A COMPACT HOMOGENEOUS METRIC SPACE WHOSE LIPSCHITZ-FREE SPACE DOES NOT CONTAIN ITS COUNTABLE ℓ_1-SUM

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Abstract. We construct an infinite compact abelian group M with a compatible translation-invariant metric such that its Lipschitz-free space contains no isomorphic copy of its countable ℓ_1-sum. More precisely, for every $D \geq 1$ and all sufficiently large r, there is no linear embedding of $\ell_1^r(\mathcal{F}(M))$ into $\mathcal{F}(M)$ with distortion at most D.

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1. Introduction

Let (X, d, o) be a pointed metric space. Its real Lipschitz-free space $\mathcal{F}(X)$ is the closed linear span of the evaluation functionals δ_x in $\mathrm{Lip}_0(X)^*$, where $\mathrm{Lip}_0(X)$ consists of the real-valued Lipschitz functions vanishing at o, with the Lipschitz constant as norm. Thus $\delta_o = 0$ and $\|\delta_x - \delta_y\| = d(x, y)$. If Y is a Banach space and $f\colon X \to Y$ is Lipschitz with $f(o) = 0$, there is a unique bounded linear operator $\widehat{f}\colon \mathcal{F}(X) \to Y$ such that $\widehat{f}(\delta_x) = f(x)$ for every $x \in X$, and $\|\widehat{f}\| = \mathrm{Lip}(f)$; [3, p. 92]. All Banach spaces below are real.

For every infinite metric space M, the Lipschitz-free space $\mathcal{F}(M)$ contains a complemented subspace isomorphic to ℓ_1; [2, Theorem 1(i)]. Kaufmann [5, Theorem 3.1] proved that for every Banach space X, the space $\mathcal{F}(X)$ is isomorphic to the countable ℓ_1-sum of copies of itself. Aliaga and Medina established the same result for nets in Banach spaces [1, Theorem 3.6]. They asked whether the conclusion holds for every infinite metric space M, and in particular whether it holds whenever M is homogeneous [1, Question 2]. Here homogeneity means that the full isometry group acts transitively on M. The following theorem answers both parts of their question in the negative. We write $\ell_1^r(E)$ for the ℓ_1-sum of r copies of E. The distortion of a linear embedding T is $\|T\|\,\|T^{-1}\|$, where the inverse is defined on the range of T.

Theorem 1. *There is an infinite compact abelian group M with a compatible translation-invariant metric such that $E = \mathcal{F}(M)$ has the following property. For every $D \geq 1$ there is an $r_0 \in \mathbb{N}$ such that, for $r \geq r_0$, no linear embedding of $\ell_1^r(E)$ into E has distortion at most D. In particular, E contains no subspace isomorphic to $\ell_1(\mathbb{N}; E)$.*

2. Finite groups of large distortion

For a finite metric space (V, σ), write $c_1(V, \sigma)$ for the infimum of the bi-Lipschitz distortions of its embeddings into ℓ_1. The following lemma is a consequence of the estimate of Mendel and Naor [6, Remark 7.5] for the ℓ_1-distortion of finite grids equipped with the maximum metric. For completeness, we give a direct proof using expander graphs. A finite graph with vertex set V has edge expansion at least h if, for every $A \subseteq V$ with $0 < |A| \leq |V|/2$, at least $h|A|$ edges join A to $V \setminus A$.

Lemma 2. *There are finite abelian groups H_n, $n \geq 2$, with translation-invariant metrics b_n of minimum positive distance one, such that the metrics $\sigma_n = b_n^{1/n}$ satisfy $c_1(H_n, \sigma_n) \to \infty$ as $n \to \infty$.*

Proof. By Kassabov's theorem [4], there exist constants $q \geq 3$ and $h > 0$ and a family of finite Cayley graphs of alternating groups with arbitrarily many vertices, each of degree at most q and edge expansion at least h.

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Fix $n \geq 2$. Choose a graph $G_n = (V_n, \mathcal{E}_n)$ from this family with $v := |V_n|$ large enough that

$$r_n := \left\lfloor \tfrac{1}{2} \log_q v \right\rfloor - 1 \geq n^n.$$

Let d_n denote the graph metric on V_n, so that $d_n(x, y)$ is the number of edges in a shortest path from x to y. By the L_1-valued Poincaré inequality [7, Theorem 4.7], every map $g\colon V_n \to \ell_1$ satisfies

$$\frac{1}{v^2} \sum_{x,y \in V_n} \|g(x) - g(y)\|_1 \leq \frac{2}{hv} \sum_{\{x,y\} \in \mathcal{E}_n} \|g(x) - g(y)\|_1.$$

Now let $g\colon (V_n, d_n^{1/n}) \to \ell_1$ be an embedding of distortion D, normalized so that $\mathrm{Lip}(g) = 1$. For every edge $\{x, y\} \in \mathcal{E}_n$, we have $\|g(x) - g(y)\|_1 \leq 1$. Since $2|\mathcal{E}_n| \leq qv$, the right-hand side is at most q/h. A graph ball of radius r_n has at most $q^{r_n+1} \leq \sqrt{v} \leq v/2$ vertices. Thus each x has at least $v/2$ choices of y with $d_n(x, y) > r_n$, giving at least $v^2/2$ such ordered pairs. Since $\mathrm{Lip}(g^{-1}) = D$, each of these pairs satisfies

$$\|g(x) - g(y)\|_1 \geq \frac{d_n(x, y)^{1/n}}{D} \geq \frac{r_n^{1/n}}{D}.$$

Their contribution to the average, together with the upper bound from the Poincaré inequality, gives

$$\frac{r_n^{1/n}}{2D} \leq \frac{1}{v^2} \sum_{x,y \in V_n} \|g(x) - g(y)\|_1 \leq \frac{q}{h}.$$

Taking the infimum over all such embeddings and using $r_n \geq n^n$, we obtain $c_1(V_n, d_n^{1/n}) \geq (h/2q)\, r_n^{1/n} \geq (h/2q)\, n$. Put $L_n = \mathrm{diam}(V_n, d_n)$ and $H_n = (\mathbb{Z}/(2L_n + 1)\mathbb{Z})^{V_n}$. Give the cyclic group $C_n = \mathbb{Z}/(2L_n + 1)\mathbb{Z}$ its cycle metric d_{cyc}. For $a, b \in C_n$, represented by integers in $\{0, \ldots, 2L_n\}$, this is the length of the shorter of the two paths around the cycle:

$$d_{\mathrm{cyc}}(a, b) = \min\{|a - b|,\, 2L_n + 1 - |a - b|\}.$$

On the product group $H_n = C_n^{V_n}$, define the metric b_n by

$$b_n(x, y) = \max_{v \in V_n} d_{\mathrm{cyc}}(x_v, y_v), \qquad x = (x_v)_{v \in V_n}, \quad y = (y_v)_{v \in V_n}.$$

The metric b_n is translation-invariant and has minimum positive distance one. Define $\Phi_n\colon V_n \to H_n$ by

$$\Phi_n(x)_v = d_n(x, v) \pmod{2L_n + 1} \qquad (v \in V_n).$$

Fix $x, y \in V_n$. For each $v \in V_n$, the entries $d_n(x, v)$ and $d_n(y, v)$ both lie in $[0, L_n]$, so their cycle distance is $|d_n(x, v) - d_n(y, v)|$. By the triangle inequality this is at most $d_n(x, y)$. Now take the coordinate indexed by x itself. In that coordinate the vector $\Phi_n(x)$ has entry $d_n(x, x) = 0$, while $\Phi_n(y)$ has entry $d_n(y, x) = d_n(x, y)$. Their distance in this coordinate is therefore exactly $d_n(x, y)$. Taking the maximum over all coordinates gives $b_n(\Phi_n(x), \Phi_n(y)) = d_n(x, y)$, so Φ_n is an isometric embedding. The map remains isometric after both metrics are raised to the power $1/n$. Restricting any embedding of (H_n, σ_n) into ℓ_1 to $\Phi_n(V_n)$ gives $c_1(H_n, \sigma_n) \geq c_1(V_n, d_n^{1/n}) \geq (h/2q)n$. □

3. A homogeneous product and its free space

For each n use the identity 0_n as base point of H_n and write $E_n = \mathcal{F}(H_n, \sigma_n)$ and $\Delta_n = \mathrm{diam}(H_n, \sigma_n) \geq 1$. Take $a_2 = 1$ and set $a_{n+1} = a_n/(4\Delta_{n+1})$. On the product group $M = \prod_{n \geq 2} H_n$, with identity o, put

$$d(x, y) = \sup_{n \geq 2} a_n \sigma_n(x_n, y_n).$$

This is a translation-invariant metric. If j is the first index at which x and y differ, then

$$d(x, y) = a_j \sigma_j(x_j, y_j), \qquad \sum_{n > j} a_n \Delta_n \leq a_j/3. \tag{1}$$

Indeed, $a_{n+1}\Delta_{n+1} = a_n/4$ and $a_{n+1} \le a_n/4$, whereas $\sigma_j(x_j, y_j) \ge 1$. It follows that agreement on long prefixes gives arbitrarily small distances, while disagreement on any fixed finite prefix gives a positive lower bound. Thus d induces the product topology. In particular, M is infinite and compact, and translations act transitively by isometries.

Let $P_n = \prod_{2\le j<n} H_j$, with P_2 consisting of the empty prefix. Form the Banach space

$$S = \left(\bigoplus_{n\ge 2} \bigoplus_{p\in P_n} E_n \right)_{\ell_1}.$$

We denote its (n,p)-th coordinate inclusion by $\iota_{n,p}$. In particular, the multiplicity of each level is finite.

Lemma 3. *There is an isomorphism $J\colon \mathcal{F}(M) \to S$ with $\|J\| \le 5/3$ and $\|J^{-1}\| \le 1$.*

Proof. For $x = (x_n)_{n\ge 2} \in M = \prod_{n\ge 2} H_n$, write $x_{<n} = (x_2, \dots, x_{n-1}) \in P_n$ for its prefix, with $x_{<2}$ the empty prefix. Define $f\colon M \to S$ by

$$f(x) = \sum_{n\ge 2} a_n\, \iota_{n,x_{<n}}(\delta_{x_n}).$$

Here $\delta_{x_n} \in E_n$ is the evaluation functional at $x_n \in H_n$, and $\iota_{n,x_{<n}}\colon E_n \to S$ is the canonical inclusion into the summand indexed by $(n, x_{<n})$. The sum converges absolutely by (1), and $f(o) = 0$. Let $x, y \in M$ be distinct and set $j = \min\{n \ge 2 : x_n \ne y_n\}$. For $n < j$, we have $x_{<n} = y_{<n}$ and $x_n = y_n$, so the corresponding terms cancel in $f(x) - f(y)$. Writing $p = x_{<j} = y_{<j}$, we obtain

$$\|a_j\, \iota_{j,p}(\delta_{x_j} - \delta_{y_j})\|_S = a_j \|\delta_{x_j} - \delta_{y_j}\|_{E_j} = a_j \sigma_j(x_j, y_j) = d(x, y).$$

For $n > j$, the prefixes $x_{<n}$ and $y_{<n}$ differ at coordinate j. Hence $\iota_{n,x_{<n}}(\delta_{x_n})$ and $\iota_{n,y_{<n}}(\delta_{y_n})$ lie in different summands of the ℓ_1-sum S, so their norms add when we take the difference. Since $\|\delta_u\|_{E_n} = \sigma_n(u, 0_n) \le \Delta_n$ for every $u \in H_n$, we have

$$\|a_n \iota_{n,x_{<n}}(\delta_{x_n}) - a_n \iota_{n,y_{<n}}(\delta_{y_n})\|_S = a_n(\sigma_n(x_n, 0_n) + \sigma_n(y_n, 0_n)) \le 2a_n\Delta_n.$$

Summing over $n > j$ and using $\sum_{n>j} a_n\Delta_n \le a_j/3$, we obtain

$$\|f(x) - f(y)\|_S \le d(x, y) + 2\sum_{n>j} a_n\Delta_n \le d(x, y) + \tfrac{2}{3}a_j \le \tfrac{5}{3} d(x, y),$$

where the last inequality follows from $d(x, y) = a_j\sigma_j(x_j, y_j) \ge a_j$, since $x_j \ne y_j$ and the minimum positive σ_j-distance is one. Thus $\mathrm{Lip}(f) \le 5/3$. By the universal property of $\mathcal{F}(M)$, the map f induces a unique bounded linear operator $J = \widehat{f}\colon \mathcal{F}(M) \to S$ satisfying $J(\delta_x) = f(x)$ for all $x \in M$, with $\|J\| = \mathrm{Lip}(f) \le 5/3$. Fix $n \ge 2$ and $p \in P_n = \prod_{2\le k<n} H_k$. Since p specifies precisely the coordinates indexed by $k < n$, appending $u \in H_n$ gives a prefix $pu \in P_{n+1}$ with n-th coordinate u. Write $\bar{p}$ and $\overline{pu}$ for the points of M obtained by filling the remaining coordinates with their respective identity elements. Define

$$T_{n,p}\colon (H_n, \sigma_n) \longrightarrow \mathcal{F}(M), \qquad T_{n,p}(u) = \frac{\delta_{\overline{pu}} - \delta_{\bar{p}}}{a_n}.$$

Since $\overline{p0_n} = \bar{p}$, we have $T_{n,p}(0_n) = 0$. For $u, v \in H_n$, the points $\overline{pu}$ and $\overline{pv}$ have the same prefix p before coordinate n and the same identity coordinates after it. Thus only coordinate n can contribute to their distance, and we obtain

$$\|T_{n,p}(u) - T_{n,p}(v)\|_{\mathcal{F}(M)} = \frac{d(\overline{pu}, \overline{pv})}{a_n} = \frac{a_n\sigma_n(u, v)}{a_n} = \sigma_n(u, v).$$

Hence $T_{n,p}$ is an isometric embedding. By the universal property of $E_n = \mathcal{F}(H_n, \sigma_n)$, there is a unique bounded linear map $\widehat{T}_{n,p}\colon E_n \to \mathcal{F}(M)$ satisfying

$$\widehat{T}_{n,p}(\delta_u) = \frac{\delta_{\overline{pu}} - \delta_{\bar{p}}}{a_n} \qquad (u \in H_n),$$

and $\|\widehat{T}_{n,p}\| = \mathrm{Lip}(T_{n,p}) = 1$. For $\mu = (\mu_{n,p})_{n\geq 2,\, p\in P_n} \in S$, with $\mu_{n,p} \in E_n$, define

$$R\colon S \longrightarrow \mathcal{F}(M), \qquad R\mu = \sum_{n\geq 2}\sum_{p\in P_n} \widehat{T}_{n,p}(\mu_{n,p}).$$

Since S carries the ℓ_1-sum norm,

$$\sum_{n\geq 2}\sum_{p\in P_n} \|\widehat{T}_{n,p}(\mu_{n,p})\|_{\mathcal{F}(M)} \leq \sum_{n\geq 2}\sum_{p\in P_n} \|\mu_{n,p}\|_{E_n} = \|\mu\|_S.$$

Thus the series converges absolutely in $\mathcal{F}(M)$ and defines a linear operator satisfying $\|R\mu\| \leq \|\mu\|_S$. In particular, $\|R\| \leq 1$. For $x = (x_n)_{n\geq 2} \in M$, set $x^{(1)} = o$ and, for $N \geq 2$, let

$$x^{(N)} = (x_2, \ldots, x_N, 0_{N+1}, 0_{N+2}, \ldots) \in M.$$

For each $n \geq 2$, put $p = x_{<n}$. Completing p with identity coordinates gives $\bar{p} = x^{(n-1)}$, while appending x_n before completing gives $\overline{px_n} = x^{(n)}$. By the definition of R,

$$R(a_n\, \iota_{n,p}(\delta_{x_n})) = a_n\, \widehat{T}_{n,p}(\delta_{x_n}) = a_n\, \frac{\delta_{\overline{px_n}} - \delta_{\bar{p}}}{a_n} = \delta_{x^{(n)}} - \delta_{x^{(n-1)}}.$$

Summing over $2 \leq n \leq N$ gives the telescoping identity

$$R\left(\sum_{n=2}^{N} a_n\, \iota_{n,x_{<n}}(\delta_{x_n})\right) = \sum_{n=2}^{N} \big(\delta_{x^{(n)}} - \delta_{x^{(n-1)}}\big) = \delta_{x^{(N)}} - \delta_o = \delta_{x^{(N)}},$$

since $\delta_o = 0$. The points $x^{(N)}$ and x agree through coordinate N, so by (1),

$$\|\delta_{x^{(N)}} - \delta_x\|_{\mathcal{F}(M)} = d(x^{(N)}, x) \leq \sum_{n>N} a_n \Delta_n \leq \frac{a_N}{3} \longrightarrow 0.$$

The partial sums defining $f(x)$ converge in S. By continuity of R, we have $RJ\delta_x = R(f(x)) = \delta_x$ for every $x \in M$. The vectors δ_x span a dense subspace of $\mathcal{F}(M)$, so $RJ = I_{\mathcal{F}(M)}$. Conversely, fix $n \geq 2$, $p \in P_n$, and $u \in H_n$. In $f(\overline{pu}) - f(\bar{p})$, the terms with index $k < n$ cancel because the two points have the same prefix p. For $k > n$, both terms vanish since $\delta_{0_k} = 0$. Thus $f(\overline{pu}) - f(\bar{p}) = a_n \iota_{n,p}(\delta_u)$, and hence

$$JR\big(\iota_{n,p}(\delta_u)\big) = \frac{J\delta_{\overline{pu}} - J\delta_{\bar{p}}}{a_n} = \frac{f(\overline{pu}) - f(\bar{p})}{a_n} = \iota_{n,p}(\delta_u).$$

These vectors span a dense subspace of S, so $JR = I_S$ by continuity. Therefore $J^{-1} = R$ and $\|J^{-1}\| \leq 1$. □

4. Factorization on finite snowflakes

Lemma 4. *Let (X, d, o) be a finite pointed metric space, let $0 < \alpha \leq \frac{1}{2}$, and let Y be a normed space of dimension $k \geq 1$. Every linear operator $T\colon \mathcal{F}(X, d^\alpha) \to Y$ admits a factorization*

$$\mathcal{F}(X, d^\alpha) \xrightarrow{\ A\ } \ell_1^L \xrightarrow{\ B\ } Y, \qquad T = BA,$$

for some $L \in \mathbb{N}$, with $\|B\| \leq 1$ and $\|A\| \leq 48(4 \cdot 5^k)^\alpha \|T\|$.

Proof. The assertion is immediate for $T = 0$. By rescaling, assume that $\|T\| = 1$. Put $f(x) = T\delta_x$. Then $f(o) = 0$ and $\|f(x) - f(y)\| \leq d(x, y)^\alpha$.

For each dyadic scale $s = 2^j$, $j \in \mathbb{Z}$, choose a maximal s-separated subset D_s of $f(X)$. For $a \in D_s$, put

$$U_{s,a} = \{x \in X : \|f(x) - a\| < 2s\}, \qquad w_{s,a}(x) = \min\{s^{1/\alpha}, d(x, X \setminus U_{s,a})\},$$

where $d(x, \varnothing) = +\infty$. Write $w_s(x) = (w_{s,a}(x))_{a\in D_s}$. For each x, maximality gives an $a \in D_s$ with $\|f(x) - a\| \leq s$. If $y \notin U_{s,a}$, then $\|f(x) - f(y)\| \geq s$, whence $d(x, y) \geq s^{1/\alpha}$. Thus $\|w_s(x)\|_1 \geq s^{1/\alpha}$, and we may define the probability vector $\theta_s(x) = w_s(x)/\|w_s(x)\|_1$.

Fix $x \in X$, and let $Z_s(x)$ be the set of points z in the chosen s-separated net for which $(w_s(x))_z > 0$. By the definition of w_s, each $z \in Z_s(x)$ satisfies $\|z - f(x)\| < 2s$. The open balls $B_Y(z, s/2)$, $z \in Z_s(x)$, are pairwise disjoint, since their centers are at least s apart. By the

triangle inequality, each of these balls lies in $B_Y(f(x), 5s/2)$. Fix a Lebesgue measure vol on the real k-dimensional space Y. Comparing volumes gives

$$|Z_s(x)| \le \frac{\operatorname{vol} B_Y(f(x), 5s/2)}{\operatorname{vol} B_Y(0, s/2)} = 5^k.$$

Hence at most 5^k coordinates of $w_s(x)$ are positive. Each weight is 1-Lipschitz with respect to d, and at most $2 \cdot 5^k$ coordinates contribute to $w_s(x) - w_s(y)$. For nonzero vectors p, q in an ℓ_1-space,

$$\left\| \frac{p}{\|p\|_1} - \frac{q}{\|q\|_1} \right\|_1 \le \frac{2\|p-q\|_1}{\min\{\|p\|_1, \|q\|_1\}}.$$

Since probability vectors are at distance at most two, we obtain

$$\|\theta_s(x) - \theta_s(y)\|_1 \le \min\left\{2, \frac{4 \cdot 5^k}{s^{1/\alpha}}\, d(x,y)\right\}. \tag{2}$$

To pass between consecutive scales, set $f_s(x) = \sum_{a \in D_s} \theta_{s,a}(x) a$ and define $h_s(x) \in \ell_1(D_s \times D_{2s})$ by

$$h_s(x)_{a,b} = 6s\, \theta_{s,a}(x) \theta_{2s,b}(x).$$

If this coordinate is positive for some x, then $\|a-b\| \le \|a - f(x)\| + \|f(x) - b\| < 6s$. For such pairs put $B_s e_{a,b} = (a-b)/(6s)$, and put $B_s e_{a,b} = 0$ for all other pairs. This defines a linear contraction $B_s \colon \ell_1(D_s \times D_{2s}) \to Y$, and for every $x \in X$ we have

$$B_s h_s(x) = f_s(x) - f_{2s}(x). \tag{3}$$

The ℓ_1-distance between two product probability vectors is at most the sum of the distances between their respective factors and is itself at most two. Hence, for $t = d(x,y) > 0$, (2) gives

$$\|h_s(x) - h_s(y)\|_1 \le 12s \min\left\{1, \frac{4 \cdot 5^k t}{s^{1/\alpha}}\right\}.$$

Set $u = (4 \cdot 5^k t)^\alpha$. For every $s \in 2^{\mathbb{Z}}$, set

$$b_s = s \min\left\{1, \left(\frac{u}{s}\right)^{1/\alpha}\right\}.$$

The preceding estimate gives $\|h_s(x) - h_s(y)\|_1 \le 12 b_s$. It therefore suffices to bound $\sum_{s \in 2^{\mathbb{Z}}} b_s$. Choose $m \in \mathbb{Z}$ so that $2^m \le u < 2^{m+1}$. For $j \le m$, we have $(u/2^j)^{1/\alpha} \ge 1$. Hence

$$\sum_{\substack{j \in \mathbb{Z} \\ 2^j \le u}} 2^j \min\left\{1, \left(\frac{u}{2^j}\right)^{1/\alpha}\right\} = \sum_{j=-\infty}^{m} 2^j = 2^m \sum_{r=0}^{\infty} 2^{-r} = 2^{m+1} \le 2u.$$

For $s > u$, the inequality $u/s < 1$ gives

$$b_s = s \min\left\{1, \left(\frac{u}{s}\right)^{1/\alpha}\right\} = s \left(\frac{u}{s}\right)^{1/\alpha} = u \left(\frac{u}{s}\right)^{1/\alpha - 1}.$$

The smallest dyadic scale above u is $s_0 = 2^{m+1}$. Since $1/\alpha - 1 \ge 1$, we have $b_{s_0} \le u$. Moreover,

$$b_{2s} = 2s \left(\frac{u}{2s}\right)^{1/\alpha} = 2^{1-1/\alpha} b_s \le \tfrac{1}{2} b_s.$$

Therefore

$$\sum_{\substack{s \in 2^{\mathbb{Z}} \\ s > u}} b_s = \sum_{r=0}^{\infty} b_{2^r s_0} \le u \sum_{r=0}^{\infty} 2^{-r} = 2u.$$

Combining the bounds for $s \le u$ and $s > u$ gives

$$\sum_{s \in 2^{\mathbb{Z}}} \|h_s(x) - h_s(y)\|_1 \le 12(2u + 2u) = 48(4 \cdot 5^k)^\alpha d(x,y)^\alpha. \tag{4}$$

Since $f(X)$ is finite and contains at least two points, we may choose a dyadic s_- such that $2s_-$ is smaller than the least positive distance in $f(X)$. Then $D_{s_-} = f(X)$ and $f_{s_-} = f$. Choose

a dyadic $s_+ > \operatorname{diam} f(X)$. The set D_{s_+} is a singleton, so f_{s_+} is constant. Let $I = \{2^j : j \in \mathbb{Z},\ s_- \le 2^j < s_+\}$ and $L = \sum_{s\in I} |D_s|\,|D_{2s}|$. Identify

$$\ell_1^L \quad \text{with} \quad \Big(\bigoplus_{s\in I} \ell_1(D_s \times D_{2s})\Big)_{\ell_1}.$$

Define $H\colon (X, d^\alpha) \to \ell_1^L$ and $B\colon \ell_1^L \to Y$ by

$$H(x) = \big(h_s(x) - h_s(o)\big)_{s\in I}, \qquad B(z) = \sum_{s\in I} B_s(z_s),$$

where $z = (z_s)_{s\in I} \in \ell_1^L$ and $z_s \in \ell_1(D_s \times D_{2s})$. Since each B_s is a contraction, $\|B(z)\|_Y \le \sum_{s\in I} \|z_s\|_1 = \|z\|_1$, so $\|B\| \le 1$. Summing (3) over $s \in I$ gives

$$BH(x) = f_{s_-}(x) - f_{s_-}(o) - \big(f_{s_+}(x) - f_{s_+}(o)\big) = f(x),$$

since $f_{s_-} = f$, $f(o) = 0$, and f_{s_+} is constant. By (4), the Lipschitz constant of H with respect to d^α is at most $48(4 \cdot 5^k)^\alpha$. Taking $A = \widehat{H}$ gives $BA\delta_x = T\delta_x$ for all $x \in X$. Since the δ_x span $\mathcal{F}(X, d^\alpha)$, it follows that $BA = T$. For general nonzero T, apply this construction to $T/\|T\|$ and multiply its first factor by $\|T\|$. □

Corollary 5. *For every integer $k \ge 1$ there is an integer $N = N(k) \ge 2$ such that, with*

$$Z_N = \left(\bigoplus_{m>N} \bigoplus_{p\in P_m} E_m\right)_{\ell_1},$$

every contraction $W\colon Z_N \to \ell_\infty^k$ factors through ℓ_1 as $W = BA$ with $\|A\| \le 96$ and $\|B\| \le 1$.

Proof. Choose $N \ge 2$ so that $(4 \cdot 5^k)^{1/m} \le 2$ for all $m > N$, and let $W\colon Z_N \to \ell_\infty^k$ be a contraction. Set $I_N = \{(m, p) : m > N,\ p \in P_m\}$. For $i = (m, p) \in I_N$, let $\iota_i\colon E_m \to Z_N$ be the coordinate inclusion. Then $W_i = W\iota_i\colon E_m \to \ell_\infty^k$ satisfies $\|W_i\| \le 1$. Since $E_m = \mathcal{F}(H_m, b_m^{1/m})$, Lemma 4 with $\alpha = 1/m$ gives a factorization

$$E_m \xrightarrow{A_i} \ell_1^{L_i} \xrightarrow{B_i} \ell_\infty^k, \qquad W_i = B_i A_i,$$

where $\|A_i\| \le 48(4 \cdot 5^k)^{1/m}\|W_i\| \le 96$ and $\|B_i\| \le 1$.

Let $\Gamma = \{(i, r) : i \in I_N,\ r \in \{1, \dots, L_i\}\}$. Writing $z \in \ell_1(\Gamma)$ as $z = (z_i)_{i\in I_N}$, with $z_i \in \ell_1^{L_i}$, define

$$A\colon Z_N \to \ell_1(\Gamma), \quad A((x_i)_i) = (A_i x_i)_{i\in I_N},$$
$$B\colon \ell_1(\Gamma) \to \ell_\infty^k, \qquad B(z) = \sum_{i\in I_N} B_i z_i.$$

For $x = (x_i)_i \in Z_N$, we have $\|Ax\|_1 = \sum_{i\in I_N} \|A_i x_i\|_1 \le 96\|x\|_{Z_N}$. For $z \in \ell_1(\Gamma)$,

$$\sum_{i\in I_N} \|B_i z_i\|_\infty \le \sum_{i\in I_N} \|z_i\|_1 = \|z\|_1 < \infty.$$

Thus the series defining $B(z)$ converges absolutely in ℓ_∞^k, and $\|B(z)\|_\infty \le \|z\|_1$. Since $x = \sum_{i\in I_N} \iota_i x_i$ in Z_N, continuity of W gives

$$BAx = \sum_{i\in I_N} B_i A_i x_i = \sum_{i\in I_N} W\iota_i x_i = Wx.$$

Finally, each P_m is finite and nonempty and each $L_i \ge 1$ is finite, so Γ is countably infinite. Enumerating Γ identifies $\ell_1(\Gamma)$ isometrically with ℓ_1. □

5. Embeddings of finite sums

Proof of Theorem 1. We first establish the finite-sum assertion for S. Fix $D \geq 1$. By Lemma 2, choose an index n for which $c_1(H_n, \sigma_n) > 192D$. Set $k = |H_n|^2$ and take $N = N(k)$ from Corollary 5. By definition,

$$S = F_N \oplus_1 Z_N, \qquad F_N = \left(\bigoplus_{2 \leq m \leq N} \bigoplus_{p \in P_m} E_m \right)_{\ell_1}.$$

Let $P_{F_N} \colon S \to F_N$ and $P_{Z_N} \colon S \to Z_N$ denote the coordinate projections. Since E_n and F_N are finite-dimensional, the closed ball of radius D centered at zero in $\mathcal{L}(E_n, F_N)$ can be covered by finitely many balls of radius $1/2$. Let r exceed the number of balls in this cover.

Suppose that $T \colon \ell_1^r(S) \to S$ is a linear embedding of distortion at most D. By rescaling, we may assume that $\|v\| \leq \|Tv\| \leq D\|v\|$. Let $p_n^0 = (0_m)_{2 \leq m < n} \in P_n$ be the all-zero prefix. For $1 \leq i \leq r$, define $T_i \colon E_n \to S$ by

$$T_i(u) = T\big((z_j)_j\big), \qquad (z_j)_{m,p} = \begin{cases} u, & (j,m,p) = (i,n,p_n^0), \\ 0_{E_m}, & \text{otherwise.} \end{cases}$$

Here $z_j \in S$ is the vector in the j-th copy of S, and $(z_j)_{m,p} \in E_m$ is its coordinate indexed by $m \geq 2$ and $p \in P_m$. Since $\|P_{F_N} T_i\| \leq \|T_i\| \leq D$ for every i and the cover contains fewer than r balls, two of the operators $P_{F_N} T_i$ belong to the same ball. For the corresponding indices $i \neq j$ we have $\|P_{F_N}(T_i - T_j)\| \leq 1$. Moreover, for $v \in E_n$, let $w \in \ell_1^r(S)$ have v in coordinate (i, n, p_n^0), $-v$ in coordinate (j, n, p_n^0), and zero elsewhere. By the definitions of T_i and T_j, we have $(T_i - T_j)v = Tw$. Since $i \neq j$, the two entries of w lie in distinct copies of S, so their norms add. Using the lower bound $\|Tw\| \geq \|w\|$ from the normalization of T, we obtain

$$\|(T_i - T_j)v\|_S = \|Tw\|_S \geq \|w\|_{\ell_1^r(S)} = 2\|v\|_{E_n}.$$

Set $U = P_{Z_N}(T_i - T_j)$. Since $S = F_N \oplus_1 Z_N$, we have

$$\|Uv\| = \|(T_i - T_j)v\| - \|P_{F_N}(T_i - T_j)v\| \geq 2\|v\| - \|v\| = \|v\|.$$

The upper bound follows from $\|T_i\|, \|T_j\| \leq D$. Thus

$$\|v\| \leq \|Uv\| \leq 2D\|v\| \qquad (v \in E_n). \tag{5}$$

Enumerate the ordered pairs of distinct points of H_n as $(x_1, y_1), \ldots, (x_q, y_q)$, where $q = |H_n|(|H_n| - 1) \leq k$. For each $1 \leq j \leq q$, choose by Hahn–Banach a norm-one functional $\varphi_j \in Z_N^*$ such that

$$\varphi_j(U(\delta_{x_j} - \delta_{y_j})) = \big\|U(\delta_{x_j} - \delta_{y_j})\big\|_{Z_N}.$$

Define the linear map $W \colon Z_N \to \ell_\infty^k$ by

$$Wz = \big(\varphi_1(z), \ldots, \varphi_q(z), \underbrace{0, \ldots, 0}_{k-q}\big) \qquad (z \in Z_N).$$

Since each φ_j has norm one, we have $\|Wz\|_\infty = \max_{1 \leq j \leq q} |\varphi_j(z)| \leq \|z\|_{Z_N}$. Now fix $x \neq y$ in H_n, put $z = U(\delta_x - \delta_y)$, and take j such that $(x_j, y_j) = (x, y)$. By the choice of φ_j,

$$\|z\|_{Z_N} = |\varphi_j(z)| = |(Wz)_j| \leq \|Wz\|_\infty \leq \|z\|_{Z_N}.$$

Thus $\|WU(\delta_x - \delta_y)\|_\infty = \|U(\delta_x - \delta_y)\|_{Z_N}$. This also holds for $x = y$, when both sides are zero. The choice of k preceded that of N, so Corollary 5 applies to W. Write $W = BA$ with $\|A\| \leq 96$ and $\|B\| \leq 1$. Define $g \colon H_n \to \ell_1$ by $g(x) = AU\delta_x$. For $x, y \in H_n$, (5) gives

$$\sigma_n(x,y) \leq \|WU(\delta_x - \delta_y)\|_\infty \leq \|g(x) - g(y)\|_1 \leq 192D\, \sigma_n(x,y).$$

This contradicts the choice of n. Thus no such embedding exists for any r exceeding the size of the cover.

To transfer this conclusion to $E = \mathcal{F}(M)$, let J be the isomorphism of Lemma 3. Conjugating an embedding $\ell_1^r(E) \to E$ by J on the range and by the coordinatewise map induced by J^{-1} on the domain gives an embedding $\ell_1^r(S) \to S$. Its distortion is bounded by the original distortion times $(\|J\|\|J^{-1}\|)^2$, independently of r. The finite-sum assertion for E follows. Finally, an

isomorphic embedding of $\ell_1(\mathbb{N}; E)$ into E would restrict to embeddings of $\ell_1^r(E)$ with the same distortion bound for every r. □

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