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ELEMENTARY SYMMETRIC FEWNOMIAL SYSTEMS: $B(2,3) \geq 13$, MAXIMUM GROWTH $\xi(2/5) \geq 13^{1/5}$, AND DIAGONAL IMPROVEMENT $B(5,5) \geq 130$

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Abstract. Let $B(n,k)$ denote the maximum number of distinct strictly positive nondegenerate real solutions of a system of n real Laurent polynomials in n variables supported on at most $n+k+1$ monomials, and let $\xi(\alpha) = \limsup_{n/(n+k)\to\alpha} B(n,k)^{1/(n+k)}$ be the asymptotic exponential root-growth constant along ray $\alpha \in (0,1)$. Across the classical published lower-bound families (Descartes, Bihan–Rojas–Sottile, El Hilany 2017, and Bihan–Santos–Spaenlehauer 2018), the largest established value of $\xi(\alpha)$ over all $\alpha \in (0,1)$ has been $7^{1/4} \approx 1.6265766$, attained on the diagonal $\alpha = 1/2$ via products of El Hilany's tropical construction $B(2,2) \geq 7$.

We construct and formally certify two completely elementary, zero-tropical symmetric bivariate Laurent systems: (i) a 6-monomial system proving $B(2,3) \geq 13$ (compared with the four-source product benchmark $C_{\mathrm{DP}}(2,3) = 7$), whose r-fold disjoint products yield $B(2r,3r) \geq 13^r$ and establish a new largest known exponential root-growth constant *off the diagonal*:

$$\xi\left(\tfrac{2}{5}\right) \;\geq\; 13^{1/5} \approx 1.6702777 \;>\; 1.6265766 \approx 7^{1/4} \qquad (+2.69\%,\;\; 13^4 = 28\,561 > 16\,807 = 7^5);$$

and (ii) a 5-monomial small-exponent *switchable* witness for $B(2,2) \geq 7$ whose deletion of a single non-constant monomial leaves 3 nondegenerate positive roots, giving $B(3,2) \geq 10$, $B(3,3) \geq 17$, and $B(n,2) \geq 3n+1$ ($\forall n \geq 2$) via universal d-fold monomial switching. Taking the disjoint product of $B(2,3) \geq 13$ and $B(3,2) \geq 10$ (sharing the constant monomial 1) also strictly improves the asymptotic diagonal bound:

$$B(5,5) \geq 130, \qquad B(5r,5r) \geq 130^r \quad (\forall r \geq 1), \qquad B(10,10) \geq 16\,900 > 16\,807 = 7^5,$$

raising the diagonal base from $\sqrt{7} \approx 2.6457513$ to $130^{1/5} \approx 2.6472117$ (+0.055%) and $\xi(1/2)$ to $130^{1/10} \approx 1.6270254$. All 19 headline theorems are machine-checked from first principles in pure Lean 4.36.0 / Mathlib4 (zero `sorry`, zero `native_decide`, strictly standard axioms) and independently certified to 260 decimal digits.

1. Introduction

Khovanskii's fewnomial theory [7] bounds the number of nondegenerate positive real solutions of a sparse polynomial system in n variables purely in terms of the number of distinct monomials $m = n+k+1$, independently of the degrees. Let $B(n,k)$ denote the supremum (over all n-tuples of real polynomials in n variables sharing at most $n+k+1$ distinct monomials) of the number of distinct strictly positive solutions $x \in \mathbb{R}^n_{>0}$ with $\det J_F(x) \neq 0$. While upper bounds were sharpened by Bihan and Sottile [4] to $B(n,k) \leq \frac{e^2+3}{4} 2^{\binom{k}{2}} n^k$, constructing sparse systems with large exponential root growth remains challenging.

To compare exponential root growth across aspect ratios $\alpha \in (0,1)$, one defines

$$\xi(\alpha) \;=\; \limsup_{\substack{n+k\to\infty \\ \frac{n}{n+k}\to\alpha}} B(n,k)^{\frac{1}{n+k}}.$$

Across the four classical published lower-bound families—Descartes' rule $B(1,k) = k+1$, the Bihan–Rojas–Sottile (BRS) linear circuits $B(n,1) = n+1$ [2], the Gale-dual constructions of Bihan,

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Santos, and Spaenlehauer [3], and El Hilany's tropical construction [5] giving $B(2,2) \geq 7$ (improving Haas [6] and Li–Rojas–Wang [8])—the largest known value of $\xi(\alpha)$ for *any* $\alpha \in (0,1)$ has been

$$\xi\left(\tfrac{1}{2}\right) \geq 7^{1/4} \approx 1.6265766, \qquad B(2m,2m)^{1/(2m)} \geq \sqrt{7} \approx 2.6457513, \tag{1}$$

attained on the diagonal $\alpha = 1/2$ by m-fold disjoint products of $B(2,2) \geq 7$. In a companion paper [1], we introduced *universal d-fold monomial switching*, a closed dimension-raising operator that strictly improves this four-source asymptotic envelope on $(0,1/2)\cup(1/2,1)$ while leaving $\alpha = 1/2$ and the global maximum $7^{1/4}$ unchanged.

1.1. **Main Results.** In this paper we construct two completely elementary symmetric bivariate Laurent systems reduced in closed form to 1-variable real functions (requiring zero tropical geometry or Viro patchworking).

Theorem 1.1 (Elementary Symmetric $B(2,3) \geq 13$ and Global Maximum of $\xi(\alpha)$)**.** *The 6-monomial ($k = 3$) bivariate Laurent system*

$$F_1(x,y) = -\frac{872683}{400000} + x^{20} + y^{20} - \frac{5016667}{5000000}\left(x^{-87}y^{23} + x^{23}y^{-87}\right) - \frac{3449251}{2500000}x^{-40}y^{-40} = 0, \tag{2}$$

$$F_2(x,y) = \left(x^{20} - y^{20}\right) + \left(x^{-87}y^{23} - x^{23}y^{-87}\right) = 0 \tag{3}$$

has at least 13 *distinct nondegenerate positive roots (6 symmetric off-diagonal pairs plus 1 diagonal root), improving the four-source published DP benchmark* $C_{\mathrm{DP}}(2,3) = 7$ *by* $+85.7\%$*. Taking r-fold disjoint products yields* $B(2r,3r) \geq 13^r$ *for all* $r \geq 1$*, hence*

$$\xi\left(\tfrac{2}{5}\right) \;\geq\; 13^{1/5} \approx 1.6702777 \;>\; 1.6265766 \approx 7^{1/4},$$

a $+2.69\%$ *improvement over the previous global maximum* (1) *(with exact integer comparison* $13^4 = 28\,561 > 16\,807 = 7^5$ *at* $n + k = 20$*), shifting the largest known value of* $\xi(\alpha)$ *off the diagonal to* $\alpha = 2/5$*.*

Theorem 1.2 (Elementary Small-Exponent Switchable $B(2,2) \geq 7$ and Lifts)**.** *The 5-monomial bivariate Laurent system with single-digit exponents*

$$p_1(x,y) = 1 + x^{-1}y - \frac{21}{20}\left(x^{-7}y^2 + xy^{-6}\right) - \frac{103}{40}x^{-1} = 0, \tag{4}$$

$$p_2(x,y) = 1 - x^{-1}y + x^{-7}y^2 - xy^{-6} = 0 \tag{5}$$

has at least 7 *distinct nondegenerate positive roots, and deleting the non-constant monomial* $w_* = x^{-1}$ *leaves at least* 3 *distinct nondegenerate positive roots. Applying universal d-fold monomial switching* [1] *yields:*

1. $B(2,2) \geq 7$ *(an elementary, zero-tropical proof of El Hilany's bound);*
2. $B(3,2) \geq 7 + 3 = 10$ *($d = 1$ lift), and iteratively* $B(n,2) \geq 3n + 1$ *for all* $n \geq 2$*;*
3. $B(3,3) \geq 2 \cdot 7 + 3 = 17$ *($d = 2$ lift), improving* $C_{\mathrm{DP}}(3,3) = 14$ *and* [1]*'s* 15*;*
4. $B(4,3) \geq 27$ *and* $B(4,4) \geq 44$ *(iterated lifts of* $B(3,3) \geq 17$*).*

Theorem 1.3 (Strict Diagonal Improvement $B(5,5) \geq 130$)**.** *Taking the disjoint product (sharing the constant monomial* 1*) of* $B(2,3) \geq 13$ *from Theorem 1.1 and* $B(3,2) \geq 10$ *from Theorem 1.2 yields an explicit sparse system of* 5 *polynomials in* 5 *variables supported on at most* 11 *monomials ($k = 5$) with at least* $13 \times 10 = 130$ *distinct strictly positive nondegenerate real roots:*

$$B(5,5) \;\geq\; 130, \qquad B(5r,5r) \;\geq\; 130^r \quad (\forall r \geq 1).$$

At $n = k = 10$ *this yields the explicit finite crossover*

$$B(10,10) \;\geq\; 130^2 = 16\,900 \;>\; 16\,807 = 7^5,$$

and asymptotically along the diagonal $\alpha = 1/2$,

$$\limsup_{n\to\infty} B(n,n)^{1/n} \;\geq\; 130^{1/5} \approx 2.6472117 \;>\; \sqrt{7} \quad (+0.055\%),$$
$$\xi\left(\tfrac{1}{2}\right) \;\geq\; 130^{1/10} \approx 1.6270254 \;>\; 7^{1/4} \quad (+0.028\%).$$

Corollary 1.4 (Additional Odd-Diagonal Bounds)**.** *Taking disjoint products of Theorem 1.3 with* $B(2,2) \geq 7$ *yields*

$$B(7,7) \geq 130 \cdot 7 = 910 \;(> 686), \qquad B(9,9) \geq 130 \cdot 7^2 = 6\,370 \;(> 4\,802).$$

Remark 1.5 (Scope, Small Even Diagonals, and Literature Boundary).
(1) Every Lean 4 theorem proves the lower-bound predicate `AttainsFewnomialBound n k R`: the existence of $\geq R$ distinct strictly positive simple ($\det J_F \neq 0$) real solutions on $\leq n+k+1$ monomials. We do not claim global 0-dimensionality of the complex solution set or sharp upper bounds ($= R$).
(2) On small even diagonal dimensions $n = k \in \{2,4,6,8\}$, products of $B(2,2) \geq 7$ $(7, 49, 343, 2401)$ remain ahead of or tied with Theorem 1.3; the first explicit even-diagonal crossover occurs at $n = k = 10$ $(16\,900 > 16\,807)$.
(3) Our automated benchmark C_{DP} covers strictly four published families (Descartes, BRS [2], El Hilany [5], and BSS [3]); our 2019–2026 literature search found no published improvement over $7^{1/4}$ or $C_{\mathrm{DP}}(2,3) = 7$. The companion preprint [1] tabulated $B(3,2) \geq 9$ and $B(3,3) \geq 15$ prior to the elementary witnesses proved here, which supersede those entries and strengthen the off-diagonal envelope around $\alpha = 2/5$.

2. General 1D Symmetric Reduction and Log-Jacobian Factorization

Both witnesses share a unified symmetric-antisymmetric structure: $F_1(y,x) = F_1(x,y)$ and $F_2(y,x) = -F_2(x,y)$. For any bivariate Laurent system (F_1, F_2), denote its logarithmic partial derivatives by

$$L_{i1}(x,y) = x\frac{\partial F_i}{\partial x}(x,y), \qquad L_{i2}(x,y) = y\frac{\partial F_i}{\partial y}(x,y) \qquad (i \in \{1,2\}).$$

Since $x, y > 0$, $\det J_F(x,y) = (xy)^{-1} \det J_{\log}(x,y)$, where $\det J_{\log} = L_{11}L_{22} - L_{12}L_{21}$.

Lemma 2.1 (Curve Chain-Rule Factorization)**.** *Let* $(x(u), y(u)) \in \mathbb{R}^2_{>0}$ *be a differentiable curve on an open interval* I *satisfying* $F_2(x(u), y(u)) = 0$ *identically on* I*, and let* $g(u) = F_1(x(u), y(u))$*. Set* $\alpha = x'(u)/x(u)$ *and* $\beta = y'(u)/y(u)$*. Then*

$$\big(L_{11}L_{22} - L_{12}L_{21}\big)(\alpha - \beta) \;=\; (L_{21} + L_{22})\, g'(u). \tag{6}$$

In particular, if $L_{21} + L_{22} \neq 0$ *and* $g'(u) \neq 0$*, then* $\det J_{\log}(x(u), y(u)) \neq 0$.

Proof. Differentiating $g(u) = F_1(x(u), y(u))$ and $0 = F_2(x(u), y(u))$ yields $g'(u) = L_{11}\alpha + L_{12}\beta$ and $0 = L_{21}\alpha + L_{22}\beta$. Then

$$(L_{21} + L_{22})(L_{11}\alpha + L_{12}\beta) - (L_{11} + L_{12})(L_{21}\alpha + L_{22}\beta) \;=\; (L_{11}L_{22} - L_{12}L_{21})(\alpha - \beta). \qquad \square$$

Lemma 2.2 (Diagonal Factorization)**.** *At any diagonal point* $(w,w) \in \mathbb{R}^2_{>0}$ *of a symmetric-antisymmetric system,* $L_{12} = L_{11}$ *and* $L_{22} = -L_{21}$*, hence* $\det J_{\log}(w,w) = -2L_{11}(w,w)L_{21}(w,w)$*. Thus* $L_{11}(w,w) \neq 0$ *and* $L_{21}(w,w) \neq 0$ *guarantee nondegeneracy.*

Because F_1 is symmetric and F_2 is antisymmetric, every nondegenerate off-diagonal root (x,y) with $x > y > 0$ produces a distinct nondegenerate symmetric twin (y,x) with $y < x$. Whenever a curve parametrization has $x(u)/y(u)$ strictly increasing in $u > 1$, any m roots $1 < u_1 < \cdots < u_m$ together with one diagonal root (w_d, w_d) are automatically pairwise distinct because their coordinate ratios x/y are strictly ordered:

$$\frac{y(u_m)}{x(u_m)} < \cdots < \frac{y(u_1)}{x(u_1)} < 1 < \frac{x(u_1)}{y(u_1)} < \cdots < \frac{x(u_m)}{y(u_m)}.$$

3. Proof of Theorem 1.1: Elementary Symmetric $B(2,3) \geq 13$

Consider the 6-monomial system (2)–(3), with exact rational parameters

$$c_0 = -\frac{872683}{400000}, \qquad p_1 = -\frac{5016667}{5000000}, \qquad p_2 = -\frac{3449251}{2500000}.$$

Because the exponents satisfy $20 - (-87 + 23) = 84 = 4 \cdot 21$ and $23 - (-87) = 110 = 10 \cdot 11$, the substitution

$$x(u) = w_{13}(u)\, t_{13}(u), \qquad y(u) = w_{13}(u)/t_{13}(u), \qquad t_{13}(u) = u^{1/10} \quad (u > 1)$$

eliminates all high powers:

$$x^{20} = w_{13}^{20} u^2, \quad y^{20} = w_{13}^{20} u^{-2}, \quad x^{-87} y^{23} = w_{13}^{-64} u^{-11}, \quad x^{23} y^{-87} = w_{13}^{-64} u^{11}, \quad x^{-40} y^{-40} = w_{13}^{-80}.$$

3.1. **Nonsingular Quotient Representation.** Setting $r_{13}(u) = w_{13}(u)^4$, the equation $F_2 = 0$ solves to

$$r_{13}(u)^{21} = \frac{u^{11} - u^{-11}}{u^2 - u^{-2}} =: R_{13}(u).$$

Although $2 \nmid 11$, the factor $u - u^{-1}$ cancels identically between numerator and denominator, yielding a completely nonsingular representation at $u = 1$:

$$R_{13}(u) = \frac{S_{10}(u)}{C(u)}, \qquad S_{10}(u) = 1 + \sum_{j=1}^{5} \big(u^{2j} + u^{-2j}\big), \qquad C(u) = u + u^{-1} \geq 2. \tag{7}$$

Setting $r_{13}(u) = R_{13}(u)^{1/21}$ and $w_{13}(u) = R_{13}(u)^{1/84}$, we obtain $F_2(x(u), y(u)) \equiv 0$ for all $u > 0$, and

$$r_{13}(u)^{20} F_1(x(u), y(u)) \;=\; G_{13}(u) \;:=\; p_2 + r_{13}(u)^4 H_{13}(u) + c_0 r_{13}(u)^{20},$$

where $H_{13}(u) = R_{13}(u)(u^2 + u^{-2}) + p_1(u^{11} + u^{-11}) = N_H(u)/C(u)$ with

$$N_H(u) = 2 + 2\sum_{j=1}^{4} \big(u^{2j} + u^{-2j}\big) - \frac{16667}{5000000}\big((u^{10} + u^{-10}) + (u^{12} + u^{-12})\big).$$

3.2. **Exact Odd-Symmetric Derivative Factorization.** Differentiating (7) yields

$$u C(u)^2 R_{13}'(u) = \sum_{m \in \{1,3,5,7,9\}} 2m\,(u^m - u^{-m}) + 9(u^{11} - u^{-11}) > 0 \qquad (\forall u > 1),$$

proving strict monotonicity of R_{13} and r_{13} on $(1, \infty)$ immediately. Differentiating $G_{13}(u)$ yields the exact odd-symmetric factorization

$$G_{13}'(u) \;=\; \frac{r_{13}(u)^4}{21\, u\, C(u)^3 R_{13}(u)} \Big(P_+(u) - P_-(u) - Q_+(u)\, r_{13}(u)^{16} \Big), \tag{8}$$

where $P_+(u)$ (10 odd pairs $k = 1, 3, \ldots, 19$), $P_-(u)$ (2 small odd pairs $k = 21, 23$), and $Q_+(u)$ (6 even pairs $k = 2, 4, \ldots, 12$) are strictly positive linear combinations of $u^k - u^{-k}$, hence individually strictly increasing on $(1, \infty)$. On any compact interval $[a, b] \subset (1, \infty)$ with rational bounds $r_{a,\mathrm{lo}} \leq r_{13}(a) \leq r_{a,\mathrm{hi}}$ and $r_{b,\mathrm{lo}} \leq r_{13}(b) \leq r_{b,\mathrm{hi}}$ (verified via 21st powers), the sign of G_{13}' throughout (a, b) is certified purely by exact rational arithmetic:

- $P_+(a) - P_-(b) - Q_+(b)\, r_{b,\mathrm{hi}}^{16} > 0 \implies G_{13}'(u) > 0$ on (a, b);
- $P_+(b) - P_-(a) - Q_+(a)\, r_{a,\mathrm{lo}}^{16} < 0 \implies G_{13}'(u) < 0$ on (a, b).

Along $F_2 = 0$, $L_{21} + L_{22} = 84(x^{20} - y^{20}) = 84 r_{13}(u)^5 (u^2 - u^{-2}) > 0$, so Lemma 2.1 certifies nondegeneracy at every zero.

3.3. **Exact Certification of All 13 Roots.** Using continued-fraction rational brackets (a, b) for u (at most 6-digit denominators) and $r_{13}(a), r_{13}(b)$ (at most 10-digit denominators), Lean 4's `norm_num` certifies all 6 off-diagonal root pairs:

Root pair	Exact bracket (a, b) for $u = t^{10}$	Approx. u^*	G'_{13}	Margin on $P_+ - P_- - Q_+ r^{16}$
#1 (×2)	(47343/46423, 65251/63983)	1.01981776	> 0	$+7.34 \times 10^{-4}$
#2 (×2)	(124161/116608, 86122/80883)	1.06477257	< 0	-1.04×10^{-3}
#3 (×2)	(82030/74289, 39346/35633)	1.10420116	> 0	$+1.41 \times 10^{-3}$
#4 (×2)	(7563/6740, 31658/28213)	1.12210683	< 0	-2.04×10^{-3}
#5 (×2)	(9821/7622, 14698/11407)	1.28850700	> 0	$+6.15$
#6 (×2)	(4070/1983, 3754/1829)	2.05246752	< 0	-1.65×10^{6}

For the 13th (diagonal) root (w, w) with $z = w^4$, we have

$$z^{20} F_1(w, w) = P_{\text{diag}}(z) := 2z^{25} + c_0 z^{20} + 2p_1 z^4 + p_2 = 0.$$

Setting $z_{\text{lo}} = 203281/187431$, exact arithmetic certifies simultaneously that

$$z_{\text{lo}}^{21} > \frac{11}{2}, \qquad P_{\text{diag}}(z_{\text{lo}}) < 0 < P_{\text{diag}}\left(\frac{3}{2}\right).$$

Hence there exists $z \in (z_{\text{lo}}, 3/2)$ with $F_1(z^{1/4}, z^{1/4}) = 0 = F_2(z^{1/4}, z^{1/4})$. At $w = z^{1/4}$, $L_{11}(w, w) = 20z^5 - 64p_1 z^{-16} - 40p_2 z^{-20} > 0$ (all three terms strictly positive), and $z^{16} L_{21}(w, w) = 20(z^{21} - \frac{11}{2}) > 0$. By Lemma 2.2, the diagonal root is nondegenerate. Since $x(u)/y(u) = u^{1/5}$ is strictly increasing on $(1, \infty)$, all 13 roots are pairwise distinct, proving Theorem 1.1.

4. Proof of Theorem 1.2: Elementary Switchable $B(2, 2) \geq 7$

Multiplying (4)–(5) by $x > 0$ yields the symmetric-antisymmetric system

$$F_1(c_0; x, y) = c_0 + x + y - \frac{21}{20}\left(x^{-6}y^2 + x^2 y^{-6}\right), \qquad F_2(x, y) = (x - y) + \left(x^{-6}y^2 - x^2 y^{-6}\right),$$

with $c_0 = -103/40$ in the ON state and $c_0 = 0$ in the OFF state (when the non-constant monomial $w_* = x^{-1}$ is erased from p_1). Parametrizing $x = wt$, $y = w/t$ $(t > 1)$ solves $F_2 = 0$ to $w_7(t) = R_7(t)^{1/5}$, where

$$R_7(t) = \frac{t^8 - t^{-8}}{t - t^{-1}} = (t + t^{-1}) + (t^3 + t^{-3}) + (t^5 + t^{-5}) + (t^7 + t^{-7}) > 0,$$

and $w_7(t)^4 F_1(c_0; w_7(t)t, w_7(t)/t) = H_7(t) + c_0 w_7(t)^4$, with

$$H_7(t) = 2 + 2(t^2 + t^{-2}) + 2(t^4 + t^{-4}) + 2(t^6 + t^{-6}) - \frac{1}{20}(t^8 + t^{-8}).$$

Along $F_2 = 0$, $L_{21} + L_{22} = 5(x - y) > 0$. Exact rational arithmetic on fifth-power bounds for w_7 certifies all 7 ON roots and all 3 OFF roots:

Root	Bracket (a, b)	Sign change	Certified deriv. sign	Type
ON #1 (×2)	(259/200, 162/125)	$G_7(a) > 0 > G_7(b)$	$5R_7 G'_7 < 0$	Off-diag pair
ON #2 (×2)	(927/500, 371/200)	$G_7(a) < 0 < G_7(b)$	$5R_7 G'_7 > 0$	Off-diag pair
ON #3 (×2)	(111/40, 347/125)	$G_7(a) > 0 > G_7(b)$	$5R_7 G'_7 < 0$	Off-diag pair
ON diag (×1)	$w \in (149/100,\ 3/2)$	$P(a) < 0 < P(b)$	$w^5 < (3/2)^5 < 8$	Diagonal
OFF #1 (×2)	(63/10, 13/2)	$H_7(a) > 0 > H_7(b)$	$H'_7 < 0$	Off-diag pair
OFF diag (×1)	$w = (21/20)^{1/5}$	exact	$w^5 = 21/20 < 8$	Diagonal

For diagonal roots, $w^4 L_{11}(w, w) = w^5 + \frac{21}{5} > 0$ and $w^4 L_{21}(w, w) = w^5 - 8 < 0$ for $w < 3/2$. This proves `switchable_B22_7_3`, and applying universal d-fold monomial switching (`SwitchableSystem.lift` [1]) yields all parts of Theorem 1.2.

5. Proof of Theorem 1.3

Proof of Theorem 1.3. Let $S_1 \subset \mathbb{Z}^2$ be the 6-monomial support of Theorem 1.1 (containing 0) and $S_2 \subset \mathbb{Z}^3$ the 6-monomial support of the $d = 1$ lift $B(3,2) \geq 10$ from Theorem 1.2 (containing 0). Embedding S_1 in coordinates (x_1, x_2) and S_2 in (x_3, x_4, x_5) yields a 5-variable Laurent system sharing the constant monomial $0 \in \mathbb{Z}^5$, so its support cardinality is at most $6+6-1 = 11 = 5+5+1$, and its positive nondegenerate root set is the Cartesian product of the 13 and 10 roots, giving 130 distinct nondegenerate positive roots. Clearing negative exponents uniformly by a single positive monomial $\prod_{j=1}^{5} x_j^{s_j}$ preserves support size and nondegenerate positive roots, establishing $B(5,5) \geq 130$. Iterating the disjoint product $r \geq 1$ times yields $B(5r, 5r) \geq 130^r$. Finally, $7^5 = 16\,807 < 16\,900 = 130^2$ implies $\sqrt{7} < 130^{1/5}$ and $7^{1/4} < 130^{1/10}$. □

6. Machine Verification and 260-Digit Independent Certification

6.1. **Pure Lean 4.36.0 Formalization.** All results in this paper are formalized in Lean 4.36.0 / Mathlib4 across seven modules (`PrimalSystem`, `RootPersistence`, `SwitchLift`, `SymmetricReduction`, `WitnessB22`, `WitnessB23`, and `DiagonalRecord`). Automated auditing via `DiagonalFewnomial/Audit.lean` confirms that all 19 headline theorems depend strictly on the standard foundational axioms (`propext`, `Classical.choice`, `Quot.sound`), with zero `sorry`, zero `admit`, and zero `native_decide`.

6.2. **Independent 260-Digit Numerical Certification.** As an independent coordinate-level check (including explicit $\varepsilon = 10^{-8}$ monomial-switching realizations in 3D and 5D), the script `scripts/verify_diagonal_records.py` refines all 170 roots across all six explicit systems in 260-digit `mpmath` arithmetic, checking both the local-scale residual $\|F(x)\|_2/(\|J_F(x)\|_F\|x\|_2)$ and the relative Newton step $\|J_F(x)^{-1}F(x)\|_2/\|x\|_2$ alongside the exact four-source DP values $C_{\mathrm{DP}}(n,k)$:

Explicit system	(n,k,m)	Roots	Max local res.	Max rel. step	$\min \lvert\det J_{\log}\rvert$	Min sep.
Witness 2: $B(2,3) \geq 13$	$(2,3,6)$	13	6.64×10^{-262}	3.31×10^{-255}	9.67×10^{-6}	0.00258
Witness 1 ON: $B(2,2) \geq 7$	$(2,2,5)$	7	2.68×10^{-262}	1.51×10^{-259}	5.31×10^{-2}	0.746
Witness 1 OFF: 3 roots	$(2,1,4)$	3	4.29×10^{-262}	2.85×10^{-260}	2.44×10^{-1}	85.59
Lifted 3D ($d=1$): $B(3,2) \geq 10$	$(3,2,6)$	10	6.35×10^{-262}	1.21×10^{-259}	2.44×10^{-9}	0.746
Lifted 3D ($d=2$): $B(3,3) \geq 17$	$(3,3,7)$	17	2.66×10^{-261}	2.05×10^{-259}	2.44×10^{-9}	0.746
Product 5D: $B(5,5) \geq 130$	$(5,5,11)$	**130**	$\mathbf{5.00 \times 10^{-262}}$	$\mathbf{2.25 \times 10^{-255}}$	2.36×10^{-14}	0.00258

References

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