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Von Neumann reduction of old and new algebras

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Abstract

We quickly review von Neumann's reduction theory, in the tracial case. We determine the center for compact groups crossed by discrete groups of automorphisms and the locally finite adjoint part for O_N^*. Half-liberated rectangular orbits have matrix fibers of size 2 at every rank. With one free side at rank one, almost every central fiber realizes the Jones basic construction of the free projective-to-sphere inclusion. The marked fibers give a geometric reformulation of the square-versus-rectangle criterion for free factors. We compare 2 half-liberation prescriptions for twisted projective spaces. The resulting real and complex Grassmannians have isomorphic tracial algebras at fixed ambient dimension. One diagonal coordinate recovers the field and both dimensions. The rank-one inclusions into the respective affine sphere algebras coincide up to tracial isomorphism exactly in even ambient dimension.

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Introduction

A factor has only the scalars in its center. At the other extreme, the center of $M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ has 2 minimal projections, and cutting by them recovers the 2 matrix algebras. What plays the role of a summand when the center is $L^\infty([0,1])$, where there are no minimal projections? Von Neumann's reduction theorem gives a precise answer. We review it in the setting of the rings of operators of Murray and von Neumann [22, 23, 31, 24], then use it to ask which features of a quantum space one can read from its von Neumann algebra.

Throughout, our von Neumann algebras have separable predual and come with a faithful normal tracial state τ. A projection p in the center $Z(A)$ specifies a measurable part of the algebra, and $\tau(p)$ is its mass. Thus we write $Z(A) = L^\infty(X, \mu)$, with μ a probability measure. In the matrix example X consists of 2 points. In our tracial setting the reduction theorem of von Neumann [32] reads:

Theorem A. *Let A be a von Neumann algebra with separable predual and faithful normal tracial state τ. Write its center, with the restriction of τ, as $L^\infty(X,\mu)$ on a standard probability space. There is a measurable field of factors (A_x,τ_x) with normalized traces whose direct integral gives*

$$A \simeq \int_X A_x \, d\mu(x), \qquad \tau(a) = \int_X \tau_x(a_x)\, d\mu(x).$$

Under this isomorphism a central function f acts as $f(x)1$ on the fiber. The field is determined up to a measure-space isomorphism modulo null sets and a measurable field of trace-preserving isomorphisms almost everywhere.

This is of course a very classical result. Our proof here is self-contained, assumes only basic knowledge of von Neumann algebras and II_1 factors, and takes 7 pages. For the reduction theory in the general case, for arbitrary von Neumann algebras, see Kadison and Ringrose [19], and for the Hilbert space methods used here see Sinclair and Smith [27].

For a countable discrete group Γ, conjugacy classes give a way of finding the center before constructing any fibers. Write Γ_{fc} for the subgroup of elements with finite conjugacy class. In the group von Neumann algebra $L(\Gamma)$ the center-valued trace of a group unitary is the average over its conjugacy class when that class is finite, and zero otherwise. Thus

$$Z(L(\Gamma)) = L(\Gamma_{fc})^\Gamma,$$

where the superscript Γ stands for the fixed points of the conjugation action of Γ on $L(\Gamma_{fc})$. For the discrete Heisenberg group this gives an entire circle of parameters. The value of the central generator specifies the commutation relation between the 2 remaining generators.

There is a corresponding question for a compact quantum group G of Kac type. Let $P = Pol(G)$ be its algebra of representative functions and $L^\infty(G)$ its von Neumann algebra in the invariant trace representation. For $a, b \in P$ the adjoint translate of b by a is $\sum_i c_i b S(d_i)$, where $\Delta(a) = \sum_i c_i \otimes d_i$ and S is the antipode. Write P_{fin} for the elements whose adjoint translates span a finite-dimensional space. For the dual of a discrete group these are precisely the linear combinations of elements of Γ_{fc}.

We take double commutants in the bounded operators on the Hilbert space of the trace representation. Does P''_{fin} contain the center of $L^\infty(G)$? We can answer this for a class of crossed products.

Theorem B. *Let a countable discrete group D act by continuous automorphisms on a compact metrizable group Y. Form $A = L^\infty(Y) \rtimes D$ using invariant probability measure on Y, and let P be its polynomial crossed product. Then*

$$Z(A) = Z(P)'' \subset P''_{fin}.$$

Here P consists of the finite sums $\sum_g f_g u_g$, where f_g is a linear combination of matrix coefficients of continuous finite-dimensional unitary representations of Y and u_g implements the action of g.

When Y is connected, only the elements of D_{fc} acting trivially on Y can occur with a nonzero coefficient in a central element. For example, taking Y to be a point gives back the group formula. For an arbitrary compact quantum group of Kac type the containment question remains open.

Our next example is the half-liberated orthogonal group O_N^*. Its coordinate matrix is orthogonal with self-adjoint entries, and any 3 entries satisfy $abc = cba$. Products of 2 entries generate its even polynomial algebra, identified with $Pol(PU_N)$, where $PU_N = U_N/\mathbb{T}$ is the projective unitary group. Entrywise complex conjugation induces an involution σ on PU_N.

Theorem C. *For $N \geq 2$ the polynomial algebra $P = Pol(O_N^*)$ satisfies*

$$P_{fin} = Pol(PU_N), \qquad Z(L^\infty(O_N^*)) = L^\infty(PU_N)^\sigma.$$

The central base is the space of conjugate pairs in PU_N, with the image of invariant probability measure. Almost every factor is $M_2(\mathbb{C})$.

Thus even polynomials give all the elements with finite-dimensional adjoint span. The center sees each conjugate pair as one point. The same pairs also occur in homogeneous spaces, where we can ask how the projective coordinates sit inside each matrix factor.

For 2 orthogonal quantum groups with coordinate matrices u, v of sizes M, N, the orbit of a rectangular identity block of size K has coordinates

$$z_{ia} = \sum_{r=1}^{K} u_{ri} \otimes v_{ra}.$$

We use the product of the invariant traces. The z_{ia} generate the affine orbit algebra, and their products $z_{ia}z_{jb}$ generate its projective subalgebra. First take both groups to be half-liberated. Write U_{MN}^K for the complex $M \times N$ partial isometries of rank K [3]. At rank one a partial isometry modulo phase is determined by its range and source lines, so $U_{MN}^1/\mathbb{T} = P_{\mathbb{C}}^{M-1} \times P_{\mathbb{C}}^{N-1}$.

Theorem D. *For $M, N \geq 2$ and $1 \leq K \leq \min(M, N)$, put $X = U_{MN}^K/\mathbb{T}$, where $\mathbb{T}$ acts by scalar multiplication, and give X its invariant probability measure. For the rectangular orbit of $O_M^* \times O_N^*$ the projective-to-affine inclusion is*

$$L^\infty(X) \ \subset \ L^\infty(X) \rtimes \mathbb{Z}_2,$$

where the generator acts by complex conjugation. Over almost every point of the central base $X/\mathbb{Z}_2$ this is the diagonal inclusion $\mathbb{C}^2 \subset M_2(\mathbb{C})$.

At rank one this matrix inclusion is the same for every pair of side lengths. Nevertheless, the second and fourth moments of the single magnitude $|z_{11}|$ determine the unordered pair $\{M, N\}$. A different way to retain information is to take the second group to be the free orthogonal group O_N^+. Its defining matrix has self-adjoint entries subject to orthogonality. Put $y_a = v_{1a}$. This row generates the free sphere algebra $Q = L^\infty(S_{\mathbb{R},+}^{N-1})$, and its products $y_a y_b$ generate the projective algebra $B = L^\infty(P_+^{N-1})$. Conjugation by the spectral sign of y_1 gives an involution α on B.

Theorem E. *Let $M \geq 2$, $N \geq 3$, and let X be complex projective space $P_{\mathbb{C}}^{M-1}$ modulo complex conjugation, with the image of uniform measure. The rank-one orbit algebra of $O_M^* \times O_N^+$ is*

$$L^\infty(X) \,\bar{\otimes}\, M_2(B).$$

In almost every central fiber the free projective algebra is embedded by $b \to diag(b, \alpha(b))$. Adjoining the sign of z_{11} gives the intermediate algebra Q, and $B \subset Q \subset M_2(B)$ is the first Jones basic construction for $B \subset Q$.

In this construction $M_2(B)$ is generated on $L^2(Q)$ by Q and the orthogonal projection onto $L^2(B)$. The specified copy of B gives a geometric reformulation of the square-versus-rectangle criterion of [29]. Write O_{MN}^{1+} for the rank-one orbit with both groups free. For $M, N \geq 3$, the factors $L^\infty(O_{MM}^{1+})$ and $L^\infty(O_{MN}^{1+})$ are isomorphic exactly when the mixed fibers for $(2, M)$ and $(2, N)$ are isomorphic by a map carrying one specified copy of the free projective algebra onto the other.

The same inclusion $B \subset Q$ also occurs on the radial real free ball. Writing its coordinates as ry_a, with an independent scalar radius r, gives $B \subset Q$ at almost every positive radius and a scalar inclusion at any atom at zero.

Our last examples start with the ordinary Grassmannian $Gr_{LN}(F)$ of L-planes in F^N, for $F = \mathbb{R}, \mathbb{C}$. Write p_{ij} for the entries of the orthogonal projection onto a plane. We write X^τ for the coordinate sign twist of a space or quantum group X. In the twist $Gr_{LN}^\tau(F)$ the entries become $q_{ij} = p_{ij}c_ic_j$, where the c_i are self-adjoint unitaries with $c_ic_j = -c_jc_i$ for $i \neq j$. At rank one these entries are the projective coordinates of the twisted sphere [4]. Signed half-commutation reverses a product of 3 generators with the sign obtained by interchanging each pair once. We can impose this relation either on the entries of the projection or on the affine coordinates of a vector spanning its range.

Theorem F. *In the real twisted projective presentation, replacing signed commutation of the projection entries by signed half-commutation gives back* $P_{\mathbb{R}}^{N-1,\tau}$. *Taking the projective algebra of the twisted half-liberated real sphere gives* $P_{\mathbb{C}}^{N-1,\tau}$. *At every rank* $1 \leq L < N$ *the real and complex twisted Grassmannians have nonisomorphic coordinate* C^**-algebras. Their spaces of characters have real dimensions* r *and* $2r$*, respectively, where* $r = \min(L, N-L)$.

A character assigns scalar values to all the coordinates. Already for $L = 1$ and $N = 2$ the 2 character spaces are a circle and a 2-sphere. We now take the invariant traces on the 2 Grassmannians, given in the model above by integrating the Clifford trace over the classical planes. The Clifford trace vanishes on $c_{i_1} \dots c_{i_k}$ when $i_1 < \dots < i_k$ and $k \geq 1$. Let D^+ be the determinant-one diagonal sign matrices, acting by changes of coordinate signs.

Theorem G. *Let* $X = Gr_{LN}(F)$, *with* $F = \mathbb{R}, \mathbb{C}$ *and* $1 \leq L < N$. *The center of* $L^\infty(Gr_{LN}^\tau(F))$ *has measured base* X/D^+. *With* $m = 2^{[(N-1)/2]}$ *we have a trace-preserving isomorphism*

$$L^\infty(Gr_{LN}^\tau(F)) \simeq L^\infty([0,1]) \bar{\otimes} M_m(\mathbb{C})$$

for Lebesgue measure and normalized matrix trace. If the diagonal coordinate q_{11} *is specified, the algebra with its trace and this element determines* F, L, N.

For Gr_{24}^τ, for instance, both fields give matrix size 2 and $\tau(q_{11}) = 1/2$. The variance of this entry is $1/12$ over $\mathbb{R}$ and $1/20$ over $\mathbb{C}$. Finally, at rank one we may discard all coordinate labels and keep just the inclusion of the projective algebra in the affine sphere algebra. In each branch this is the inclusion generated by the products $x_i x_j$ inside the algebra generated by the sphere coordinates x_i.

Theorem H. *For* $N \geq 2$, *the projective-to-affine inclusions of the twisted real sphere and of the twisted half-liberated real sphere are isomorphic as inclusions with trace exactly when* N *is even. For even* N *their common description is the diagonal block inclusion*

$$M_m(\mathbb{C}) \oplus M_m(\mathbb{C}) \ \subset \ M_{2m}(\mathbb{C}),$$

tensored on both sides with $L^\infty([0,1])$, *where* $m = 2^{[(N-1)/2]}$ *and the* 2 *blocks have equal weight.*

For odd N the half-liberated cover still has these 2 diagonal blocks. The real cover instead embeds a single $M_m(\mathbb{C})$ diagonally in 2 copies of itself over the same diffuse probability space. Thus the affine cover can distinguish 2 projective algebras which, taken on their own, are isomorphic.

The paper is organized as follows. Sections 1 and 2 construct the fibers from the center-valued trace and prove von Neumann's reduction theorem. Sections 3 and 4 treat group algebras, crossed products and orbit stabilizers. Sections 5 and 6 pass from Peter–Weyl theory to the adjoint action for compact quantum groups. Sections 7 and 8 study rectangular orbits and radial free balls. Sections 9 and 10 compute the central fibers of twisted real Grassmannians. Sections 11 and 12 compare the 2 projective constructions and their von Neumann algebras.

1 From the center-valued trace to the fibers

Our aim here is to provide a quick proof of von Neumann's reduction theorem [32] in the tracial case. Write $Z = Z(A)$. For a central projection p, multiplication by p splits A into the summands pA and $(1-p)A$. An element central in pA commutes also with $(1-p)A$, since both products with that summand vanish. Thus $Z(pA) = pZ$. In particular pA is a factor exactly when p is a minimal nonzero projection of Z. This already proves the reduction theorem for a purely atomic center: sum over its atoms and use on pA the trace $a \to \tau(a)/\tau(p)$. Faithfulness gives positive weights, whose sum is one.

Consider what gets lost on taking a scalar trace. In $M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ put $a = (b, c)$ and give the first summand weight s, with $0 < s < 1$. The 2 traces one encounters are

$$\tau(a) = s\, tr_2(b) + (1 - s)\, tr_3(c), \qquad E(a) = \big(tr_2(b)1_2, tr_3(c)1_3\big),$$

where the matrix traces are normalized. From $E(a)$ one obtains the scalar trace for every choice of s. For a fixed s, knowing just $\tau(a)$ leaves the 2 matrix traces undetermined. Knowing also $\tau(pa)$ for each central projection p determines both. This last observation survives even when there are no atoms.

Here and below one can realize (Z, τ) as $L^\infty(X, \mu)$ with integration. To recall why the usual probability spaces suffice, the unit ball of A in its trace metric is separable. Indeed the predual hypothesis makes the ball compact metrizable in the weak star topology. A countable dense set in this topology has rational convex hull dense in the trace metric: bounded weak star convergence implies weak convergence of the corresponding L^2 vectors, and a convex set has the same weak and norm closures in Hilbert space. The same argument applies to Z. Spectral cuts at rational levels of a countable dense self-adjoint set give countably many projections p_j generating Z. A single bounded self-adjoint operator $h = \sum_{j \geq 1} 2/3^j \cdot p_j$ generates them all: the spectral functions reading the ternary digits $0, 2$ recover the projections. The scalar spectral measure of h under τ gives the claimed model on a Borel subset of the real line, with its measure completed. Functions equal almost everywhere represent the same central element.

Theorem 1.1. *There is a unique linear map $E : A \to Z$ characterized by*

$$\tau(za) = \tau(zE(a))$$

for $a \in A$ and $z \in Z$. It is positive, faithful, normal, unital and Z-linear. It fixes Z pointwise and satisfies $E(ab) = E(ba)$.

Proof. Put $\langle a, b \rangle = \tau(b^* a)$ on A and complete to $L^2(A, \tau)$. Orthogonal projection of a onto $L^2(Z, \tau)$ gives a function $g \in L^2(X, \mu)$. Testing against the indicator of a measurable set S gives

$$\int_S g\, d\mu = \tau(1_S a).$$

When $0 \leq a \leq c1$, the right side belongs to $[0, c\mu(S)]$. Testing the imaginary part and the level sets of the real part gives $0 \leq g \leq c$ almost everywhere. A general a is a linear combination of 4 positive elements. Thus its projection is bounded and defines an element $E(a)$ of Z. Orthogonality gives the asserted identity for all z. Testing a difference against its own adjoint proves uniqueness.

The projection fixes Z pointwise. Testing against z after multiplying by another central element gives Z-linearity. Cyclicity of τ gives $\tau(zab) = \tau(zba)$, hence the tracial identity. If $a \geq 0$ and $E(a) = 0$, then $\tau(a) = 0$ and $a = 0$. For a bounded increasing net $a_i \uparrow a$ of positive elements, $\sup_i E(a_i) \leq E(a)$ and the traces of the 2 sides agree by normality of τ. Faithfulness gives equality, proving normality. □

We call E the center-valued trace, normalized by $E(1) = 1$. Dixmier's approximation theorem [14] gives $E(a)$ as a norm limit of averages of unitary conjugates. The following L^2 argument suffices here and shows why E is independent of the scalar trace.

Lemma 1.2. *For each $a \in A$ there are finite convex averages of unitary conjugates of a converging in $L^2(A, \tau)$ to $E(a)$. Each average has operator norm at most $\|a\|$.*

Proof. Take the closed convex hull in $L^2(A, \tau)$ of the elements uau^* with u unitary in A. It has a unique element b of least Hilbert space norm. This element still belongs to the operator norm ball of radius $\|a\|$. Indeed any L^2-convergent sequence in that ball has a weak star convergent subnet there. Testing against $c \in A$ by $d \to \tau(c^* d)$ identifies the subnet limit with the L^2 limit.

Conjugation by a unitary preserves the hull and the Hilbert space norm, so $ubu^* = b$ for all u. Thus $b \in Z$. Orthogonal projection onto $L^2(Z)$ takes every conjugate of a to $E(a)$ and is continuous in the Hilbert space norm. It therefore takes b to $E(a)$. As b is central it is its own projection. □

A bounded sequence converging in L^2 converges weak star: each weak star cluster point agrees with the limit on the tests just used. Consequently every normal tracial state ρ has $\rho(a) = \rho(E(a))$, by applying ρ to the averages. In particular, for another faithful normal tracial state ρ the map E has $\rho(za) = \rho(E(za)) = \rho(zE(a))$ for every $z \in Z$. Uniqueness in Theorem 1.1 gives the very same E. Conversely, composing any normal state on Z with E gives a normal tracial state on A. Choosing a faithful one selects a probability measure in the measure class of the center.

Retain the following form of the defining identity. For $a, b \in A$ the scalar product of their restrictions to S is

$$\langle 1_S a, 1_S b\rangle = \int_S E(b^*a)(x)\, d\mu(x).$$

We know the integral over every measurable S. What would it mean to use its integrand as a scalar product at x?

There is a small trap in this question. Fix x in the ordinary unit interval with Lebesgue measure. The functions 0 and $1_{\{x\}}$ represent the same element of the center and take different values at x. Even the expression $f(x)$ for a central f needs a convention. Choosing representatives of countably many functions lets us impose countably many identities simultaneously off one null set. The rest of the construction will be by completion.

Let D be a countable unital $*$-algebra over $\mathbb{Q} + i\mathbb{Q}$ in A, dense in $L^2(A, \tau)$. For example take the algebra over these scalars generated by a countable trace-metric dense subset of the operator norm unit ball, together with 1. We can enlarge it countably many times. For later use we make one particular enlargement now.

Lemma 1.3. *We may choose D so that for each $d \in D$ there is a sequence c_n of finite rational convex averages of udu^*, with unitaries $u \in D$, satisfying*

$$\sum_{n\geq 1} \|c_n - E(d)\|_2^2 < \infty.$$

The algebra can at the same time contain a countable set dense in the unit ball of A in the trace metric.

Proof. Start with the algebra just described. For each of its elements d, Lemma 1.2 supplies averages within 2^{-n} of $E(d)$ in L^2. The coefficients may be approximated by positive rational numbers summing to one, at additional cost at most 2^{-n}. Adjoin all the unitaries occurring in these averages, take the rational $*$-algebra they generate with the old one, and repeat. At each round there are only countably many choices. An element added today can require new conjugating unitaries tomorrow. Taking the union lets us make all these choices before fixing a point of the base. In the union every element has been treated at some finite round, and the squared error bounds form a convergent series. □

For each $d \in D$ choose a measurable representative f_d of $E(d)$. The indices include the products e^*d. There is a common conull set on which $d \to f_d(x)$ is rational complex linear, respects the involution, sends 1 to 1, is tracial and takes d^*d to a positive number or zero. For example $E(de) = E(ed)$ gives one null exceptional set for each pair d, e, and there are only countably many pairs. We impose one further countable list of inequalities:

$$f_{b^*d^*db}(x) \leq \|d\|^2 f_{b^*b}(x) \qquad (b, d \in D).$$

They follow from $b^*d^*db \leq \|d\|^2 b^*b$ and positivity of E. The norm on the right is the operator norm in A. For the remainder we work on this conull set, retaining the name X.

Definition 1.4. *At x form vectors $[d]_x$ with scalar products*

$$\langle [d]_x, [e]_x \rangle_x = f_{e^* d}(x).$$

The space H_x is obtained by dividing by the vectors of length zero and completing. We write $\Omega_x = [1]_x$ for its distinguished unit vector.

To spell out this completion, positivity for rational complex combinations implies positivity of each finite Gram matrix for arbitrary complex coefficients by continuity. We may therefore take the complex span with this form. Cauchy–Schwarz shows that a vector of length zero is orthogonal to every vector. The countable set of sections $x \to [d]_x$ is dense in every completed space. Even the dimension may depend on x: with $A = M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ the 2 Hilbert spaces have dimensions 4 and 9.

Definition 1.5. *A section ξ, with $\xi(x) \in H_x$, is measurable if*

$$x \longrightarrow \langle \xi(x), [d]_x \rangle_x$$

is measurable for each $d \in D$. The Hilbert space direct integral $\int_X H_x \, d\mu(x)$ consists of such sections for which $\int_X \|\xi(x)\|^2 \, d\mu(x) < \infty$, identified when equal almost everywhere. Its scalar product is the integral of the fiber scalar products.

Here are some details behind this definition. Enumerate the vectors $[d]_x$ and apply Gram–Schmidt in that order, writing zero whenever the next residual has length zero. All coefficients are measurable, including the indicator of the set where division by a residual norm is allowed. This produces sections e_j whose nonzero values form an orthonormal basis at each x. The norm squared of a measurable section is the sum of the squared moduli of its coordinates against the e_j. Conversely any measurable square-summable list of coordinates, zero where the corresponding e_j vanishes, gives a section by summing the series. Thus the proposed direct integral is complete: in these coordinates it is the Hilbert direct sum of the spaces $L^2(\{e_j \neq 0\}, \mu)$. In particular, pointwise limits of measurable sections in fiber norm are measurable.

Theorem 1.6. *Sending each $d \in D$ to the section $x \to [d]_x$ extends uniquely to a unitary operator*

$$L^2(A, \tau) \simeq \int_X H_x \, d\mu(x)$$

which intertwines multiplication by each $f \in Z$ with pointwise multiplication by f.

Proof. Start on finite sums of vectors fd, with $f \in Z$, $d \in D$. Send fd to $x \to f(x)[d]_x$. For $f, g \in Z$ and $d, e \in D$, the scalar product on the left is

$$\tau(e^* g^* f d) = \int_X f(x) \overline{g(x)} f_{e^* d}(x) \, d\mu(x).$$

This proves that the rule respects every relation between finite sums and extends isometrically, since D itself is dense. If a section is orthogonal to its range, testing against all $1_S[d]_x$ shows that its coefficient against $[d]_x$ vanishes almost everywhere. Intersecting the conull sets for all d gives pointwise orthogonality to a dense set. Thus the section is zero. The intertwining identity holds on the initial sums and hence everywhere. □

The theorem explains a point that is easy to miss when drawing a direct integral as a sum. On a diffuse part of X, a vector supported at one point represents zero in the global Hilbert space. Yet Ω_x has norm one at every point in our construction. To see its contribution globally one takes the section $1_S \Omega$, whose squared norm is $\mu(S)$. Positive-measure sets are the places where a fiberwise calculation can be tested.

2 Von Neumann's reduction theorem

We first put operators on the Hilbert spaces just constructed. We then answer separately the questions of recovering all measurable sections and of obtaining factors. Throughout the argument $\|\cdot\|_{2,x}$ will refer to the trace at x.

Proposition 2.1. *For $d, b \in D$ the rule $\pi_x(d)[b]_x = [db]_x$ defines bounded operators of norm at most $\|d\|$. The von Neumann algebra*

$$A_x = \pi_x(D)''$$

has faithful normal tracial state $\tau_x(T) = \langle T\Omega_x, \Omega_x\rangle_x$. Its trace Hilbert space identifies with H_x by $T \to T\Omega_x$.

Proof. The bound is the pointwise inequality in Section 1. The adjoint rule follows from $(d^*e)^* = e^*d$, so left multiplication is a $*$-representation on the completion. Traciality of f at x makes $[b]_x \to [b^*]_x$ an antilinear isometric involution. Conjugating left multiplication by this involution gives bounded right multiplication (with the adjoint on the multiplier). Right multiplications commute with A_x and their values on Ω_x are dense. Hence $T\Omega_x = 0$ for $T \in A_x$ implies $T = 0$. The vector state is faithful and normal. It is tracial on the generating algebra by construction. For each argument separately the trace identity is weak operator continuous with the other argument fixed; applying weak density first in one variable and then in the other proves it on A_x. Finally the vectors $\pi_x(d)\Omega_x$ are dense by definition. □

For these tracial fibers there is a convenient test for operator measurability. For an essentially uniformly bounded field $T_x \in A_x$ it is enough to test the vector section $T_x\Omega_x$. Indeed $T_x[b]_x$ is right multiplication by b applied to that section. Right multiplication preserves measurability: its adjoint is right multiplication by b^*, which takes each test vector to another test vector. Measurability on all measurable sections now follows by expansion in the Gram–Schmidt basis and pointwise convergence. Conversely one can take the measurable input Ω_x. The operator norm of the field is measurable as a supremum of the ratios $\|T_x[d]_x\|/\|[d]_x\|$ with zero denominators omitted. Such a field acts boundedly on the direct integral by pointwise application. This is the meaning of the operator direct integral used here.

We will need a way to keep an operator norm bound while approximating in the tracial L^2 norm. We give the elementary estimate because approximating vectors alone would not control the norms of the operators we assemble.

Lemma 2.2. *In a tracial von Neumann algebra let $b = u|b|$ be the polar decomposition and put $b_1 = u\min(|b|, 1)$. Then*

$$\|b_1 - t\|_2 \le \|b - t\|_2$$

for every contraction t. Moreover $b_1 = b\,g(b^*b)$ for a continuous function g on $[0, \infty)$.

Proof. Set $h = (|b| - 1)_+$. We have $b - b_1 = uh$ and $b_1^*uh = h$. Since t^*u is a contraction, the real part of $\tau(t^*uh)$ is at most $\tau(h)$, as is seen by applying the positive functional $v \to \tau(h^{1/2}vh^{1/2})$. Thus the real part of $\langle b-b_1, b_1-t\rangle$ is positive or zero. Expand the square of $b-t = (b-b_1)+(b_1-t)$. The functional calculus formula uses $g(s) = 1/\max(1, \sqrt{s})$. □

We can now answer the question of which fields occur. Denote by L_a left multiplication by a on $L^2(A, \tau)$. We use its faithful normal representation of A; normality follows by testing bounded increasing nets on the dense vectors $b \in A$ with the normal functionals $a \to \tau(b^*ab)$.

Theorem 2.3. *Under the unitary of Theorem 1.6, the operators L_a are precisely the fields*

$$x \longrightarrow T_x \in A_x$$

that are measurable and essentially bounded, with fields identified almost everywhere. Multiplication and adjoints are computed in the fibers.

Proof. For $d \in D$, left multiplication acts as $\pi_x(d)$: test both actions on the dense vectors fb used in Theorem 1.6. For a contraction $a \in A$, choose contractions $d_n \in D$ with $\|d_n - a\|_2 < 2^{-n}$. The sections $[d_n]_x$ converge in H_x for almost every x, since the integral of $\sum_n \|[d_{n+1} - d_n]_x\|$ is finite. At such a point the operators $\pi_x(d_n)$ also converge on every $[b]_x$, by bounded right multiplication. Their uniform bound then gives strong convergence on H_x to a contraction $a_x \in A_x$. This is a measurable field. Bounded convergence on vector sections shows that L_{d_n} converges strongly to this field. Globally $L_{d_n} \to L_a$ strongly as well, by testing on $b \in A$ and using $\|(d_n - a)b\|_2 \leq \|b\| \, \|d_n - a\|_2$. The resulting field is unique almost everywhere, since its action on the single section Ω determines it by Proposition 2.1.

Conversely let T_x be a measurable field of contractions. Enumerate D as b_j and clip each b_j to a contraction c_j by Lemma 2.2. The field of c_j is the clipping of $\pi_x(b_j)$: the continuous functional calculus formula is obtained by uniform polynomial approximation on $[0, \|b_j\|^2]$, globally and in every fiber. The vectors $\pi_x(b_j)\Omega_x$ are dense in H_x. Applying the clipping inequality to the target T_x shows that the vectors $(c_j)_x\Omega_x$ approximate $T_x\Omega_x$ arbitrarily closely at almost every x.

Fix n. At x take the first index j giving error less than 2^{-n}. Each error is measurable, so the sets S_j where an index is chosen form a measurable partition, up to a null set. The sum $a_n = \sum_j 1_{S_j} c_j$ is a contraction in A: its summands act on orthogonal central summands, and the partial sums converge strongly. Its field acts as $(c_j)_x$ on S_j. The vector $a_n\Omega$ therefore lies within 2^{-n} in L^2 of the section $T_x\Omega_x$. The global unit ball is L^2-closed by the argument in Lemma 1.2. Hence the limit vector is $a\Omega$ for a contraction $a \in A$. The first half of the proof and separation by Ω_x now give $a_x = T_x$ almost everywhere. Scaling removes the unit ball restriction. Products act pointwise, and the scalar product integral gives the pointwise adjoint identity. □

We have reconstructed A, but the argument up to this point has not shown that any A_x is a factor. Knowing that every measurable central section is scalar would suggest choosing a nonscalar central element wherever one exists. Such a choice would itself have to be measurable. The tracial setting allows a different proof. We already chose all the averages we shall need before constructing the Hilbert spaces.

Theorem 2.4. *Almost every A_x is a finite factor. Under the isomorphism of Theorem 2.3 the normalized fiber traces satisfy*

$$E(a)(x) = \tau_x(a_x), \qquad E(b^*a)(x) = \langle a_x\Omega_x, b_x\Omega_x\rangle_x$$

almost everywhere for each $a, b \in A$.

Proof. First take $d \in D$ and the averages c_n in Lemma 1.3. The central element $E(d)$ acts as multiplication by $f_d(x)$. The Hilbert space isometry gives

$$\int_X \sum_n \|(c_n)_x - f_d(x)1\|_{2,x}^2 \, d\mu(x) = \sum_n \|c_n - E(d)\|_2^2 < \infty.$$

Thus at almost every x the scalar $f_d(x)1$ is an $L^2(A_x, \tau_x)$ limit of convex averages of unitary conjugates of $\pi_x(d)$. The unitaries in these averages were included in D, so their images really are unitaries in A_x. Intersect over $d \in D$. We now have a single conull set on which the assertion holds for every test element.

Fix one such x. Take a central $z \in A_x$ with $\tau_x(z) = 0$. Pairing z against any conjugate of $\pi_x(d)$ gives the same answer as pairing it against $\pi_x(d)$, by centrality and traciality. It follows by taking the averages and their limit that

$$\langle [d]_x, z\Omega_x\rangle_x = \tau_x(z^*\pi_x(d)) = f_d(x)\tau_x(z^*) = 0.$$

The vectors $[d]_x$ are dense, so $z\Omega_x = 0$. Faithfulness gives $z = 0$. Subtracting the scalar trace of an arbitrary central element proves $Z(A_x) = \mathbb{C}1$. Notice the order of the choices: only after fixing a good x did we take z. There is no measurability condition on this individual z.

Finally the construction sends the vector a to $a_x\Omega_x$. Testing on $1_S\Omega$ gives

$$\int_S \tau_x(a_x)\,d\mu(x) = \tau(1_S a) = \int_S E(a)\,d\mu$$

for every measurable S. This proves the first identity. Apply it to b^*a to obtain the second. □

In particular the integral of the fiber traces is τ. The scalar product constructed in Section 1 now has an interpretation entirely inside the finite factors. The null set in the displayed identity may depend on a, b. The fixed countable algebra in the proof is what permitted us to assert factoriality on one conull set.

It remains to specify what it means for these factors to be determined by A. We take measurable fields in the separable sense of Section 1: countably many measurable vector sections span a dense subspace in each fiber. In a tracial field of algebras this structure on the trace Hilbert spaces is given by countably many bounded measurable algebra sections whose complex spans are fiberwise dense in the tracial L^2 norm. This convention is satisfied by our sections $[d]_x$. An isomorphism of fields is measurable when it transports measurable sections in both directions.

Theorem 2.5. *For a fixed identification of (Z,τ) with $L^\infty(X,\mu)$, any 2 trace-preserving decompositions of (A,τ) into factors inducing that identification are isomorphic over X. The fiber isomorphisms carry the 2 sections of each $a \in A$ to one another almost everywhere.*

Proof. Write B_x for the factors of a second decomposition and d'_x for the sections there of elements of D. For each $d, e \in D$, testing on all measurable S gives the same integrals of the 2 Gram coefficients, because both integrals are $\tau(1_S e^* d)$. On one conull set all these coefficients agree and the rational algebra relations for D hold in both models. Consequently $[d]_x \to d'_x\Omega'_x$ extends to an isometry on H_x.

Here is why its image fills $L^2(B_x)$. Choose countably many bounded measurable algebra sections total in the second field in the sense just described. Each is the image of a global element by the meaning of decomposition. Approximate each in global L^2 by elements of D, choosing squared errors summable along its approximating sequence. Integration of the squared errors implies pointwise L^2 convergence almost everywhere along each sequence. Outside the union of the exceptional sets the image of our isometry contains every chosen vector in its closure and is therefore the whole fiber. The resulting unitaries U_x form a measurable field: on the countable total sections their values are the specified measurable vectors, and Gram–Schmidt and passage to limits give the assertion on all sections, in both directions.

These unitaries send Ω_x to Ω'_x and intertwine left multiplication by each d. In the second field the operators d'_x generate B_x almost everywhere. Indeed their vectors are total by the preceding argument, and clipping approximants as in Lemma 2.2 gives bounded strong approximation to every contraction in B_x. Thus conjugation by U_x identifies the generated algebras and preserves their traces. On the direct integrals it carries the 2 copies of every $d \in D$ to each other. The bounded approximation of Theorem 2.3 extends this to each $a \in A$. □

We have so far identified the 2 centers with functions on the same space. Suppose instead that the 2 decompositions are given over different spaces. Which change of variables on the base must precede the fiber isomorphisms?

Proposition 2.6. *A trace-preserving isomorphism between 2 direct integrals of finite factors identifies their probability bases modulo null sets.*

Proof. In both models the diagonal algebras are their centers. To see this directly in a factor field, a central section commutes almost everywhere with a countable generating class of sections; off one null set it is central in the fiber and hence scalar. The scalar is measurable by taking its fiber trace. Thus an isomorphism preserving the global trace identifies the 2 measure algebras

and their probability measures. On standard probability spaces this identification is implemented by a point map modulo null sets. Concretely, choose a bounded real Borel coordinate separating points on the first standard Borel space (realize it as a Borel subset of $[0,1]$). Its image in the other algebra is a measurable function with the same law. Spectral cuts show that pulling back by this function implements the isomorphism on all Borel sets. Doing the same for the inverse gives inverse maps almost everywhere, as can be checked on countably many rational cuts. □

We have identified the base together with its measure. What happens to that measure if we keep the algebra and choose another trace?

Proposition 2.7. *Changing the faithful normal tracial state changes the measure on the base by its Radon–Nikodym density and preserves the normalized traces in the fibers.*

Proof. The new restriction to Z is $h\,d\mu$ with $h > 0$ almost everywhere and $\int h\,d\mu = 1$. The center-valued trace stays the same by Section 1. The very same measurable field, equipped with its original normalized fiber traces and the new probability measure, now integrates to the new trace:

$$\rho(a) = \rho(E(a)) = \int_X h(x)\tau_x(a_x)\,d\mu(x).$$

For an atom this is precisely the freedom to choose the weight s in our opening calculation. □

A central algebra of diagonal functions can distinguish every point of an interval and still miss a central projection. The following example exhibits that projection and its weight.

Proposition 2.8. *Let R be a II_1 factor with separable predual and normalized trace τ_R. Over $[0,1]$ with Lebesgue measure take*

$$A = L^\infty([0,1]) \mathbin{\bar\otimes} \big(M_2(\mathbb{C}) \oplus R\big).$$

Fix $0 < s < 1$ and on the constant algebra use $s\,tr_2 + (1-s)\tau_R$. The central reduction has 2 copies of the interval, carrying respectively the constant factors $M_2(\mathbb{C})$ and R, with measures $s\,dt$ and $(1-s)\,dt$.

Proof. If we diagonalize only the displayed $L^\infty([0,1])$, the fiber at t is $M_2(\mathbb{C}) \oplus R$. In particular the constant section $(1_2, 0)$ is a central projection which this choice of diagonal misses. The whole center gives 2 copies of $[0,1]$, with measures $s\,dt$ and $(1-s)\,dt$. The fibers on the first copy are $M_2(\mathbb{C})$ and on the second copy they are R. For a section $a(t) = (b(t), c(t))$ the central function is

$$E(a)(t,1) = tr_2(b(t)), \qquad E(a)(t,2) = \tau_R(c(t)).$$

Integrating on either measurable part of either copy reproduces the trace of the corresponding central cut. This is exactly the test that characterized E before any fiber had been constructed. □

Taking the full center gives factor fibers, and the trace fixes their weights. We can now state the reduction theorem with its uniqueness assertion.

Theorem 2.9. *Let A be a von Neumann algebra with separable predual and faithful normal tracial state τ. Write its center, with the restriction of τ, as $L^\infty(X,\mu)$ on a standard probability space. There is a measurable field of factors (A_x, τ_x) with normalized traces whose direct integral gives*

$$A \simeq \int_X A_x\,d\mu(x), \qquad \tau(a) = \int_X \tau_x(a_x)\,d\mu(x).$$

Under this isomorphism a central function f acts as $f(x)1$ on the fiber. The field is determined up to a measure-space isomorphism modulo null sets and a measurable field of trace-preserving isomorphisms almost everywhere.

Proof. Theorem 1.6 constructs the unitary and its diagonal action. Theorem 2.3 identifies all bounded measurable operator sections, and Theorem 2.4 gives finite factors with the stated traces. For uniqueness, identify the 2 bases by the preceding measure-space argument and apply Theorem 2.5 over this common base. □

3 Group von Neumann algebras

For a countable discrete group Γ we use the left translations λ_g on $l^2(\Gamma)$ and the trace $\tau(a) = \langle a\delta_1, \delta_1\rangle$. Thus the Fourier coefficients of a are the coordinates of $a\delta_1$. They are square summable. We begin with the classical finite-group Plancherel formula [26].

Theorem 3.1. *For a finite group G, the central reduction of $L(G)$ has base $Irr(G)$. At π the factor is $M_{d_\pi}(\mathbb{C})$ and its mass is $d_\pi^2/|G|$, where $d_\pi = \dim \pi$. The center-valued trace at π is the normalized matrix trace of the Fourier transform at π.*

Proof. In the regular representation, Peter–Weyl theory [26] puts d_π copies of π on the space of its matrix coefficients. Thus λ_g becomes the direct sum of the matrices $\pi(g)$ with these multiplicities. Schur orthogonality shows that the Fourier map from the group algebra onto the sum of matrix algebras is bijective. On the regular representation the normalized operator trace takes the value $\delta_{g,1}$ on λ_g, so it agrees with τ. A block projection has rank d_π^2, giving its mass. Normalizing on that block gives the final assertion. □

For an infinite group the regular representation can have diffuse center. The classical conjugacy class formula still computes the projection onto that center [13, 7].

Proposition 3.2. *Let C_g be the conjugacy class of g in a countable group Γ. The center-valued trace in $L(\Gamma)$ satisfies*

$$E(\lambda_g) = \begin{cases} |C_g|^{-1} \sum_{k \in C_g} \lambda_k & (|C_g| < \infty), \\ 0 & (|C_g| = \infty). \end{cases}$$

In particular $L(\Gamma)$ is a factor exactly when every conjugacy class except $\{1\}$ is infinite.

Proof. The Fourier coefficients of any central element are constant on each conjugacy class, by invariance under conjugation. A square summable function constant on an infinite class vanishes there. On a finite class the displayed average is central and has the same scalar product against every central element as λ_g. On an infinite class this scalar product vanishes. Orthogonal projection therefore gives the formula. A nonidentity finite class supplies a nonscalar central element. In the absence of such classes every central vector is supported at 1. □

This formula singles out a characteristic subgroup of the ambient group.

Definition 3.3. *The FC-center $H = \Gamma_{fc}$ consists of the elements with finite conjugacy class in Γ. A group is FC when all its elements have this property.*

Products and inverses of elements with finite conjugacy class again have finite conjugacy class, and automorphisms preserve the class sizes. This proves the characteristic subgroup assertion. The ambient group matters: an infinite cyclic subgroup of the free group F_2 is abelian, while none of its nonidentity elements has finite conjugacy class in F_2. A finitely generated FC-group has center of finite index, because the centralizers of a finite generating set are finite-index subgroups and their intersection is the center. For the group theory see [25, 30]. The classical description of the center is as follows [7].

Theorem 3.4. *Let H be the FC-center of a countable group Γ. Then*

$$Z(L(\Gamma)) = L(H)^\Gamma = Z(L(H))^\Gamma.$$

Here Γ acts on $L(H)$ by conjugation.

Proof. Every central element has Fourier coefficients supported on H, by the preceding proposition. The trace-preserving expectation onto $L(H)$ is orthogonal projection in the trace Hilbert space and retains exactly the coefficients in H. Thus every central element belongs to $L(H)$. Conversely an element there fixed by conjugation commutes with every λ_g, hence with $L(\Gamma)$. In particular it commutes with $L(H)$, proving the second equality. □

For the representation theory surrounding this formula see [13]. Bekka describes the central measure itself in [7]. On the extreme normalized characters of H take the Plancherel measure and pass to the orbits of the closure of the conjugation action of Γ on H. This closure in the topology of pointwise convergence is compact, since it is a closed subgroup of the product of the permutation groups of the finite conjugacy classes. The image measure is the central measure for Γ. Almost every factor representation is obtained by taking the direct integral of the character representations over one orbit with its invariant probability measure and inducing the resulting representation from H to Γ.

Theorem 3.5. *The restricted direct sum $\Gamma = \bigoplus_{n\geq 1} S_3$ is FC. The base of its central reduction is the infinite product of the 3-point probability space with masses $1/6, 1/6, 2/3$. Almost every fiber is the hyperfinite II_1 factor.*

Proof. Conjugation preserves the finite support of a group element, so its class is finite. The regular trace is the product of the coordinate traces. On one copy of S_3 the trivial, sign and standard representations have dimensions $1, 1, 2$. The finite group formula gives the indicated 3 central weights. At a sequence of labels take the tracial infinite tensor product of the selected matrix blocks. On finite tensors this prescription respects products, adjoints and the integrated trace by the finite group formula and independence. Cylinder indicators belong to the image as products of central projections and generate the base algebra. Fiberwise the finite tensors generate the selected blocks at every finite stage. Consequently the trace representation gives the full direct integral: finite tensors with rational coefficients give countably many fiberwise total sections, and multiplication by base indicators gives dense simple approximations as in Theorem 2.3.

With product probability one the standard label occurs infinitely often: the probability of avoiding it for k consecutive coordinates is $(1/3)^k$. Discarding scalar tensor legs leaves the infinite tracial tensor product of copies of $M_2(\mathbb{C})$. The trace expectations onto its finite tensor stages converge in the norm $\|\cdot\|_2$ by density. They send a central element to a scalar at each stage by bimodularity. Thus the limit is scalar. The increasing matrix stages give hyperfiniteness and have unbounded dimension, so this finite factor has type II_1. Factoriality identifies the base above as the full center. □

In the restricted direct sum, a point of the base records one representation label for each copy of S_3. The discrete Heisenberg group gives a different kind of parameter. Here the value of a single central unitary specifies the commutation relation in the fiber.

Theorem 3.6. *Let Γ be the discrete Heisenberg group with generators u, v, w, where w is central and $uv = wvu$. The center of $L(\Gamma)$ is $L(\langle w\rangle)$. Its base is the circle with uniform measure. At almost every $t \in [0, 1)$ the fiber is generated by unitaries U, V with $UV = e^{2\pi i t}VU$ and trace zero on $U^m V^n$ for $(m, n) \neq (0, 0)$.*

Proof. Every element has a unique form $u^m v^n w^k$, as one sees in the upper unitriangular integer matrices of size 3. Conjugation by u or v varies k by n or $-m$. Hence all classes outside $\langle w\rangle$ are infinite. The preceding formula and density give the center, and Fourier transformation sends w to the circle coordinate. Central expectation kills every monomial with $(m, n) \neq (0, 0)$. Consequently the fiber vectors $U^m V^n \Omega$ are orthonormal, and they span by the construction in Section 1. Their multiplication and trace determine the fiber representation by left multiplication. For irrational

t, conjugation by U and by V acts on these vectors, in the conjugation representation, with eigenvalues $e^{2\pi itn}$ and $e^{-2\pi itm}$. Only the constant vector is fixed by both. This proves factoriality at every irrational parameter in this model. The rational parameters form a null set. □

4 Crossed products and ergodic decomposition

For a probability preserving action of a countable group Γ on a standard probability space (Y, ν) we write $u_g f u_g^* = f \circ g^{-1}$. In $L^\infty(Y) \rtimes \Gamma$ the trace of a finite sum $\sum_g f_g u_g$ is $\int f_1 \, d\nu$. Its squared trace norm is $\sum_g \|f_g\|_2^2$, by multiplication of the sum with its adjoint. Hence every element has unique Fourier coefficients in this Hilbert space expansion.

Theorem 4.1. *Suppose the action is essentially free, meaning that each $g \neq 1$ fixes a null set. Then the center of $L^\infty(Y) \rtimes \Gamma$ is $L^\infty(Y)^\Gamma$. Given the ergodic decomposition $\nu = \int_X \nu_x \, d\mu(x)$ over the invariant functions, its central fibers are*

$$L^\infty(Y, \nu_x) \rtimes \Gamma$$

almost everywhere. The fiber trace on a finite sum is $\int f_1 \, d\nu_x$.

Proof. For an element commuting with multiplication by b, comparison of Fourier coefficients gives $f_g(b - b \circ g^{-1}) = 0$. Take b from a countable class of indicators separating points of Y. For $g \neq 1$ the support of f_g lies in the fixed set of g and hence is null. The surviving coefficient commutes with the translations exactly when it is invariant. This identifies the center.

For the fiber calculation take indicators from a countable algebra generating the Borel sets and close that algebra under the action. Rational finite Fourier sums with these coefficient functions are total in the trace Hilbert spaces for ν and for each ν_x. Their products and adjoints are given by the same group action. Disintegrating the scalar trace, also after multiplication by each base indicator, identifies the fiber scalar products in Section 1 with those computed using ν_x. Completion gives the asserted representation. Freeness passes to almost every component by disintegrating the countably many fixed sets. The center computation on each such component gives scalars by ergodicity. □

The rational parameters omitted from the almost-everywhere statement for the Heisenberg group can still be examined individually. For a rational rotation every point has the same nontrivial stabilizer, whose generator contributes a central unitary in the crossed product.

Proposition 4.2. *For the rotation action of $\mathbb{Z}$ on the circle through angle $2\pi t$, the crossed product with uniform measure realizes the tracial Heisenberg fiber at t. If $t = p/q$ in lowest terms, the center of this rotation algebra is generated by V^q, U^q, where V is the circle coordinate and $UVU^* = e^{2\pi it}V$.*

Proof. Choose the direction of rotation to give the displayed covariance relation. The vectors $U^m V^n$ are orthonormal for the crossed product trace and total by Fourier expansion, giving the same trace representation as the monomial construction above. At a rational parameter the 2 conjugation eigenvalues are simultaneously one exactly on $q\mathbb{Z}^2$. The corresponding monomials generate the stated central algebra. Every central element belongs to it, since the orthogonal projection onto its trace Hilbert space is its trace-preserving conditional expectation [27]. □

We turn to finite orbits. The points of an orbit account for a matrix algebra. To find what remains in each diagonal corner we must keep track of the group elements that fix one point. This is the finite-orbit case of Green's imprimitivity theory [16].

Theorem 4.3. *Let a countable discrete group D act by measure-preserving transformations on a standard probability space (Y, ν) with almost every orbit finite. Give the orbit space X its image probability measure. Over an orbit O of size m, choose $y \in O$ and put $J = D_y$. The tracial fiber of $L^\infty(Y) \rtimes D$ over O is*

$$M_m(\mathbb{C}) \,\bar{\otimes}\, L(J)$$

with the normalized matrix trace times the regular trace of J. Its further central reduction is obtained by reducing $L(J)$.

Proof. Choose a Borel coordinate separating points on an invariant conull model of Y. Each finite orbit has a least point. Countability of D makes this selection Borel, giving a model for X. Disintegrate ν over X. Each $g \in D$ preserves the base and the measure, hence preserves the conditional probabilities by uniqueness. Countability and transitivity give uniform measure on almost every orbit.

For a finite Fourier sum $a = \sum_g f_g u_g$, its squared trace norm is the integral over X of the orbitwise values

$$\sum_g \frac{1}{m} \sum_{z \in O} |f_g(z)|^2.$$

Restriction is therefore isometric in L^2. On each orbit the indicators from a countable generating algebra of Borel sets span all point projections. With the group translations they generate the orbit crossed product, so these sections are fiberwise total. Multiplying by base indicators and applying Theorems 1.6 and 2.3 gives the full measurable field.

Order the orbit starting at y, write p_i for the point projections and choose g_i taking y to point i, with $g_1 = 1$. These choices are measurable using coordinate order and the first possible g_i in a fixed enumeration of D starting at 1. The operators $v_i = u_{g_i} p_1$ have initial projection p_1 and final projections p_i, so $v_i v_j^*$ are matrix units. The corner is spanned weakly by $p_1 u_g p_1$ for $g \in J$, the other compressions being zero. The surviving unitaries have normalized trace $\delta_{g,1}$ and identify the corner with $L(J)$. The matrix units give the tensor product, whose center consists of scalar matrices over $Z(L(J))$. □

When the stabilizer is finite we can finish the calculation with the finite group formula of Section 3.

Corollary 4.4. *On an orbit of size m with finite stabilizer J, the central factors have sizes $m d_\pi$ and relative masses $d_\pi^2/|J|$, for $\pi \in Irr(J)$.*

Proof. In the corner the central projection indexed by π has trace $d_\pi^2/|J|$ and cuts out $M_{d_\pi}(\mathbb{C})$. Tensoring with $M_m(\mathbb{C})$ multiplies the matrix size by m and preserves this weight for the normalized trace. □

5 Compact quantum groups of Kac type

Let G be a compact matrix quantum group of Kac type. Write h for its Haar trace, $L^\infty(G)$ for its tracial completion, and u^α for representatives of the irreducible unitary corepresentations. We distinguish 2 operations involving exactly the same matrix coefficients.

Theorem 5.1. *The coefficients of u^α span a space of dimension d_α^2 in $L^2(G)$, where $d_\alpha = \dim u^\alpha$. On the discrete dual, the same index set gives the central decomposition*

$$l^\infty(\widehat{G}) = \prod_{\alpha \in Irr(G)} M_{d_\alpha}(\mathbb{C}).$$

For a faithful normal tracial state on this product the masses are arbitrary strictly positive weights summing to one. When G is finite the normalized regular trace on the dual gives masses $d_\alpha^2 / \sum_\beta d_\beta^2$.

Proof. The Kac form of Schur orthogonality [35] says that the coefficients $\sqrt{d_\alpha}\, u^\alpha_{ij}$ over all indices form an orthonormal basis of $L^2(G)$. The coproduct sends u^α_{ij} to $\sum_k u^\alpha_{ik} \otimes u^\alpha_{kj}$. Consequently the coefficient functionals dual to these basis elements before normalization multiply as matrix units within each fixed α, and multiply to zero between distinct indices. The bounded product of the resulting matrix blocks is the discrete dual algebra. Its minimal central projections are the block identities. This proves the formula for every choice of scalar trace. In finite dimension left multiplication on a matrix block of size d_α contains d_α copies of its defining representation. Taking the normalized regular trace gives the weight of each block. □

For $G = \widehat{\Gamma}$ all irreducible corepresentations are one-dimensional. For instance, when $\Gamma = S_3$ the discrete dual has 6 scalar blocks, one per permutation, while $L^\infty(\widehat{S_3}) = \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$ has 3 blocks, indexed by the trivial, sign and standard representations. We compute the central trace on the compact side directly from conjugacy.

Proposition 5.2. *For the compact quantum group $G = \widehat{\Gamma}$ of a countable discrete group, the Peter–Weyl spaces are the individual lines $\mathbb{C}\lambda_g$. Writing C_g for the conjugacy class of g, the center-valued trace satisfies*

$$E(\lambda_g) = \begin{cases} \dfrac{1}{|C_g|} \displaystyle\sum_{k \in C_g} \lambda_k & (|C_g| < \infty), \\ 0 & (|C_g| = \infty) \end{cases}$$

for every $g \in \Gamma$.

Proof. The coproduct sends λ_g to $\lambda_g \otimes \lambda_g$, and the group basis is orthonormal and total for the regular trace $\delta_{g,1}$. These are already the coefficients of inequivalent one-dimensional corepresentations. In the compact algebra multiplication sends the lines indexed by g, k into the line indexed by gk. Its von Neumann algebra is $L(\Gamma)$, so Section 3 applies. In particular an infinite group with all nonidentity classes infinite has infinitely many Peter–Weyl lines and a one-point central reduction. □

Let a countable discrete group D act by continuous group automorphisms on a compact metrizable group Y. In the crossed product construction of Wang [33] the compact quantum group G has $C(G) = C(Y) \rtimes D$, with each implementing unitary group-like. On a finite Fourier sum h integrates its identity coefficient against normalized Haar measure on Y. Indeed left and right invariance follow on $C(Y)$ from the group integral, and both invariance identities vanish in every nonidentity degree. Its trace representation gives $L^\infty(G) = L^\infty(Y) \rtimes D$. See also Fima, Mukherjee and Patri [15].

Proposition 5.3. *Let $K = \mathbb{Z}_2$ act on $Y = S_3$ by conjugation by (12). For the associated compact quantum group G, the algebra $L^\infty(G)$ has 4 scalar factors of mass $1/12$ each and 2 factors $M_2(\mathbb{C})$ of mass $1/3$ each.*

Proof. The coproduct on $C(S_3)$ comes from group multiplication and sends the implementing involution u to $u \otimes u$. By the crossed product trace formula, $h(f_0 + f_1 u)$ is the uniform average of f_0. The identity and (12) are fixed points, each of mass $1/6$. Each contributes $L(\mathbb{Z}_2)$, giving 2 scalar factors of equal relative weight. The other 2 transpositions form one free orbit, and the 2 3-cycles another. Each orbit has mass $2/6$ and trivial stabilizer. Theorem 4.3 gives the 2 matrix factors. □

In the tetrahedral example of [28], a finite quotient determines the orbits, but the kernel of that quotient contributes a diffuse part of the center.

Theorem 5.4. *Let $D = \mathbb{Z}^3 \rtimes \mathbb{Z}$, where the generator t of $\mathbb{Z}$ cycles the standard basis e_1, e_2, e_3. Let D act on SO_5 by conjugation through the homomorphism $D \to A_4$ sending e_1 to $(12)(34)$ and t to (123), with A_4 permuting the first 4 coordinates. Put $C(G) = C(SO_5) \rtimes D$. Write X for the space of A_4-conjugation orbits in SO_5, equipped with the image of Haar probability measure. Then*

$$L^\infty(G) \simeq L^\infty(X \times \mathbb{T}^4) \bar{\otimes} M_{12}(\mathbb{C})$$

with uniform probability measure on the torus and normalized matrix trace. In particular the central fibers have matrix size 12 almost everywhere.

Proof. The fixed points of $a \in A_4$ form its centralizer in SO_5. For $a \neq 1$ this is a proper closed subgroup because SO_5 has trivial center. A closed subgroup of positive measure has finite index, since all its cosets have the same measure. Such a subgroup is open and hence is the whole connected group. Removing the centralizers of the 11 nonidentity elements therefore leaves a conull set on which each orbit has 12 points.

The stabilizer in D of every remaining point is the same group $J = \ker(D \to A_4)$. Passing to the quotient of A_4 by the Klein 4 group shows that the exponent of t must be divisible by 3. On $\mathbb{Z}^3$ the kernel is the index-4 lattice Λ consisting of triples whose coordinates have the same parity: the 3 images in $\mathbb{Z}_2^2$ sum to zero and any 2 generate it. Since t^3 acts trivially on $\mathbb{Z}^3$, we obtain

$$J = \Lambda \times \langle t^3 \rangle \simeq \mathbb{Z}^4.$$

Choose a measurable orbit representative as in Theorem 4.3. At each representative the corner unitaries are indexed by this fixed J, so the matrix units and these unitaries identify the field with the constant field $M_{12}(\mathbb{C}) \otimes L(J)$ over X. Fourier transformation on J identifies its regular trace with uniform probability on $\mathbb{T}^4$. This gives the product measure and identifies the full center. □

The lattice shift of [28] gives a different comparison. In this example a virtual character has zero restriction to every torus, and we can compute its center-valued trace from 2 coordinates.

Theorem 5.5. *Let $D = \mathbb{Z}^r$, with $r \geq 1$, act by shifts on $Y = S_3^D$, and put $C(G) = C(Y) \rtimes D$. Then $L^\infty(G)$ is a II_1 factor. Fix a standard basis vector e of D, and write s, c for the sign character and the standard 2-dimensional character of S_3. The virtual character*

$$P(x) = (1 - s(x_0))(1 + s(x_e) - c(x_e))$$

restricts to zero on every group-dual torus of G and has center-valued trace $E(P) = 1$.

Proof. The uniform product measure on Y is invariant under group translation. For $g \neq 0$, a configuration fixed by g has equal entries at $0, g, \ldots, ng$. These independent entries agree with probability 6^{-n}. Thus the action is essentially free. Given 2 cylinder sets, shifting one sufficiently far makes their coordinate sets disjoint and their intersection measure the product of their measures. Approximation by cylinders gives mixing, hence ergodicity. The center computation in Section 4 gives a factor. Its trace is finite and its diffuse subalgebra $L^\infty(Y)$ rules out finite matrix size.

For the torus assertion, the restriction criterion of [28] reduces the test on Y to shift-invariant abelian subgroups. All coordinates of an element of such a subgroup commute pairwise, because the element commutes with each of its shifts. The 2 factors defining P are respectively twice the indicator of the transpositions and 3 times the indicator of the 3-cycles. A transposition and a 3-cycle in S_3 never commute. Hence the restriction vanishes. Under product measure the entries at 0 and e are independent, giving

$$h(P) = 6 \cdot \frac{3}{6} \cdot \frac{2}{6} = 1.$$

The normalized center-valued trace on a factor is its scalar trace times the identity, proving the last assertion. □

A magic unitary is a square matrix of projections which are pairwise orthogonal in each row and column and sum to one there. The universal such matrix gives S_N^+ [34]. A pair of reflections already gives a quantum permutation subgroup whose central reduction is diffuse.

Theorem 5.6. *Put $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2$ with generators s, t. Its compact dual is a quantum subgroup of S_4^+. With its own normalized invariant trace the subgroup algebra has the central reduction*

$$L(\Gamma) \simeq L^\infty((0,\pi), d\theta/\pi) \bar{\otimes} M_2(\mathbb{C}).$$

At θ the product st has eigenvalues $e^{i\theta}, e^{-i\theta}$ and s interchanges their eigenspaces.

Proof. For $p = (1+s)/2$ form the magic block with diagonal entries p and off-diagonal entries $1-p$. Take a second block with $q = (1+t)/2$. Their block diagonal sum is magic and its entries generate the full group C^*-algebra. The coproduct sends p to $p \otimes p + (1-p) \otimes (1-p)$, and likewise for q. Hence the quotient map intertwines coproducts.

Write $r = st$. The group is generated by r, s with $srs = r^{-1}$, and r has infinite order by the reduced word description. Fourier transformation realizes $L(\Gamma)$ as the crossed product of the uniform circle by inversion. Apart from the 2 endpoints every orbit consists of $e^{i\theta}$ and $e^{-i\theta}$ for $0 < \theta < \pi$. A circle function becomes its 2 diagonal values, and s becomes the matrix interchanging them. The indicators of the 2 semicircles give the diagonal matrix units. Multiplication by s supplies the off-diagonal ones. Invariant functions give all scalar functions on the interval. The integrated normalized matrix trace agrees with the group trace on the basis $r^k, r^k s$, proving the asserted tracial identification. □

Inversion identifies the angles θ and $-\theta$. Their common cosine gives a single real coordinate on the central base.

Corollary 5.7. *Let $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2$, with canonical generators s, t. The center of $L(\Gamma)$ is generated by $c = (st + ts)/2$. With respect to the canonical trace, the distribution of c is the arcsine law*

$$\frac{dx}{\pi\sqrt{1-x^2}}$$

on $(-1, 1)$.

Proof. In the matrix model c is $\cos\theta$ times the identity. Cosine is one-to-one on the chosen interval, so c generates the full center. Substitution in $d\theta/\pi$ gives the density. □

The trace used here is that of the quantum subgroup. Pulling it back to the universal permutation algebra gives the displayed example in 4 letters. The center computed above is that of $L(\Gamma)$.

6 The FC problem for quantum groups

Theorem 3.4 recovers the center from group elements whose conjugates span a finite-dimensional space. For quantum groups we use the Hopf adjoint action. For harmonic analysis and character algebras in the finite-dimensional semisimple case see [1]. Let P be the polynomial Hopf $*$-algebra of a compact quantum group G of Kac type, with antipode S and counit ε.

Definition 6.1. *The locally finite part P_{fin} is the set of $b \in P$ for which the elements*

$$a \cdot b = \sum a_{(1)} b S(a_{(2)})$$

span a finite-dimensional vector space as a runs through P. We use Sweedler notation $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ for the coproduct.

The adjoint translates satisfy $(a \cdot b)^* = S(a)^* \cdot b^*$. As a runs through P, so does $S(a)^*$, and hence P_{fin} is closed under adjoints. Its double commutant is therefore the von Neumann algebra it generates.

In a group algebra this part is exactly $\mathbb{C}[\Gamma_{fc}]$. A finite-dimensional space of group polynomials is supported on a common finite set, which for the conjugation span must contain the conjugates of every support element. The converse follows by taking the union of their finite classes. Joseph and Letzter [18] studied this locally finite part for quantized enveloping algebras. For general Hopf algebras, Kolb, Lorenz, Nguyen and Yammine [20] prove that P_{fin} is a left coideal subalgebra and, under virtual cocommutativity, a Hopf subalgebra.

Question 6.2. *For an arbitrary compact quantum group G of Kac type, with polynomial Hopf algebra P, do we have*

$$Z(L^\infty(G)) \subset P''_{fin}$$

in the representation associated to the invariant trace?

The n-i.c.c. conditions of Krajczok and Sołtan [21] reduce to ordinary i.c.c. for discrete groups and imply factoriality of $L^\infty(G)$ when imposed on $\widehat{G}$. The question above also concerns algebras with nontrivial center. For an ordinary discrete group the sum over any finite conjugacy class belongs to the center of the group algebra. These class sums span a dense subspace of the trace Hilbert space of the center, by the Fourier calculation in Section 3. Thus one can recover the von Neumann center even from central polynomials. We prove the corresponding assertion for compact groups acted on by discrete groups of automorphisms.

Theorem 6.3. *Let a countable discrete group D act by continuous automorphisms on a compact metrizable group Y. Put $A = L^\infty(Y) \rtimes D$ and let P consist of the finite Fourier sums with representative functions on Y as coefficients. Then*

$$Z(A) = Z(P)'' \subset P''_{fin}.$$

Here P_{fin} is taken for the Hopf adjoint action of the polynomial crossed product, and the double commutants are taken on $L^2(A)$.

Proof. Write $a \in Z(A)$ in trace Hilbert space coordinates as $\sum_{g \in D} f_g u_g$, where $u_g b u_g^* = \alpha_g(b)$. Commutation with every function and every group unitary gives

$$f_g(b - \alpha_g(b)) = 0, \qquad f_{kgk^{-1}} = \alpha_k(f_g).$$

Testing the first relation on a countable class of continuous functions separating points gives $f_g = 0$ almost everywhere outside $Y_g = \{y : gy = y\}$. The second gives equal coefficient norms on each conjugacy class. Since the squares of these norms sum to $\|a\|_2^2$, only finite classes can have a nonzero coefficient. It suffices to approximate one such class at a time.

Fix g with $f_g \neq 0$, and let D_g be its centralizer. Then f_g is D_g-invariant. Write $L^2(Y)$ as the orthogonal sum of its Peter–Weyl coefficient spaces. The automorphisms from D_g permute these finite-dimensional spaces, preserving their Hilbert space norms. The norms of the projections of f_g are therefore constant on each orbit of spaces. Square summability again forces the projections to vanish on the infinite orbits. Projecting onto successively larger finite unions of the remaining orbits approximates f_g in L^2 by D_g-invariant representative functions.

We must keep these approximants supported on Y_g. This is where the group structure of Y enters. Its fixed set Y_g is a closed subgroup of positive measure and hence has finite index. It is open, and its indicator is a representative function: it is the diagonal coefficient at the identity coset of the permutation representation on Y/Y_g. Multiplying the approximants by this indicator preserves D_g-invariance and can only decrease their distance to f_g. Denote a resulting polynomial by p.

Choose representatives k of the finitely many left cosets of D_g in D. Transport p around the class to form

$$b_p = \sum_{k \in D/D_g} \alpha_k(p) u_{kgk^{-1}}.$$

This sum is independent of the representatives by D_g-invariance. Conjugation by a group unitary permutes its terms, and each coefficient is supported on the fixed set of its group element. Hence b_p commutes with the functions and the group unitaries. It belongs to $Z(P)$, and its squared distance to the part of a on this class is $[D : D_g]\|p - f_g\|_2^2$. Truncating first to finitely many classes now approximates a by elements of $Z(P)$ in L^2.

Since P generates A, we have $Z(P) \subset Z(A)$. The trace-preserving expectation onto $Z(P)''$ is orthogonal projection in L^2. It fixes the approximants and hence fixes a, proving equality. Finally, for $z \in Z(P)$ the Hopf adjoint action is $b \cdot z = \varepsilon(b)z$, by the antipode identity, where ε is the counit. Thus every central polynomial belongs to P_{fin}. □

For a connected base the fixed-set condition on a surviving coefficient has a group-theoretic formulation.

Corollary 6.4. *With the assumptions of Theorem 6.3, suppose Y is connected. Write K for the kernel of the action and $H = K \cap D_{fc}$. Then*

$$Z(A) = \left(L^\infty(Y) \mathbin{\bar{\otimes}} L(H)\right)^D,$$

where D acts on functions by the given automorphisms and on $L(H)$ by conjugation.

Proof. For every surviving degree g, the open subgroup Y_g must equal the connected group Y. Thus $g \in K$, and the finite-class condition also gives $g \in D_{fc}$. The trace-preserving expectation retaining the degrees in H consequently fixes every central element. Since H acts trivially on Y, the resulting subalgebra is the displayed tensor product. Each of its elements commutes with $L^\infty(Y)$, and invariance under D is precisely commutation with the remaining generators. □

Taking Y to be a point recovers Theorem 3.4. For the tetrahedral action the kernel J lies in D_{fc}, since J is abelian and has finite index. Here the surviving group degrees recover exactly the lattice whose dual supplies the torus in Theorem 5.4.

For connected Y we can determine which coefficients and group degrees have finite-dimensional adjoint span.

Theorem 6.5. *Let a countable discrete group D act by continuous automorphisms on a connected compact metrizable group Y. Put $P = Pol(Y) \rtimes D$ algebraically and $H = \ker(D \to Aut(Y)) \cap D_{fc}$. Let $B \subset Pol(Y)$ consist of the functions whose D-translates span a finite-dimensional space. Then*

$$P_{fin} = B \otimes \mathbb{C}[H],$$

where $p \otimes g$ is represented by pu_g. In particular, if the action factors through a finite group, B consists of all representative functions on Y.

Proof. Let $z = \sum_g p_g u_g$ have finite-dimensional adjoint span. A basis of that span involves only finitely many group degrees. Conjugating z by the u_d shows that every degree g with $p_g \neq 0$ has finite conjugacy class. Project the translates by the centralizer D_g onto degree g. This gives the translates of p_g by D_g in a finite-dimensional space. Since D_g has finite index, taking the finitely many cosets shows $p_g \in B$.

The adjoint action of a representative function f gives another restriction. The coefficient at g of $f \cdot z$ is the function

$$y \longrightarrow p_g(y) f\big(y(g^{-1}y)^{-1}\big),$$

where $g^{-1}y$ denotes the image of y under the automorphism. Indeed the coproduct evaluates f on a product, the antipode inverts the second factor, and moving that factor past u_g applies α_g. Write $q_g(y) = y(g^{-1}y)^{-1}$. On the open set where $p_g \neq 0$, the function q_g can take only finitely many values. Otherwise choose arbitrarily many distinct values. Representative functions separate points and form a unital algebra, so their restrictions to any finite set give all functions on that set. Evaluating at preimages and multiplying by the nonzero values of p_g would give arbitrarily large dimension in degree g.

The equality $q_g(x) = q_g(y)$ holds exactly when $y^{-1}x$ is fixed by g. Thus each level set is a coset of Y_g. Finitely many such cosets cover the nonempty open set where $p_g \neq 0$. This set has positive measure, so Y_g has positive measure and, by connectedness, equals Y. This proves $g \in H$.

Conversely, for $p \in B$ and $g \in H$, the group conjugates of pu_g span a finite-dimensional space: all coefficients belong to the translate span of p, and only finitely many degrees occur. Every such degree acts trivially on Y. The displayed formula then shows that the adjoint action of each f on this space is multiplication by $f(1)$. The group unitaries and the representative functions generate P, proving local finiteness. If the action has finite image, every function has only finitely many translates. □

The half-liberated orthogonal group gives a test with a diffuse center. Its fundamental matrix v is orthogonal, its entries are self-adjoint, and any 3 entries satisfy $abc = cba$. Use the faithful Hopf algebra model of Bichon and Dubois-Violette [9] inside the polynomial crossed product of U_N by complex conjugation. If $z_{ij}(u) = u_{ij}$ and s implements conjugation, the generators are $v_{ij} = z_{ij}s$. Their products $v_{ij}v_{kl} = z_{ij}\overline{z}_{kl}$ generate the even part, identified with the representative functions on PU_N in [6]. Write σ for the induced involution on PU_N and on its functions.

Theorem 6.6. *Let $N \geq 2$, put $P = Pol(O_N^*)$, and identify the even part of P with $Pol(PU_N)$. Then*

$$P_{fin} = Pol(PU_N), \qquad Z(L^\infty(O_N^*)) = L^\infty(PU_N)^\sigma.$$

The central base is the space of conjugate pairs $\{x, \sigma(x)\}$ with the image of uniform measure on PU_N. Almost every factor is $M_2(\mathbb{C})$.

Proof. For an even polynomial f, the adjoint action of v_{ij} is

$$\sum_k v_{ik} f v_{jk} = \sigma(f) \sum_k v_{ik} v_{jk} = \delta_{ij}\sigma(f).$$

The generators therefore preserve the span of $f, \sigma(f)$ under the adjoint action. This places the entire even part in P_{fin}.

Write any polynomial in the model as $f + bs$, with f even and b a representative function on U_N. For $a \in Pol(PU_N)$ the odd coefficient of its adjoint action on this polynomial is

$$u \longrightarrow b(u)\, a\big(x\sigma(x)^{-1}\big), \qquad x = [u] \in PU_N.$$

Indeed, moving the second leg of the coproduct past s applies conjugation, and applying the antipode inverts its argument. If $b \neq 0$, the set where it is nonzero projects onto a nonempty open subset of PU_N. Representative functions interpolate arbitrary values on a finite set of distinct points. Thus finite-dimensional adjoint span would force $x\sigma(x)^{-1}$ to take only finitely many values on this open set. Each level set is a coset of the fixed subgroup of σ, by the calculation just made for connected groups. This subgroup is proper: the class of $diag(i, 1, \ldots, 1)$ differs from its complex conjugate. As PU_N is connected, the fixed subgroup has measure zero. No finite union of its cosets contains an open set. This excludes every nonzero odd coefficient.

To pass to von Neumann algebras, the invariant trace on the ambient crossed product restricts to the invariant trace on the Hopf subalgebra, by invariance and uniqueness. The even functions

generate $L^\infty(PU_N)$. The first entry z_{11} is nonzero almost everywhere since the first column of a uniformly distributed unitary is uniform on the unit sphere. The sign of the self-adjoint operator v_{11} is therefore the involution $w = (z_{11}/|z_{11}|)s$. It implements σ on the even algebra. Each $v_{ij}w = z_{ij}\overline{z}_{11}/|z_{11}|$ is a bounded function invariant under scalar rotations of U_N and hence belongs to $L^\infty(PU_N)$. These observations give

$$L^\infty(O_N^*) \simeq L^\infty(PU_N) \rtimes \mathbb{Z}_2.$$

Indeed w belongs to the generated algebra, all the original generators belong to the algebra on the right, and the trace vanishes on every term fw by the ambient crossed product formula. The fixed subgroup of σ is null, so almost all orbits have 2 points and trivial stabilizer. Theorem 4.3 gives the center and the stated factors. □

The calculation depends on the action of conjugation on the projective group. The same proof gives a class of examples in which the compact group underlying the matrix model can be varied.

Corollary 6.7. *Let $K \subset U_N$ be a closed subgroup stable under entrywise complex conjugation. In the crossed product of $Pol(K)$ by that involution take the Hopf $*$-algebra P generated by the elements $z_{ij}s$. Write Y for the image of K in PU_N and σ for the induced involution. If Y is connected and σ is nontrivial there, then*

$$P_{fin} = Pol(Y), \qquad Z(P'') = L^\infty(Y)^\sigma.$$

The double commutant is taken in the invariant trace representation. Almost every factor over the orbit space of σ is $M_2(\mathbb{C})$.

Proof. Even words give precisely the algebra generated by $z_{ij}\overline{z}_{kl}$. These are the coefficients of the representation of Y on matrices by conjugation. This representation is faithful on Y, so its coefficients and their conjugates generate $Pol(Y)$ by Peter–Weyl. The adjoint action formulas in Theorem 6.6 depend only on unitarity and the crossed product multiplication; they restrict to K. The image in Y of the nonzero set of an odd coefficient is open because $K \to Y$ is a quotient homomorphism. The fixed subgroup of σ in Y is again proper and has measure zero. The same interpolation argument excludes all odd coefficients.

For the measurable involution one can dispense with any assumption on the zeros of a particular matrix entry. Partition Y by the first nonzero entry in the first column, reading from top to bottom. These measurable pieces are well defined on projective classes and are preserved by σ. On the piece indexed by i use the sign of $z_{i1}s$. The sum of these signs cut down by their respective projections is an involution w implementing σ on $L^\infty(Y)$. Each $z_{kl}sw$ is an even measurable function, since the scalar phase cancels on each piece. As in Theorem 6.6, the trace therefore identifies P'' with $L^\infty(Y) \rtimes \mathbb{Z}_2$. Its generic orbits have size 2. □

By the classification of Bichon and Dubois-Violette [9], every nonclassical closed quantum subgroup of O_N^* admits such a model. The corollary applies whenever the projective image Y is connected and conjugation acts nontrivially on it. For a general G, central polynomials always lie in P_{fin}. The argument for crossed products proves that they generate the von Neumann center in that class. Whether the inclusion of Question 6.2 holds for every compact quantum group of Kac type remains open here.

7 Quantum homogeneous spaces

Let G have a quantum subgroup H with polynomial quotient map π. The right coset algebra on polynomial functions consists of the elements fixed by the right coaction, that is, those a satisfying $(id \otimes \pi)\Delta(a) = a \otimes 1$. We take its von Neumann closure for the restriction of h from G.

Theorem 7.1. *Suppose a surjective group map $\Gamma \to \Lambda$ has kernel K. For $G = \widehat{\Gamma}$ and $H = \widehat{\Lambda}$ we have*

$$L^\infty(G/H) = L(K) \subset L(\Gamma).$$

For the quotient of $\mathbb{Z}_2 * \mathbb{Z}_2$ sending both reflections to the generator of $\mathbb{Z}_2$, the center of the coset algebra is the full circle algebra. The inclusion of the ambient center into this center is pullback by the orbit map for inversion.

Proof. The right coaction sends λ_g to $\lambda_g \otimes \lambda_{\bar{g}}$, where $\bar{g}$ is the image in Λ. Linear independence shows that the fixed polynomials have support in K. Their trace is the regular trace of K, proving the first assertion. In the reflection example the kernel consists of the even words, precisely the powers of $r = st$. Its algebra is abelian with uniform circle spectrum. By Theorem 5.6 the ambient center consists of the functions taking the same value at r and r^{-1} in the Fourier model. □

We can thus decompose the same coset algebra over either of 2 centers. The map between the bases pairs a circle point with its inverse.

Proposition 7.2. *In the reflection example, reducing the coset algebra over the ambient center gives diagonal matrices $\mathbb{C}^2$ almost everywhere. Its own central reduction has scalar fibers. The conditional distribution on each generic inversion orbit gives weight $1/2$ to each point.*

Proof. At θ the coset generator r is diagonal with distinct eigenvalues. It therefore generates both diagonal projections in that fiber. Restricting the normalized matrix trace gives the 2 equal weights. Splitting each pair of points recovers the circle with its uniform measure. Over the circle the algebra acts by scalar multiplication. □

We now take the rectangular orbit construction of [3, 29] and vary its 2 acting groups. All the algebras below are generated by the orbit coordinates in the product of the trace representations. This convention specifies both the homogeneous space and its measure. In particular the calculation requires no identification with a universal algebra given by relations.

Definition 7.3. *Let $1 \le K \le \min(M, N)$ and let u, v be the fundamental matrices of 2 orthogonal quantum groups of sizes M, N. Take their tensor product with the product trace. The rank-K orbit algebra is generated by*

$$z_{ia} = \sum_{r=1}^{K} u_{ri} \otimes v_{ra}$$

for $1 \le i \le M$ and $1 \le a \le N$. Its projective algebra is generated by the products $z_{ia} z_{jb}$.

These coordinates are the orbit of the rectangular matrix with an identity block of size K. Indeed their coproduct in the product quantum group is $\Delta(z_{ia}) = \sum_{jb} z_{jb} \otimes (u_{ji} \otimes v_{ba})$. We first take O_M^*, O_N^* on the 2 sides. Write $V = U_{MN}^K$ for the complex $M \times N$ partial isometries of rank K and $X = U_{MN}^K / \mathbb{T}$ for their quotient by scalar multiplication. Give V the invariant probability measure of the transitive left and right unitary action. Complex conjugation gives an involution σ on X.

Theorem 7.4. *For $M, N \ge 2$ and every admissible K, the projective and affine orbit algebras of the half-liberated rectangle form the tracial inclusion*

$$L^\infty(X) \ \subset \ L^\infty(X) \rtimes \mathbb{Z}_2,$$

where the generator acts by σ. The center of the affine algebra has base X/σ with its image probability measure. Over this base the inclusion is the diagonal $\mathbb{C}^2 \subset M_2(\mathbb{C})$ almost everywhere.

Proof. Apply the model of Section 6 on both legs, with independent uniform unitaries U, W and implementing symmetries s, t. Let J be the rectangular matrix with the indicated identity block. Put $T = U^t J W$ and $d = s \otimes t$. Then T has the stated invariant law on V and $z_{ia} = T_{ia} d$. Thus $z_{ia} z_{jb} = T_{ia} \overline{T}_{jb}$. The full table of these products is the rank-one matrix formed from the vector of entries of T and its adjoint. It determines T up to one common phase. Its coefficients therefore separate points of the compact space X and generate its full function algebra by Stone–Weierstrass.

Almost surely $T_{11} \neq 0$. Indeed T_{11} is a nonzero real analytic function of the pair of unitaries on the connected manifold $U_M \times U_N$. The sign w of z_{11} is consequently the involution $(T_{11}/|T_{11}|)d$. It acts by conjugation of the point of X. Each $z_{ia} w$ is a bounded function on X, and the odd Fourier coefficient has trace zero in the product model. The projective algebra and w thus generate the asserted tracial crossed product.

A point fixed on X has a representative real up to phase. In particular it satisfies $Im(T_{11}\overline{T}_{21}) = 0$. This analytic equation is nontrivial for every K in the statement. Start with J, rotate its first column in the first 2 real row coordinates to make both entries nonzero, and multiply the second row by i. The zero set upstairs has measure zero, hence so does the fixed set on X. Theorem 4.3 gives 2 equally weighted projective points and the full matrix algebra in each generic orbit. □

At higher rank the 2 supporting planes give a natural smaller algebra. This is the counterpart of the 2 Grassmannian legs inside the free rectangles of [29]. The information lost by passing to the planes can be described explicitly.

Proposition 7.5. *Suppose $1 \leq K < \min(M, N)$ in Theorem 7.4. In its even algebra let D be generated by the entries of the range and source projections of T. Then D is the function algebra, in the invariant probability representation, on*

$$Y = Gr_{KM}(\mathbb{C}) \times Gr_{KN}(\mathbb{C}).$$

Over Y/σ the affine algebra has fiber $M_2(L^\infty(PU_K))$ almost everywhere, with D giving the constant diagonal algebra $\mathbb{C}^2$ in that fiber. The measure on PU_K is uniform.

Proof. The range projection TT^* has entries $\sum_a z_{ia} z_{ja}$. The source projection T^*T is obtained by reversing the column indices in the corresponding sums over i. Both projections are unchanged by a common scalar phase on T. The distribution of their pair is the product of the uniform Grassmannian measures, since the left and right unitaries in the orbit model are independent. Projection entries determine the 2 planes.

Given the planes, a partial isometry is a unitary from the source plane to the range plane. Modulo its common scalar the set of choices is a copy of PU_K after choosing orthonormal bases. Such bases can be chosen measurably by applying Gram–Schmidt to the projections of the standard coordinate vectors, retaining the first linearly independent vectors. The conditional measure on the unitary between these bases is uniform. Indeed, sampling the planes independently and filling in the map between them by an independent uniform unitary gives a law invariant under $U_M \times U_N$; on the transitive orbit that law is unique. Quotienting by scalars gives the asserted conditional measure.

Conjugation on Y has 2-point orbits almost everywhere. For example, the set of real range planes satisfies $Im((TT^*)_{12}) = 0$. This analytic condition on the first unitary is nontrivial, as one sees by rotating one basis vector from the coordinate K-plane towards its orthogonal complement and multiplying one of these coordinates by i, with a coordinate permutation if needed. Choose one pair of planes over each generic orbit in Y/σ, and use conjugation to identify the 2 copies of the remaining PU_K space. The involution now exchanges those copies. Crossing by it gives the stated matrix algebra. Functions of the 2 planes give the 2 constant diagonal entries. □

Thus for $K = 1$ the supporting planes already specify the projective point. For $K \geq 2$ there is still an entire projective unitary to integrate over before one obtains the central factors. This

accounts for the difference between reduction over the supporting planes modulo conjugation and reduction over the full center.

At rank one the coordinates have the form $x_i \otimes y_a$ for 2 independent half-liberated rows. Bichon's model of the half-liberated sphere [8] identifies the projective row space with complex projective space. Now $X = \mathbb{C}P^{M-1} \times \mathbb{C}P^{N-1}$ by the rank-one factorization. The projective matrix determines the line spanned by each unit vector, and the phase of either vector can be chosen independently. We can now ask the dimension question of [29] for the magnitude inclusion. Here magnitudes are the absolute values of the rectangle entries and their conjugates by words in the coordinate signs.

Theorem 7.6. *For $K = 1$ and $M, N \geq 2$, every magnitude belongs to the center of the half-liberated rectangular algebra. Up to trace-preserving isomorphism, the inclusion of the magnitude algebra in the affine algebra is the same for all such side lengths. If one also specifies the magnitude $|z_{11}|$, its second and fourth moments determine $\{M, N\}$.*

Proof. Use independent complex unit vectors a, b and put $p_i = |a_i|^2$, $q_j = |b_j|^2$. Then $|z_{ij}|^2 = p_i q_j$ is invariant under simultaneous conjugation, hence central. Conjugating it by signs fixes it. Row and column sums recover p_i, q_j, so the magnitude algebra consists precisely of functions of the pair (p, q).

A uniform complex unit vector is obtained by normalizing independent standard complex Gaussian entries. Their squared moduli are independent exponential variables. Changing variables to their sum and their proportions gives a constant density on the probability simplex for the proportions. Their phases are independent uniform circle variables independent of the proportions. Passing to a projective vector removes one common phase, leaving independent relative phases. Thus, over the product S of the 2 probability simplices, the remaining probability space in X is a constant torus $\mathbb{T}^{M+N-2}$. Conjugation fixes S and inverts this torus. Its fixed points are null. Choosing a measurable representative of each pair gives the inclusion

$$L^\infty(S) \otimes 1 \;\subset\; L^\infty(S) \mathbin{\bar{\otimes}} L^\infty(\mathbb{T}^{M+N-2}/\sigma) \mathbin{\bar{\otimes}} M_2(\mathbb{C}).$$

Both probability spaces on the right are standard and diffuse. Each can be identified in measure with the unit interval, taking the first leg to the magnitude algebra. This proves the assertion about the inclusion.

For the extra observable, integration on the simplex gives $\mathbb{E}(p_1) = 1/M$ and $\mathbb{E}(p_1^2) = 2/(M(M+1))$, and similarly for q_1. Independence gives

$$\tau(|z_{11}|^2) = \frac{1}{MN}, \qquad \tau(|z_{11}|^4) = \frac{4}{MN(M+1)(N+1)}.$$

The first moment displayed determines MN, and the second then determines $(M+1)(N+1)$, hence $M + N$. The resulting sum and product determine the unordered pair. □

For the free rectangle of [29], the magnitude inclusion itself recovers the side lengths when $M, N \geq 3$. Here the 2 inclusions for $(3, 3)$ and $(3, 4)$ are isomorphic, although either coordinate law distinguishes them. To see what remains of the free problem when the center is still available, take one half-liberated side and one free side.

Theorem 7.7. *Take a rank-one orbit of $O_M^* \times O_N^+$ with $M \geq 2$, $N \geq 3$. Let $B = L^\infty(P_+^{N-1})$ be the projective algebra of the free row, and α conjugation on B by the sign of its first coordinate. Put $X = P_{\mathbb{C}}^{M-1}/\sigma$ with the image of uniform projective measure. The mixed orbit algebra is tracially isomorphic to*

$$L^\infty(X) \mathbin{\bar{\otimes}} M_2(B).$$

Over almost every central point the natural free projective leg $1 \otimes B$ is represented by the inclusion $b \to \mathrm{diag}(b, \alpha(b))$ in $M_2(B)$.

Proof. Write x_i, y_a for the half-liberated and free row entries. Products of 2 rectangular entries recover both projective legs because

$$\sum_a (x_i \otimes y_a)(x_j \otimes y_a) = x_i x_j \otimes 1, \qquad \sum_i (x_i \otimes y_a)(x_i \otimes y_b) = 1 \otimes y_a y_b.$$

Conversely each such product belongs to their tensor product. Write $D = L^\infty(\mathbb{C}P^{M-1})$ for the first projective leg. The sign of $x_1 \otimes y_1$ is $w = sgn(x_1) \otimes sgn(y_1)$, since both entries have zero kernel by the half-liberated row model and [29]. The first sign acts on D by complex conjugation, as in Theorem 7.4. On the second leg the row decomposition of [29] gives the involution α and shows that B is a factor and α is outer. Each $(x_i \otimes y_a)w$ lies in $D \bar{\otimes} B$: on each row $x_i\, sgn(x_1)$ or $y_a\, sgn(y_1)$ is the bounded strong limit of even products obtained by using $x_1/(|x_1| + \varepsilon)$ or $y_1/(|y_1| + \varepsilon)$. The odd summand has zero trace by the sign symmetry of the first row. Thus the algebra is the crossed product of $D \bar{\otimes} B$ by the simultaneous involution $\sigma \otimes \alpha$.

On projective space conjugation has 2-point orbits almost everywhere: its fixed points are real projective space, of measure zero. Choose one point per pair measurably. On a pair, a coefficient takes 2 values b_0, b_1 in B. Represent it by $diag(b_0, \alpha(b_1))$, and represent w by the matrix interchanging the 2 coordinates. Covariance follows from $\alpha^2 = 1$. The matrices obtained are all of $M_2(B)$, and their normalized matrix trace agrees with the orbit average of the traces on B. The constant coefficient $1 \otimes b$ gives $diag(b, \alpha(b))$. Because B is a factor, this is the central decomposition. □

The sign of the first rectangular coordinate gives the intermediate algebra in the following index-2 tower.

Proposition 7.8. *In a mixed central fiber, adjoining $w = sgn(z_{11})$ to the specified copy of B gives the free sphere algebra $Q = L^\infty(S^{N-1}_{\mathbb{R},+})$. The resulting tower*

$$B \subset Q \subset M_2(B)$$

is the first Jones basic construction for the free projective-to-sphere inclusion, with normalized traces.

Proof. Write $s = sgn(y_1)$ in the free row. By [29] we have $Q = B \rtimes \mathbb{Z}_2$ with $sbs = \alpha(b)$. On the right B-module Q use coordinates $b_0 + sb_1$. Left multiplication by b is $diag(b, \alpha(b))$ and left multiplication by s interchanges the 2 coordinates. These are precisely the marked algebra and the fiber value of w. In these coordinates the trace-preserving expectation onto B is projection onto the first component. Thus the Jones projection is e_{11}. Adjoining it to the displayed representation of Q produces the 2 matrix units off the diagonal and arbitrary coefficients from B in the first corner, hence all of $M_2(B)$. The matrix trace gives e_{11} trace $1/2$, as required by the index-2 basic construction of Jones [17]. □

The marked inclusion is a diagonal subfactor in the sense of Bisch, Das and Ghosh [10]. Even the unmarked fiber determines B up to isomorphism, since any projection of normalized trace $1/2$ in $M_2(B)$ cuts out a copy of B. To recover the involution we keep the embedding of B given by the orbit coordinates.

Theorem 7.9. *For 2 mixed rectangles as in Theorem 7.7, their factor fibers with their specified free projective legs are isomorphic exactly when there is an isomorphism between the 2 projective factors carrying the class of one involution in the outer automorphism group to the class of the other.*

Proof. For an outer involution α of a factor B, the relative commutant of $b \to diag(b, \alpha(b))$ in $M_2(B)$ is $\mathbb{C}e_{11} + \mathbb{C}e_{22}$. Indeed a diagonal intertwiner is scalar in each corner. For an off-diagonal

intertwiner its 2 absolute squares are scalar, so a nonzero one would give a unitary implementing α.

Write B, C for the 2 projective factors. Let an isomorphism of the marked fibers induce $\theta : B \to C$ on their specified legs, and let β be the second involution. It permutes the 2 minimal relative commutant projections. The flip in $M_2(C)$ preserves the marked leg and induces β on it, so after composing with the flip if necessary we may assume the corners are preserved. The image of e_{12} is then ve_{12} for a unitary $v \in C$. The map on the first corner is θ. On the second it is $\beta\theta\alpha^{-1}$, by restriction to marked diagonal elements. Transport between corners by the matrix unit gives

$$\theta = Ad(v)\,\beta\theta\alpha^{-1}.$$

Thus $\theta\alpha\theta^{-1}$ and β differ by an inner automorphism. Conversely, given this relation, apply θ entrywise and conjugate by $diag(1, v^*)$. This sends $diag(b, \alpha(b))$ to $diag(\theta(b), \beta(\theta(b)))$ and proves sufficiency. □

We can now express the square-versus-rectangle criterion of [29] in terms of marked mixed fibers.

Corollary 7.10. *Let $M, N \geq 3$. The mixed $(2, M)$ and $(2, N)$ rectangles, with half-liberated side of length 2, have isomorphic central fibers with their specified free projective legs if and only if*

$$L^\infty(O_{MM}^{1+}) \simeq L^\infty(O_{MN}^{1+})$$

as unmarked factors.

Proof. By Theorem 7.9, the marked fibers are isomorphic exactly when an isomorphism of the projective factors identifies the outer classes of their row involutions. By the involution lemma of [29], an inner perturbation of an outer involution which is again an involution is inner conjugate to it. Thus the condition is precisely conjugacy of the row actions. The square-versus-rectangle theorem of [29] identifies this with the displayed isomorphism. □

Whether the free $(3, 3)$ and $(3, 4)$ factors are isomorphic remains open. In the mixed model this is the question whether a fiber isomorphism preserves the specified projective legs. The entirely half-liberated magnitude inclusions in these dimensions are isomorphic by Theorem 7.6.

8 Central reduction of free balls

The real free ball $B_{\mathbb{R},+}^N$ and its radial density $Nr^{N-1}\,dr$ were introduced in [12]. For this density the radial tensor product and its center were computed in [29]. We allow an arbitrary law of the radius and reduce the projective and affine algebras together. The coordinates x_i and their products $x_i x_j$ give the inclusion

$$\{x_i x_j \mid 1 \leq i, j \leq N\}'' \;\subset\; \{x_1, \ldots, x_N\}''.$$

Which inclusion do we see at a given radius? Write $x_i = ru_i$, where the u_i are the self-adjoint coordinates of the real free sphere, with the invariant trace h obtained from one row of O_N^+. Give $r \in [0, 1]$ a probability law ν independent of the row. Since $\sum_i u_i^2 = 1$, the operator $R = (\sum_i x_i^2)^{1/2}$ is multiplication by r. Its spectral projections already belong to the smaller algebra. We can therefore reduce both sides over the same radius.

Theorem 8.1. *For any radial law ν, let $C \subset A$ be the radial ball inclusion above and put $B = \{u_i u_j \mid 1 \leq i, j \leq N\}'' \subset Q = \{u_1, \ldots, u_N\}''$. Over the positive radii, the inclusion $C \subset A$ is*

$$L^\infty((0, 1], \nu) \,\bar{\otimes}\, B \;\subset\; L^\infty((0, 1], \nu) \,\bar{\otimes}\, Q$$

with the identity on the radial functions and the given inclusion on the sphere algebras. If ν has an atom at zero, it contributes the summand $\mathbb{C} \subset \mathbb{C}$ with mass $\nu(\{0\})$.

Proof. Put $p = 1_{\{0\}}(R)$ and $q = 1 - p$. Every bounded function of R belongs to C. Each coordinate vanishes on p, so $pC = pA = \mathbb{C}p$. On q, division by the radius recovers the sphere: the uniformly bounded operators $(R + \varepsilon)^{-1}x_i$ converge strongly to qu_i as ε decreases to zero. Likewise $(R^2 + \varepsilon)^{-1}x_ix_j$ converges strongly to qu_iu_j. Together with the radial functions these limits generate the indicated tensor products. Conversely $x_i = ru_i$ and $x_ix_j = r^2u_iu_j$ belong to the respective tensor products on q. □

The calculation up to this point uses only the sphere relation and the choice of the radial law. To recognize this decomposition over the radius as the central reduction, we need to know the centers of the 2 algebras on the sphere.

Corollary 8.2. *For $N \geq 3$ the 2 radial algebras have common center $\{R\}''$. At almost every positive radius their central fiber inclusion is*

$$L^\infty(P_+^{N-1}) \;\subset\; L^\infty(S_{\mathbb{R},+}^{N-1}),$$

of index 2. For the radial density $Nr^{N-1}\,dr$ of [12], these fibers have full measure.

Proof. The projective and sphere algebras $B \subset Q$ are factors and their inclusion has index 2 by [29]. The positive-radius tensor products therefore have exactly the radial functions as center. At an atom at zero the scalar inclusion has index 1. The center-valued trace of a coordinate word is obtained by taking its normalized trace in the fiber. For $k \geq 1$ this gives

$$E(x_{i_1} \dots x_{i_k})(r) = r^k h(u_{i_1} \dots u_{i_k}).$$

The same expression gives zero at the origin, where all the coordinates vanish. □

9 Grassmannians as affine orbits

Let us look at an ordinary real plane from the viewpoint of affine homogeneous spaces. Its orthogonal projection is a matrix of real coordinates. Moving the plane moves this matrix, and the same prescription makes sense when the entries of the orthogonal group cease to commute. We first specify which integration functional comes with these coordinates.

Definition 9.1. *Let v be a unitary representation of dimension n of a compact quantum group G of Kac type. For a nonempty index set $I \subset \{1, \dots, n\}$, take*

$$x_i = \frac{1}{\sqrt{|I|}} \sum_{a \in I} v_{ai}, \qquad L^\infty(X_{G,I}) = \{x_i, x_i^* \mid 1 \leq i \leq n\}'' \subset L^\infty(G),$$

with the restriction of the invariant trace. The representation used in the notation $X_{G,I}$ will always be specified.

These are the measured affine homogeneous spaces of [2]. If the coefficients of v generate a proper C^*-subalgebra of $C(G)$, one can use the compact matrix quantum group with precisely this coefficient algebra. The invariant state restricts to its invariant state. This observation allows us to use tensor representations.

Proposition 9.2. *The coproduct gives the orbit coordinates the right coaction*

$$\Delta(x_i) = \sum_j x_j \otimes v_{ji}.$$

The fixed algebra consists of the scalars. Any affine presentation of the same orbit with its prescribed invariant state gives the same tracial von Neumann algebra.

Proof. The coproduct formula follows by summing the formula for the coefficients of v. If $\Delta(b) = b \otimes 1$, applying $id \otimes h$ gives $b = h(b)1$. For the last assertion let α be the coordinate map of the presentation into $C(G)$. Its invariant state is $h\alpha$ by [2]. The map $a \to \alpha(a)$ preserves the scalar product $(a, b) \to h(\alpha(b)^*\alpha(a))$. It therefore induces a unitary from the GNS Hilbert space onto the L^2-closure of the coordinate algebra and intertwines left multiplication. Taking weak closures gives the assertion. □

Write Gr_{LN} for the Grassmannian of unoriented L-planes in $\mathbb{R}^N$, with $1 \leq L < N$. Realize it as the real algebraic set

$$X = \{P \in M_N(\mathbb{R}) \mid P^t = P,\ P^2 = P,\ Tr(P) = L\}.$$

We equip it with the invariant probability measure μ. Put $J = diag(1_L, 0_{N-L})$. In the tensor square of the defining representation of O_N, index the basis by pairs and choose $I = \{(a, a) \mid 1 \leq a \leq L\}$. Its normalized index vector is the matrix $J/\sqrt{L}$.

Theorem 9.3. *For this choice the affine homogeneous space is Gr_{LN}. Its coordinates, multiplied by $\sqrt{L}$, are the entries of the projection*

$$P(g) = g^t J g, \qquad p_{ij}(g) = \sum_{a=1}^{L} g_{ai} g_{aj}.$$

Its central reduction is over (Gr_{LN}, μ) itself, with scalar fiber at almost every plane.

Proof. The coefficient of the tensor square from (a, a) to (i, j) is $g_{ai}g_{aj}$ in our row convention. Thus the index sum gives the indicated projection. Every plane has an orthonormal basis which extends to one of $\mathbb{R}^N$, so every projection in X occurs. The entries separate the projections, and their unital algebra is closed under conjugation. By Stone–Weierstrass its uniform closure is $C(X)$. The image of the invariant probability on O_N is μ, so the weak closure is $L^\infty(X, \mu)$. Its center is the whole algebra. □

The same projection J and the same tensor-square index set give an affine orbit for the twisted orthogonal group O_N^τ of [5]. The entries of its fundamental orthogonal matrix u are self-adjoint. Any 2 distinct entries on a common row or column anticommute, and entries on different rows and columns commute.

Definition 9.4. *The twisted real Grassmannian Gr_{LN}^τ is the affine orbit whose measured coordinate algebra is*

$$A = L^\infty(Gr_{LN}^\tau) = \{q_{ij} \mid 1 \leq i, j \leq N\}'', \qquad q = u^t J u.$$

Here $q_{ij} = \sum_{a=1}^{L} u_{ai}u_{aj}$, and the trace is inherited from O_N^τ.

The matrix q is a projection in $M_N(A)$ and $\sum_i q_{ii} = L$. Its off-diagonal entries satisfy $q_{ji} = q_{ij}^* = -q_{ij}$. The order of the 2 indices will matter even though the underlying classical projection is symmetric.

Let Cl_N denote the complex Clifford algebra on self-adjoint unitaries $c_1, \ldots, c_N$ which anticommute in distinct indices. Write Cl_N^0 for its even part. The increasing products c_S, indexed by subsets S, form a basis. Indeed, reordering gives spanning, and signed left multiplication on the vector space with basis the subsets gives the regular representation. Its normalized trace t is 1 on the empty word and zero on every nonempty basis word. It is faithful, since $t(c_S^* c_T) = \delta_{ST}$.

Lemma 9.5. *For the invariant trace on Gr_{LN}^τ, the coordinates q_{ij} have the joint $*$-distribution of*

$$p_{ij}(P) c_i c_j \quad (P \in X = Gr_{LN})$$

in $L^\infty(X, \mu) \bar{\otimes} Cl_N^0$, with trace $\int_X t(\cdot)\, d\mu$.

Proof. The matrices with entries $g_{ij} \otimes c_i \otimes c_j$ and $u_{ij} \otimes c_i \otimes c_j$ are orthogonal. The entries of the first satisfy the twisted sign rules, and those of the second are commuting and self-adjoint. Thus the first represents $C(O_N^\tau)$ by a map θ, and the second represents $C(O_N)$. Composing on the first leg and regrouping gives $u_{ij} \otimes d_i \otimes d_j$, with $d_i = c_i \otimes c_i$. These commuting involutions have linearly independent square-free products by the Clifford basis. Their algebra therefore admits the character $d_i \to 1$. Applying it on both legs recovers u_{ij} and proves faithfulness of θ.

Let h' be ordinary integration times $t \otimes t$ in this model. For words of equal length I, J put $g_{IJ} = \prod_s g_{i_s j_s}$ and $c_I = \prod_s c_{i_s}$, in the indicated order. The corresponding word u_{IJ} satisfies

$$h'(u_{IJ}) = m_{IJ}\, t(c_I) t(c_J), \qquad m_{IJ} = \int_{O_N} g_{IJ}.$$

Left and right sign changes force $m_{IJ} = 0$ unless each index occurs evenly in each word. Whenever it does, c_I, c_J are scalars of modulus one. Apply $id \otimes h'$ to the coproduct of u_{IJ} and evaluate the first leg in the model at g. The result is

$$\sum_K g_{IK}(g)\, m_{KJ}\, c_I \otimes c_K t(c_K) t(c_J) = m_{IJ}\, c_I \otimes 1\, t(c_J).$$

Indeed $c_K t(c_K) = 1$ on all surviving words and ordinary group invariance sums the scalar coefficients. The last expression equals $h'(u_{IJ})1$. Faithfulness gives right invariance, which characterizes the invariant state. Finally $\theta(q_{ij}) = p_{ij} \otimes 1 \otimes c_i c_j$. Pushing the ordinary group measure to X proves the lemma. □

Let $D \subset O_N$ be the diagonal sign matrices. Conjugation by $d \in D$ sends p_{ij} to $d_i d_j p_{ij}$. The corresponding automorphism α_d of Cl_N^0 sends $c_i c_j$ to $d_i d_j c_i c_j$.

Lemma 9.6. *On a conull open subset of Gr_{LN} all off-diagonal entries p_{ij} are nonzero. The stabilizer of each of these planes in D is $\{1, -1\}$.*

Proof. For $i \neq j$ take the plane spanned by $(e_i + e_j)/\sqrt{2}$ and $L - 1$ coordinate vectors outside this pair. At its projection $p_{ij} = 1/2$. The Grassmannian is connected, real analytic, and its invariant probability is smooth in local coordinates. A nonzero real analytic function on such a manifold has a null zero set. Intersecting the finitely many complements gives the assertion on the entries. If $dPd = P$ there, then $d_i d_j = 1$ for every pair, so all the signs agree. □

10 Central fibers of twisted Grassmannians

Put $X = Gr_{LN}$ and $Y = X/D$. The generic stabilizer $\{1, -1\}$ acts trivially on even Clifford words. We give each quotient of X the image of its invariant probability measure μ.

Proposition 10.1. *The twisted orbit algebra is the algebra of measurable essentially bounded fields $f : X \to Cl_N^0$ satisfying*

$$f(dPd) = \alpha_d(f(P)) \qquad (d \in D)$$

almost everywhere. It contains $L^\infty(Y)$ as scalar fields. Over Y its tracial fiber is (Cl_N^0, t) almost everywhere.

Proof. The fields $p_{ij} c_i c_j$ satisfy the covariance condition. To recover all scalar fields on Y, take the polynomials in the p_{ij} and average over D. The surviving monomials have every index occurring evenly. In the corresponding product of twisted coordinates the Clifford word is then 1 or -1. Thus every invariant polynomial times 1 belongs to A. Polynomial approximation on X followed by averaging gives $C(Y)$, and weak closure gives $L^\infty(Y)$.

On the conull set of Lemma 9.6, choose a measurable representative $P(y)$ from each sign orbit, for example the lexicographically least list of entries. The stabilizer acts trivially on Cl_N^0, so restriction to these representatives identifies all covariant fields with all fields on Y valued in Cl_N^0. Under this identification A contains the scalar functions and the fields $p_{ij}(P(y))c_ic_j$. For $i \neq j$ multiply the latter by the scalar reciprocal on $|p_{ij}(P(y))| \geq 1/n$ and by zero elsewhere. The resulting bounded fields converge strongly to c_ic_j. These products generate Cl_N^0, giving every field. Finally t is unchanged by the sign action, so restriction carries the trace to integration of t on Y. □

The central base depends on how a change of representative permutes the simple summands of Cl_N^0. Let $D^+ = \{d \in D \mid \det d = 1\}$ be the determinant-one sign matrices.

Theorem 10.2. *Let $1 \leq L < N$, put $X = Gr_{LN}$ and $m = 2^{[(N-1)/2]}$. The center of $L^\infty(Gr_{LN}^\tau)$ has measured base X/D^+, and its central fibers are $M_m(\mathbb{C})$ with normalized matrix trace almost everywhere. The natural map*

$$X/D^+ \longrightarrow X/D$$

is one-to-one for odd N and 2-to-1 almost everywhere for even N. In the latter case the conditional weights are $1/2, 1/2$.

Proof. For $|S|$ even, commuting c_S past c_ic_j produces the sign $(-1)^{|S\cap\{i,j\}|}$ for $i \neq j$. As the basis words are simultaneous eigenvectors for conjugation by the c_ic_j, a central word must contain either both or neither of i, j for every pair. Only the empty set and the full set pass this test. If N is odd the full set is absent from the even algebra. The center is scalar, and the dimension 2^{N-1} gives the stated matrix size. Moreover d and $-d$ act identically on planes and have opposite determinants, so $X/D^+ = X/D$.

For even N put $w = i^{N/2}c_1 \dots c_N$. This is a central self-adjoint unitary and $t(w) = 0$. The projections $(1 \pm w)/2$ are the 2 minimal central projections. Each summand is simple, and a change of one coordinate sign exchanges them, giving equal dimensions 2^{N-2} and the asserted matrix size. Each has trace weight $1/2$.

In the covariant field description $\alpha_d(w) = (\det d)w$. A central field assigns scalar values to the 2 labels $+$ and $-$, interchanging them precisely for sign changes of negative determinant. Its value on the $+$ label thus gives a function on X/D^+. Conversely such a function specifies the $-$ value by applying any sign change of negative determinant. Integration of the 2 values with equal weights gives the image of μ on X/D^+, since sign changes preserve μ. Finally both elements of the generic stabilizer $\{1, -1\}$ have positive determinant in even dimension, so the 2 lifts are distinct almost everywhere. □

Consider the first plane Grassmannian with both rank and corank at least 2. We can see the splitting directly in its measured coordinate algebra, without labelling planes by hand.

Corollary 10.3. *For Gr_{24}^τ the algebra over almost every coordinate sign orbit is $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$. Its full center is generated over $L^\infty(Gr_{24}/D)$ by the symmetry*

$$s = sgn(q_{12}q_{34}).$$

Its spectral projections distinguish the 2 lifts of the sign orbit with equal conditional weights.

Proof. In the model, $q_{12}q_{34} = p_{12}p_{34}c_1c_2c_3c_4$. The scalar coefficient is real and nonzero almost everywhere by Lemma 9.6. The Clifford product is a central self-adjoint unitary with trace zero. Taking the spectral sign therefore yields, on each generic orbit representative, one of the 2 signs of this unitary. Together with the scalar fields this generates both central summands by Proposition 10.1. Theorem 10.2 gives their sizes and weights. □

The coordinate fields also have a meaning at the exceptional planes. For this purpose take the continuous C^*-algebra generated by $P \to p_{ij}(P)c_ic_j$ in $C(X) \otimes Cl_N^0$. Individual points can be evaluated in this algebra. Its weak closure for the invariant measure is the algebra of Lemma 9.5.

Proposition 10.4. *Given $P \in Gr_{LN}$, join 2 distinct indices i, j when $p_{ij}(P) \neq 0$. If the connected components of this graph have sizes $n_1, \ldots, n_l$, the evaluation algebra at P is*

$$Cl_{n_1}^0 \otimes \ldots \otimes Cl_{n_l}^0,$$

with $Cl_1^0 = \mathbb{C}$. The sign stabilizer of P consists precisely of the choices of one scalar sign on each component.

Proof. The diagonal coordinates evaluate to scalars. The off-diagonal coordinates generate c_ic_j for the edges, since their coefficients are nonzero complex numbers. Along a path, multiplying the successive Clifford pairs cancels the interior generators, leaving the pair at its endpoints. Thus all pairs in each component occur, and these generate its even Clifford algebra. Even words on disjoint components commute. Products of their respective basis words give distinct Clifford basis words up to sign, so the resulting tensor product map is injective. Finally $dPd = P$ says exactly that the signs agree at the 2 endpoints of each edge. □

For instance, in Gr_{24} the coordinate plane spanned by e_1, e_2 has scalar evaluation. At the plane spanned by $(e_1+e_2)/\sqrt{2}$ and $(e_3+e_4)/\sqrt{2}$ the graph has edges 12 and 34. The 2 commuting Clifford pairs each generate $\mathbb{C}^2$, so evaluation gives $\mathbb{C}^4$. Both types of plane belong to the null set excluded in Lemma 9.6. Almost every plane gives the 2 matrix blocks of Corollary 10.3.

The ordinary Grassmannian has real dimension $L(N-L)$. We finish by asking whether its rank can still be read from the abstract tracial algebra of the twisted space.

Corollary 10.5. *For fixed N, all the algebras $L^\infty(Gr_{LN}^\tau)$ with $1 \leq L < N$ are isomorphic as tracial von Neumann algebras. With the projection coordinates specified, the rank is recovered from any one diagonal trace by*

$$L = N\tau(q_{ii})$$

for each $1 \leq i \leq N$.

Proof. The invariant probability on each Grassmannian is nonatomic, and the quotient by a finite group still has no atoms. The measured central bases are therefore all isomorphic to the unit interval with Lebesgue measure. Choosing orbit representatives as in Proposition 10.1 identifies each field on its central base with a constant matrix field of the size given by Theorem 10.2. For even N one chooses a fixed matrix identification for each of the 2 simple summands. This gives the tracial isomorphisms. Finally q_{ii} is p_{ii} in the model. Coordinate permutations give all p_{ii} the same integral and their sum is the constant L. □

11 Half-liberation in affine and projective coordinates

Where should half-commutation be imposed in projective geometry? A line is given by a unit vector up to phase or by its projection entries. The projective diagram of [4] assigns the twisted complex projective space to the twisted half-liberated real sphere. We compare this affine prescription with imposing signed half-commutation on the projection entries, using the coordinate sign twist of [2].

Let $F = \mathbb{R}, \mathbb{C}$, let $1 \leq L < N$, and put $X = Gr_{LN}(F)$. Write p_{ij} for the entries of its orthogonal or Hermitian projection. Inside $C(X) \otimes Cl_N^0$ let $C(Gr_{LN}^\tau(F))$ be generated by $q_{ij} = p_{ij}c_ic_j$. For $L = 1$ we also write $P_F^{N-1,\tau}$ for this space. The real construction is the one of Section 9. To read the signs, set $\epsilon_{ii} = 1$, $\epsilon_{ij} = -1$ for $i \neq j$, and put

$$\chi(ij, kl) = \epsilon_{ik}\epsilon_{il}\epsilon_{jk}\epsilon_{jl}.$$

Theorem 11.1. *The complex twisted Grassmannian has the universal C^*-presentation*

$$q = q^* = q^2, \qquad Tr(q) = L, \qquad q_{ij}q_{kl} = \chi(ij,kl)q_{kl}q_{ij}.$$

The real twisted Grassmannian is obtained by adjoining $q_{ji} = \epsilon_{ij}q_{ij}$. At $L = 1$ these are the projective coordinate algebras of the corresponding twisted spheres.

Proof. With $D_c = diag(c_1, \ldots, c_N)$ we have $q = D_c p D_c$ and $D_c^2 = 1$. The diagonal entries are unchanged. Commuting 2 Clifford pairs gives the displayed sign. Over the real Grassmannian $p_{ji} = p_{ij}$, which gives the extra relation.

Conversely let Q be any representation of the proposed relations, and tensor with a fresh Clifford algebra. The entries $b_{ij} = Q_{ij} \otimes c_i c_j$ commute, give a projection of trace L, and are symmetric in the real case. The joint spectrum therefore consists of classical rank-L projections over F. There is a representation of $C(X)$ on this tensor product sending p_{ij} to b_{ij}. Apply it to the function leg of the defining model for q. After regrouping, its image is $Q_{ij} \otimes d_i d_j$, where $d_i = c_i \otimes c_i$. The character $d_i \to 1$ of Lemma 9.5 recovers Q. Every representation thus factors through the model. For the last assertion the twisted sphere coordinates are $y_i = z_i c_i$, where z is an ordinary unit vector. This Clifford model is faithful by the same untwisting argument applied to the sphere relations. The products $y_i y_j^* = z_i \bar{z}_j c_i c_j$ give the asserted algebra because $z \to zz^*$ is onto $Gr_{1N}(F)$. □

The involution needed for a real affine lift merits care. Conjugation of the ordinary complex sphere, acting trivially on the Clifford leg, gives a complex-linear $*$-automorphism κ with $\kappa(y_i) = y_i^*$. On projective coordinates it gives

$$\kappa(q_{ij}) = \epsilon_{ij} q_{ji}.$$

Plain transposition can destroy the projection equation. At the line through $(1,1,1)$ in dimension 3 the model gives $q_{ij} = c_i c_j/3$. Thus $q^t = (2/3)1_3 - q$, whose possible eigenvalues $2/3, -1/3$ are incompatible with a projection.

There are now 2 sets of generators on which to try the triple relation. On the sphere they are the coordinates of a vector. On the projective space they are the entries of a projection, and the sum of its diagonal entries is already a nonzero scalar. This trace relation will distinguish the 2 calculations.

Theorem 11.2. *The projective algebra of the sign twist of the half-liberated real sphere is $C(P_{\mathbb{C}}^{N-1,\tau})$. Half-liberating the twisted sphere by the crossed-product construction with κ gives the same affine and projective generators. In the presentation of $Gr_{LN}^{\tau}(F)$, replacing the pair relations by*

$$abc = \chi(a,b)\chi(a,c)\chi(b,c)cba$$

for entries a, b, c leaves the presented algebra unchanged. Here $\chi(a,b)$ is the sign attached to the 2 index pairs, and in the real presentation one retains signed reality.

Proof. Bichon's faithful model [8] realizes the untwisted half-liberated real sphere by $z_i s$ in $C(S_{\mathbb{C}}^{N-1}) \rtimes \mathbb{Z}_2$, where s implements conjugation. Twisting these generators gives $(z_i s)c_i$. Alternatively, form the crossed product of the twisted complex sphere algebra by κ and put $x_i = y_i s$. Both constructions sit in the tensor product of the ordinary crossed product with Cl_N and give the same elements there, since s commutes with c_i. The embedding of the twisted crossed product in this model is faithful by the conditional expectation onto its identity coefficient. We have $x_i^* = x_i$, $\sum_i x_i^2 = 1$ and $x_i x_j = y_i y_j^*$. Reversing $x_i x_j x_k$ gives a minus sign when i, j, k are all distinct and a plus sign otherwise, by moving the Clifford generators past one another. These are the twisted half-sphere relations of [4]. Conversely attaching a fresh c_i to each generator of their universal presentation restores ordinary half-commutation. Bichon's model and the character of the doubled Clifford generators then give faithfulness as in Theorem 11.1.

For the projective prescription, take $b = q_{rr}$. Both signs involving b are 1. Summing the resulting triple relations over r gives $Lac = L\chi(a,c)ca$. Since $L > 0$, this recovers every pair relation. The pair relations imply all the triple relations by 3 interchanges. □

At higher rank the same distinction occurs for the span of a frame. Use the sign twist of O_N^* in the same row convention as Section 9.

Corollary 11.3. *For $1 \le L < N$ the rank-L projection orbit of $O_N^{*,\tau}$ is $Gr_{LN}^{\tau}(\mathbb{C})$, with the trace obtained from the invariant probability on the complex Grassmannian and the Clifford trace.*

Proof. In the faithful model of Bichon and Dubois-Violette [9] used in Section 6, the coefficients of O_N^* are $u_{ai}s$, with $u \in U_N$ and s implementing conjugation. Twisting on the row and column sign degrees gives $c_a \otimes u_{ai}s \otimes c_i$. This realization is faithful by doubling the Clifford legs as in Lemma 9.5. In the product of 2 entries on row a the row Clifford factors cancel. Summing over $a \le L$ gives

$$1 \otimes \left(\sum_{a=1}^{L} u_{ai}\bar{u}_{aj} \right) \otimes c_i c_j.$$

The middle matrix runs over all Hermitian rank-L projections as u runs over U_N, with invariant law for a uniform u. The trace agrees on homogeneous words: sign invariance kills nonzero degrees and the degree-zero coefficient has its ordinary integral, with the scalar Clifford sign of the word. This gives the stated orbit model and trace. □

For every interior rank the 2 signed presentations differ on their specified coordinates. Take the complex plane spanned by $(e_1 + ie_2)/\sqrt{2}$ and $L-1$ further coordinate vectors outside this pair. Its projection has $p_{12} = -i/2$ and $p_{21} = i/2$, so the field $q_{21} + q_{12}$ is nonzero at this plane.

Even the abstract spaces can be distinguished by scalar points. A scalar point here means a unital $*$-character, with the topology of pointwise convergence on the generators.

Proposition 11.4. *Put $r = \min(L, N-L)$. The spaces of scalar points of $Gr_{LN}^{\tau}(\mathbb{R})$ and $Gr_{LN}^{\tau}(\mathbb{C})$ have real dimensions r and $2r$, respectively. In particular the 2 coordinate C^*-algebras are nonisomorphic for every $1 \le L < N$. At $L = 1$, $N = 2$ the 2 compact spaces are a circle and a 2-sphere.*

Proof. At a scalar point 2 off-diagonal entries with precisely one common index have product zero by the sign rule. The nonzero off-diagonal pairs therefore form disjoint edges. A permutation of coordinates writes the scalar projection as singleton blocks $0, 1$ and Hermitian 2×2 projections with nonzero off-diagonal entry. Each such block has rank and corank one. Conversely every such block matrix satisfies the sign and projection equations, provided its off-diagonal entries are purely imaginary in the real case. This proves completeness by Theorem 11.1.

A block has diagonal entries $a, 1-a$ and off-diagonal entry b with $|b|^2 = a(1-a)$. Including its 2 diagonal endpoints gives a 2-sphere over $\mathbb{C}$ and a circle when b is purely imaginary. There are finitely many choices of edges and singleton blocks. With s edges the stratum is a product of s such spaces with their endpoints removed. The largest possible s is r, attained by taking r disjoint blocks and filling the remaining trace with singletons. This gives the stated dimensions of the finite semialgebraic unions, also their topological dimensions. For $N = 2$ every pair of projective generators commutes by the sign rule, so the character spaces describe the full algebras. □

There is also a symmetry check. We use projective isometries in the coordinate sense of [2, 11]: a compact quantum subgroup of O_N^+ coacts by $q_{ij} \to \sum_{ab} q_{ab} \otimes u_{ai}u_{bj}$, and its projective version is generated by the quadratic coefficients.

Proposition 11.5. *In rank one the projective quantum isometry groups of the* 2 *signed spaces are*

$$PO_N^\tau, \qquad PU_N^\tau$$

in the real and complex cases, respectively. On projective coefficients the twist uses the restriction of the coordinate sign cocycle to the even degrees.

Proof. Label the coordinate sign degrees by e_i in $\mathbb{Z}_2^N$. In the ordered Clifford basis the multiplication sign is the bicharacter $\sigma(s,t) = (-1)^{\sum_{i>j} s_i t_j}$. The twist multiplies a homogeneous product of degrees s, t by $\sigma(s,t)$. Its involution on degree s acquires the sign $\sigma(s,s)$. On a quantum group containing the diagonal signs there are left and right degrees. A coefficient u_{ai} has bidegree (e_a, e_i) and the product sign is the product of the left and right signs, the inverse bicharacter being the same here.

These rules transport the quadratic coaction. In the twisted vector-space identification $q_{ij} = \sigma(e_i, e_j)p_{ij}$. On a term indexed by a, b the sign $\sigma(e_a, e_b)$ occurs once on the space and once on the group, leaving precisely $\sigma(e_i, e_j)$. More generally on multiplying 2 homogeneous coaction terms their intermediate degree signs occur twice and cancel. This proves multiplicativity, and the same cancellation for the diagonal signs $\sigma(s,s)$ preserves the involution. The coproduct remains the same linear map by cancellation of the intermediate degrees. The twisted fundamental matrix stays orthogonal by the row and column sums, so this gives a coaction in the prescribed class. Twisting twice restores the original coaction.

For either the original or twisted space, the diagonal sign matrices themselves act by $p_{ij} \to d_i d_j p_{ij}$ or $q_{ij} \to d_i d_j q_{ij}$. Hence its universal acting group contains this diagonal subgroup and can be twisted back. Universality in both directions identifies the maximal groups under twisting. Passing to their projective coefficients commutes with this operation, since each quadratic coefficient is homogeneous of even left and right degrees and is multiplied only by a sign. The untwisted projective groups are PO_N, PU_N by [2, 11], which gives the result. □

12 Von Neumann algebras of the 2 branches

Use the invariant probability μ on $X = Gr_{LN}(F)$ and the normalized Clifford trace t. On $C(Gr_{LN}^\tau(F))$ take the restriction of $\int_X t(\cdot)\, d\mu$. This is the twisted orbit trace: in the sign grading, integration vanishes on every nonzero degree by sign invariance, and on degree zero the ordered Clifford basis vector is 1. Thus the trace is ordinary integration under the linear identification by degrees. The invariant integral remains invariant under the transported coaction, whose underlying linear map is unchanged. We first compare the resulting tracial von Neumann algebras. We then put back a single coordinate, and finally the affine sphere above a projective point.

On a sign orbit the entries of a representative projection supply scalar coefficients, while the Clifford pairs supply the matrices. Passing from real to complex planes changes the coefficients. To determine the central fibers we must see which Clifford pairs can be recovered wherever those coefficients are nonzero.

Theorem 12.1. *For $F = \mathbb{R}, \mathbb{C}$, $1 \le L < N$, let $X = Gr_{LN}(F)$ and $m = 2^{[(N-1)/2]}$. The central base of $L^\infty(Gr_{LN}^\tau(F))$ is X/D^+ with image probability, and almost every central fiber is $M_m(\mathbb{C})$. As an abstract tracial algebra it is therefore isomorphic to*

$$L^\infty([0,1]) \,\bar{\otimes}\, M_m(\mathbb{C})$$

with Lebesgue measure and normalized matrix trace.

Proof. For the complex Grassmannian use $|p_{ij}|^2$ in the analytic argument of Lemma 9.6. The same real test plane makes this function nonzero, and the complex Grassmannian is connected with smooth invariant measure. Thus all the off-diagonal coordinates are nonzero almost surely and

the sign stabilizer is $\{1, -1\}$. Polynomials in the projection entries are closed under conjugation because $\bar{p}_{ij} = p_{ji}$. Averaging them over D gives the scalar orbit functions as in Proposition 10.1. At a measurable orbit representative one can divide by the complex number p_{ij} on $|p_{ij}| \geq 1/n$. The bounded truncated Clifford pairs recover the full even algebra on almost every orbit. The proof of Theorem 10.2 now gives the same central labels and traces over X/D^+ as in the real case. In both cases the finite quotient of X is standard and nonatomic, and the orbit representatives identify the fibers measurably with fixed matrices. □

Restriction to real planes gives a quotient on continuous coordinates. Does it extend to the von Neumann algebras?

Proposition 12.2. *For every $1 \leq L < N$ the canonical quotient $C(Gr^{\tau}_{LN}(\mathbb{C})) \to C(Gr^{\tau}_{LN}(\mathbb{R}))$ admits no normal unital extension between the orbit von Neumann algebras.*

Proof. In the complex model the positive central element

$$h = (q_{12} + q_{21})^*(q_{12} + q_{21}) = 4\, Im(p_{12})^2\, 1$$

has support 1. Indeed the imaginary part is real analytic, and the test plane following Theorem 11.2 makes it nonzero. On the real model $h = 0$. A normal unital extension would preserve the support projection, for example by taking the increasing limit of $h(h + 1/n)^{-1}$. This would send 1 to 0. □

How much information does a single diagonal coordinate add to the abstract algebra? Write $V(a) = \tau(a^2) - \tau(a)^2$ for the variance of a self-adjoint element. The mean and variance suffice.

Theorem 12.3. *Let $A = L^{\infty}(Gr^{\tau}_{LN}(F))$, let $a = q_{11}$, and put $b = 1/2$ for $F = \mathbb{R}$ and $b = 1$ for $F = \mathbb{C}$. The law of a is the beta law of parameters $bL, b(N - L)$. Its first 2 moments give*

$$u = \tau(a) = \frac{L}{N}, \qquad V(a) = \frac{u(1-u)}{bN + 1}.$$

The tracial isomorphism class of (A, a) determines the field F, the ambient dimension N and the rank L, even when none is specified.

Proof. The Clifford factors cancel on the diagonal, so $a = p_{11}$. An invariant plane is spanned by the first L rows of a uniform orthogonal or unitary matrix. Its first column has the law of a vector of independent standard real or complex Gaussian variables divided by its length, by rotational invariance. Thus $a = T/(T + U)$, where T, U are sums of squared Gaussian lengths over L and $N - L$ indices. These independent gamma variables have the stated shapes and a common scale. Changing variables to the sum and the ratio gives density proportional to $x^{bL-1}(1 - x)^{b(N-L)-1}$ on $(0, 1)$. Its beta integrals give the 2 moment formulas.

The moments recover $bN = u(1 - u)/V(a) - 1$ and $bL = u \cdot bN$. The only possible ambiguity is a complex system in dimension n and a real one in dimension $2n$. Here $n \geq 2$. Their central matrix sizes are respectively $2^{[(n-1)/2]}$ and 2^{n-1} by Theorem 12.1. They differ, and central matrix size is preserved by a von Neumann algebra isomorphism. With b determined, both integers follow. □

For Gr^{τ}_{24} the 2 fields give the same central matrix size 2 and the same mean $1/2$. The diagonal entry is uniform on $[0, 1]$ over $\mathbb{R}$ and has density $6x(1 - x)$ over $\mathbb{C}$. Its variances are respectively $1/12$ and $1/20$.

In rank one we can instead retain the whole affine cover and discard all coordinate labels. Take the even-to-affine tracial inclusion in each case: the even algebra is generated by $x_i x_j$ and the affine algebra by the x_i. In the real branch take the twisted ordinary sphere. In the complex projective branch take the twisted half-liberated real sphere of Theorem 11.2. Use uniform sphere measure, Clifford trace, and the identity-coefficient trace in the crossed product.

Theorem 12.4. *Fix $N \geq 2$ and $m = 2^{[(N-1)/2]}$. Up to tracial isomorphism the 2 rank-one inclusions just defined coincide exactly when N is even. After tensoring each displayed inclusion with the diffuse probability algebra $L^\infty([0,1])$, their descriptions are as follows. The half-liberated cover gives*

$$M_m(\mathbb{C}) \oplus M_m(\mathbb{C}) \ \subset \ M_{2m}(\mathbb{C})$$

as diagonal blocks for every N. The real cover gives the same blocks for even N, and the diagonal copy of $M_m(\mathbb{C})$ in $M_m(\mathbb{C}) \oplus M_m(\mathbb{C})$ for odd N. Each pair of summands has weights $1/2, 1/2$.

Proof. For the real sphere take the positive orthant as sign-orbit representative. The squares $x_i^2 = z_i^2$ in the even algebra distinguish these orbits and generate every scalar function there. Off the coordinate hyperplanes, division by $|z_i|$ recovers c_i from the affine generators $z_i c_i$ on the orthant, using bounded truncations. Division by $|z_i z_j|$ recovers $c_i c_j$ on the even side. Restriction to the orthant is faithful and trace preserving with the image measure, since each other sign choice acts by the corresponding trace-preserving Clifford automorphism. The inclusion is therefore $Cl_N^0 \subset Cl_N$ over a diffuse base.

For even N the full Clifford algebra has scalar center: an even central basis word must be empty, and an odd one would have to contain every index. Its dimension gives $Cl_N = M_{2m}(\mathbb{C})$. The 2 even blocks of Theorem 10.2 have equal trace, hence give the diagonal blocks. For odd N the full volume word, multiplied by a phase, is a central odd self-adjoint unitary of trace zero. Multiplication by it identifies the even and odd vector spaces. Thus Cl_N consists of 2 copies of its simple even algebra, with the latter embedded diagonally.

For the half-liberated cover write $X = P_{\mathbb{C}}^{N-1}$, let A be the even algebra and realize $x_i = z_i c_i s$ as in Section 11. Since $z_1 \neq 0$ almost surely, the spectral sign of x_1 is the symmetry $v = (z_1/|z_1|)c_1 s$. On an even field its conjugation is

$$\beta(f)(P) = c_1 f(\bar{P}) c_1.$$

In particular it preserves A by the covariance formula of Proposition 10.1, which holds over X by Theorem 12.1. Each $x_i v$ belongs to A, being $q_{i1}/|z_1|$ off its null denominator with bounded truncations. Thus A and v generate the affine algebra. Since Av has zero trace in the odd crossed-product coefficient, the inclusion is $A \subset A \rtimes \mathbb{Z}_2$ with action β.

Put $d = diag(-1, 1, \ldots, 1)$. On even Clifford words conjugation by c_1 is α_d. Consequently $\beta(f)(P) = f(d\bar{P}d)$, and on the central base from Theorem 12.1 it sends $[P]$ to $[d\bar{P}d]$. A fixed orbit would give $p_{12} = \pm\bar{p}_{12}$. Almost surely $Im(p_{12}^2) \neq 0$, since this real analytic function is nonzero, for example on the line spanned by $(1, e^{\pi i/4}, 0, \ldots, 0)$. The action on the central base is therefore free almost everywhere. Choose one representative of each resulting pair. The 2 central fibers of A are $M_m(\mathbb{C})$, and v identifies them and exchanges the 2 central projections. Multiplying v by these projections gives the 2 off-diagonal matrix units. Thus the crossed product consists of the indicated full matrices over the pair quotient, with the 2 even blocks of equal weight. The representative choice trivializes the inclusion over this nonatomic standard base. For odd N the larger algebras of the 2 covers have different central matrix sizes, which proves the last distinction. □

References

[1] N. Andruskiewitsch and S. Natale, Harmonic analysis on semisimple Hopf algebras, *Algebra i Analiz* **12** (2000), 3–27.

[2] T. Banica, Affine noncommutative geometry, arXiv:2012.10973 (2026).

[3] T. Banica, Liberation theory for noncommutative homogeneous spaces, *Ann. Fac. Sci. Toulouse Math.* **26** (2017), 127–156.

[4] T. Banica, Liberations and twists of real and complex spheres, *J. Geom. Phys.* **96** (2015), 1–25.

[5] T. Banica, J. Bichon and B. Collins, The hyperoctahedral quantum group, *J. Ramanujan Math. Soc.* **22** (2007), 345–384.

[6] T. Banica and R. Vergnioux, Invariants of the half-liberated orthogonal group, *Ann. Inst. Fourier* **60** (2010), 2137–2164.

[7] B. Bekka, The Plancherel formula for countable groups, *Indag. Math.* **32** (2021), 619–638.

[8] J. Bichon, Half-liberated real spheres and their subspaces, *Colloq. Math.* **144** (2016), 273–287.

[9] J. Bichon and M. Dubois-Violette, Half-commutative orthogonal Hopf algebras, *Pacific J. Math.* **263** (2013), 13–28.

[10] D. Bisch, P. Das and S.K. Ghosh, The planar algebra of diagonal subfactors, in Quanta of maths, Clay Math. Proc. **11** (2010), 23–47.

[11] Claude O and Teo B, Italian projective geometry goes free, zenodo:24448353 (2026).

[12] Codex and Teo, The magic of free spheres, zenodo:24434373 (2026).

[13] J. Dixmier, C^*-algebras, North-Holland (1982).

[14] J. Dixmier, Les anneaux d'opérateurs de classe finie, *Ann. Sci. École Norm. Sup.* **66** (1949), 209–261.

[15] P. Fima, K. Mukherjee and I. Patri, On compact bicrossed products, *J. Noncommut. Geom.* **11** (2017), 1521–1591.

[16] P. Green, The structure of imprimitivity algebras, *J. Funct. Anal.* **36** (1980), 88–104.

[17] V.F.R. Jones, Index for subfactors, *Invent. Math.* **72** (1983), 1–25.

[18] A. Joseph and G. Letzter, Local finiteness of the adjoint action for quantized enveloping algebras, *J. Algebra* **153** (1992), 289–318.

[19] R.V. Kadison and J.R. Ringrose, Fundamentals of the theory of operator algebras, II. Advanced theory, American Mathematical Society (1997).

[20] S. Kolb, M. Lorenz, B. Nguyen and R. Yammine, On the adjoint representation of a Hopf algebra, *Proc. Edinb. Math. Soc.* **63** (2020), 1092–1099.

[21] J. Krajczok and P.M. Sołtan, On certain invariants of compact quantum groups, *J. Funct. Anal.* **291** (2026), 111468.

[22] F.J. Murray and J. von Neumann, On rings of operators, *Ann. of Math.* **37** (1936), 116–229.

[23] F.J. Murray and J. von Neumann, On rings of operators, II, *Trans. Amer. Math. Soc.* **41** (1937), 208–248.

[24] F.J. Murray and J. von Neumann, On rings of operators, IV, *Ann. of Math.* **44** (1943), 716–808.

[25] B.H. Neumann, Groups with finite classes of conjugate elements, *Proc. London Math. Soc.* **1** (1951), 178–187.

[26] F. Peter and H. Weyl, Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe, *Math. Ann.* **97** (1927), 737–755.

[27] A.M. Sinclair and R.R. Smith, Finite von Neumann algebras and masas, Cambridge University Press (2008).

[28] TB 2.0 and GPT 6.2, Maximal torus theory for crossed product quantum groups, zenodo:24267186 (2026).

[29] Teo 2.0 and GPT 6.2, The free von Neumann algebra zoo, zenodo:24636473 (2026).

[30] M.J. Tomkinson, FC-groups, Pitman (1984).

[31] J. von Neumann, On rings of operators, III, *Ann. of Math.* **41** (1940), 94–161.

[32] J. von Neumann, On rings of operators. Reduction theory, *Ann. of Math.* **50** (1949), 401–485.

[33] S. Wang, Tensor products and crossed products of compact quantum groups, *Proc. London Math. Soc.* **71** (1995), 695–720.

[34] S. Wang, Quantum symmetry groups of finite spaces, *Comm. Math. Phys.* **195** (1998), 195–211.

[35] S.L. Woronowicz, Compact quantum groups, in Symétries quantiques (Les Houches, 1995), North-Holland (1998), 845–884.