^2 \leq 4s^2 \int_{\partial B_s} |\nabla_{\tan} w_1|^2$ integrated over $[r/2, r]$ gives $I_{w_1}(r) \geq ca^2$.

(b) Since $k \geq 2$ and $I_{w_j}(r_1) > 0$ for $j \neq i$, the maximum principle forces $w_j > 0$ somewhere on ∂B_r, so w_i vanishes at some point of ∂B_r and every component of $\{w_i > 0\} \cap \partial B_r$ is a proper open arc with zero endpoints. On each such arc of angular length $\alpha \leq \theta_i(r)$, Wirtinger's inequality gives $\int w_i^2 \leq (\alpha r/\pi)^2 \int |\nabla_{\tan} w_i|^2$. By Cauchy–Schwarz and $2ab \leq a^2 + b^2$,

$$I_{w_i}(r) \leq \int_{\partial B_r} w_i |\partial_r w_i| \leq \frac{\theta_i(r) r}{2\pi} \int_{\partial B_r} \big(|\partial_r w_i|^2 + |\nabla_{\tan} w_i|^2\big) = \frac{\theta_i(r) r}{2\pi} I'_{w_i}(r).$$

Since $\sum_{i=1}^k \theta_i(r) \leq 2\pi$, Cauchy–Schwarz gives $\sum_{i=1}^k 2\pi/\theta_i(r) \geq k^2$, yielding (b).

(c) Near z_0, $w_i(z_0 + re^{i\vartheta}) = r|\nabla h_i(z_0)|(\cos(\vartheta - \vartheta_i))_+ + O(r^2)$, so $I_{w_i}(r) = (\pi/2)|\nabla h_i(z_0)|^2 r^2 + O(r^3)$. By (b) with $k = 2$, $I_{w_1}(r)I_{w_2}(r)/r^4$ increases from $(\pi/2)^2|\nabla h_1(z_0)|^2|\nabla h_2(z_0)|^2$. Thus $(r/2)(m_1 + m_2)'(r) \geq I_{w_1}(r) + I_{w_2}(r) \geq 2\sqrt{I_{w_1}(r)I_{w_2}(r)} \geq \pi r^2 |\nabla h_1(z_0)||\nabla h_2(z_0)|$. Dividing by $r/2$ and integrating over $[0, R]$ gives (c). □

4.2. **Nontangential escape and three-phase target coalescence.** Work now in first inverse units ($\rho_d = 1$, $F_d(0) = 0$, $F_d'(0) = 1$) with $dD(F_d) \leq J < \infty$. Let $f_d : \mathbb{D} \to \mathbb{C}$ be the normalized univalent inverse branch ($f_d(0) = 0$, $f_d'(0) = 1$). Passing to a subsequence (by compactness of normalized univalent functions and Proposition 3.1), we have $f_d \to f$ locally uniformly on $\mathbb{D}$, and the joint empirical law of $(u, v) = (c/d, F_d(c))$ converges weakly to a compactly supported probability measure μ. Let $S \subset \mathbb{T}$ be the compact set of boundary points across which f admits no local conformal continuation.

Proposition 4.2 (Nontangential escape at boundary singularities)**.** *For every $A \in S$, $f(z) \to \infty$ as $z \to A$ in every fixed Stolz cone. In particular, S is nonempty (every Hausdorff limit point of first singular targets of f_d lies in S).*

Proof. Fix $A \in S$ and $t \in (0, 1)$; set $w := tA$, $\tau := 1 - t$, and $b_d := f_d(w)$. The Taylor radius r_d of the inverse of F_d at w through b_d satisfies $r_d \geq \tau$; we claim $r_d \to \tau$. Otherwise along a subsequence the inverse continues univalently to $D(w, \tau + \varepsilon)$, and schlicht compactness produces a conformal continuation of f across A, contradicting $A \in S$. Let v_d be any first singular target of this recentered inverse ($|v_d - w| = r_d$). Since f_d is analytic on $\mathbb{D}$, $v_d \notin \mathbb{D}$, so $r_d \to \tau$ forces $v_d \to A$.

Apply Proposition 2.4 ($\delta = 1/4$) at the shifted mark b_d (with derivative $1/f_d'(w)$):

$$\frac{1}{25d} < \sum_{|F_d(c) - v_d| < r_d/4} \frac{|F_d(c) - w|^2\, |f_d'(w)|^2}{|c - b_d|^2}.$$

On this target disc $|F_d(c)| \geq 1 - \tau/4 - o(1)$, so $|c|/d \geq (1-\tau/4)/(2J)$ stays bounded away from 0, while $b_d/d \to 0$ and $f_d'(w) \to f'(tA)$. Passing to the weak limit μ gives

$$\frac{1}{25} \leq |f'(tA)|^2 \int_{|v-A|\leq \tau/4} |v - tA|^2 |u|^{-2}\, d\mu(u,v) \leq |f'(tA)|^2 \cdot \Big(\frac{5\tau}{4}\Big)^2 \cdot \Big(\frac{J}{1-\tau/4}\Big)^2.$$

Hence $\tau |f'(tA)| \geq 3/(25J)$ for all $t \in (0,1)$. In any fixed Stolz cone ending at A, Koebe's distortion theorem compares $(1-|z|^2)|f'(z)|$ from below to a positive multiple of $(1-t^2)|f'(tA)|$, and Koebe's quarter theorem then keeps $f(z)$ at a uniformly positive distance from $\partial f(\mathbb{D})$. Thus no subsequence of $f(z)$ can converge to a finite point (which would lie in $f(\mathbb{D})$, contradicting $|z| \to 1$), so $f(z) \to \infty$. Conversely, every limit of first singular targets of f_d satisfies the same estimate along $[0, A)$, hence lies in S. □

Proposition 4.3 (Three-phase coalescence of first targets)**.** *Keep first inverse units with $dD(F_d) \leq J$. Suppose b_1, b_2 are two (possibly distinct) regular marks with $|b_i|$ and $|w_i|$ (where $w_i := F_d(b_i)$) bounded and $|F_d'(b_i)|, t_i$ bounded above and away from 0, such that the inverse branch through b_i is univalent on $D(w_i, t_i)$. Let $v_i \in \partial D(w_i, t_i)$ satisfy $|v_i| \geq a > 0$ and $|v_1 - v_2| \geq \delta > 0$, with radial endpoints z_i. Then for all sufficiently large d,*

$$\min(|z_1|, |z_2|) \leq C\sqrt{d}\,(\log d)^5, \tag{13}$$

where C depends only on J, a, δ and the uniform bounds on $b_i, w_i, F_d'(b_i), t_i$. Consequently, if both v_i are first singular targets, they must coalesce ($|v_1 - v_2| \to 0$); in particular $|S| = 1$.

Proof. Set $Q := F_d'$ and $R := \min(ad/[100(J+1)], |z_1|/8, |z_2|/8)$; we show $R \leq C\sqrt{d}(\log d)^5$ (we may assume $R \to \infty$). On B_{2R} (writing $B_r := B(0,r)$) every critical point satisfies $|F_d(c)| \leq a/50$, and $|z_1|, |z_2| \geq 8R$.

We construct three disjoint subharmonic phases on B_{2R}. Write $\xi_i := (F_d - w_i)/t_i$ and $e_i := (v_i - w_i)/t_i \in \mathbb{T}$. On the i-th inverse sheet $\varphi_i(D(w_i, t_i))$, set $H_i := [\mathrm{Re}((e_i + \xi_i)/(e_i - \xi_i)) - T]_+$ and extend by 0 elsewhere; by properness of the polynomial F_d and Carathéodory's theorem (the boundary of the algebraic domain $\varphi_i(D(w_i, t_i))$ is locally connected), φ_i extends continuously to $\overline{D(w_i, t_i)}$ with unique boundary point z_i over v_i, and since z_i lies outside B_{2R}, H_i is continuous, locally Lipschitz, and subharmonic on B_{2R}. Put

$$\Phi := \log^+\big|(2/a^4)\, Q(F_d - v_1)^2 (F_d - v_2)^2\big|.$$

Then $\Phi(0) \geq \log 2$. On the i-th sheet, Koebe distortion gives $|Q| \leq 8|Q(b_i)|/(1-|\xi_i|)$, while $H_i > 0$ gives (via $\mathrm{Re}\,\frac{e_i+\xi_i}{e_i-\xi_i} = \frac{1-|\xi_i|^2}{|e_i-\xi_i|^2} > T$) $|F_d - v_i| \leq 2t_i/T$ and $|F_d - v_i|^2 \leq 2t_i^2(1-|\xi_i|)/T$; fixing T sufficiently large makes H_1, H_2, Φ have pairwise disjoint positive sets with $\{H_i > 0\} \subset \{|F_d - v_i| < \delta/8\}$. Testing at $\xi_i = te_i$ gives a fixed r_0 such that H_1, H_2, Φ are each bounded below by a fixed positive constant inside $B_{r_0/4}$.

To switch phases when H_i grows too large, given a radius S_0 and parameters $h, M > 0$ with $hS_0 \leq M$, put $X := F_d - zQ$ (so $X' = -zQ'$), $Z := Q/h$, $u := -\log|Z|$, and define on $\{|Z| < 1\} \cap B_{S_0}$

$$N_i(z; h, M) := \Big[u - \frac{u + 1 - |Z|}{32} \sum_{\tau \in \{\pm 1, \pm i\}} \exp\Big(\frac{\mathrm{Re}(\tau(X - v_i))}{5M}\Big)\Big]_+,$$

extended by 0 on $\{|Z| \geq 1\}$. Putting $\psi_0(r) := -\log r + 1 - r$ and $f_\tau := \mathrm{Re}(\tau(X - v_i))/M$, we have $|\nabla f_\tau| = |zQ'|/M \leq |Z'|$, $\Delta\psi_0(|Z|) = -|Z'|^2/|Z|$, and $|Z|\psi_0(|Z|) \leq 1$, so off $\{Q = 0\}$

$$\Delta\big(e^{f_\tau/5}\psi_0(|Z|)\big) \leq \frac{e^{f_\tau/5}|Z'|^2}{|Z|}\Big(-1 + \frac{2(1+|Z|)}{5} + \frac{|Z|\psi_0(|Z|)}{25}\Big) \leq 0.$$

Thus N_i is subharmonic off $\{Q = 0\}$, and hence across $\{Q = 0\}$ whenever N_i vanishes in a neighborhood of the zeros of Q. Where $N_i > 0$, we have $|X - v_i| \leq 5\sqrt{2}M \log 32$ and hence

$|F_d - v_i| < 26M$; for $M \le m := \min(a, \delta, 1)/1000$ this excludes all critical points in B_{2R} (where $|F_d(c)| \le a/50$). Thus N_i is continuous, locally Lipschitz, and subharmonic on B_{S_0}, obeys $0 \le N_i \le \log^+(h/|Q|)$, and satisfies $N_i \ge (\log 2)/2$ whenever $0 < |Q| \le h/2$ and $|X - v_i| \le M$. Set $m_0 := m/100$, $h_1 := m/(2R)$, and $N_i^{\mathrm{fin}} := N_i(\cdot; h_1, m)$ on B_{2R}; note $\Phi = 0$ on $\{N_i^{\mathrm{fin}} > 0\}$ for large R.

By Proposition 3.2(a), $\#\{|c| \le s\} \le C\sqrt{ds}$ for all $s > 0$, so $\sum \log(1 + 2R/|c|) \le C\sqrt{dR}$. Thus $\log^+ |Q|$ and $\log^+ |F_d|$ are $O(\sqrt{dR})$ on B_{2R}, and the normalized area L^2-norm $(\mathrm{avg}_{B_R}(\log^- |Q|)^2)^{1/2}$ is also $O(\sqrt{dR})$ (at most $C\sqrt{dR}$ copies in B_{2R} each have bounded rescaled L^2-norm, while distant copies contribute $\le 2R/|c|$, summing to $O(\sqrt{dR})$ by partial summation from Proposition 3.2(a)). By Lemma 4.1(a), $I_\Phi(R/16)$ and $I_{N_i^{\mathrm{fin}}}(R/16)$ are at most CdR.

Set $L := \log d$. For each $i \in \{1, 2\}$, follow H_i up to $R/4$ until the first radius $\rho \ge r_0$ where $\max_{\overline{B}_\rho} H_i \ge L^6$. If no hit occurs, keep H_i on $[r_0, R/16]$ (energy $\le CL^{12}$). If a hit occurs, at some $z^* \in \overline{B}_\rho$ we have $|F_d(z^*) - v_i| \le 2t_i/L^6$. Follow the straight target segment from $F_d(z^*)$ to v_i on the i-th branch until reaching radius 2ρ (possible since $|z_i| \ge 8R$); the source arc has length $\ge \rho$ and target length $\le 2t_i/L^6$, giving $y \in \overline{B}_{2\rho}$ with $0 < |Q(y)| \le C/(\rho L^6)$ and $|F_d(y) - v_i| \le C/L^6$.

- If $\rho \ge R/L^4$, then $|Q(y)| \le h_1/2$ and $|X(y) - v_i| < m$, so $N_i^{\mathrm{fin}}(y) \ge (\log 2)/2$; discard $[\rho/2, 8\rho]$ (stopping at $\rho/2$ if $8\rho > R/16$) and switch from H_i to N_i^{fin} (using continuity of N_i^{fin} at $y \in \overline{B}_{2\rho}$ to apply Lemma 4.1(a) on $B_{8\rho}$).
- If $\rho < R/L^4$, introduce the intermediate phase $N_i^{\mathrm{int}} := N_i(\cdot; m_0/S_1, m_0)$ on B_{S_1} with $S_1 := \rho L^4$ (note $\Phi = 0$ on $\{N_i^{\mathrm{int}} > 0\}$ for large d). Then $N_i^{\mathrm{int}}(y) \ge (\log 2)/2$, so $I_{N_i^{\mathrm{int}}}(8\rho) \ge c$ by Lemma 4.1(a) (using Φ). On $[8\rho, S_1/8]$, $rI'/I \ge 1$ (since $\Phi > 0$ somewhere on each circle by the maximum principle, forcing the disjoint phase N_i^{int} to vanish there), so $I_{N_i^{\mathrm{int}}}(S_1/8) \ge cS_1/\rho = cL^4$, forcing $\max_{\overline{B}_{S_1/4}} N_i^{\mathrm{int}} \ge c'L^2 > 2\log d$. At the first radius $\sigma \in [8\rho, S_1/4]$ where N_i^{int} hits $2\log d$, we have $|Q| \le (m_0/S_1)d^{-2} \le h_1/2$ (since $R = O(d)$ and $S_1 d^2 \gg d$) and $|X - v_i| < m$, triggering $N_i^{\mathrm{fin}} \ge (\log 2)/2$. We discard $[\rho/2, 8\rho]$ and $[\sigma/2, 8\sigma]$ (stopping at $\sigma/2$ if $8\sigma > R/16$; if $\sigma \le 16\rho$, the discarded intervals overlap and we switch directly from H_i to N_i^{fin}), and switch successively from H_i (energy $\le CL^{12}$) to N_i^{int} (energy $\le CL^2$) to N_i^{fin} on the remaining subintervals.

The discarded intervals have total logarithmic length $O(1)$. On the remaining subset of $[r_0, R/16]$, three disjoint active phases have energies starting at $\ge c > 0$. By Lemma 4.1(b) ($k = 3$) and telescoping each function's monotone energy,

$$9\log R - O(1) \le 3\log(dR) + 2(12\log L + 2\log L) + O(1),$$

giving $6\log R \le 3\log d + 28\log\log d + O(1)$, i.e., $R \le C\sqrt{d}(\log d)^{14/3}$. This proves (13).

If both v_i are first singular targets, both endpoints are critical with $|z_i| \ge ad/J$, impossible for large d. For $A_1, A_2 \in S$, recentering at $w_i = A_i/2$ as in Proposition 4.2 gives $t_i \to 1/2$ and $v_i \to A_i$, forcing $A_1 = A_2$. Thus $|S| = 1$. □

4.3. **The twice-punctured covering and logarithmic tract.** Let Ω be the connected component containing 0 of the open set where F_d converges locally uniformly along our subsequence to a finite holomorphic function F.

Proposition 4.4 (Covering property and logarithmic tract)**.** *Let A be the unique point of S from Proposition 4.3. Then Ω is simply connected and unbounded, $F(0) = 0$, $F'(0) = 1$, and*

$$F : \Omega \setminus F^{-1}(\{0, A\}) \longrightarrow \mathbb{C} \setminus \{0, A\}$$

is a surjective unbranched covering. Moreover, the component of $\{0 < |F - A| < 1\}$ in Ω containing the marked radial lift $f(tA)$ $(0 < t < 1)$ is parametrized injectively by $\mathbb{H}$ via a holomorphic map

$\Psi : \mathbb{H} \to \Omega$ *satisfying*

$$F(\Psi(s)) = A(1 - e^{-s}), \qquad \Psi(x) = f\big((1 - e^{-x})A\big) \quad (x > 0),$$

and Ψ *extends holomorphically across* $s = 0$ *with* $\Psi(0) = 0$ *and* $|\Psi'(0)| = 1$.

Proof. By Rouché's theorem on compact discs inside $f(\mathbb{D})$, $f(\mathbb{D}) \subset \Omega$ with $F \circ f = \mathrm{id}$, so $F(0) = 0$, $F'(0) = 1$, and Ω is unbounded (Proposition 4.2). By the maximum principle on pairwise differences $F_d - F_{d'}$, Ω is simply connected. Every critical point of F in Ω is a limit of critical points of F_d (by Hurwitz), hence has value 0 since $|F_d(c)| \le J|c|/d \to 0$.

Take any $y \in \Omega$ with $W := F(y) \in \mathbb{C} \setminus \{0, A\}$. Then $F'(y) \neq 0$, so local inverses h_d of F_d near W through $y_d \to y$ (paired with the mark $f_d(A/2)$ targeting A) satisfy the hypotheses of Proposition 4.3, with first inverse radii bounded away from 0 by Rouché's theorem and bounded above by $|W| + O(1)$ because Proposition 3.1 bounds all critical values $|F_d(c)| \le DR = O(1)$. Hence every limit of their first singular targets must be 0 or A, so the first inverse radii of h_d have $\liminf \ge \operatorname{dist}(W, \{0, A\})$. By schlicht compactness and Rouché's theorem, the inverse germ of F at y continues univalently over $D(W, \operatorname{dist}(W, \{0, A\}))$ with image inside Ω. Lifting arbitrary paths in $\mathbb{C} \setminus \{0, A\}$ from $f(\mathbb{D}) \setminus \{0\}$ gives surjectivity, and discs of radius $\operatorname{dist}(W, \{0, A\})/4$ are evenly covered, proving the covering property.

Restricting this covering to $D(A, 1) \setminus \{A\}$, the marked component T over $0 < |F - A| < 1$ is either a finite cyclic cover $F(T_0(\xi)) = A(1 - \xi^m)$ ($\xi \in \mathbb{D} \setminus \{0\}$) or the universal cover $F(\Psi(s)) = A(1 - e^{-s})$ ($s \in \mathbb{H}$). In the finite-degree case, injectivity of T_0 excludes an essential singularity at $\xi = 0$ (Casorati–Weierstrass), nontangential escape $f(tA) \to \infty$ excludes a removable singularity, and a pole would place a neighborhood of ∞ inside simply connected Ω, forcing $\Omega = \mathbb{C}$ and boundedness of F near ∞, contradicting $F'(0) = 1$. Hence T is the universal cover, yielding Ψ with $\Psi(x) = f((1 - e^{-x})A)$ for $x > 0$; since f is conformal near 0 with $f(0) = 0$, $f'(0) = 1$, Ψ extends holomorphically across $s = 0$ with $\Psi(0) = 0$ and $|\Psi'(0)| = 1$. □

5. Sharp asymptotic constant and rigidity

Proof of Theorem 1.2, part 1: $\lim_{d\to\infty} d\, D_{\min}(d) = 1$. Shifted powers $P(z) = ((1 + z)^d - 1)/d$ give $d\, D_{\min}(d) \le 1$. Conversely, take any subsequence with $d\, D(F_d) \to q < \infty$ in first inverse units (with first singular targets $A_d \to A$), and extract $f_d \to f$, μ, Ω, F, A, and Ψ as in Propositions 4.2–4.4. We prove

$$1 \le \int_{v=A} |u|^{-2}\, d\mu(u, v) \le q^2\, \mu(\{v = A\}), \tag{14}$$

which immediately yields $q \ge 1$.

Fix $\delta \in (0, 1/2)$ and put $S := \log(1/\delta)$. Choose regular levels $r_d \uparrow 1$ of $|F_d - A_d|$, put $b_d := f_d((1 - r_d)A_d) \to 0$, and let U_d be the adjacent component of $\{|F_d - A_d| < r_d\}$. Let $V_d \subset U_d$ be the component of $\{|F_d - A_d| < \delta\}$ containing $y_0 := \Psi(S + 1)$ (for large d, $y_0 \in V_d \subset U_d$ by local convergence along the radial lift). Every critical copy $c \in K_d := \{c \in V_d : F_d'(c) = 0\}$ has $|F_d(c)| \ge 1 - \delta$, hence $|c| \ge (1 - \delta)/D$. Set

$$u_d(z) := \sum_{c \in K_d} G_{U_d}(z, c).$$

First, we claim that the Carathéodory kernel of V_d at y_0 is $T_S := \Psi(\{\operatorname{Re} s > S\})$ along every further subsequence. Every compact subset of T_S connected to y_0 eventually lies in V_d by convergence on Ω. Conversely, on any kernel domain U containing y_0, $|F_d| \le 1 + \delta$, so normality identifies the limit along a further subsequence with F on $U \subset \Omega$ (applying Proposition 4.4 to that subsequence, whose tract parametrization agrees with Ψ on the real ray and hence everywhere). Since $U \setminus F^{-1}(A)$ is connected and maps into $D(A, \delta) \setminus \{A\}$, it lies in T_S. Indeed, F cannot take the value A in U: every critical point of F in Ω has value 0 (Proposition 4.4), so any preimage $z_* \in U$ of A would

be simple, and a punctured disc $B(z_*,\varepsilon)\setminus\{z_*\}$ would form a one-sheeted cover of $D(A,\eta)\setminus\{A\}$ inside the connected infinite-sheeted universal cover T_S, a contradiction; hence $U\subset T_S$. Since V_d is simply connected, omits 0, and contains a fixed disc about y_0, Carathéodory's kernel theorem [17, Theorem 1.8] gives convergence of the conformal radii of V_d to those of T_S.

On V_d, $(F_d-A_d)/\delta$ in disc coordinates is a finite Blaschke product with critical copies K_d; let β_d denote the hyperbolic derivative of $(F_d-A_d)/\delta : V_d\to\mathbb{D}$. By Lemma 2.3, $\beta_d\ge 2|C|/(1+|C|^2)\ge|C|$ (trivially also if $K_d=\emptyset$, where $\beta_d\equiv 1$), and $G_{U_d}\ge G_{V_d}$ gives $u_d\ge -\log\beta_d$. For any $Y_d\to\Psi(x)$ ($x>S$ real), kernel convergence gives $\beta_d(Y_d)\to(x-S)/\sinh(x-S)$ (the hyperbolic derivative of $s\mapsto e^{-(s-S)}$ from $\{\operatorname{Re}s>S\}$ to $\mathbb{D}$), so

$$\liminf_{d\to\infty} u_d(Y_d)\ge\log\frac{\sinh(x-S)}{x-S}. \tag{15}$$

Next, fix $x>S$ and $L>x$. Because Ψ extends holomorphically across $s=0$ and maps $\{|s-L|\le L\}\setminus\{0\}$ into Ω where $F'\neq 0$, Rouché's theorem transfers Ψ to holomorphic maps Ψ_d on a neighborhood of $\{|s-L|\le L\}$ satisfying $F_d(\Psi_d(s))=A_d(1-r_de^{-s})$ and $\Psi_d\to\Psi$, with $\Psi_d(0)=b_d$ and $|\Psi_d'(0)|=r_d/|F_d'(b_d)|\to 1$. For large d, $\Psi_d(\{|s-L|<L\})\subset U_d\setminus K_d$ (it is bounded while K_d is at distance $\asymp d$) and $\Psi_d(x)\in V_d$. Thus $h_d(s):=u_d(\Psi_d(s))$ is positive harmonic on $\{|s-L|<L\}$ and vanishes at 0. Mapping $\{|s-L|<L\}$ onto $\mathbb{H}$ via $s\mapsto s/(1-s/(2L))$ and applying Harnack's inequality between $\varepsilon\downarrow 0$ and x yields

$$h_d'(0)\ge\Big(1-\frac{x}{2L}\Big)\frac{h_d(x)}{x}.$$

Because $|F_d(\Psi_d(s))-A_d|=r_de^{-\operatorname{Re}s}$, the real ray $s>0$ is normal to ∂U_d at b_d, so $h_d'(0)=|\Psi_d'(0)|\sum_{c\in K_d}P_i(b_d,c)$. By exterior comparison on F_d-A_d, $P_o(b_d,\infty)\ge|F_d'(b_d)|/(dr_d)$. Lemma 2.1 gives $P_i(b_d,c)\le(dr_d/|F_d'(b_d)|)|c-b_d|^{-2}$, whence

$$\Big(1-\frac{x}{2L}\Big)\frac{h_d(x)}{x}\le h_d'(0)\le d\Big(\frac{r_d}{|F_d'(b_d)|}\Big)^2\sum_{c\in K_d}\frac{1}{|c-b_d|^2}. \tag{16}$$

Taking $d\to\infty$ with (15), then $L\to\infty$, then $x\to\infty$ at fixed δ, we obtain

$$\liminf_{d\to\infty}\partial_{\mathrm{in}}u_d(b_d)\ge 1\qquad\text{and}\qquad\int_{|v-A|\le\delta}|u|^{-2}\,d\mu(u,v)\ge 1. \tag{17}$$

Letting $\delta\downarrow 0$ gives the first inequality in (14); the second follows from $|v|\le q|u|$ on the support of μ. □

Proof of Theorem 1.2, part 2: angular rigidity and exponential limit. Assume now that $q=1$, i.e., $d\,D(F_d)\to 1$. By (14) and $|v|\le|u|$ on the support of μ, we have $v=A$ and $|u|=1$ μ-a.e. Let σ be the source marginal of μ (supported on $\mathbb{T}$). Fix $\delta\in(0,1/2)$ and retain r_d,b_d,U_d,u_d from above.

Step 1: Angular rigidity ($\sigma=\delta_{u_0}$). Define macroscopic phases on $\mathbb{C}$:

$$p_d(z):=d^{-1}\log^+\frac{|F_d(dz)-A_d|}{r_d},\qquad n_d(z):=d^{-1}u_d(dz)\ \text{ on } U_d/d\ \text{(zero outside)}.$$

Near b_d/d (away from K_d/d, which has radius $\ge(1-\delta)/2$), p_d,n_d satisfy Lemma 4.1(c) with slopes $a_d:=|F_d'(b_d)|/r_d\to 1$ and $k_d:=\partial_{\mathrm{in}}u_d(b_d)$ ($\liminf k_d\ge 1$ by (17)). Integrating Lemma 4.1(c) over $B(b_d/d,t)$ (fixed small $t>0$) gives

$$\operatorname{avg}_{B(b_d/d,t)}\big(p_d^2+n_d^2\big)\ge\frac{a_dk_dt^2}{4}. \tag{18}$$

Because $\overline{U_d}$ is a continuum containing b_d and a pole of u_d at distance $\asymp d$, with $|F_d-A_d|\le 1$, the continuum Markov bound (12) gives $\sup_{U_d}|F_d'|\le Cd$. Thus $\log(Cd)-\log|F_d'|$ is nonnegative superharmonic on U_d with at least the logarithmic poles of u_d, so $0\le n_d(z)\le[d^{-1}\log(Cd)-d^{-1}\log|F_d'(dz)|]^+$. By Proposition 3.2(b) (uniform integrability of $\log|c/R|$), $d^{-1}\log|F_d'(dz)|\to$

$H(z) := \int \log|z-u|\,d\sigma(u) - \int \log|u|\,d\sigma(u)$ in local area L^2, so the majorant of n_d converges in L^2 to H^-. Let $R_d := \max_c |c| \asymp d$ (Proposition 3.1). Since $|A_d| = 1$, shifting $F_d(dz)$ by $-A_d$ and replacing r_d by a fixed $B_0 > 2\sup(R_d/d)$ changes $d^{-1}\log^+$ by $O(1/d)$, and rescaling from R_d to d cancels identically in H; thus Proposition 3.3 gives $p_d \to H^+$ locally uniformly.

Passing to the lim sup in (18) yields $\operatorname{avg}_{B(0,t)} H^2 \geq t^2/4$. Since σ is supported on $\mathbb{T}$, H is harmonic near 0 with $H(0) = 0$ and $|\nabla H(0)| = |\int u^{-1}\,d\sigma(u)|$, so $\operatorname{avg}_{B(0,t)} H^2 = |\nabla H(0)|^2 t^2/4 + O(t^3)$. Letting $t \downarrow 0$ forces $|\int_{\mathbb{T}} u^{-1}\,d\sigma(u)| \geq 1$, so $\sigma = \delta_{u_0}$ for a single $u_0 \in \mathbb{T}$. Because $u_0 \in \mathbb{T}$ has modulus 1 and zero variance on every extracted subsequence (with compact macroscopic support from Proposition 3.1), the empirical variance of c/d tends to 0 and the modulus of its centroid tends to 1 along the full sequence.

Step 2: Physical-scale energy saturation ($F_d(z) \to A(1-e^{-z/A})$ on $\mathbb{C}$). Now define the *unscaled* physical phases $\tilde{p}_d := \log^+(|F_d - A_d|/r_d)$ and $\tilde{n}_d := u_d$ on U_d (zero outside), with energies $I_{\tilde{p}_d}(s), I_{\tilde{n}_d}(s)$ on $B(b_d, s)$ and $J_d(s) := I_{\tilde{p}_d}(s) I_{\tilde{n}_d}(s)/s^4$. By Lemma 4.1(c), $J_d(s) \geq (\pi^2/4) a_d^2 k_d^2$. For any fixed $s > 0$ and small fixed $t > 0$, integrating $(r/2)(m_{\tilde{p}_d} + m_{\tilde{n}_d})' \geq 2r^2\sqrt{J_d(s)}$ from s to $T = td$ and averaging over $B(b_d, T)$ gives

$$\frac{1}{d^2}\operatorname{avg}_{B(b_d,td)}(\tilde{p}_d^2 + \tilde{n}_d^2) \geq \frac{\sqrt{J_d(s)}}{2\pi}\, t^2\Big(1 - \frac{s^2}{t^2 d^2}\Big)^2.$$

The left-hand side has $\limsup_{d\to\infty} \leq \operatorname{avg}_{B(0,t)} H^2 = t^2/4 + O(t^3)$ (since $H(z) = \log|1 - z/u_0|$). Letting $d \to \infty$ and then $t \downarrow 0$ yields $\limsup_{d\to\infty} \sqrt{J_d(s)} \leq \pi/2$, hence $J_d(s) \to \pi^2/4$ for every fixed $s > 0$.

On a small disc $B(0, s_2)$ inside Ω where $F \neq A$ and $F' \neq 0$, set $l(z) := \log((F(z) - A)/(-A))$ ($l(0) = 0$, $l'(0) = -1/A$) and $p := (\operatorname{Re} l)^+$. Let $\theta_d(s)$ denote the angular length of $\{\tilde{p}_d > 0\}$ on $\partial B(b_d, s)$, and let $\theta(s)$ denote the angular length of $\{p > 0\}$ on $\partial B(0, s)$. On $[s_1, s_2]$ ($s_1 > 0$), Lemma 4.1(b) gives

$$0 \leq \int_{s_1}^{s_2} \Big(\frac{sI'_{\tilde{p}_d}(s)}{I_{\tilde{p}_d}(s)} + \frac{2\pi}{2\pi - \theta_d(s)} - 4\Big)\frac{ds}{s} \leq \log\frac{J_d(s_2)}{J_d(s_1)} \to 0.$$

By analytic convergence $\tilde{p}_d(b_d + \cdot) \to p$ and Fatou's lemma, $sI'_p(s)/I_p(s) + 2\pi/(2\pi - \theta(s)) = 4$ for a.e. $s \in [s_1, s_2]$, forcing $\theta(s) = \pi$ and $I'_p(s) = 2I_p(s)/s$ a.e. Since $\int_{\partial B_s} p^2/s^2 \leq \int_{\partial B_s} |\nabla_{\tan} p|^2$ (for $\theta(s) = \pi$) and $I_p(s) = \int_{\partial B_s} p\partial_r p$, we get $\int_{\partial B_s}(\partial_r p - p/s)^2 \leq I'_p(s) - 2I_p(s)/s = 0$. Thus $p(sz)/s$ is independent of s and equals $(\operatorname{Re}(l'(0)z))^+$ on a half-disc. By harmonic continuation, $l(z) = -z/A$ and $F(z) = A(1 - e^{-z/A})$ on Ω.

Finally, $E(z) := A(1 - e^{-z/A})$ is a covering from $Z := \mathbb{C} \setminus E^{-1}(\{0, A\})$ onto $\mathbb{C} \setminus \{0, A\}$, and so is $E|_{\Omega\cap Z}$ by Proposition 4.4. Lifting any path in Z from a point of $\Omega \cap Z$ forces $Z \subset \Omega$; simple connectivity of Ω fills the discrete lattice $E^{-1}(\{0, A\})$, so $\Omega = \mathbb{C}$. Because every subsequence has a further extraction converging locally uniformly to $A(1-e^{-z/A})$ and the model depends continuously on $A \in \mathbb{T}$, we obtain $F_d(z) - A_d(1 - e^{-z/A_d}) \to 0$ locally uniformly on $\mathbb{C}$ (with all derivatives by Cauchy's estimates) along the full sequence. Since E' is nowhere zero, Hurwitz's theorem forces every individual critical distance $|c_j| \to \infty$. □

We also record the bounded-spread strictness needed to guarantee inner critical copies in Section 6.

Proposition 5.1 (Bounded-spread strictness)**.** *For every $K \geq 1$ there exists $d_0(K)$ such that if $d \geq d_0(K)$, the critical points of P are not all coincident, and $\max|c_j|/\min|c_j| \leq K$, then $d\,D(P) > 1$.*

Proof. Suppose not: take a sequence of noncoincident weak failures ($dD \leq 1$) with radial spread $\leq K$. By Theorem 1.2 and Proposition 3.1, in first inverse units $u_l := c_l/d$ satisfy $1/(2K) \leq |u_l| \leq C$ (for large d, as $\liminf \max|u_l| \geq 1$) and concentrate at $u_0 \in \mathbb{T}$ along an extracted subsequence.

Let $\mu_1 := n^{-1}\sum u_l^{-1} \to u_0^{-1}$ and normalize $\alpha_l := u_l^{-1}/\mu_1$: then $n^{-1}\sum \alpha_l = 1$, $|\alpha_l| \le M$, and $0 < v := n^{-1}\sum |\alpha_l - 1|^2 \to 0$.

Let $\mathcal{G} := \{l : \operatorname{Re}\alpha_l \ge 3/4\}$, pick $j \in \mathcal{G}$ maximizing $|\alpha_l|$, and set $e := |\alpha_j| - 1 \ge 0$. Every $l \notin \mathcal{G}$ has $\operatorname{Re}(1-\alpha_l) > 1/4$ while every $l \in \mathcal{G}$ has $\operatorname{Re}(\alpha_l - 1) \le e$, so $\#(\mathcal{G}^c)/n \le 4e$. On $\mathcal{G}$, $\sum_{\mathcal{G}} \operatorname{Re}(\alpha_l - 1) \ge 0$ gives $n^{-1}\sum_{\mathcal{G}} |\alpha_l - 1|^2 \le 2e + e^2$, while $\mathcal{G}^c$ contributes $\le 4e(M+1)^2$. Thus $|\alpha_j| \ge 1 + c_M v$ ($c_M > 0$).

Put $\beta := 1/\alpha_j$. The normalized secant at c_j is $I = \int_0^1 \prod_l (1 - t\beta\alpha_l)\, dt$, compared with $J := \int_0^1 (1-t\beta)^n\, dt = \alpha_j[1 - (1-\beta)^d]/d$. Interpolate by $w_l(a) := \beta[1 + a(\alpha_l - 1)]$ ($a \in [0,1]$) and $Q_a(t) := \prod_l (1 - t w_l(a))$. Since $\log|1 - tw| \le -t\operatorname{Re} w + t^2|w|^2/2$ and $2\operatorname{Re}\beta - |\beta|^2(1+v) = (2\operatorname{Re}\alpha_j - 1 - v)/|\alpha_j|^2 \ge 1/(4M^2)$ for small v, we have $|Q_a(t)| \le e^{-cnt}$ on $[0,1]$ (and $\le C_M e^{-cnt}$ after omitting one or two factors). On $[0, 1/4]$, subtracting the zero sum $\sum(\alpha_l - 1) = 0$ (via $1/(1 - tw_l(a)) - 1/(1 - t\beta) = at\beta(\alpha_l - 1)/[(1 - tw_l(a))(1 - t\beta)]$) gives

$$\partial_a Q_a(t) = -\frac{at^2}{1 - t\beta} \sum_l [\beta(\alpha_l - 1)]^2 \prod_{k \ne l} (1 - t w_k(a)),$$

whose integral over $[0, 1/4] \times [0,1]$ is $O_M(v/n^2)$. On $[1/4, 1]$, $\partial_a Q_0 = 0$ and $|\partial_a^2 Q_a(t)| \le C_M n^2 v e^{-cn/4}$ by Cauchy–Schwarz. Hence $d|I - J| = o(v)$. Finally $|1-\beta|^2 \le 1 - 1/(2M^2)$ and $|1-\beta|^2 \le (16/9)nv$, so $|1-\beta|^d = o(v)$, giving $d|I| \ge |\alpha_j| - o(v) \ge 1 + c_M v - o(v) > 1$, a contradiction. □

6. Resolution for all sufficiently large degrees

Suppose for contradiction that Theorem 1.1 fails: there exists a sequence of polynomials F_d of degrees $d \to \infty$ with noncoincident critical points and $d\,D(F_d) \le 1$ (weak failures). Work in first inverse units ($\rho_d = 1$, with first singular targets $A_d \in \mathbb{T}$). By Theorem 1.2 and Proposition 5.1, the least critical distance $r_* := \min_j |c_j|$ satisfies $r_* \to \infty$ and $r_*/d \to 0$. Fix a small constant $\eta \in (0, 1/4)$, call critical copies with $|c| < \eta d$ *inner* (let k be their number), and the remaining $N := d - 1 - k$ copies *outer*. By Theorem 1.2, $k = o(d)$. Passing to a subsequence on which $A_d \to A$ and the empirical joint law converges to $\delta_{(u_0, A)}$ by Theorem 1.2(b), let μ_{out} be the mean reciprocal of the outer copies; since $u_l = c_l/d$ on the outer annulus stays bounded and concentrates at $u_0 \in \mathbb{T}$ with $k = o(d)$, we have $d\mu_{\text{out}} \to 1/u_0$ and $|d\mu_{\text{out}}| \to 1$. Rescale by $z = \mu_{\text{out}} \cdot$ (old coordinate) and $G(z) := \mu_{\text{out}} F_d(z/\mu_{\text{out}})$: then $G(0) = 0$, $G'(0) = 1$, D is unchanged, and $G' = BE$ where

$$B(z) = \prod_{l=1}^{N} (1 - \alpha_l z), \qquad E(z) = \prod_{i=1}^{k} (1 - \gamma_i z),$$

with outer reciprocals α_l having mean 1, $|\alpha_l| \le A_0$, $v := N^{-1}\sum |\alpha_l - 1|^2 \to 0$, $v_c := N^{-1}\sum(\alpha_l - 1)^2 \to 0$, and inner sites $x_i = 1/\gamma_i$ satisfying $|x_i| \le 1/(1 + c_0)$ for a fixed constant $c_0 > 0$ (since $\eta < 1/4$). Every critical point x obeys $|G(x)| \le |x|/d$.

6.1. **Divergence of the inner population.**

Proposition 6.1 (Logarithmically many inner copies)**.** *Along the weak-failure sequence above, $k \ge c_\eta \log(2 + r_*)$ for all sufficiently large d; in particular $k \to \infty$.*

Proof. Suppose along a subsequence $k = o(\log(2 + r_*))$. Let $r := \min_i |x_i|$ (so $Nr \sim r_* \to \infty$, $r \to 0$, $k = o(\log(Nr))$) and $T := \sum_{i=1}^k |\gamma_i| \le k/r = o(N)$.

Put $H(z) := -N^{-1}\log B(z)$ ($H(0) = 0$, $H'(0) = 1$). Since $|\alpha_l| \le A_0$, fix $a \in (0, 1/4)$ so small that H has a univalent inverse ψ on $\{|\zeta| \le 8a\}$ with $|\psi'| \le 2$ and $\psi'(0) = 1$. Call x_i *small* if $|x_i| \le a$ (eventually nonempty), set $\zeta_i := H(x_i)$ ($|\zeta_i| \le 2a$), and put $W_2 := \sum_{\text{small}} |\gamma_i|^2$. For each small x_i, integrate $G'dz = e^{-N\zeta} E(\psi(\zeta))\psi'(\zeta)\, d\zeta$ from 0 to $4a$, across to $4a + \zeta_i$, and back to ζ_i. On $[0, 4a]$, $N\int_0^{4a} \to 1$ by dominated convergence ($T/N \to 0$). On the connecting segment $\operatorname{Re}\zeta \ge 2a$, $|E(\psi(\zeta))| \le (1 + C_a/r)^k = e^{o(N)}$, so the contribution is $o(1/N)$. On the return segment $\zeta_i + t$

$(0 \le t \le 4a)$, factor out $B(x_i) = e^{-N\zeta_i}$: the i-th factor of E is $\le 2t/|x_i|$, while every other factor is $\le (1 + |x_i|/|x_l|)e^{2t/|x_l|}$; integrating $Nte^{-Nt+2Tt}$ gives $O(1/N)$. Since $N|G(x_i)| \le a$, we obtain

$$|B(x_i)| \ge \frac{cN|x_i|}{\prod_{l\ne i}(1 + |x_i|/|x_l|)}.$$

Comparing with $\log|B(x_i)| \le N|x_i|^2[(1+v)/2 - \operatorname{Re}\gamma_i]$, multiplying by $|\gamma_i|^2$, and summing over small i (bounding the denominator sum by kW_2: for small $l \ne i$, $\log(1 + |x_i|/|x_l|) \le |x_i|/|x_l|$ and Cauchy–Schwarz contribute $\sum_{\text{small}\,i,l} |\gamma_i||\gamma_l| \le k_{\text{small}}W_2$; for nonsmall l, $|x_i|/|x_l| < 1$ and $\log 2 < 1$ contribute $\le k_{\text{nonsmall}}W_2$, with $k_{\text{small}} + k_{\text{nonsmall}} = k$) yields

$$\sum_{\text{small}} \Big(\frac{1+v}{2} - \operatorname{Re}\gamma_i\Big) \ge \frac{W_2}{N}\big(\log(cNr) - k\big).$$

For any nonsmall inner site ($a \le |x_j| \le 1/(1 + c_0)$), if $\operatorname{Re}\gamma_j \ge 3/4$ then along $z = tx_j$ we have $|B(tx_j)| \le e^{-c_a Nt}$ and $|E(tx_j)| \le e^{Tt}$, giving $NG(x_j) = 1 + o(1)$, contradicting $N|G(x_j)| \le 1/(1 + c_0)$. Thus every nonsmall inner copy has $\operatorname{Re}\gamma_j < 3/4$, and for small v,

$$k - \operatorname{Re}\sum_{i=1}^{k}\gamma_i \ge \frac{k}{4} + \frac{W_2}{N}\big(\log(cNr) - k\big). \tag{19}$$

Now pick an outer copy with $\operatorname{Re}\alpha_j \ge 1$ and put $\beta := 1/\alpha_j$ ($|\beta| \le 1$, $|1 - \beta| \le \kappa < 1$). By the same interpolation as in Proposition 5.1 (the factor $|E(t\beta)| \le e^{Tt}$ absorbs at most half the decay e^{-cNt}), the secant $I = \int_0^1 B(t\beta)E(t\beta)\,dt$ agrees with $\int_0^1 (1 - t\beta)^N E(t\beta)\,dt$ up to $O(v/N)/d$. Substituting $z = t\beta$, completing the segment from β to 1 at cost $O(e^{-c'N})$, and expanding $E(z) = \sum_{s=0}^{k}(-1)^s e_s(\gamma)z^s$ against the Beta integrals $\int_0^1 (1 - z)^N z^s\,dz = s!/(N + 1)\cdots(N + s + 1)$ (with $s!|e_s(\gamma)| \le T^s$ and $T > k$) gives

$$d|I| \ge |\alpha_j|\Big(1 + \frac{k - \operatorname{Re}\sum\gamma_i}{N} - \frac{CT^2}{N^2}\Big) - O\Big(\frac{v}{N}\Big) - O(e^{-c'N}).$$

Since $T^2 \le k(W_2 + k/a^2)$ and $k = o(\log(Nr))$, the term $(W_2/N^2)\log(cNr)$ from (19) dominates CkW_2/N^2, and $k/(4N)$ absorbs $O(k^2/(a^2N^2)) + O(v/N) + O(e^{-c'N})$ (as $k \ge 1$, $v \to 0$). Hence $d|I| > 1$, contradicting weak failure. □

6.2. **Outer coordinate, estimates for F, and the positive-phase barrier.** Fix an outer critical point $\beta = 1/\alpha_j$ with $\operatorname{Re}\alpha_j \ge 1$ (so $|\beta| \le 1$, $|G(\beta)| \le 1/d$). Define

$$Z := \frac{1}{G(\beta)}, \qquad F(z) := 1 - Z\,G(z) = Z\int_z^{\beta} B(\zeta)E(\zeta)\,d\zeta.$$

Then $F(0) = 1$, $F(\beta) = 0$, $F' = -ZBE$, and $|Z| \ge d = N + k + 1$.

Counting and $|Z| \sim N$. Set $L := \log(2+N/k)$. By Proposition 3.2(a), $n(r) := \#\{i : N|x_i| \le r\} \le \min(k, C\sqrt{Nr})$ (equivalently the j-th smallest $N|x_i|$ is $\ge cj^2/N$). Therefore $\Lambda(\rho) := \sum_{i=1}^{k}\log(1 + \rho/(N|x_i|)) \le Ck + k\log(1+\rho N/k^2)$ for all $\rho > 0$; in particular $\Lambda(CN) = o(N)$. Fix a large constant C_1 (chosen in order below) and set $\rho_{\text{in}} := C_1kL = o(N)$. For all $\rho \ge \rho_{\text{in}}$,

$$\frac{\Lambda(C\rho + 4k) + Ck}{\rho} \le \varepsilon_1,$$

where $\varepsilon_1 = O((1+\log C_1)/C_1) \to 0$ as $C_1 \to \infty$. The convex quadratic majorant $m(z) := -N\operatorname{Re}z + N(1 + v)|z|^2/2$ satisfies $\log|B| \le m$ and $m(\beta') \le -c_3N$ for every outer $\beta' = 1/\alpha$ with $\operatorname{Re}\alpha \ge 3/4$; since $|E| \le e^{\Lambda(N|z|)} = e^{o(N)}$ on $|z| \le C$, $G(\beta)$ and $G(\beta')$ differ by $O(e^{-c_3N/2})$, and concentration at A (Theorem 1.2) gives $|Z|/N \to 1$. At every inner site x_i, $|F(x_i) - 1| = |ZG(x_i)| \le (1 + o(1))|x_i|$, so $|F(x_i)| \ge c_4 > 0$.

The s-coordinate. Set $b := -\log B$, $\nu := b'$, and $\sigma := b + \log(\nu/N)$. Then $\sigma(0) = 0$, $\sigma'(0) = N + 1 + v_c$, $\sigma''(0) = N(1 + v_c) + O(1)$, $\sigma''' = O(N)$, and $B = e^{-\sigma}\nu/N$. Fix $h_0 \le 1/(16A_0^2)$ with $h_0 \ll c_3$ so that σ is univalent with $N/2 \le |\sigma'| \le 2N$ on $\{|z| \le 4h_0\}$ and its image contains $\{|s| < 2h_0N\}$. Write $s = \sigma(z)$, $z(s) := \sigma^{-1}(s)$, $\Omega_0 := \{|s| < R_0\}$ with $R_0 := h_0N/100$, $s_i := \sigma(x_i)$ whenever $x_i \in z(\{|s| \le 2R_0\})$, and $D_i(s) := N|z(s) - x_i|$. On $\{|s| < 2h_0N\}$, we abbreviate $F(s) := F(z(s))$ and $E(s) := E(z(s))$, and set

$$g := -\log|F|, \qquad \mathcal{L} := \log|F| + \operatorname{Re} s \quad \text{(subharmonic on } \Omega_0\text{)}.$$

In Ω_0 the only critical points of F are inner sites $s_i = \sigma(x_i)$, where $|F(s_i) - 1| \le 1.1|x_i|$ and $g(s_i) \le 2|x_i| \le 4|s_i|/N < 1$. By Theorem 1.2, near 0 the branch $w := -\log F$ ($w(0) = 0$) satisfies $w = \lambda_0 s + o(1)$ and $dw/ds = \lambda_0 + o(1)$ on $\{|s| \le S'\}$ for some $S' \to \infty$ ($S' \le \rho_{\text{in}}$, no critical sites in $\{|s| \le S'\}$), where $\lambda_0 = Z/\sigma'(0)$ has $|\lambda_0| \to 1$.

Lemma 6.2 (Estimates for F and $\mathcal{L}$ in the s-plane)**.** *For all sufficiently large d:*

(a) *For $|s| \le 2R_0$,*

$$F(s) = \frac{Z}{N} e^{-s} \int_0^{h_0N} e^{-t} E(s+t)\big(1 + O(1/N)\big)\, dt + O(e^{-h_0N/3}),$$

and $\mathcal{L}(s) \le \log|Z/N| + \log|E(s)| + \sum_i \log(1 + 4k/D_i(s)) + C \le \Lambda(2|s| + 4k) + Ck + C$. Consequently $\mathcal{L} \le \varepsilon_1\rho$ on $\{|s| \le 32\rho\}$ for $\rho_{\text{in}}/2 \le \rho \le R_0/32$ (absorbing a factor of 2 in C_1), and every inner site $x_i \in z(\{\rho_{\text{in}}/8 \le |s| \le 2R_0\})$ satisfies $\operatorname{Re} s_i \le 8\varepsilon_1|s_i|$.

(b) *Fix $\delta_1 \ge 64\varepsilon_1$. On $\{\operatorname{Re} s \ge \delta_1|s|,\ \rho_{\text{in}} \le |s| \le R_0\}$, every $D_i(s) \ge c\delta_1 \max(|s|, N|x_i|)$, $\sum_i 1/D_i \le Ck/(\delta_1|s|)$, $F = (Z/N)e^{-s}E(s)(1 + O(k/(\delta_1|s|)) + O(1/N))$, $|\mathcal{L}| \le C(\delta_1)\varepsilon_1|s|$, and for $\operatorname{Re} s \ge 2\delta_1|s|$, $dw/ds = 1 + O(k/(\delta_1^2|s|))$ for any branch $w = -\log F$.*

(c) *Let $R_1 := \varrho_1 N$ with ϱ_1 small ($2^{14}R_1 \le R_0$), and $\Lambda_0 := \Lambda(R_1) \le CkL$. Write $\mathcal{L}_\Psi := \mathcal{L} - \log|Z/N| - \log|E|$. Then the area averages over $\{|s| \le 2R_1\}$ of $|\mathcal{L} - \Lambda_0|^2$, $|\log|E| - \Lambda_0|^2$, and $\mathcal{L}_\Psi^2$ are all $\le Ck^2$, with C independent of ϱ_1.*

Proof. (a) In $F(s) = Z\int_{z(s)}^{\beta} BE\, dz$, integrate horizontally from s to $s + h_0N$ using $BE\, dz = N^{-1}e^{-s'}E(\nu/\sigma')\, ds'$ and $\nu/\sigma' = 1 + O(1/N)$, then vertically to h_0N inside $\operatorname{Re} s' \ge 0.98h_0N$, and finally along the segment from $z(h_0N)$ to β, where $m \le -h_0N/2$ and $|E| = e^{o(N)}$. On $[0, h_0N]$, $|E(s+t)/E(s)| \le \prod_i(1 + 2t/D_i)$; splitting $e^{-t} = e^{-t/2}\prod_i e^{-t/(2k)}$ and using $\sup_t e^{-t/(2k)}(1 + 2t/D_i) \le 1 + 4k/D_i$ gives the bound for $\mathcal{L}$ (for the $O(e^{-h_0N/3})$ term, $1/|E| \le \prod_i(1 + 4k/D_i)(1 + N|x_i|/k)$ and $k\log(1 + N/k) = o(N)$). Using $D_i \le 2|s| + N|x_i|$ yields $\mathcal{L} \le \Lambda(2|s| + 4k) + Ck + C$, whence $\mathcal{L} \le \varepsilon_1\rho$ on $|s| \le 32\rho$; at an inner site $|F(s_i)| \ge c_4$ then forces $\operatorname{Re} s_i \le 8\varepsilon_1|s_i|$ (applying the Λ-bound directly at $|s_i|$ when $|s_i| > R_0$).

(b) In the cone $\operatorname{Re} s \ge \delta_1|s|$, every inner site x_i satisfies either $x_i \in z(\{|s'| \le |s|/8\})$, or $x_i \in z(\{\rho_{\text{in}}/8 \le |s'| \le 2R_0\})$ (where $\operatorname{Re} s_i \le 8\varepsilon_1|s_i| \le \frac{1}{8}\delta_1|s_i|$), or $x_i \notin z(\{|s'| \le 2R_0\})$, giving $D_i(s) \ge c\delta_1 \max(|s|, N|x_i|)$. Thus $|E(s+t)/E(s) - 1| = O(t\sum 1/D_i)$ when $t\sum 1/D_i \le 1$, while for $t\sum 1/D_i > 1$ we have $|E(s+t)/E(s)| \le \exp(2t\sum 1/D_i) \le e^{t/2}$ (since $\sum 1/D_i \le C/(\delta_1C_1L) \le 1/4$), yielding the asymptotics for F, $\mathcal{L}$, and $w' = -F'/F$.

(c) On $\{|s| \le 2^{10}R_1\}$, $\mathcal{L} \le \Lambda_0 + Ck$ by (a), while at $p = R_1$ we have $\mathcal{L}(p) \ge \Lambda_0 - Ck$ by (b) (and similarly for $\log|E|$). Rescaling by $s = R_1\zeta$, the nonnegative superharmonic function $\tilde{v}_0(\zeta) := \Lambda_0 + Ck - \mathcal{L}(R_1\zeta)$ satisfies $\tilde{v}_0(1) \le Ck$ on $\{|\zeta| \le 2^{10}\}$. Writing its Riesz decomposition on $B(0,8)$ as $\tilde{v}_0(\zeta) = h(\zeta) + \int_{|\xi|<8} G_{B(0,8)}(\zeta,\xi)\, d\mu_0(\xi)$ with $h \ge 0$ harmonic on $B(0,8)$ and $\mu_0 \ge 0$, splitting the integral into $|\xi| \le 4$ and $4 < |\xi| < 8$ yields a nonnegative harmonic function $\tilde{h}(\zeta) := h(\zeta) + \int_{4<|\xi|<8} G_{B(0,8)}(\zeta,\xi)\, d\mu_0(\xi)$ on $B(0,4)$ with $\tilde{h}(1) \le \tilde{v}_0(1) \le Ck$, while $G_{B(0,8)}(1,\xi) \ge \log(3/2)$ for $|\xi| \le 4$ gives $\mu_0(\{|\xi| \le 4\}) \le Ck$; Harnack's inequality bounds $\tilde{h} \le Ck$ on $\{|\zeta| \le 2\}$, and since $0 \le G_{B(0,8)}(\zeta,\xi) \le \log(9/|\zeta - \xi|)$ for $|\zeta| \le 2$ and $|\xi| \le 4$, Minkowski's integral inequality bounds

the area L^2-norm of $\tilde{v}_0$ on $\{|\zeta| \le 2\}$ by Ck, with C independent of ϱ_1. The same holds for $\log|E|$, and Minkowski's inequality bounds $\mathcal{L}_\Psi$. □

Fix $\delta := 1/100$, and take C_1 so large that ε_1 and the errors in Lemma 6.2(b) (with $\delta_1 = \delta/4$) are $\le 10^{-4}\delta$ for $|s| \ge \rho_{\text{in}}$. Define the right sector and the positive-phase basin (Figure 4):

$$\Sigma := \{\rho_{\text{in}} < |s| < R_0,\ |\arg s| < \pi/2 - \delta\},$$
$$\mathfrak{B} := \{s \in \Omega_0 : g(s) > 0,\ \text{the ascending } \nabla g\text{-trajectory reaches } \Sigma \text{ in } \Omega_0\}.$$

On Σ, $g > 0$ and $\nabla g = 1 + O(\delta/100)$, so ascending trajectories ($\operatorname{Im} w$ constant) move rightward and exit Σ only through $|s| = R_0$. Clearly $\mathfrak{B}$ is open, contains no zero of F, and $\{g > 0\} \setminus \mathfrak{B}$ is invariant under ascending trajectories in Ω_0.

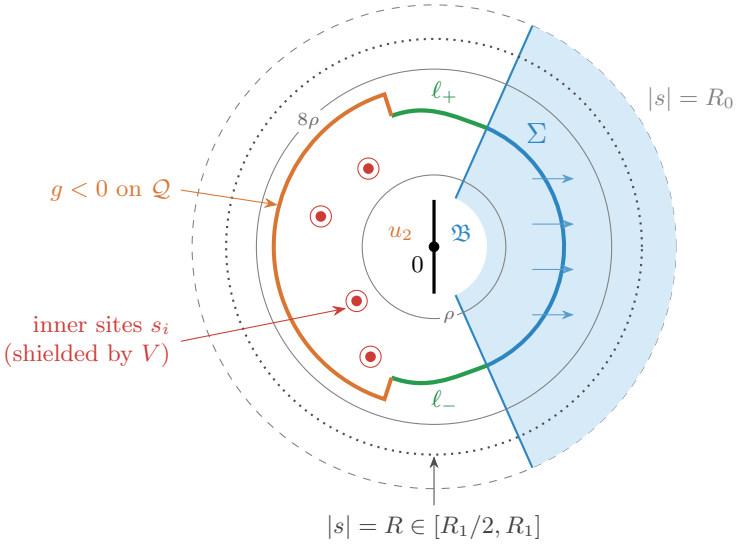

FIGURE 4. Two-phase ACF geometry in the s-plane (Section 6). The closed blocking curve $\mathcal{Q}$ winds around 0 using an arc in $\Sigma \subset \mathfrak{B}$, two trajectories $\ell_\pm$, and arcs where $g < 0$, confining every component of $\{g > 0\} \setminus \mathfrak{B}$. Inner critical sites s_i lie near or left of the imaginary axis ($\operatorname{Re} s_i \le 8\varepsilon_1|s_i|$) and are cut off from $u_1 = (g - V)_+$ by the positive harmonic barrier V.

Lemma 6.3 (Blocking curve)**.** *For every $\rho \in [2\rho_{\text{in}}, R_0/64]$, the open annulus $\mathcal{A} = \{\rho < |s| < 8\rho\}$ contains a closed curve $\mathcal{Q}$ winding once around 0, composed of an arc in Σ, two trajectories $\ell_\pm$ ($\operatorname{Im} w$ constant) starting on $\arg s = \pm(\pi/2 - \delta)$ at radii in $(3\rho, 5\rho)$ and ending on $\arg s = \pm(\pi/2 + \delta_\pm)$ with $\delta_\pm \in (\delta, 2\delta)$, and curves on which $g < 0$. Consequently, no connected set $K \subset \{g > 0\} \setminus \mathfrak{B}$ meets both $\{|s| = \rho\}$ and $\{|s| = 8\rho\}$.*

Proof. (i) The nonnegative superharmonic function $v_0 := \varepsilon_1\rho - \mathcal{L}$ on $\{|s| \le 32\rho\}$ satisfies $v_0(4\rho) \le C\varepsilon_1\rho$ by Lemma 6.2(b). By the Riesz decomposition on $B(0, 32\rho)$, on $\{|s| \le 16\rho\}$ the function v_0 decomposes as a logarithmic potential of $\le C\varepsilon_1\rho$ zeros plus a harmonic function $\le C\varepsilon_1\rho$. By Cartan's lemma [14, Lecture 11], outside discs of total radius $\le \delta\rho/100$, the potential is $\le C\varepsilon_1\rho\log(C/\delta)$, so $\mathcal{L} \ge -\frac{1}{2}\rho\sin\delta$ on $\mathcal{A}$ outside these discs (for $\varepsilon_1 \ll \delta$). The discs meeting $\mathcal{A}$ cast angular shadows from 0 of total size $< \delta$. Whenever $g \ge 0$ and $|\arg s| \ge \pi/2 + \delta$ in $\mathcal{A}$, we have $\mathcal{L} = \operatorname{Re} s - g \le -\rho\sin\delta$, so every such point lies inside the discs. Pick $r \in (\rho, 8\rho)$ and $\delta_\pm \in (\delta, 2\delta)$ so that $\{|s| = r\}$ and the ray segments of $\arg s = \pm(\pi/2 + \delta_\pm)$ inside $\mathcal{A}$ avoid all discs meeting $\mathcal{A}$ (hence carry $g < 0$ on $|\arg s| \ge \pi/2 + \delta$).

(ii) Put $\mathbb{A} := \{\rho < |s| < 8\rho,\ \pi/2-\delta < \arg s < \pi/2+\delta_+\}$. From $te^{i(\pi/2-\delta)}$ ($t \in (3\rho, 5\rho)$), follow the *descending* trajectory of g: since $\nabla g \approx 1$, it enters $\mathbb{A}$ transversely and cannot re-cross its starting ray. Along the starting segment, $y = \operatorname{Im} w$ has $dy/dt \geq 0.9$, filling an interval of length $\geq 1.8\rho$; distinct y give disjoint trajectories, and only finitely many hit critical points. On $\mathbb{A}$, $g \geq -(8\rho \sin 2\delta + \varepsilon_1\rho)$ by Lemma 6.2(a), while at the start $g \leq 5.1\delta\rho$, so g drops by $\Delta g \leq 23\delta\rho$. If no trajectory exited through $\arg s = \pi/2 + \delta_+$, each would travel length $L_y \geq 2\rho$ inside $\mathbb{A}$; by Cauchy–Schwarz $L_y^2 \leq \Delta g \int dg/|\nabla g|^2$, so the swept area would be $\int dy \int dg/|\nabla g|^2 \geq 1.8\rho \cdot 4\rho^2/(23\delta\rho) \geq 31\rho^2$, exceeding $\operatorname{Area}(\mathbb{A}) \leq 96\delta\rho^2 < \rho^2$. Thus some trajectory ℓ_+ exits through $\arg s = \pi/2 + \delta_+$, and symmetrically ℓ_- exits below.

(iii) Connect ℓ_+, the upper ray to $|s| = r$, the arc $\{|s| = r,\ \arg s \in [\pi/2 + \delta_+,\ 2\pi - (\pi/2 + \delta_-)]\}$, the lower ray, ℓ_-, and an arc in $\Sigma \cap \{3\rho \leq |s| \leq 5\rho\}$ to form $\mathcal{Q}$. Any $K \subset \{g > 0\} \setminus \mathfrak{B}$ misses $\Sigma \subset \mathfrak{B}$, misses the arcs with $g < 0$, and misses $\{g > 0\} \cap \ell_\pm \subset \mathfrak{B}$ (points there ascend along $\ell_\pm$ into Σ). □

Lemma 6.4 (Univalent logarithm on $\mathfrak{B}$)**.** *Set $S := S'/4$. Every point with $w = x + iy$ in the local chart at 0 satisfying $0 < x \leq 2$ and $|y| \leq S$ lies in $\mathfrak{B}$. Extending $w = -\log F$ from 0 into Σ along the ascending trajectory of the point with $w = 2$ in the local chart at 0, continuing on simply connected Σ, and pulling back along trajectories defines a univalent holomorphic map on $\mathfrak{B}$ with $\operatorname{Re} w = g > 0$.*

Proof. Let $c(y)$ be the point with $w = 2 + iy$ ($|y| \leq S$). Its ascending trajectory cannot hit a critical point (since $g < 1$ at all inner sites in Ω_0) and cannot exit Ω_0 outside Σ (by Lemma 6.3 at $\rho = 2\rho_{\text{in}}$). Hence it either enters Σ or terminates at a zero of F in Ω_0. Both alternatives are open in y (with the terminal zero fixed). If the second held, all $|y| \leq S$ would reach the same zero $q \in \Omega_0$. Every zero of F in Ω_0 is simple (as $|F| \geq c_4$ at all critical sites), so F is injective near q. Taking $y = 0$ and $y = 2\pi$ (for $S > 2\pi$), $c(0)$ and $c(2\pi)$ have the same F-value and ascend to q along the same target ray; continuing back along this ray (which meets no critical point with $g \geq 2$) forces $c(0) = c(2\pi)$, contradicting injectivity of the local w-chart. Thus every $c(y)$ enters Σ, so $\{0 < x \leq 2,\ |y| \leq S\} \subset \mathfrak{B}$.

For univalence on $\mathfrak{B}$: every trajectory in Σ exits through the outer arc $|s| = R_0$, where $\partial_\vartheta \operatorname{Im} w = R_0 \operatorname{Re}(e^{i\vartheta} w') \geq R_0(\sin\delta - O(\delta/100)) > 0$. Thus each value of $\operatorname{Im} w$ corresponds to at most one exit point and hence one trajectory; since $\operatorname{Re} w = g$ increases strictly along trajectories, w is injective on $\mathfrak{B}$. Moreover, the trajectory of $c(0)$ crosses $\mathcal{Q}$ ($\rho = 2\rho_{\text{in}}$) inside Σ or on $\ell_\pm$, so it enters Σ at radius $\leq 32\rho_{\text{in}}$. □

Lemma 6.5 (Boundary of $\mathfrak{B}$ and harmonic barrier)**.** *Fix $R \in [R_1/2, R_1]$ and a large constant K_2.*

(a) *For every $s_0 \in \partial\mathfrak{B}$ with $g(s_0) > 0$ and $|s_0| \leq R$, the ascending trajectory of s_0 ends at an inner site $s_i = \sigma(x_i) \in \Omega_0$; the boundary limit $w(s_0)$ exists from each side of $\partial\mathfrak{B}$ and satisfies $|w(s_0) - 2\pi i m| \leq 1.5|x_i|$ for an integer m with $|m| \geq S/20$. If $|s_0| \leq K_2\rho_{\text{in}}$ then $|x_i| \leq CK_2\rho_{\text{in}}/N$; if $|s_0| \geq K_2\rho_{\text{in}}$ then $|x_i| \leq C|m|/N$.*

(b) *Let $\mathcal{M}$ be the finite set of integers m arising in (a), put $\rho_m := 1.5 \max|x_i|$ over sites with label m, and define the positive harmonic barrier*

$$V := \sum_{m \in \mathcal{M}} \rho_m^2 \operatorname{Re} \frac{1}{w - 2\pi i m} \qquad \text{on } \mathfrak{B}.$$

Set $u_1 := (g - V)_+$ on $\mathfrak{B} \cap \{|s| \leq R\}$ (and 0 elsewhere) and $u_2 := \log^+ |F|$. Then u_1, u_2 satisfy Lemma 4.1(c) on $\{|s| \leq R\}$, $u_1 \leq g_+$, and

$$|\nabla h_1(0)|\,|\nabla h_2(0)| = (1 - \eta_V)\,\frac{|Z|^2}{|N + 1 + v_c|^2}, \qquad \eta_V := \sum_{m \in \mathcal{M}} \frac{\rho_m^2}{(2\pi m)^2} = o\Big(\frac{k}{N}\Big).$$

Proof. (a) The ascending trajectory from s_0 cannot reach Σ or a zero of F (both basins are open) and cannot exit Ω_0 (Lemma 6.3), so it ends at an inner critical point $s_i = \sigma(x_i) \in \Omega_0$ with $g(s_0) < g(s_i)$. Along it $\operatorname{Im} w$ is constant, and $w_i := w(s_i)$ satisfies $e^{-w_i} = F(s_i)$ with $|F(s_i) - 1| \le 1.1|x_i|$ and $|x_i| \le 2R_0/N = h_0/50 \ll 1$, whence $|w_i - 2\pi i m| \le 1.5|x_i|$ for some $m \in \mathbb{Z}$; moving back along the real decrease of g from $g(s_i)$ to $g(s_0) > 0$ keeps $|w(s_0) - 2\pi i m| \le 1.5|x_i|$. Since $0 < \operatorname{Re} w(s_0) < g(s_i) < 1$, Lemma 6.4 and univalence of w force $|\operatorname{Im} w(s_0)| \ge S$, so $|m| \ge S/20$. By Lemma 6.3 at $\rho = \max(|s_0|, 2\rho_{\mathrm{in}})$, the trajectory stays inside $|s| \le 16\max(|s_0|, \rho_{\mathrm{in}})$, bounding $|x_i| \le 2|s_i|/N$. When $\rho_0 := |s_0| \ge K_2\rho_{\mathrm{in}}$ (with K_2 chosen large enough that $32\rho_{\mathrm{in}} \ll 3\rho$ for $\rho := \rho_0/16$), apply Lemma 6.3 at $\rho = \rho_0/16$. Trajectories from $\mathfrak{B}$ near s_0 lie outside $\mathcal{Q}$ and cannot cross the $g < 0$ arcs or the trajectories $\ell_\pm$, so they must enter Σ through $\arg s = \pm(\pi/2 - \delta)$ at radius $\ge 3\rho$ before meeting the arc of $\mathcal{Q}$ in $\Sigma \cap \{3\rho \le |s| \le 5\rho\}$. Since $\operatorname{Im} w = 0$ at the entry point of $c(0)$ into Σ (radius $\le 32\rho_{\mathrm{in}}$, from the proof of Lemma 6.4) and $dw/ds = 1 + O(\delta/100)$ on Σ, we obtain $|\operatorname{Im} w| \ge 2\rho$, so $|m| \ge c\rho_0$ and $|x_i| \le C|m|/N$.

(b) At $s_0 \in \partial\mathfrak{B}$ with $g(s_0) > 0$, $|w - 2\pi i m| \le \rho_m$ gives $\rho_m^2 \operatorname{Re}(1/(w - 2\pi i m)) = \rho_m^2 g/|w - 2\pi i m|^2 \ge g$, so $g - V \le 0$. At $g(s_0) = 0$, $u_1 \le g \to 0$. Hence u_1 is continuous, subharmonic, and locally Lipschitz on $\{|s| \le R\}$ (for each fixed polynomial, ∇u_1 is bounded on $\{u_1 > 0\}$ since $|w - 2\pi i m| > \rho_m$ there and zeros of F stay off $\mathfrak{B}$). Near 0, $\mathfrak{B} = \{g > 0\}$ and $g - V = \operatorname{Re} w \cdot (1 - \sum \rho_m^2/|w - 2\pi i m|^2)$, giving $h_1 = \operatorname{Re}(w - \sum \rho_m^2/(w - 2\pi i m))$, $h_2 = -g$, and $|\nabla h_1(0)| = (1 - \eta_V)|\nabla g(0)|$ with $|\nabla g(0)| = |Z|/|\sigma'(0)| = |Z|/|N + 1 + v_c|$. On Σ, $|\operatorname{Im} w| = O(R_0) = O(N)$, so $|\mathcal{M}| = O(N)$. Splitting $\mathcal{M}$ by (a) gives

$$\eta_V \le \sum_{|m| \ge S/20} \frac{(CK_2\rho_{\mathrm{in}}/N)^2}{(2\pi m)^2} + O(N) \cdot \frac{C}{N^2} \le \frac{C(kL/N)^2}{S} + \frac{C}{N} = o\Big(\frac{k}{N}\Big),$$

because $S \to \infty$, $(k/N)L^2 \to 0$, and $k \to \infty$ (Proposition 6.1). □

6.3. **First Fourier coefficient and completion of the proof.**

Proof of Theorem 1.1. Fix the constants in order: A_0 (from η), h_0, then ϱ_1 small enough that $C\varrho_1 < 1/4$ below, then C_1 large enough that $\varepsilon_1 \le \varrho_1^3$ and the sector error is $\le 10^{-4}\delta$, then K_2; finally let $d \to \infty$.

Since $u_1^2 + u_2^2 \le g^2 = (R\cos\vartheta - \mathcal{L})^2$, choose $R \in [R_1/2, R_1]$ ($R \ne |s_i|$ for all $x_i \in z(\{|s| \le 2R_0\})$) by Markov's inequality so that the ϑ-averages on $\{|s| = R\}$ of $|\mathcal{L} - \Lambda_0|^2$, $\mathcal{L}_\Psi^2$, $|\mathcal{L}_\Psi|$, and $(\mathcal{L}_\Psi)_+$ are at most 10 times their area averages on $\{|s| \le R_1\}$. Expanding g^2 gives

$$\int_0^{2\pi} (u_1^2 + u_2^2)(Re^{i\vartheta})\, d\vartheta \le \pi R^2 - 2R \int_0^{2\pi} \cos\vartheta\, \mathcal{L}(Re^{i\vartheta})\, d\vartheta + \int_0^{2\pi} \mathcal{L}(Re^{i\vartheta})^2\, d\vartheta.$$

Write $\mathcal{L} = \log|Z/N| + \sum_{i=1}^k \log|1 - z/x_i| + \mathcal{L}_\Psi$. We bound each term in turn:

(i) *Square term.* By Lemma 6.2(c), $\int \mathcal{L}^2\, d\vartheta \le C(\Lambda_0^2 + k^2) = O(k^2L^2) = o(kN)$.

(ii) *Residual $\mathcal{L}_\Psi$.* Put $\delta_1 := \varrho_1^2 \ge 64\varepsilon_1$. On $\{\cos\vartheta \ge \delta_1\}$, $\mathcal{L}_\Psi = O(k/(\delta_1 R) + 1/N)$ by Lemma 6.2(b). On $\{\cos\vartheta \le -\delta_1\}$, $-\cos\vartheta > 0$ and $(\mathcal{L}_\Psi)_+ \le \sum_i \log(1 + 4k/D_i) + C$ (Lemma 6.2(a)), whose area average on $\{|s| \le R_1\}$ is $\le Ck^2/R_1 + C = o(k)$ (since $\log(1 + 4k/D_i) \le 4k/D_i$ and the normalized area average of $1/D_i$ is $O(1/R_1)$). On $\{|\cos\vartheta| \le \delta_1\}$, $\int |\mathcal{L}_\Psi|\, d\vartheta \le Ck$ by Lemma 6.2(c). Altogether, $-2R\int \cos\vartheta\, \mathcal{L}_\Psi\, d\vartheta \le (C\varrho_1 + o(1))R^2k/N$.

(iii) *Inner sites with $x_i \in z(\{|s| < R\})$ (count k_{in}).* Write $\log(1 - z/x_i) = \log(1 - s/s_i) + \phi_i(s)$, where ϕ_i is holomorphic on $\{|s| \le R\}$ with

$$\phi_i'(0) = \frac{1}{s_i} - \frac{1}{\sigma'(0)x_i} = -\frac{\sigma''(0)}{2\sigma'(0)^2} + O\Big(\frac{|x_i|}{N}\Big) = -\frac{1}{2N}\big(1 + o(1) + O(\varrho_1)\big).$$

The first cosine coefficients of $\log|1-s/s_i|$ and $\operatorname{Re}\phi_i$ on $\{|s|=R\}$ are $-\operatorname{Re}s_i/R$ and $R\operatorname{Re}\phi_i'(0)$, respectively. By Lemma 6.2(a), $\operatorname{Re}s_i \le 8\varepsilon_1 R+\rho_{\text{in}} \le (C\varrho_1+o(1))R^2/N$, so each enclosed site contributes

$$-2R\int_0^{2\pi}\cos\vartheta\,\log\Big|1-\frac{z}{x_i}\Big|\,d\vartheta = 2\pi\operatorname{Re}s_i+\frac{\pi R^2}{N}\big(1+o(1)+O(\varrho_1)\big)\le \frac{\pi R^2}{N}\big(1+C\varrho_1+o(1)\big).$$

(iv) *Inner sites with $x_i\notin z(\{|s|\le R\})$ (count $k_{\text{out}}=k-k_{\text{in}}$).* Here $\log|1-z/x_i|$ is harmonic on $\{|s|\le R\}$, so $-2R\int_0^{2\pi}\cos\vartheta\log|1-z/x_i|\,d\vartheta = 2\pi R^2\operatorname{Re}(\gamma_i/\sigma'(0)) = (2\pi R^2/N)(\operatorname{Re}\gamma_i+o(1))$. We claim $\operatorname{Re}\gamma_i\le 3/4$: if $x_i\in z(\{|s|\le 2R_0\})$, then $\operatorname{Re}\sigma(x_i)\le 8\varepsilon_1|s_i|$ expands as $\operatorname{Re}x_i+\frac12\operatorname{Re}(x_i^2)+O(|x_i|^3)+o(1)|x_i|\le 16\varepsilon_1|x_i|$, giving $\operatorname{Re}\gamma_i\le 1/2+O(h_0)+C\varepsilon_1/\varrho_1+o(1)<3/4$ (dividing by $|x_i|^2$ with $|x_i|\ge c\varrho_1$ and $\operatorname{Re}(x_i^2)\ge -|x_i|^2$); if $x_i\notin z(\{|s|\le 2R_0\})$, then $|x_i|\ge ch_0$, and $\operatorname{Re}\gamma_i\ge 3/4$ would make the quadratic majorant satisfy $m(x_i)\le -N|x_i|^2/5$ and (by convexity of m) $m\le -cN$ on the segment from x_i to β, forcing $F(x_i)\to 0$ against $|F(x_i)|\ge c_4$. Thus each outside inner site contributes at most $(3/2+o(1))\pi R^2/N$.

Summing (i)–(iv) gives

$$\int_0^{2\pi}(u_1^2+u_2^2)(Re^{i\vartheta})\,d\vartheta \le \pi R^2\Big[1+\frac{k_{\text{in}}+\frac32 k_{\text{out}}}{N}+(C\varrho_1+o(1))\frac{k}{N}\Big]\le \pi R^2\Big[1+\Big(\frac32+C\varrho_1+o(1)\Big)\frac{k}{N}\Big].$$

On the other hand, Lemma 4.1(c) and Lemma 6.5(b) give the lower bound

$$\int_0^{2\pi}(u_1^2+u_2^2)(Re^{i\vartheta})\,d\vartheta \ge \pi R^2\big(1-o(k/N)\big)\frac{|Z|^2}{|N+1+v_c|^2}.$$

Because $|Z|\ge d=N+k+1$, $v_c\to 0$, $k\to\infty$, and $k=o(N)$, we have

$$\frac{|Z|^2}{|N+1+v_c|^2}\ge \frac{(N+k+1)^2}{(N+1+|v_c|)^2}=1+\big(2-o(1)\big)\frac{k}{N}.$$

Comparing the two bounds gives $2-o(1)\le 3/2+C\varrho_1+o(1)$, contradicting $C\varrho_1<1/4$ for large d. Thus no noncoincident weak failure exists for $d\ge d_0$. When all critical points coincide, $P(z)=A(z-c)^d+B$ satisfies $D(P,a)=1/d$ by direct integration of $(1-t)^{d-1}$. □

7. Centered critical circle and the uniform annulus

We give a self-contained, computer-free proof of Theorem 1.4, combining radial derivative integration along the first inverse lift with a finite-sampling expansion of the elementary symmetric polynomials in critical-point deficits.

Proof of Theorem 1.4. Normalize $P(0)=0$ and $P'(0)=1$, write $n:=d-1$, and let $c_1,\dots,c_n$ be the critical copies of P with $r:=\min_j|c_j|$. Put $u_j:=r/c_j$ (so $|u_j|\le 1$), $m:=n^{-1}\sum_{j=1}^n u_j$, $S:=|m|\in[0,1]$, and $V:=n^{-1}\sum_{j=1}^n|u_j-m|^2$. Note that $P'(z)=\prod_{j=1}^n(1-z/c_j)=\prod_{j=1}^n(1-u_jz/r)$.

Step 1: Radial lift bounds for the first inverse radius ρ. Let v be a first singular target ($|v|=\rho$) and let $\gamma(t)=f(tv)$ $(0\le t<1)$ be its radial lift, terminating at a first boundary critical point c_* with $|P(c_*)|=\rho$ (so the set defining R_{hit} is nonempty with $R_{\text{hit}}\le|c_*|$, and by definition $D(P)\ge\rho/R_{\text{hit}}$). Since $|c_*|\ge r$, γ crosses every circle $\{|z|=xr\}$ $(0<x<1)$, and its image $[0,v]$ is a straight segment of length ρ; hence $\rho=\int_\gamma|P'(z)|\,|dz|\ge r\int_0^1\min_{|z|=xr}|P'(z)|\,dx$.

At $z=xre^{i\theta}$, set $a_j:=1-\operatorname{Re}(e^{i\theta}u_j)\ge 0$. Then $\sum_{j=1}^n a_j=n(1-\operatorname{Re}(e^{i\theta}m))\ge n(1-S)$, and $|1-u_jz/r|\ge 1-x+xa_j$ for each j. Expanding and retaining only the terms of degree 0 and 1 in the nonnegative quantities xa_j yields

$$|P'(xre^{i\theta})|\ge\prod_{j=1}^n(1-x+xa_j)\ge(1-x)^n+x(1-x)^{n-1}\sum_{j=1}^n a_j\ge(1-x)^n+n(1-S)x(1-x)^{n-1}.$$

Integrating over $x \in [0,1]$ (using $\int_0^1 (1-x)^n\,dx = 1/d$ and $\int_0^1 nx(1-x)^{n-1}\,dx = 1/d$) gives

$$\frac{d\rho}{r} \geq 2 - S. \tag{20}$$

If all critical points lie on $\{|z| = r\}$, then every first boundary critical point c_* has $|c_*| = r$, so $d|P(c_*)/c_*| = d\rho/r \geq 2 - S \geq 1$. Equality forces $S = 1$, which with $|u_j| = 1$ forces all u_j to coincide, proving part (a).

For large n, we record a second radial bound. For $0 \leq x \leq 1/2$ and $|w| \leq 1$, the logarithm series gives $\log|1 - xw| \geq -x\,\mathrm{Re}\,w - \sum_{k=2}^{\infty} x^k/k \geq -x\,\mathrm{Re}\,w - x^2$, since $x^2/[2(1-x)] \leq x^2$. Summing over j gives $\log|P'(xre^{i\theta})| \geq -nxS - nx^2$. Substituting $y = nx$, for any fixed $H > 0$ and $n \geq 2H$ we obtain

$$\frac{n\rho}{r} \geq \int_0^H \exp\Bigl(-Sy - \frac{y^2}{n}\Bigr)\,dy. \tag{21}$$

Step 2: Finite-sampling estimate at a selected critical point. Fix any index $a \in \{1,\dots,n\}$ and set $U := |u_a| > 0$, $v_j := 1 - u_j/u_a$, $\mu := n^{-1}\sum_{j=1}^n v_j = 1 - m/u_a$, and $\sigma^2 := n^{-1}\sum_{j=1}^n |v_j - \mu|^2 = V/U^2$. Let $b_k := e_k(v_1,\dots,v_n)/\binom{n}{k}$ be the normalized elementary symmetric polynomial ($b_0 = 1$, $b_1 = \mu$, and $b_n = 0$ because $v_a = 0$). Substituting $z = tc_a$ into $P(c_a)/c_a = \int_0^1 \prod_{j=1}^n (1 - t + tv_j)\,dt$ and evaluating the Beta integrals $\int_0^1 (1-t)^{n-k}t^k\,dt = 1/[d\binom{n}{k}]$ gives

$$F := \frac{d\,P(c_a)}{c_a} = \sum_{k=0}^{n} b_k.$$

Assume $n \geq 3$ and put $B^2 := \frac{n}{n-2}\cdot n^{-1}\sum_{j=1}^n |v_j|^2$. We claim that whenever $B < 1$,

$$\left|\frac{m}{u_a}F - 1\right| \leq \frac{\sigma^2}{(n-1)(1-B)^2}. \tag{22}$$

Indeed, for $1 \leq k \leq n-1$, let $(I_1,\dots,I_{k+1})$ be a uniformly random ordered $(k+1)$-tuple of distinct indices in $\{1,\dots,n\}$, and put $X_l := v_{I_l}$. Appending an independent uniform index Y to $(I_1,\dots,I_k)$ produces a distinct index with probability $(n-k)/n$ and repeats one of $I_1,\dots,I_k$ with probability k/n; by symmetry of the first k draws, $\mu b_k = \frac{n-k}{n}b_{k+1} + \frac{k}{n}\mathbb{E}[X_1^2 X_2 \cdots X_k]$. Subtracting $b_{k+1} = \mathbb{E}[X_1 X_{k+1} X_2 \cdots X_k]$ and symmetrizing $I_1 \leftrightarrow I_{k+1}$ gives

$$\mu b_k - b_{k+1} = \frac{k}{2n}\mathbb{E}\bigl[(X_1 - X_{k+1})^2 X_2 \cdots X_k\bigr].$$

Conditional on (I_1, I_{k+1}), Maclaurin's inequality for nonnegative numbers bounds $|\mathbb{E}[X_2\cdots X_k \mid I_1, I_{k+1}]|$ by the $(k-1)$-th power of the mean of the remaining $n-2$ values $|v_j|$, which by Cauchy–Schwarz is at most B^{k-1} (for $k = 1$ the empty product is 1). Since $\mathbb{E}[|X_1 - X_{k+1}|^2] = 2n\sigma^2/(n-1)$, we get $|\mu b_k - b_{k+1}| \leq kB^{k-1}\sigma^2/(n-1)$. Summing over $1 \leq k \leq n-1$ telescopes to

$$\sum_{k=1}^{n-1}(\mu b_k - b_{k+1}) = \mu(F-1) - (F - 1 - \mu) = 1 - (1-\mu)F = 1 - \frac{m}{u_a}F,$$

and $\sum_{k=1}^{\infty} kB^{k-1} = (1-B)^{-2}$ yields (22).

Step 3: Choosing the witness index. Fix $\lambda \in (0, 1/2]$ and choose a maximizing $|u_j|^2 - \lambda|u_j - m|^2$. Comparing with the average over $j = 1,\dots,n$ (which equals $S^2 + (1-\lambda)V$) gives

$$U^2 - S^2 \geq (1-\lambda)V + \lambda|u_a - m|^2.$$

Hence $U \geq S$, and since $n^{-1}\sum |v_j|^2 = (V + |u_a - m|^2)/U^2$ and $\lambda \leq 1/2 \leq 1 - \lambda$,

$$B^2 \leq \frac{n}{n-2}\cdot\frac{1 - S^2/U^2}{\lambda} \leq \frac{n}{n-2}\cdot\frac{1 - S^2}{\lambda}. \tag{23}$$

If $V > 0$ and $S > 0$, then $1 - S/U = (U^2 - S^2)/[U(U+S)] \geq (1-\lambda)\sigma^2/2$. Whenever $B < 1$ and

$$\frac{1}{(n-1)(1-B)^2} < \frac{1-\lambda}{2}, \tag{24}$$

(22) implies $|(m/u_a)F| \geq 1 - \frac{\sigma^2}{(n-1)(1-B)^2} > 1 - (1-\lambda)\sigma^2/2 \geq S/U = |m/u_a|$, whence $|F| > 1$ and $dD(P) > 1$. If $V = 0$, all critical points coincide and $D(P) = 1/d$.

Step 4: Proof of (b) and (c). For (b), assume $d \geq 8$ ($n \geq 7$) and $R_{\text{hit}}/r \leq 101/100$. If $S < 99/100$, then (20) gives $dD(P) \geq d\rho/R_{\text{hit}} \geq (2-S)/(R_{\text{hit}}/r) > 1$. If $S \geq 99/100$ and $V > 0$, choose $\lambda = 1/3$; then (23) gives

$$B^2 \leq \frac{7}{5} \cdot 3 \cdot \frac{199}{10000} = \frac{4179}{50000} < \Big(\frac{29}{100}\Big)^2,$$

so $(n-1)(1-B)^2 > 6(71/100)^2 = 3.0246 > 3 = 2/(1-\lambda)$, verifying (24) and proving (b).

For (c), fix $1 \leq M < \sqrt{2}$ and choose s with $1/\sqrt{2} < s < 1/M$. If $S \geq s$ and $V > 0$, take $\lambda = 1/2$: then (23) gives $B^2 \leq \frac{n}{n-2} \cdot 2(1-s^2)$, which stays bounded strictly below 1 for all large n, so $(n-1)(1-B)^2 \to \infty > 4 = 2/(1-\lambda)$ and (24) holds uniformly for $n \geq n_1(M)$. If $S \leq s$, choose $H > 0$ so that $\int_0^H e^{-sy}\,dy > M$ (possible since $1/s > M$). Then (21) gives $d\rho/r > n\rho/r \geq e^{-H^2/n}\int_0^H e^{-sy}\,dy > M \geq R_{\text{hit}}/r$ for all $n \geq n_2(M)$, so $dD(P) > 1$. □

8. Sharp ramified Blaschke bound and multiplicity floor

In this section we sharpen Lemma 2.3 when every critical point of a finite Blaschke product has multiplicity at least $m \geq 1$, establish two elementary parameter inequalities for the half-disc geometry of Lemma 2.5 (which are also used in Section 9), and deduce Theorem 1.5.

Lemma 8.1 (Sharp ramified Blaschke derivative bound)**.** *Let $B : \mathbb{D} \to \mathbb{D}$ be a finite Blaschke product of degree ≥ 2 whose critical points in $\mathbb{D}$ all have multiplicity at least $m \geq 1$ in B'. Let C be a finite Blaschke product whose zeros are the critical copies of B with full multiplicity, and set $U := -\log|C|$ and $\beta(z) := (1-|z|^2)|B'(z)|/(1-|B(z)|^2)$. Then*

$$\beta(z) \geq \frac{(m+1)\sinh(U(z)/m)}{\sinh\big((1+1/m)U(z)\big)} \qquad (z \in \mathbb{D}), \tag{25}$$

with the right-hand side interpreted as 0 at zeros of C. In particular, if $B(0) = 0$, $B'(0) \neq 0$, and the local inverse of B at 0 continues along $[0,\sigma)$ to a critical point over $\sigma \in (0,1)$, then

$$|C(0)|^{1+1/m} \leq \sigma. \tag{26}$$

Both inequalities are sharp for the power map $z \mapsto z^{m+1}$ composed with disc automorphisms.

Proof. Away from the zeros of C, pick any local analytic branch $G := C^{1/m}$ (mapping into $\mathbb{D}$); its hyperbolic derivative $\gamma := (1-|z|^2)|G'|/(1-|G|^2)$ is independent of the branch. At a zero a of C of order $k \geq m$, $|G'| = O(|z-a|^{k/m-1})$, so γ extends continuously across all zeros of C (smoothly to a positive value if $k = m$, where G extends holomorphically with $G'(a) \neq 0$, and by 0 if $k > m$). Since C is a finite Blaschke product, $\gamma \to 1$ as $|z| \to 1$. With $T := \frac{1}{4}(1-|z|^2)^2\Delta$, at any interior maximum of $\gamma > 1$ (necessarily away from zeros of C of order $k > m$ and zeros of G', where $\gamma = 0$; at any zero a of C of order $k = m$, G extends holomorphically with $G'(a) \neq 0$, so $T\log\gamma = \gamma^2 - 1$ holds there as well) we would have $T\log\gamma = \gamma^2 - 1 > 0$, impossible. Hence $\gamma \leq 1$ on $\mathbb{D}$, which off the zeros of C reads

$$(1-|z|^2)|\nabla U| \leq 2m\sinh(U/m). \tag{27}$$

Put $F(u) := (m+1)\sinh(u/m)/\sinh((1+1/m)u)$. Direct differentiation gives

$$m^2\sinh^2(u/m)\,(\log F)''(u) = F(u)^2 - 1 < 0,$$

since $\sinh(v)/v$ is strictly increasing on $(0,\infty)$. Because U is harmonic off the zeros of C, the chain rule and (27) yield $T\log F(U) = (\log F)''(U)\cdot\frac{1}{4}(1-|z|^2)^2|\nabla U|^2 \geq F(U)^2-1$. Writing $w := \log(\beta/F(U))$, we obtain $Tw \leq \beta^2 - F(U)^2$ off the zeros of C. Expressing $F(U) = (m+1)|C|(1-|C|^{2/m})/(1-|C|^{2+2/m})$, since B'/C is holomorphic and nonvanishing while $|C|^{2/m}$ is smooth at order-m zeros (as a positive smooth multiple of $|z-a|^2$) and has leading radial exponent $2k/m > 2$ at zeros of order $k > m$, w extends C^2 across all critical points of B, with boundary limit 0 on $\mathbb{T}$. On $\{w<0\}$, w is superharmonic, so the minimum principle forces $w \geq 0$ on $\mathbb{D}$, proving (25).

For (26), equip source and target discs with the hyperbolic metric $2|dz|/(1-|z|^2)$. Along the radial lift of $[0,\sigma)$ parametrized by source hyperbolic arclength l, (27) gives $|dU/dl| \leq m\sinh(U/m)$, while the target speed is $\beta \geq F(U)$. Since $\frac{d}{du}\log\tanh(\frac{m+1}{2m}u) = \frac{m+1}{m\sinh((1+1/m)u)}$, we obtain

$$\frac{d}{dl}\log\tanh\Big(\frac{m+1}{2m}U\Big) \leq \frac{m+1}{m\sinh((1+1/m)U)}\cdot m\sinh(U/m) = F(U) \leq \beta.$$

Integrating from $l=0$ to the critical endpoint (where $U\to+\infty$ and $\log\tanh\to 0$, while total target hyperbolic length equals $2\operatorname{artanh}\sigma$) gives $2\operatorname{artanh}\sigma \geq -\log\tanh(\frac{m+1}{2m}U(0))$, which simplifies to $|C(0)|^{1+1/m} \leq \sigma$. For $B(z) = z^{m+1}$ recentered from a real point $r\in(0,1)$ to 0, $|C(0)| = r^m$ and $\sigma = r^{m+1}$, attaining equality in both (25) and (26). □

Lemma 8.2 (Two-sided parameter bounds for the half-disc map)**.** *For $x\in(0,1)$, retain $A = 8-3x^2$, $s = A/8$, $t = s-1/2$, $T_0 := s/x^2 - 1/2$, $w = s/x - 1/2$, and σ from Lemma 2.5, and set $q := (1-\sqrt{\sigma})/(1+\sqrt{\sigma})$. Then*

$$\frac{x}{6} \leq q \leq \frac{7x}{25} \qquad (0<x<1). \tag{28}$$

Proof. Recall $U_{\mathrm{hd}}(z) := 1/z - z/T_0^2$ and $\sigma = (U_{\mathrm{hd}}(t)-U_{\mathrm{hd}}(w))/(U_{\mathrm{hd}}(t)+U_{\mathrm{hd}}(w))$. Put $z_0 := U_{\mathrm{hd}}(w)/U_{\mathrm{hd}}(t) = (1-\sigma)/(1+\sigma) = 2q/(1+q^2)$, so $q \geq z_0/2$. Since $0<t<w<T_0$,

$$z_0 = \frac{t}{w}\cdot\frac{T_0-w}{T_0-t}\cdot\frac{T_0+w}{T_0+t} \geq \frac{t}{w}\cdot\frac{T_0-w}{T_0-t}.$$

Substituting $T_0 - w = s(1-x)/x^2$ and $T_0 - t = s(1-x^2)/x^2$ gives $(T_0-w)/(T_0-t) = 1/(1+x)$, while $t/w = x(A-4)/(A-4x)$. Because $3(A-4)-(1+x)(A-4x) = (1-x)(4-2x^2)+x^3 > 0$ on $(0,1)$, we have $z_0 \geq x/3$ and hence $q \geq x/6$.

For the upper bound, $q \leq 7x/25$ is equivalent to $\sigma \geq [(25-7x)/(25+7x)]^2$. Subtracting the right-hand side from σ and clearing the strictly positive denominators on $(0,1)$ leaves a numerator proportional to

$$x\big(2709x^7 + 5733x^6 - 4911x^5 - 18343x^4 + 8332x^3 + 12932x^2 - 8800x + 2400\big).$$

Substituting $x = y/(1+y)$ $(y>0)$ and multiplying by $(1+y)^8$ yields the exact polynomial

$$2400y + 8000y^2 + 10532y^3 + 24992y^4 + 52305y^5 + 37772y^6 + 2870y^7 + 52y^8,$$

whose coefficients are all strictly positive, so the numerator never vanishes on $(0,1)$. Evaluating the difference $\sigma - [(25-7x)/(25+7x)]^2$ at $x = 1/2$ gives the positive value $555692/29127285 > 0$, proving $q \leq 7x/25$ throughout $(0,1)$. □

Proof of Theorem 1.5. For $h\in(0,1)$ with $x := \sqrt{h}$, let $V \subset C_h \subset C_0$ be the half-disc preimage component in the proof of Theorem 1.3, with marked height $X \geq J(w)$ and critical heights x_j. Because P is unbranched on V away from the critical points of P, every critical point of the normalized finite Blaschke product $B : \mathbb{D}\to\mathbb{D}$ on V has multiplicity at least m. By Lemma 8.1,

$|C(0)| \le \sigma^{m/(m+1)}$; setting $a := \operatorname{artanh} q$ gives $\sqrt{\sigma} = (1-q)/(1+q) = e^{-2a}$ and hence $|C(0)| \le e^{-2\lambda a}$, where $\lambda = 2m/(m+1) \in [1,2)$. By Schwarz–Pick and superadditivity of artanh as in (5),

$$\sum_{c\in V} \frac{x_c}{X} \ge \frac{1-|C(0)|}{1+|C(0)|} \ge \tanh(\lambda a).$$

Since $\lambda - 1 \in [0,1)$ and tanh is concave on $[0,\infty)$, we have $\tanh((\lambda-1)a) \ge (\lambda-1)\tanh a = (\lambda-1)q$; by the tangent-hyperbolic addition formula,

$$\tanh(\lambda a) = \frac{\tanh a + \tanh((\lambda-1)a)}{1+\tanh a\tanh((\lambda-1)a)} \ge \frac{\lambda q}{1+(\lambda-1)q^2}.$$

Applying Lemma 8.2 ($q^2 \le \frac{49}{625}h$) together with the rational lower bound $[2J(w)/s^2]q > (12/25)(1-h)$ from Lemma 2.5 upgrades (9) to

$$M(h) = d\sum_{c\in C_h} \frac{\rho^2}{|c|^2} > \frac{12}{25}(1-h)\cdot \frac{\lambda}{1+\frac{49}{625}(\lambda-1)h} \qquad (0<h<1).$$

By the layer-cake identity of Theorem 1.3, $d\sum_{c\in C_0}|P(c)/c|^2 \ge \int_0^1 2h\,M(h)\,dh$. With respect to the probability density $6h(1-h)\,dh$ on $[0,1]$, the mean of h is $1/2$; applying Jensen's inequality to the convex function $h \mapsto [1+\frac{49}{625}(\lambda-1)h]^{-1}$ yields

$$d\sum_{c\in C_0}\left|\frac{P(c)}{c}\right|^2 > \frac{4}{25}\cdot\frac{\lambda}{1+\frac{49}{1250}(\lambda-1)} = A_m.$$

Since C_0 contains at most $d-1$ critical copies, $D(P) > \sqrt{A_m}/d$. Because A_m increases strictly with λ (hence with m), for $m \ge 4$ ($\lambda \ge 8/5$) we have $A_m \ge A_4 = 1600/6397 > 1/4$, giving $D(P) > 1/(2d)$. □

9. Effective large-degree linear improvements

While Theorem 1.2 establishes $\lim_{d\to\infty} d\,D_{\min}(d) = 1$ via compactness (with an ineffective threshold), refining Lemma 2.3 with explicit finite-disc curvature barriers yields both parts of Theorem 1.6 completely effectively.

9.1. Sixth-power curvature comparison and $D > 11/(20d)$ for $d \ge 10^{75}$.

Lemma 9.1 (Sixth-power Blaschke curvature comparison)**.** *Let $B : \mathbb{D} \to \mathbb{D}$ be a finite Blaschke product of degree ≥ 2 with $B'(0) \ne 0$, hyperbolic derivative $\beta(z) = (1-|z|^2)|B'(z)|/(1-|B(z)|^2)$, and critical Blaschke product C. Suppose every zero u_j of C satisfies $1-|u_j|^2 \le 10^{-65}$. Write $U_0 := -\log|C(0)|$ and $f(v) := v/\sinh v$. With $k := 17/16$ and $\alpha := 19/20$,*

$$\beta(0) \ge \frac{f(k\sqrt{\alpha}\,U_0)}{1+3\cdot 10^{-4}}. \tag{29}$$

Proof. Take $r_0 \in (0,1)$ with $1-r_0^2 = 10^{-60}$, so C is zero-free on $\{|z| \le r_0\}$ and $U := -\log|C|$ is positive harmonic there. Put $T := \frac14(1-|z|^2)^2\Delta$, $Y := f(kU)^2$, $A_0 := 1-\alpha+\alpha Y$, and $m := f(kU)^\alpha$. Because all zeros of C lie outside $\{|z| < R\}$ with $1-R^2 = 10^{-65}$ (so U is positive harmonic on $\{|z|<R\} \supset \{|z|\le r_0\}$), half-plane Schwarz–Pick on the R-disc bounds $(1-|z|^2)^2|\nabla U|^2/(4U^2) \le [R(1-|z|^2)/(R^2-|z|^2)]^2 \le (1-10^{-5})^{-2} \le 1+4\cdot 10^{-5}$; combined with $T\log\beta = \beta^2-1$ and $U^2(\log f(kU))'' = Y-1$, this gives $T\log(\beta/m) \le \beta^2 - A_0 + e$ with $e := 4\cdot 10^{-5}$.

Set $w := \log(\beta/m)$ and $\tilde v := 1-e^{-6w}$. On $\{\tilde v \le 0\}$, put $y := e^w \le 1$; dropping the nonpositive gradient-square term in $T\tilde v$ gives $T\tilde v \le 6y^{-6}(y^2Y^\alpha - A_0 + e)$. Minimizing $1-\alpha+\alpha Y - y^2Y^\alpha$ over $Y \in [0,1]$ yields its minimum $(1-\alpha)(1-y^{40}) \ge (1-\alpha)(1-y^6)$ at $Y = y^{40}$. Hence on $\{\tilde v \le 0\}$,

$$T\tilde v \le \big(6(1-\alpha) - 6e\big)\tilde v + 6e \le c\,\tilde v + 6e, \qquad c := \frac{149}{500}.$$

By Lemma 2.3, $\beta \geq \operatorname{sech} U$, which with $\alpha k - 1 = 3/320$ implies $m/\beta \leq 64$ on $\{|z| \leq r_0\}$ (for $U \leq 1$, $m \leq 1$ and $\cosh 1 < 2$; for $U \geq 1$, $\cosh U \leq \frac{3}{5}e^U$ and $\sinh(kU) \geq \frac{2}{5}e^{kU}$ give $m/\beta \leq \frac{3k}{2} U e^{-3U/320} < 64$), so $\tilde{v} \geq -64^6$.

Define the radial barrier $G(r) := 10^{-3} + 64^6[(1-r_0^2)/(1-r^2)]^p$ with $p := 6/25$. Since $T(1-r^2)^{-p} = p(1+pr^2)(1-r^2)^{-p}$ and $p(1+p) = 186/625 < c$, we have $TG \leq c(G - 10^{-3})$. On $|z| = r_0$, $\tilde{v} + G \geq 0$, and at any negative interior minimum we would have $\tilde{v} < 0$ and $T(\tilde{v}+G) \leq c(\tilde{v}+G) + 6e - c\cdot 10^{-3} < 0$, impossible. Thus $\tilde{v}(0) + G(0) \geq 0$. Since $64^6 < 7\cdot 10^{10}$ and $10^{-60\cdot 6/25} = 10^{-14.4}$, we get $G(0) < 0.0017$ and hence $m(0)/\beta(0) \leq (1 + G(0))^{1/6} < 1 + 3 \cdot 10^{-4}$.

Finally, $\psi(t) := \log(\sinh(\sqrt{t})/\sqrt{t})$ is concave on $t \geq 0$ with $\psi(0) = 0$: the derivative of $(\coth u - 1/u)/u$ is nonpositive because $u \sinh(2u)/2 + u^2 - \cosh(2u) + 1 = \sum_{j \geq 3} (j/2 - 1) 2^{2j} u^{2j}/(2j)! \geq 0$. Concavity gives $\psi(\alpha t) \geq \alpha \psi(t)$, i.e., $f(v)^\alpha \geq f(\sqrt{\alpha}\, v)$, proving (29). □

Proof of Theorem 1.6(a). Assume $dD \leq 1$ (otherwise $D > 1/d > 11/(20d)$ immediately). On the band $h \in [1/20, 19/20]$ ($x = \sqrt{h} \in [1/\sqrt{20}, \sqrt{19/20}]$), let B be the half-disc auxiliary finite Blaschke product from the proof of Theorem 1.3, with marked height $X \geq J(w) > w - t = s(1-x)/x \geq (5/8)(1 - \sqrt{19/20}) > 1/64$, critical heights x_j, pseudohyperbolic target distance σ, and $q = (1 - \sqrt{\sigma})/(1 + \sqrt{\sigma})$. By Corollary 2.2 and $dD \leq 1$, $|P(c)| > \rho/20$ forces $|c| > d\rho/20$, giving $x_j \leq 200/d$ and hence $y_j := x_j/X \leq 12800/d$. By Schwarz–Pick, $|u_j| \geq (1 - y_j)/(1 + y_j)$, so $1 - |u_j|^2 \leq 4y_j \leq 51200/d$.

By Lemma 8.2, $x/6 \leq q \leq 7x/25 < 7/25$ on $[1/20, 19/20]$; since $x \geq 1/\sqrt{20} > 1/5$, we have $q > 1/30$, $1 - \sigma = 4q/(1+q)^2 \geq q > 1/30$, $\ell := 2 \operatorname{artanh} \sigma = \log((1+\sigma)/(1-\sigma)) \leq \log(2/q) < \log 60 < 6$, and (by Lemma 2.3) $U_0 = -\log|C(0)| \geq 2 \operatorname{artanh}(q) > 1/15 > 1/160$. Writing $l := 2 \operatorname{artanh}|z|$ for hyperbolic source distance, every critical point has $l(u_j) > \log(d/51200) > 160 > 12$ for $d \geq 10^{75}$, so the lifted radial path exits $\{l \leq 12\}$ before reaching its critical endpoint. For every $l \leq 12$, the disc automorphism identity preserves $1 - |u_j'|^2 \leq e^{12}(1 - |u_j|^2) \leq e^{12} \cdot 51200/d < 10^{-65}$ (as $e^{12} < 190000$ and $190000 \cdot 51200 < 10^{10}$).

By Harnack's inequality on the zero-free disc $\{|z| < R_*\}$ (where $R_* := 1 - 25600/d$, since $|u_j| \geq (1 - y_j)/(1 + y_j) \geq 1 - 2y_j \geq R_*$), $U(z) \leq U_0 e^l/(1 - 25600e^{12}/d)$ for $l \leq 12$, where $25600e^{12}/d < 10^{-4}$; setting $K := 259/250$ satisfies $(k\sqrt{\alpha})^2 = 5491/5120 < [K(1-10^{-4})]^2$. Applying Lemma 9.1 at every recentered point along the lifted inverse path up to its first crossing of $l = 12$ gives $\beta(z) \geq f(KU_0 e^l)/(1 + 3 \cdot 10^{-4})$. Integrating this lower speed against source arclength (which dominates $|dl|$) up to $l = 12$, where $G_1(v) := 2 \operatorname{artanh}(e^{-v})$ satisfies $G_1'(v) = -1/\sinh v$ and $G_1(KU_0 e^{12}) < 10^{-8}$ (as $KU_0 e^{12} > 40$), yields

$$G_1(KU_0) \leq (1 + 3 \cdot 10^{-4})\ell + G_1(KU_0 e^{12}) < \ell + \delta, \qquad \delta := \frac{181}{100000}.$$

Viewing q as a function of ℓ, we have $(\log q)' = -(1+\sigma)/(2\sqrt{\sigma})$; since $q \leq 7/25$ gives $\sqrt{\sigma} \geq 9/16 > 1/4$, $|(\log q)'| \leq 17/8$ on $[\ell, \ell + \delta]$. Thus $q' := e^{-17\delta/8} q > (996/1000) q$ satisfies $KU_0 > 4 \operatorname{artanh}(q') \geq 4q'$. Since $y_j < 10^{-3}$, factorwise Schwarz–Pick gives $U_0 \leq 2 \sum_j \operatorname{artanh}(y_j) \leq 2(10001/10000) \sum_j y_j$, whence

$$\sum_{c \in V} \frac{x_c}{X} > \frac{2 \cdot (996/1000)}{(259/250)(10001/10000)}\, q > \frac{48}{25}\, q.$$

Substituting this upgraded factor into (9) gives $M(h) > \frac{48}{25} \cdot \frac{12}{25}(1 - h)$ throughout $h \in [1/20, 19/20]$. Integrating the layer-cake identity of Theorem 1.3 over $[1/20, 19/20]$ retains fraction $1971/2000$ of the full integral $\int_0^1 2h(1-h)\, dh = 1/3$, yielding

$$d(d-1)D^2 \geq d \sum_{c \in C_0} \left| \frac{P(c)}{c} \right|^2 > \frac{1971}{2000} \cdot \frac{48}{25} \cdot \frac{4}{25} = \frac{378432}{1250000} > \Big(\frac{11}{20}\Big)^2,$$

and therefore $D(P) > 11/(20d)$. □

9.2. **Adjustable-exponent curvature comparison and Theorem 1.6(b).**

Lemma 9.2 (Adjustable-exponent Blaschke comparison)**.** *Let $B : \mathbb{D} \to \mathbb{D}$ be a finite Blaschke product of degree ≥ 2 with $B'(0) \neq 0$, hyperbolic derivative β, critical Blaschke product C, $U := -\log|C|$, and $f(u) := u/\sinh u$. Suppose every zero u_j of C satisfies $1 - |u_j|^2 \leq E$, and let $\varepsilon \in (0, 1/100]$ satisfy $\sqrt{E} \leq \varepsilon/8$. Then*

$$\beta(0) \geq \frac{f\big((1+2\varepsilon)U(0)\big)}{1 + (16/\varepsilon)E^{\varepsilon/8}}. \tag{30}$$

Proof. Set $\alpha := 1-\varepsilon$, $k := 1+2\varepsilon$, and $g := f(kU)^\alpha$. On the closed subdisc $\{1-|z|^2 \geq \sqrt{E}\}$, writing $R := \sqrt{1-E}$, we have $|z|^2 \leq 1 - \sqrt{E} < 1 - E = R^2$, so U is positive harmonic and half-plane Schwarz–Pick on the R-disc gives $(1-|z|^2)^2|\nabla U|^2/(4U^2) \leq [R(1-|z|^2)/(R^2-|z|^2)]^2 \leq 1 + 4\sqrt{E}$ (using $(1-b)^{-2} \leq 1 + 4b$ for $0 \leq b \leq 1/4$). With $T := \frac{1}{4}(1-|z|^2)^2\Delta$, $Y := f(kU)^2$, and $A_0 := 1 - \alpha + \alpha Y$, we obtain $T\log(\beta/g) \leq \beta^2 - A_0 + 4\sqrt{E}$.

Put $V_0 := 1 - g/\beta$. On $\{V_0 \leq 0\}$, write $y := \beta/g \leq 1$: then $TV_0 \leq y^{-1}T\log y \leq yY^\alpha - y^{-1}A_0 + 4\sqrt{E}\,y^{-1}$, and since $y \leq 1$ and $Y^\alpha \leq 1 - \alpha + \alpha Y = A_0$ by Young's inequality, we have $yY^\alpha \leq A_0$, giving $TV_0 \leq (A_0 - 4\sqrt{E})V_0 + 4\sqrt{E} \leq (\varepsilon/2)V_0 + 4\sqrt{E}$ (since $A_0 \geq 1 - \alpha = \varepsilon$ and $4\sqrt{E} \leq \varepsilon/2$). By Lemma 2.3, $\beta \geq \operatorname{sech} U$. For $U \leq 1$, $g/\beta \leq \cosh 1 < 2$; for $U \geq 1$, $\sinh(kU) \geq e^{kU}/3$ and $\alpha k - 1 = \varepsilon - 2\varepsilon^2 \geq \varepsilon/2$ give $g/\beta \leq 4Ue^{-\varepsilon U/2} \leq 8/\varepsilon$. Thus $V_0 \geq -8/\varepsilon$ on $\{1-|z|^2 = \sqrt{E}\}$.

Introduce $G(r) := (8\sqrt{E}/\varepsilon) + (8/\varepsilon)[\sqrt{E}/(1-r^2)]^p$ with $p := \varepsilon/4$. Since $p(1+p) \leq \varepsilon/2$, we have $TG \leq (\varepsilon/2)(G - 8\sqrt{E}/\varepsilon)$, while $V_0 + G \geq 0$ on the boundary $1 - r^2 = \sqrt{E}$. At any negative interior minimum, $T(V_0+G) \leq (\varepsilon/2)(V_0+G) < 0$, impossible. Evaluating $V_0(0) + G(0) \geq 0$ gives $g(0)/\beta(0) \leq 1 + (16/\varepsilon)E^{\varepsilon/8}$; since $g \geq f(kU)$ (as $f \leq 1$ and $\alpha < 1$), (30) follows. □

Proof of Theorem 1.6(b). Fix $\varepsilon \in (0, 1/100]$ and $d \geq (1/\varepsilon)^{100/\varepsilon}$; assume $dD \leq 1$ (else $dD > 1 > \sqrt{L(\varepsilon)}$ immediately). On the band $h \in [\varepsilon, 1-\varepsilon]$ ($x = \sqrt{h}$), the proof of Theorem 1.3 gives $X \geq w - t = s(1-x)/x \geq 5\varepsilon/16$, while Corollary 2.2 and $|P(c)| > h\rho \geq \varepsilon\rho$ give $x_j \leq 1/(2\varepsilon^2 d)$. Thus $y_j := x_j/X \leq 2/(\varepsilon^3 d)$ and $1 - |u_j|^2 \leq 8/(\varepsilon^3 d)$ uniformly on $[\varepsilon, 1-\varepsilon]$.

With $l := 2\operatorname{artanh}|z|$ as above, stop the lifted radial path at its first exit from the hyperbolic ball $\{l \leq S\}$ of radius $S := 4\log(1/\varepsilon)$ (all critical points satisfy $|u_j| \geq 1 - 4/(\varepsilon^3 d)$ while every point of $\{l \leq S\}$ satisfies $1 - |z| \geq e^{-S} = \varepsilon^4 > 4/(\varepsilon^3 d)$, so the critical endpoint lies strictly outside). At any z with $l \leq S$, recentering gives squared radial deficit at most $E_z \leq e^S \cdot 8/(\varepsilon^3 d) \leq \varepsilon^{-8}/d$, so $E_z^{\varepsilon/8} \leq \varepsilon^{(100-8\varepsilon)/8} \leq \varepsilon^{12}$ and $\sqrt{E_z} \leq \varepsilon/8$. Hence Lemma 9.2 applies throughout $\{l \leq S\}$ with denominator at most $1+\varepsilon^8$. Harnack on the zero-free disc $\{|z| < 1 - 4/(\varepsilon^3 d)\}$ gives $U(z) \leq U_0 e^l/(1 - 4/(\varepsilon^7 d)) \leq U_0 e^l/(1-\varepsilon^2)$, so setting $K := (1+2\varepsilon)/(1-\varepsilon^2) \leq 1 + 3\varepsilon$ gives $\beta(z) \geq f(KU_0 e^l)/(1+\varepsilon^8)$.

With $\ell := 2\operatorname{artanh}\sigma$ and $G_1(v) := 2\operatorname{artanh}(e^{-v})$, integrating along the lift to $\{l = S\}$ gives

$$G_1(KU_0) \leq (1+\varepsilon^8)\ell + G_1(KU_0 e^S).$$

By Lemma 8.2, $q \geq x/6 \geq \sqrt{\varepsilon}/6$, so $\ell \leq \log(2/q) \leq \log(12/\sqrt{\varepsilon}) \leq 1/\varepsilon$ and $U_0 \geq 2\operatorname{artanh}(q) \geq \sqrt{\varepsilon}/3$; thus $KU_0 e^S \geq \frac{1}{3}\varepsilon^{-7/2} \geq 8\log(1/\varepsilon)$ and $G_1(KU_0 e^S) \leq 3\varepsilon^8$, yielding $G_1(KU_0) \leq \ell + \delta$ with $\delta := 2\varepsilon^7$.

Set $\sigma' := \tanh((\ell+\delta)/2)$ and $q' := (1 - \sqrt{\sigma'})/(1 + \sqrt{\sigma'})$, so $e^{-KU_0} \leq \sigma'$. Since $\sigma' - \sigma \leq \delta/2$ and $(1-\sigma')/(1-\sigma) \geq e^{-\delta}$ (as $\frac{d}{dy}\log(1 - \tanh(y/2)) \geq -1$), we get

$$\frac{q'}{q} \geq \frac{e^{-\delta}}{(1 + \sqrt{\delta/2})^2} \geq 1 - \delta - 2\sqrt{\delta/2} \geq 1 - \varepsilon^2.$$

Hence $KU_0 \geq 4\,\mathrm{artanh}(q') \geq 4(1-\varepsilon^2)^3 q(1+q^2/3)$, while $y_j \leq 2/(\varepsilon^3 d) \leq \varepsilon/2$ gives $U_0 \leq 2(1+\varepsilon^2)\sum_j y_j$. Using $1/K \geq 1-3\varepsilon$ and $(1-3\varepsilon)(1-\varepsilon^2)^4 \geq 1-4\varepsilon$ gives

$$\sum_{c\in V} \frac{x_c}{X} \geq 2(1-4\varepsilon)q\Big(1+\frac{q^2}{3}\Big).$$

Since $q^2/3 \geq h/108$ by Lemma 8.2, inserting this into (9) yields

$$M(h) > \frac{24}{25}(1-4\varepsilon)(1-h)\Big(1+\frac{h}{108}\Big) \qquad (\varepsilon \leq h \leq 1-\varepsilon).$$

Integrating $2h\,M(h)$ over $[\varepsilon, 1-\varepsilon]$ retains fraction $1-6\varepsilon^2+4\varepsilon^3$ of $\int_0^1 2h(1-h)\,dh = 1/3$, and by symmetry $h \leftrightarrow 1-h$ the weighted average of h on $[\varepsilon,1-\varepsilon]$ with respect to $h(1-h)$ is exactly $1/2$, contributing $1+\frac{1}{216} = \frac{217}{216}$. Thus

$$d(d-1)D^2 \geq d\sum_{c\in C_0}\left|\frac{P(c)}{c}\right|^2 > \frac{8}{25}\cdot\frac{217}{216}(1-4\varepsilon)(1-6\varepsilon^2+4\varepsilon^3) = L(\varepsilon),$$

proving $dD > \sqrt{L(\varepsilon)}$. Taking $\varepsilon = 1/1000$ gives $(1/\varepsilon)^{100/\varepsilon} = 10^{300\,000}$ and $(1-4/1000)(1-6\cdot 10^{-6}) > 216/217$, whence $L(1/1000) > 8/25$ and $D > 2\sqrt{2}/(5d)$. Taking $\varepsilon := 200\log\log d/\log d$ for sufficiently large d satisfies $\varepsilon \leq 1/100$ and $(100/\varepsilon)\log(1/\varepsilon) \leq \frac{1}{2}\log d$; since $(1-4\varepsilon)(1-6\varepsilon^2+4\varepsilon^3) > (1-3\varepsilon)^2$ on $(0,1/100]$ and $3\cdot 200\sqrt{217/675} < 350$, we obtain $d\,D_{\min}(d) > \sqrt{\frac{8}{25}\cdot\frac{217}{216}} - 350\frac{\log\log d}{\log d}$. □

10. Discussion

Before this work, the dual Smale mean value conjecture of Ng [15] and Dubinin and Sugawa [6] was known only for degrees $d \leq 7$ [24, 6, 9, 10], while the best unconditional lower bound in general degree was $(d-1/2)^{1/d}/d^2$ [11]. The theorems proved in this expanded version resolve the conjecture for every sufficiently large degree $d \geq d_0$ (Theorem 1.1), establish the sharp asymptotic constant $\lim d\,D_{\min}(d) = 1$ with rigidity (Theorem 1.2), upgrade all-degree lower bounds to linear order $D(P,a) > 2/(5d)$ (Theorem 1.3), settle the centered-circle ($d \geq 2$) and uniform-annulus ($R_{\mathrm{hit}}/r \leq 1.01$ for $d \geq 8$; $M < \sqrt{2}$ for large d) classes (Theorem 1.4), prove the sharp ramified multiplicity floor $D(P,a) > 1/(2d)$ for multiplicity ≥ 4 (Theorem 1.5), and establish explicit effective large-degree improvements $11/(20d)$ ($d \geq 10^{75}$) and $2\sqrt{2}/(5d)$ ($d \geq 10^{300\,000}$) without compactness (Theorem 1.6).

Together with $d \leq 7$ [10], the general conjecture now stands open only on the finite intermediate range $8 \leq d < d_0$. Closing that finite range invites complementary lines of attack: making the compactness steps in Sections 4–6 effective (or finding a direct finite-degree version of the two-phase logarithmic-area comparison in Section 6), pushing low-degree certification beyond $d = 7$, and narrowing the gap between our effective linear constants ($2/5$, $11/20$, $2\sqrt{2}/5$) and 1 in arbitrary degree.

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