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VANISHING LEHMER GAPS IN TORSION-FREE GROUPS

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Abstract. We answer Chung and Thom's entropy gap question for torsion-free amenable groups in the negative. For each prime p we construct a three-generated, torsion-free, residually p solvable group Γ_p. On this fixed group there are connected principal actions with positive entropies tending to zero. The defining group-ring elements have three terms with coefficients 1 or -1 and logarithmic Fuglede–Kadison determinants asymptotic to $\sqrt{\pi/24}\,p^{-r/2}$. The same construction gives all four of Lück's group Lehmer constants equal to one for Γ_p, Grigorchuk's torsion-free group of intermediate growth, the infinite braid group B_∞, and Brin's braided Thompson group BV. The determinant calculation comes from a cyclic weighted shift over a finite-index abelian subgroup. For the amenable examples every entropy in $[0,\infty]$ occurs on the infinite torus, with a Bernoulli Haar action at every positive entropy.

Note on related work. Time constraints limited the scope and depth of the literature search. Readers who identify substantial overlap with their own publicly available manuscripts are invited to contact the author at 3565586261@qq.com.

1. Introduction

For a countable discrete amenable group G and a countable left $\mathbb{Z}G$-module M, let $\rho_G(M)$ be the entropy of the algebraic action of G on the compact Pontryagin dual $\widehat{M}$. The topological and Haar measure entropies agree [6, Section 3]. Chung and Thom asked [6, Question 4.7] whether for each torsion-free amenable G there is a constant $c_G > 0$ with $\rho_G(M) > c_G$ whenever $\rho_G(M) > 0$. For $G = \mathbb{Z}$ the question is equivalent to the classical Lehmer problem [13, 6].

Lück formulated a system of Lehmer gap questions for integral group-ring matrices of arbitrary groups [16, Problems 3.3 and 12.1]. For $f \in \mathbb{Z}G$, write r_f for right multiplication on $\ell^2(G)$ and $\Delta_G(f)$ for its Fuglede–Kadison determinant with respect to the canonical group trace. A weak isomorphism is an injective operator with dense image. In Lück's notation $\Lambda(G)$ is the infimum of the determinants strictly greater than one over all integral group-ring matrices. The superscript w restricts the infimum to square matrices

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inducing weak isomorphisms. The subscript 1 restricts it to scalar elements. In particular, the four constants are bounded above by $\Lambda_1^w(G)$ and below by 1, using the extended-real convention for the infima.

For each $f \in \mathbb{Z}G$, let

$$X_f = \widehat{\mathbb{Z}G/\mathbb{Z}Gf}.$$

When G is countable amenable and r_f is a weak isomorphism, Li and Thom's entropy–determinant theorem gives

$$\rho_G(\mathbb{Z}G/\mathbb{Z}Gf) = \log \Delta_G(f). \tag{1.1}$$

We use the scalar case of their theorem [14, Theorem 1.2]. Inversion of the group index conjugates right convolution by f to left convolution by f^*. The determinant and the weak-isomorphism property are preserved by taking the adjoint.

Theorem 1.1 (A fixed group with no positive gap)**.** *For every prime p there exists a three-generated torsion-free solvable group Γ_p of derived length at most three that is residually p. There are elements $b, t_r \in \Gamma_p$, with b independent of $r \geq 1$, for which*

$$f_r = 1 - t_r + bt_r$$

induces a weak isomorphism. Its determinant satisfies

$$\log \Delta_{\Gamma_p}(f_r) > 0, \qquad \lim_{r\to\infty} p^{r/2} \log \Delta_{\Gamma_p}(f_r) = \sqrt{\frac{\pi}{24}}. \tag{1.2}$$

The underlying compact group of each X_{f_r} is isomorphic to $\mathbb{T}^{\mathbb{N}}$, and the Haar action is Bernoulli. In particular the entropies of these connected principal actions give a negative answer to Chung and Thom's Question 4.7. All four of Lück's Lehmer constants of Γ_p equal one.

Theorem 1.2 (Classical examples)**.** *All four group Lehmer constants equal one for Grigorchuk's torsion-free group of intermediate growth $\widetilde{\mathfrak{G}}$, the infinite braid group B_∞, and Brin's finitely presented torsion-free braided Thompson group BV. For each of these groups, scalar weak isomorphisms with three coefficients in $\{1, -1\}$ witness the conclusion.*

Theorem 1.3 (Connected entropy range)**.** *For $G = \Gamma_p$ and for $G = \widetilde{\mathfrak{G}}$, the entropies of principal algebraic actions on $\mathbb{T}^{\mathbb{N}}$ are dense in $[0, \infty)$. Every $h \in [0, \infty]$ is realized by an algebraic action on this torus that is mixing of all orders with respect to Haar measure. For $h > 0$ the action may be chosen to be Bernoulli.*

The calculation is made on a group H_n with a normal abelian subgroup of index n. The finite-index Fourier representation gives a matrix symbol whose determinant is linear in the torus coordinate from the central element t^n. Integrating that coordinate gives the exact expectation of the positive part of a centered sum of n independent logarithms, divided by n. We then realize this calculation for unbounded n inside a fixed group. The group H_n

also has an explicit Laurent module presentation for the principal action, which gives connectedness and supplies the entropy range application.

Deninger's work relates Fuglede–Kadison determinants in crossed products to Mahler measures and Lyapunov exponents [8, Theorem 6]. The weighted-shift formula there applies on almost every finite ergodic orbit of the periodic action considered here. We give the direct finite-index calculation, including its integrability, to exhibit the small positive determinants and their asymptotics. Dasbach and Lalín studied Mahler measures under changes of the underlying group [7]. Ben Aribi obtained upper bounds on group Lehmer constants using free-group and three-manifold examples [2]. Miles and Björklund obtained entropy range theorems using arbitrarily large finite normal subgroups [17, Theorem 4.1]. The group embeddings below give a different source of small positive values for connected actions of torsion-free amenable groups. Kammeyer's determinant-approximation counterexample [12] concerns one fixed element of the Heisenberg group and its images in finite quotients. The preprint [18] concerns the lower bound at one for unrestricted integral group-ring matrices.

2. The periodic determinant

For $n \geq 2$, let A_n be the free abelian group on $x_0, \dots, x_{n-1}$, and define

$$H_n = A_n \rtimes \langle t \rangle, \qquad t x_j t^{-1} = x_{j+1}, \qquad F_n = 1 - (1 - x_0)t. \tag{2.1}$$

Indices are read modulo n, and $\langle t \rangle \cong \mathbb{Z}$. The subgroup $K_n = A_n \times \langle t^n \rangle$ is free abelian of rank $n+1$ and has index n in H_n.

Proposition 2.1. *Let $Z_1, \dots, Z_n$ be independent Haar variables on the unit circle and put*

$$Y_j = \log|1 - Z_j|, \qquad S_n = Y_1 + \cdots + Y_n.$$

Then r_{F_n} is a weak isomorphism of determinant class and

$$\log \Delta_{H_n}(F_n) = \frac{1}{n}\mathbb{E}[(S_n)^+] = \frac{1}{2n}\mathbb{E}|S_n|, \qquad x^+ = \max(0, x). \tag{2.2}$$

In particular,

$$0 < \log \Delta_{H_n}(F_n) \leq \frac{\pi}{2\sqrt{12n}}, \qquad \lim_{n\to\infty} \sqrt{n} \log \Delta_{H_n}(F_n) = \sqrt{\frac{\pi}{24}}. \tag{2.3}$$

Proof. Write $\mathbb{Z}H_n$ as a free left $\mathbb{Z}K_n$-module with basis $1, t, \dots, t^{n-1}$. Right multiplication by $(1 - x_0)t$ sends t^j to $(1 - x_j)t^{j+1}$, using the defining conjugation rule of H_n. At the last index t^n enters the coefficient of the first basis element. Apply Fourier transform on K_n, writing z_j for the dual coordinate of x_j and u for that of t^n. With fiber basis $e_0, \dots, e_{n-1}$, the shift $W = W(z, u)$ acts by

$$W e_j = (1 - z_j) e_{j+1} \quad (j < n-1), \qquad W e_{n-1} = u(1 - z_{n-1}) e_0.$$

Thus r_{F_n} is represented by $I - W$. In the ordinary determinant expansion only the diagonal term and the full cycle can contribute. Their signs give

$$\det(I - W(z,u)) = 1 - uP(z), \qquad P(z) = \prod_{j=0}^{n-1}(1 - z_j). \tag{2.4}$$

For each z, the right side vanishes for at most one value of u. The matrix and its adjoint are therefore invertible almost everywhere, and the corresponding operators have trivial kernels. This proves the weak isomorphism assertion.

For a bounded operator in the right group von Neumann algebra, each diagonal vector state at a coset representative t^j equals the vector state at the identity, since the operator commutes with left translations. Under the Fourier decomposition the mean of these n vector states is the integral of the ordinary matrix trace divided by n. This applies to bounded functions of $|r_{F_n}|$ by the spectral calculus.

For fixed z, Jensen's formula in u gives

$$\int_{\mathbb{T}} \log|1 - uP(z)|\, dm_{\mathbb{T}}(u) = \log^+|P(z)|,$$

where $m_{\mathbb{T}}$ is Haar probability and $\log^+ 0 = 0$. The positive part of the integrand is bounded by $\log(1 + 2^n)$, and the displayed integral is nonnegative. The integral of the negative part in u is therefore bounded by the same constant, uniformly in z. Hence $\log|\det(I - W)|$ is integrable. To check integrability for the determinant defined using singular values, let $s_1, \dots, s_n$ be the singular values of $I - W$. They are bounded above uniformly. Almost everywhere they are positive and $\sum_j \log s_j = \log|\det(I - W)|$. The sum of their positive logarithmic parts is bounded, so the sum of their negative logarithmic parts is integrable as well. Truncating the singular-value logarithms in the trace and taking limits now gives

$$\log \Delta_{H_n}(F_n) = \frac{1}{n}\int_{\mathbb{T}^{n+1}} \log|1 - uP(z)|\, dm(z,u) = \frac{1}{n}\mathbb{E}[(S_n)^+]. \tag{2.5}$$

Each Y_j has mean zero by Jensen's formula. Its variance is $\sigma^2 = \pi^2/12$. The L^2 Fourier expansion in the angular variable is

$$\log|1 - e^{\mathrm{i}\theta}| = -\sum_{k\geq 1} \frac{\cos(k\theta)}{k}.$$

For $0 < a < 1$ the logarithm with $e^{\mathrm{i}\theta}$ replaced by $ae^{\mathrm{i}\theta}$ has the coefficients $-a^k/k$, which converge in the square-summable norm as $a \uparrow 1$. Pointwise convergence off $\theta = 0$ identifies the L^2 limit. Parseval's identity then gives $\sigma^2 = \frac{1}{2}\sum_{k\geq 1} k^{-2} = \pi^2/12$. Since $\mathbb{E}S_n = 0$, its expected positive part is half its expected absolute value. The probability that all Z_j lie in an arc about -1 on which $|1 - Z_j| > 1$ is positive. This proves strict positivity in (2.3). Cauchy–Schwarz gives $\mathbb{E}|S_n| \leq \sigma\sqrt{n}$.

Finally $S_n/\sqrt{n}$ converges in law to a centered normal variable of variance σ^2. Its second moments are uniformly bounded, so the positive parts are

uniformly integrable. Their expectations converge to $\sigma/\sqrt{2\pi} = \sqrt{\pi/24}$. Substitution into (2.2) proves the limit. □

Lemma 2.2 (Passage to an overgroup)**.** *If $H \leq G$ and $f \in \mathbb{Z}H$, weak isomorphism of r_f and its Fuglede–Kadison determinant agree in H and G. Consequently, if G contains copies of H_n for unbounded n, then*

$$\Lambda(G) = \Lambda^w(G) = \Lambda_1(G) = \Lambda_1^w(G) = 1.$$

Proof. Each subspace $\ell^2(sH)$, for a representative s of a left coset of H, reduces r_f and carries a copy of the right convolution on H. This proves the assertion about kernels and dense range. The spectral distribution of $|r_f|$ for the canonical trace is measured at the identity, which lies in the copy for H. Thus the distributions and hence the determinants agree. This is the scalar version of subgroup induction for the determinant [16, Lemma 5.1]. Now use the elements F_n of Proposition 2.1 in the chosen overgroup. Their determinants are strictly above one and converge to one, and they qualify for $\Lambda_1^w(G)$. □

Corollary 2.3 (Optimal support size)**.** *For a torsion-free group, any positive log determinant strictly less than* $\log 2$ *of a scalar weak isomorphism over $\mathbb{Z}G$ requires at least three support terms. The examples of Proposition 2.1 have three support terms for every n.*

Proof. A nonzero monomial has determinant the absolute value of its integer coefficient. Multiplying a binomial by a group unit preserves the determinant and gives $c + dg$ with $g \neq 1$ and $c, d \in \mathbb{Z} \setminus \{0\}$. The subgroup $\langle g \rangle$ is infinite cyclic. Apply Lemma 2.2 and the one-variable Jensen formula to get $\Delta_G(c + dg) = \max\{|c|, |d|\}$. Thus all positive logarithms in these cases are at least $\log 2$. □

3. A fixed residually p group

Let p be a prime. Put $E = \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}e_i$, written additively, and

$$W = E \rtimes \langle s \rangle, \qquad s e_i s^{-1} = e_{i+1}.$$

For $i \neq 0$ let $d_i = p^{v_p(i)}$, where $v_p(i)$ is the valuation of $|i|$. Define

$$C = \bigoplus_{i \neq 0} d_i \mathbb{Z} e_i \leq E, \qquad X = W/C.$$

Here W/C denotes the set of left cosets, with the left translation action of W. Let $B = \bigoplus_{x \in X} \mathbb{Z} b_x$ with W permuting the displayed basis, and set

$$\Gamma_p = B \rtimes W. \tag{3.1}$$

We use multiplicative notation for elements of B in the group ring, and $b = b_C$.

Lemma 3.1. *The group Γ_p is generated by b, e_0, s, is torsion-free and solvable of derived length at most three, and is residually p. For $r \geq 1$, the assignment $x_0 \mapsto b,\ t \mapsto t_r = e_{p^r}$ embeds H_{p^r} in Γ_p.*

Proof. The elements e_0, s generate W. The conjugates of b by W give the entire basis of B. The successive factors $B, E, \langle s\rangle$ are torsion-free abelian. An element of finite order has trivial image in each successive quotient, which proves torsion-freeness. The same series proves the assertion on derived length.

For $t_r = e_{p^r}$, its coset orbit at C has length exactly p^r. In fact $t_r^k C = C$ is equivalent to $ke_{p^r} \in C$, which holds exactly for $p^r \mid k$. The conjugates $t_r^j b t_r^{-j}$ for $0 \le j < p^r$ are therefore distinct basis vectors of B and are permuted cyclically by t_r. Since the image of t_r in W has infinite order, $\langle t_r\rangle$ intersects the subgroup generated by these basis vectors trivially. The resulting subgroup has the presentation (2.1).

It remains to show residual p-finiteness. For $k \ge 1$ let

$$W_k = (\mathbb{Z}/p^k\mathbb{Z})^{\mathbb{Z}/p^k\mathbb{Z}} \rtimes (\mathbb{Z}/p^k\mathbb{Z}),$$

where the last factor shifts the coordinate labels. The homomorphism $\pi_k : W \to W_k$ folds the indices modulo p^k, adds the folded coefficients modulo p^k, and reduces the exponent of s modulo p^k. Folding intertwines the shifts, so this is a group homomorphism. Every nonidentity element of W has nontrivial image for all sufficiently large k. For a base element, its finitely many original indices are distinct modulo p^k once k is large, and a nonzero coefficient survives modulo p^k.

Set $C_k = \pi_k(C)$. For every $w \in W \setminus C$, we claim $\pi_k(w) \notin C_k$ for all sufficiently large k. This follows from the shift coordinate when w has nonzero shift exponent. Otherwise choose an index i at which the base vector of w violates the defining conditions of C, and take k large enough to separate the support indices. If $i \neq 0$, take $k > v_p(i)$. All integers congruent to i mod p^k then have the same valuation as i. Thus the coefficient of any element of C_k at this residue lies in $d_i(\mathbb{Z}/p^k\mathbb{Z})$, whereas the coefficient of $\pi_k(w)$ does not. If $i = 0$, the coefficient of any element of C_k at residue zero is zero, since every contributing index in C is a nonzero multiple of p^k and carries a coefficient divisible by p^k. The nonzero coefficient of w at zero survives for large k.

It follows that the quotient map of W-sets $X \to X_k = W_k/C_k$ distinguishes every prescribed finite set of distinct cosets for all sufficiently large k. Indeed, distinct cosets vC, wC give $v^{-1}w \notin C$ and apply the preceding claim. Sum coefficients of B over the fibers of this map and reduce modulo p^k. By equivariance the construction gives a homomorphism

$$\Gamma_p \longrightarrow (\mathbb{Z}/p^k\mathbb{Z})^{X_k} \rtimes W_k.$$

The target is a finite p-group. An element projecting nontrivially to W is eventually detected in the second factor. For a nonzero element of B, the finitely many support cosets are eventually separated and at least one nonzero coefficient survives in the first factor. This is residual p-finiteness. □

By Lemmas 2.2 and 3.1, the f_r in Theorem 1.1 are weak isomorphisms with the determinants in (1.2). Solvability gives amenability, so (1.1) gives their positive entropies and their limit zero. This proves the negative answer to Question 4.7 and the four equalities for Γ_p. Section 4 proves the claims about the underlying compact groups and Haar actions.

A countable metabelian example containing all the H_n is

$$\left(\bigoplus_{n\geq 2} A_n\right) \rtimes \langle t\rangle,$$

where t cyclically permutes the basis in each A_n. It is torsion-free and locally virtually abelian. Indeed the supports of any finite set of elements meet only finitely many blocks, and a common power of t acts trivially on those blocks. The construction (3.1) obtains the small determinants within a finitely generated group with prescribed residual prime.

4. The principal actions on the infinite torus

Proposition 4.1. *If a countable group G contains H_n, the compact dual of $\mathbb{Z}G/\mathbb{Z}GF_n$ is isomorphic as a group to $\mathbb{T}^{\mathbb{N}}$. The algebraic G-action on it with Haar measure is Bernoulli.*

Proof. First take $G = H_n$. Write $M_n = \mathbb{Z}H_n/\mathbb{Z}H_nF_n$ and work over the commutative Laurent ring $\mathbb{Z}K_n = \mathbb{Z}[x_0^{\pm 1}, \dots, x_{n-1}^{\pm 1}, u^{\pm 1}]$, where $u = t^n$. The generators v_j are the classes of the t^j. Right multiplication by F_n gives the relations

$$v_j = (1 - x_j)v_{j+1} \quad (j < n-1), \qquad v_{n-1} = u(1 - x_{n-1})v_0.$$

Each equation eliminates its left-hand generator with coefficient one. Starting from the last equation and going backward gives a single generator v_0 with a single relation $(1 - uP)v_0 = 0$, where $P = \prod_j (1 - x_j)$. Thus the module is the quotient by the principal ideal $(1 - uP)$. Equivalently,

$$M_n \simeq R_n = \mathbb{Z}[x_0^{\pm 1}, \dots, x_{n-1}^{\pm 1}, P^{-1}], \qquad u \text{ acts by } P^{-1}. \tag{4.1}$$

The isomorphism follows from the universal property of localization, since the relation says that u is the inverse of P.

The additive structure in (4.1) can be checked integrally. In one variable the ring $\mathbb{Z}[x^{\pm 1}, (1-x)^{-1}]$ has an additive basis

$$1, \quad x^j, \quad x^{-j}, \quad (1-x)^{-j} \qquad (j \geq 1).$$

For nonnegative a, b, the ideals (x^a) and $((1-x)^b)$ are comaximal in $\mathbb{Z}[x]$. If $a, b > 0$, expand $(x + (1-x))^{a+b-1}$ to see the comaximality. Hence a fraction with denominator $x^a(1-x)^b$ is a sum of two fractions with the separate denominators. Polynomial division over $\mathbb{Z}$ by these polynomials, whose leading coefficients are units, gives the stated span. Taking principal parts at 0 and 1 and then the polynomial part gives linear independence. Inverting P inverts every factor $1 - x_j$. The ring R_n is the tensor product

over $\mathbb{Z}$ of the n one-variable rings, by their presentations as localizations. Tensoring their bases shows R_n is free abelian of countably infinite rank. Its compact dual is $\mathbb{T}^{\mathbb{N}}$.

We next show that the K_n-action is Bernoulli. The quotient R_n in (4.1) is a domain, so the unique associated prime of M_n over $\mathbb{Z}K_n$ is $(1-uP)$. Indeed, every nonzero element of R_n has zero annihilator in R_n, and hence has annihilator $(1-uP)$ in $\mathbb{Z}K_n$. This prime action has entropy

$$\int_{\mathbb{T}^{n+1}} \log|1-uP(z)|\,dm(z,u) = n\log\Delta_{H_n}(F_n) > 0.$$

Here one can apply (1.1) on K_n to the scalar symbol $1-uP$, whose zero set has measure zero, and use (2.5). The associated-prime criterion of Lind, Schmidt and Ward [15, Theorem 6.5] gives completely positive entropy for the K_n-action. The theorem of Rudolph and Schmidt [20] makes this algebraic $\mathbb{Z}^{n+1}$-action Bernoulli.

Let $Y = \widehat{M}_n$. The nontrivial Bernoulli action of K_n is essentially free. Each nonidentity element of H_n has a nonidentity positive power in K_n, because K_n has finite index and H_n is torsion-free. The fixed set of such an element is contained in that of its power. Thus the H_n-action is essentially free as well. Choose left coset representatives S for H_n/K_n. The co-induction of the K_n-action to H_n is the action on Y^S with product Haar measure which, for $hs = s'k$ with $k \in K_n$, moves coordinate y_s to ky_s at s'. This action is Bernoulli, since a Bernoulli representation of the K_n-action reindexes the coordinates into a single H_n-shift. The map

$$(y_s)_{s\in S} \longmapsto \sum_{s\in S} sy_s$$

is a continuous surjective homomorphism and carries product Haar measure to Haar measure. It intertwines the H_n-actions, since the contribution $s'(ky_s)$ equals $h(sy_s)$. The sum is finite. The original H_n-action is consequently an essentially free factor of a Bernoulli action of the amenable group H_n. The amenable Bernoulli factor theorem [19, 11] gives Bernoullicity of the original action.

For the general overgroup G, the right $\mathbb{Z}H_n$-module $\mathbb{Z}G$ is free on left coset representatives. The isomorphism

$$\mathbb{Z}G/\mathbb{Z}GF_n \simeq \mathbb{Z}G \otimes_{\mathbb{Z}H_n} M_n$$

follows by tensoring the presentation of M_n. As an abelian group this is a direct sum of copies of M_n, one for each left coset, and it is free of countably infinite rank. Its dual action is the co-induction of the H_n-action to G. Co-induction of a Bernoulli shift with base D gives the Bernoulli G-shift with the same base, by writing the indices of D^G as copies of H_n indexed by the left cosets. This also applies at infinite index and proves the proposition. □

Taking $G = \Gamma_p$ in Proposition 4.1 completes Theorem 1.1.

5. Groups containing periodic subgroups

5.1. **The Grigorchuk example.** Let $\widetilde{\mathfrak{G}}$ denote the torsion-free group of intermediate growth introduced by Grigorchuk [10, Section 5]. Its quotient $\mathfrak{G}$ is the first Grigorchuk group, an infinite finitely generated torsion 2-group acting faithfully and level transitively on the rooted binary tree.

Proposition 5.1. *The group $\widetilde{\mathfrak{G}}$ contains H_{2^r} for every $r \geq 1$.*

Proof. Write $\xi : \widetilde{\mathfrak{G}} \to \mathfrak{G}$ for the quotient map and X for the $\mathfrak{G}$-orbit of the ray 1^∞. We use Bartholdi and Erschler's description of the kernel [1, Section 5.1, Lemma 5.7 and its proof]. It has the form

$$N = \ker \xi = \langle z \rangle \times \bigoplus_{x \in X} C_x, \qquad C_x \simeq \mathbb{Z}^3.$$

Here z is the central square of the root generator, and the C_x are the conjugates of the subgroup at the base ray. In the description of [1], the stabilizer of the base ray in the quotient has preimage centralizing the subgroup at that ray. Thus a lift taking the base ray to x defines C_x by conjugation, and conjugation by any t sends C_x to $C_{\xi(t)x}$. The root-generator exponent sum $\varepsilon : \widetilde{\mathfrak{G}} \to \mathbb{Z}$ is a homomorphism with $\varepsilon(z) = 2$, and $\varepsilon(C_x) = 0$ for every x.

Since $\mathfrak{G}$ acts faithfully on a rooted tree, the actions on the finite levels give residual finiteness. A finitely generated residually finite 2-group of bounded exponent is finite by the restricted Burnside theorem [23]. Thus the element orders of $\mathfrak{G}$ are unbounded, and taking powers yields an element g of order $n = 2^r$ for every r. The boundary orbit X is dense by level transitivity, so the action of $\mathfrak{G}$ on X is faithful. If every g-orbit in X had size smaller than n, then $g^{n/2}$ would fix all of X. Hence some $x \in X$ has orbit of length n.

Choose a lift t of g and multiply it by a power of z so that $\varepsilon(t) \neq 0$. Take $a \neq 1$ in C_x. The conjugates $t^j a t^{-j}$ for $0 \leq j < n$ lie in distinct direct summands of N. They commute and freely generate a free abelian group of rank n. Moreover $t^n \in N$, and N is abelian, so $t^n a t^{-n} = a$. Thus t acts by the required cycle. Its cyclic subgroup intersects the abelian group just constructed trivially, as that abelian group is in $\ker \varepsilon$ and $\varepsilon(t) \neq 0$. The resulting subgroup is H_n. □

5.2. **Braids.**

Proposition 5.2. *For every $n \geq 2$, the braid group B_{2n} contains a copy of H_n. Hence B_∞ and BV contain H_n for every n.*

Proof. Place the $2n$ punctures in n disjoint small disks, each with two punctures. In the mapping class group model for B_{2n}, let a_j be the positive full twist of the pair in the j-th disk. These twists commute by disjointness of supports. They generate a free abelian group A on the a_j, since the linking number for the j-th pair on this pure braid subgroup detects just the exponent of a_j.

Take the n-strand braid $\sigma_1 \cdots \sigma_{n-1}$ and replace each strand by a disk carrying two parallel strands. Such a motion may be obtained by following the braid paths of the centers with sufficiently small disks. Its braid t cyclically permutes the disks, so it conjugates their positive full twists by the same cycle, after labeling them in that order. On the stabilizer of the disk system, collapse each disk to one puncture. The induced motion of the n disks gives a homomorphism to B_n, killing every a_j and sending t to $\sigma_1 \cdots \sigma_{n-1}$. The latter has infinite order because its exponent sum is $n-1$. Thus t has infinite order and no nonidentity power of t is in A. The subgroup generated by A and t is H_n.

The ordinary strand inclusions place the B_{2n} in B_∞. In BV, fixing a tree with $2n$ leaves and placing a braid between two copies of the tree gives a subgroup isomorphic to B_{2n}. Expansion is cabling and is injective, as one can forget all but one strand of each cabled strand. We use the tree–braid construction of Brin [4, 5, 3]. □

The braid groups are torsion-free, and BV is torsion-free and finitely presented [5, 22]. Propositions 5.1 and 5.2, together with Lemma 2.2, prove Theorem 1.2.

6. The entropy range

We prove the following general form of Theorem 1.3.

Proposition 6.1. *Suppose that G is countable amenable and contains H_n for unbounded n. Then the entropies of principal algebraic actions on $\mathbb{T}^{\mathbb{N}}$ are dense in $[0,\infty)$. Every $h \in [0,\infty]$ is the entropy of an algebraic G-action on $\mathbb{T}^{\mathbb{N}}$ which is mixing of all orders. At each positive entropy the action can be chosen Bernoulli with respect to Haar measure.*

6.1. **Positive entropy.** Choose a sequence of embedded blocks with strictly increasing size, and write f_r for the associated three-term elements and $h_r = \log \Delta_G(f_r)$. Then $h_r > 0$ and $h_r \to 0$ by Proposition 2.1 and Lemma 2.2. Let $R = \mathbb{Z}G$. For any $m \geq 2$, the sequence of left R-modules

$$0 \longrightarrow R/Rf_r^{m-1} \xrightarrow{\cdot f_r} R/Rf_r^m \longrightarrow R/Rf_r \longrightarrow 0 \tag{6.1}$$

is exact. For injectivity, $af_r = bf_r^m$ implies $(a - bf_r^{m-1})f_r = 0$, so $a = bf_r^{m-1}$ by injectivity of r_{f_r}. The image is Rf_r/Rf_r^m, exactly the kernel of the last map. The last term is free abelian by Proposition 4.1. Thus (6.1) splits as abelian groups. By induction the compact dual of R/Rf_r^m is $\mathbb{T}^{\mathbb{N}}$ for all $m \geq 1$. The entropy addition formula [6, Section 3] gives entropy mh_r. Since $h_r \to 0$, these positive integer multiples give a dense set of entropies in $[0,\infty)$.

To realize a given finite $h > 0$ exactly, let $R_0 = h$, and set

$$m_r = \lfloor R_{r-1}/h_r \rfloor, \qquad R_r = R_{r-1} - m_r h_r.$$

Here $0 \leq R_r < h_r$, so $h = \sum_r m_r h_r$. Use the product of m_r copies of X_{f_r} for each r with $m_r > 0$. This nonempty finite or countable product of infinite

tori is again $\mathbb{T}^{\mathbb{N}}$. Its entropy is the sum of the factor entropies. Indeed, finite subproducts have additive entropy and their dual modules exhaust the dual of the countable product. Entropy is continuous on these increasing unions [6, Proposition 3.2]. Countably many copies of one positive-entropy factor give $h = \infty$. By Proposition 4.1 all these factors are Bernoulli under Haar measure, and their product is Bernoulli with the product base. Bernoulli shifts are mixing of all orders. The amenable Bernoulli isomorphism theorem [19] also identifies any Bernoulli G-shift of the same entropy with the model just constructed.

6.2. **Zero entropy.** We first record how mixing behaves under co-induction. For $J \leq G$ and a discrete left $\mathbb{Z}J$-module M, suppose the algebraic action on $\widehat{M}$ is mixing of all orders. Then so is the dual of $\mathbb{Z}G \otimes_{\mathbb{Z}J} M$. To prove this, the Fourier criterion for mixing of all orders says that for any fixed list of nonzero characters $m_1, \ldots, m_k$, the relations $\sum_i g_i m_i = 0$ eventually fail when all $g_i^{-1} g_j$ for $i \neq j$ leave every finite set. This follows by integrating products of characters. Bounded measurable functions can be approximated in L^1 by continuous functions of uniformly bounded norm, and continuous functions by character polynomials in the uniform norm. These approximations extend the character criterion to the mixing identities for bounded measurable functions. Use the direct sum description of the induced module on the cosets G/J. Each m_i has finite coset support. If the Fourier relation held along a separated sequence, pass to a subsequence for which the pattern of equalities among the finitely many translated support cosets is fixed. Project onto the translated coset of a chosen nonzero coordinate. For a representative v of this coset, the projected equation has the form

$$\sum_i h_i m_i' = 0, \qquad h_i = v^{-1} g_i s_i \in J,$$

with fixed nonzero $m_i' \in M$ and fixed representatives s_i. Each contributing character occurs at most once because multiplication by g_i sends distinct cosets to distinct cosets. A single term cannot vanish. For distinct terms, $h_i^{-1} h_j = s_i^{-1} g_i^{-1} g_j s_j$ leaves every finite set of J. Thus the equation contradicts mixing of all orders on $\widehat{M}$.

Choose $J \simeq \mathbb{Z}^2$ in one of the blocks of G. It has infinite index in G. Indeed, G also contains a subgroup $\mathbb{Z}^3$, and a finite-index subgroup of $\mathbb{Z}^3$ cannot embed in $\mathbb{Z}^2$. Let θ be a root of the irreducible polynomial $x^3 - 3x + 1$. On $\mathbb{Z}[\theta]$, a free abelian group of rank three, let the two generators of J act by multiplication by θ and $1 - \theta$. Their norms are both -1, so both act as automorphisms of $\mathbb{Z}[\theta]$. These units are multiplicatively independent. At the real embedding with $\theta \in (-2, -1)$ both absolute values are greater than one, while at the embedding with $\theta \in (1, 2)$ the first is greater than one and the second is less than one. A relation $\theta^u (1 - \theta)^v = 1$ with nonzero exponents would require opposite signs of u, v at the first embedding and

equal signs at the second. If one exponent is zero, the first embedding forces the other to be zero as well.

Given two nonzero characters in the domain $\mathbb{Z}[\theta]$, multiplicative independence shows that at most one element of J carries one to the other. Thus the dual action on $\mathbb{T}^3$ is mixing. By the theorem of Schmidt and Ward [21] it is mixing of all orders. It has zero J-entropy. Choose a common bound $L > 1$ on the Lipschitz constants of the two torus automorphisms and their inverses. For the square $F_N = [-N, N]^2 \cap \mathbb{Z}^2$ all acting maps have Lipschitz constant at most L^{2N}. For any fixed $\epsilon > 0$, a net of mesh ϵL^{-2N} in $\mathbb{T}^3$ spans the corresponding dynamical metric at scale ϵ. Its logarithmic size is $O_\epsilon(N)$, which is $o(|F_N|)$. The Følner formula for topological entropy gives zero, and Haar entropy is zero as well.

Co-induce this action to G. Its underlying group is $(\mathbb{T}^3)^{G/J} \simeq \mathbb{T}^{\mathbb{N}}$. Co-induction preserves entropy for countable amenable groups [9], so the G-action has zero entropy. It is mixing of all orders by the argument above. This completes Proposition 6.1. The groups Γ_p and $\widetilde{\mathfrak{G}}$ satisfy its hypotheses, the latter by Proposition 5.1 and subexponential growth. This proves Theorem 1.3.

AI use disclosure

OpenAI's Codex was used to develop parts of the mathematical argument, audit the proofs and references, and draft and revise the manuscript.

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