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INJECTIVE HOMOCLINIC CODINGS FOR ALGEBRAIC GROUP ACTIONS

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Abstract. We prove Hayes's conjecture on the almost-everywhere injectivity of the specified homoclinic coding for semi-lopsided principal algebraic actions. This includes the case of a square-summable formal inverse with centered digits. We also answer Li and Liu's question on positively column-lopsided matrices by proving that almost every input has a singleton fiber in the full digit space, under their original hypotheses on the digit law. The general result concerns integral group-ring matrices with an acyclic dependency graph and an absolute coefficient comparison matrix of spectral radius at most one. At spectral radius one, a square-summable formal inverse and independent uniform digits give a measurable isomorphism with Haar measure, with centered odd alphabets in the square-summable setting and arbitrary consecutive alphabets when the inverse is summable.

Note on related work. Time constraints limited the scope and depth of the literature search. Readers who identify substantial overlap with their own publicly available manuscripts are invited to contact the author at 3565586261@qq.com.

1. Introduction

An integral group-ring matrix gives an algebraic action on the kernel of a torus convolution. A formal inverse of the matrix can define a coding of this action from a Bernoulli shift. Two questions are whether the coding carries the given digit law to Haar measure and whether its input can be recovered from its output. Both concern the specified convolution map.

Lind and Schmidt studied explicit Bernoulli codings for free-group actions and proposed a scalar lopsided conjecture [5, Conjecture 8.1]. Hayes constructed the Haar factor map for semi-lopsided scalar elements under two formal inverse hypotheses and conjectured that the map is injective modulo null sets [2, Corollary 5.2 and Conjecture 1]. His weak equivalence theorem for formal square-summable matrix inverses gives a separate relation to Bernoulli shifts [1, Theorem 1.1]. Li and Liu proved singleton fibers for positively row-lopsided matrices, allowing dependence between digits at the same group site. Their result includes the strictly lopsided scalar cases. They also

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proved the Haar pushforward for row- and column-lopsided matrices with the uniform product digit law and asked for the corresponding singleton-fiber theorem in the column-lopsided case [3, Theorem 1.1, Corollary 1.2 and Proposition 5.1]. The symbolic representations of Li and Schmidt [4, Theorem 3.18] use a symbolic space constructed for intrinsically ergodic principal actions of amenable groups.

1.1. **Setting and statements.** Let Γ be a countable group with identity e, and let $q \geq 1$. Put $[q] = \{1, \dots, q\}$ and $\mathcal{V} = \Gamma \times [q]$. For a matrix $A = \sum_{s \in \Gamma} A_s s \in M_q(\mathbb{Z}\Gamma)$ use column notation for the right-shift operator

$$(T_A u)_{v,i} = \sum_{j,s} (A_s)_{ij} u_{vs,j}. \tag{1.1}$$

The same formula acts on $\mathbb{T}^{\mathcal{V}}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Write $X_A = \ker(T_A : \mathbb{T}^{\mathcal{V}} \to \mathbb{T}^{\mathcal{V}})$ and let m_{X_A} be its normalized Haar measure. The group acts by the left shift $(g \cdot u)_{v,i} = u_{g^{-1}v,i}$. Throughout the paper the matrices have the form

$$A = M - F, \qquad M = \operatorname{diag}(m_1, \dots, m_q), \qquad F_{ij} = \sum_s a_{ij,s} s, \qquad m_i \in \mathbb{Z}_{\geq 1}. \tag{1.2}$$

Define $C_{ij} = \sum_s |a_{ij,s}|$ and $B = M^{-1}C$. The dependency graph on $\mathcal{V}$ has an arrow $(v,i) \to (vs,j)$ for every nonzero $a_{ij,s}$. Our acyclicity hypothesis means that there is no nonempty directed closed walk in this graph. When it holds, we write $w \preceq z$ if z is reachable from w, including the length-zero path, and $w^{\uparrow} = \{z : w \preceq z\}$. A set is *forward closed* if it contains $w^{\uparrow}$ for each of its vertices w.

For each i, let I_i be an interval of m_i consecutive integers and let $Y = (\prod_i I_i)^{\Gamma}$. A real formal inverse H of A satisfies $AH = HA = I_q$ as coefficient identities. Each product is defined because A has finite support. For a summable inverse, the homoclinic coding $\Phi_A : Y \to X_A$ is $d \mapsto \pi T_H d$, where π is reduction modulo one in every coordinate. For a square-summable inverse and product centered digits, the real convolution is defined in coordinate-wise L^2. Section 2 gives the construction and shows that the equivariant coding is well defined on an invariant conull set.

Theorem 1.1 (Product digit coding)**.** *Suppose that the dependency graph of A is acyclic, that $\rho(B) \leq 1$, and that A has a square-summable formal inverse. Give all vertices independent uniform digits. If the inverse is only assumed square summable, assume that all m_i are odd and take*

$$I_i = \left\{ -\frac{m_i - 1}{2}, \dots, \frac{m_i - 1}{2} \right\}.$$

If the inverse is summable, use any intervals of m_i consecutive integers. Then Φ_A is a measure-preserving isomorphism from the product digit shift onto (X_A, m_{X_A}).

Theorem 1.2 (Singleton fibers in the strict case)**.** *Suppose that the dependency graph is acyclic and $\rho(B) < 1$. Then A has a summable inverse. Give Y the product over group sites of a law on $\prod_i I_i$ with uniform coordinate marginals. Assume either that the digits at a site are independent or that no nonempty directed path returns to its starting group site, even at a different type. For almost every $d \in Y$,*

$$\Phi_A^{-1}(\Phi_A(d)) = \{d\} \quad \text{in the entire space } Y.$$

In particular the coding is a measurable conjugacy with its pushforward measure. For the uniform product digit law, Φ_A gives an almost topological conjugacy onto X_A with Haar measure.

Here an almost topological conjugacy is a measure-preserving equivariant homeomorphism between invariant dense G_δ sets of full measure. The two stated results allow different notions of injectivity. Theorem 1.1 recovers the digits on a conull set. Theorem 1.2 also rules out exceptional preimages of a typical coded point.

Corollary 1.3 (Hayes's conjecture)**.** *Let $f = me - \sum_{s\in S} a_s s \in \mathbb{Z}\Gamma$, where $m \geq 1$, $m \geq \sum_s |a_s|$, and the semigroup generated by S using nonempty words omits e. Suppose either that m is odd and f has a square-summable formal inverse, or that f has a summable formal inverse. In the first case use the uniform law on the centered m-digit alphabet. In the second case one may take the uniform law on $\{1, \dots, m\}$. The homoclinic coding in Hayes's Conjecture 1 is an isomorphism onto (X_f, m_{X_f}).*

Corollary 1.4 (The column-lopsided question)**.** *Li and Liu's Theorem 1.1 holds for positively column-lopsided matrices with its original digit-law hypothesis. For independent uniform digits the corresponding almost topological conjugacy with Haar measure holds as well.*

1.2. **Proof outline.** For the strict theorem we bound the positive and negative integer differences possible in a fiber by looking at each forward cone. Integer differences also appear in Li and Liu's proof for row-lopsided matrices [3, §4]. Averaging the rounded inequalities for the extrema over the digit at the cone's base gives a vector inequality $b \leq Bb$. In the critical case we use the same first-moment inequality for two inputs sampled conditionally on the same output. The conditional Haar law on a forward-closed set justifies averaging the base digit in that pair. At spectral radius one, equality in the first-moment inequality orders the signed differences of the successor lifts. Conditional quantiles then give two linear equations. A comparison of their square-summable solutions forces the conditional quantiles to be constant. The proof proceeds first on irreducible type blocks and then recovers the blocks successively.

2. Formal inverses and their support

Coefficients of a square-summable column are indexed by $(g, i) \in \mathcal{V}$. If $L_s b$ denotes left translation $(L_s b)(g) = b(s^{-1}g)$ on coefficient functions, then

left multiplication by A on a column b is the operator with coordinates $(Ab)_i(g) = \sum_{j,s}(A_s)_{ij}b_j(s^{-1}g)$. In particular it is bounded on $\ell^2(\mathcal{V})$. Let F_+ be formed from F by replacing every $a_{ij,s}$ by $|a_{ij,s}|$, and put $A_+ = M - F_+$. The identities below use finite convolution whenever one factor has finite support.

Lemma 2.1. *Suppose the dependency graph is acyclic and $\rho(B) \le 1$. Left multiplication by A on $\ell^2(\mathcal{V})$ is injective with dense range. A one-sided square-summable formal inverse is two-sided. If H is such an inverse, its entry at (i, j, g) can be nonzero only when $(e, i) \preceq (g, j)$.*

Suppose additionally that B is irreducible. If a square-summable column $h^+_{\cdot j}$ solves $A_+h^+_{\cdot j} = e_j\delta_e$, where e_j is the j-th coordinate vector, then

$$|H_{ij,g}| \le h^+_{ij,g} \qquad (i \in [q],\ g \in \Gamma). \tag{2.1}$$

The support and column statements also hold for a single type with no arrows.

Proof. First take an irreducible block and write $\rho = \rho(B)$. The case $q = 1$ with $F = 0$ has $H = m_1^{-1}e$ and is immediate. Otherwise $\rho > 0$. Choose positive right and left Perron vectors r, l with $Br = \rho r$ and $l^{\mathsf{T}}B = \rho l^{\mathsf{T}}$. Put $p_i = l_i r_i$ and $Q = M^{-1}F$, $Q_+ = M^{-1}F_+$. For a coefficient column b define $x_i = b_i/r_i$. The weights in the i-th row of Qb/r_i have absolute values $\lambda_{ij,s} = m_i^{-1}|a_{ij,s}|r_j/r_i$. They sum to ρ, and $\sum_{i,s} p_i\lambda_{ij,s} = \rho p_j$. Using the Hilbert-space inequality for a weighted average of the signed translates $L_s x_j$ gives

$$\sum_i p_i\|(Qb)_i/r_i\|_2^2 \le \rho\sum_{i,j,s} p_i\lambda_{ij,s}\|x_j\|_2^2 = \rho^2\sum_i p_i\|x_i\|_2^2. \tag{2.2}$$

The same bound holds for Q_+.

For $\rho < 1$ both operators are strict contractions in this weighted norm. For $\rho = 1$, if $Qb = b$, equality in (2.2) gives $x_i = \operatorname{sgn}(a_{ij,s})L_s x_j$ along every type edge with label s. Each type in this block lies on a directed type cycle. If its labels multiply to g, any finite order of g would produce a directed closed walk in $\mathcal{V}$ by iterating the cycle. Thus g has infinite order, and composition along the cycle gives $x_i = \pm L_g x_i$. The absolute values of the coefficients would be constant on each infinite orbit of $\langle g\rangle$, so square summability gives $x_i = 0$. The same argument applies to Q_+ with positive signs. For a contraction the fixed spaces of the operator and its Hilbert-space adjoint coincide. Indeed, when one fixes x, the inner product of the other applied to x with x is $\|x\|^2$, and equality in Cauchy–Schwarz gives the claim. Therefore $I-Q$ and $I-Q_+$ have trivial kernels and dense ranges. The same conclusions hold for A and A_+, using the invertible diagonal matrix M and equivalence of the weighted norm to the original norm.

For $0 < t < 1$, the resolvent of Q exists in the weighted norm and $\|(1 - t)(I - tQ)^{-1}\| \le 1$. The resolvent identity gives $(1 - t)(I - tQ)^{-1} \to 0$ strongly, first on $(I - Q)\ell^2(\mathcal{V})$ and then on its closure. Indeed for $u =$

$(I-Q)v$, $(I-tQ)^{-1}u = v-(1-t)(I-tQ)^{-1}Qv$ is bounded in norm as $t\uparrow 1$. Consequently, if $Ah = e_j\delta_e$, then

$$h = \lim_{t\uparrow 1}(I-tQ)^{-1}M^{-1}e_j\delta_e \quad \text{in } \ell^2(\mathcal{V}). \tag{2.3}$$

The n-th power in the resolvent series contributes only along paths of length n in the dependency graph. Coefficientwise convergence in (2.3) proves the support assertion on the block. The analogous resolvent for Q_+ has nonnegative coefficients and dominates the absolute coefficients term by term. Applying (2.3) to both column equations gives (2.1).

For general B, order the strongly connected components of its type graph so that an edge between components points from a smaller to a larger index. A strongly connected component is a maximal set of types mutually reachable by directed paths. Then A is block upper triangular and each diagonal block has the properties just established, including the trivial single-type case. The equations $Ab = 0$ can be solved from the last block upward, giving zero in every block. The standard Hilbert adjoint of A is block lower triangular with injective diagonal blocks, by dense range for each diagonal block. It is therefore injective by forward substitution, so A has dense range.

If $AH = I_q$, then $A(HA - I_q) = 0$, and injectivity on square-summable columns gives $HA = I_q$. If $HA = I_q$ first, take the group-ring adjoint. Left multiplication by A^* on the coefficient Hilbert space is the Hilbert adjoint just considered. Applying the previous argument to $A^*H^* = I_q$ gives the reverse identity as well.

It remains to verify the support for the full matrix. The equations $AH = I_q$ and injectivity of the diagonal blocks give zero for every block H_{rs} with $r > s$, by backward substitution within each column block. Each H_{rr} is the formal inverse of A_{rr}. For a fixed column block s, the entries for $r < s$ satisfy

$$A_{rr}H_{rs} = \sum_{k>r} F_{rk}H_{ks}.$$

The right-hand side uses only finite convolutions and is square summable. Apply (2.3) column by column. Formula (2.3) holds with an arbitrary square-summable right-hand side in place of $e_j\delta_e$, by the same resolvent identity. Consider a coordinate (g, i) of this column, whose source index is in block s. The coefficient of the right-hand side at (g, i) is supported on labels of paths beginning at type i that first cross to a larger type block and end at the source type. This follows by downward induction in r. Powers of Q_{rr} in the resolvent prepend paths within block r. Thus every approximant has the required support, and the square-summable limit preserves it. □

In passing from a formal inverse to a random field, let D have independent uniform coordinates on the prescribed alphabets. Write $\mu_i = \mathbb{E}D_{e,i}$ and $\sigma_i^2 = \mathrm{Var}(D_{e,i})$. Take $c_0 = 0$ for centered digits. In the summable case, take $c_0 = (\sum_s H_s)\mu$, regarded as a constant field. Then $T_A c_0 = \mu$, by the augmentation of $AH = I_q$ in the summable case and directly in the centered

case. The formula

$$U_{v,i} = c_{0,i} + \sum_{j,s} H_{ij,s}(D_{vs,j} - \mu_j) \tag{2.4}$$

converges in L^2 at every coordinate by orthogonality. Choose a version at each (e, i) and translate it to all group sites. The support statement in Lemma 2.1 allows these versions to be measurable in their respective forward cones. Finite convolution commutes with coordinatewise L^2 convergence. Thus $T_A U = D$ almost surely at all vertices, by the formal identity $AH = I_q$, and $X = \pi U$ belongs to X_A. The variance of this real lift is

$$\mathbb{E}(U_{e,i} - c_{0,i})^2 = \sum_j \sigma_j^2 \|H_{ij}\|_2^2. \tag{2.5}$$

For summable H, (2.4) agrees with the ordinary continuous coding on all of Y.

3. Full fibers in the strict case

Proof of the singleton-fiber assertion of Theorem 1.2. If $\rho(B) < 1$, the matrix Neumann series

$$H = \sum_{n \geq 0} (M^{-1}F)^n M^{-1}$$

converges entrywise in ℓ^1. Indeed the matrix of the entrywise ℓ^1 norms of the n-th term is bounded by $B^n M^{-1}$. The series is an inverse and each of its entries has the path support described above.

For $d, d' \in Y$ with $\Phi_A(d) = \Phi_A(d')$, the array $u = T_H(d' - d)$ is integral and satisfies $d' = d + T_A u$. Choose an integer K larger than $\|T_H(d' - d)\|_\infty$ for every pair in Y, using the summability of H and the bounded digit intervals. On $w^\uparrow$ consider every array u of integers in $[-K, K]$ for which $d + T_A u$ has the prescribed digits throughout that cone. All equations on the cone use only vertices in the cone. Define $p_w(d)$ to be the largest feasible value of u_w and $n_w(d)$ the largest feasible value of $-u_w$. The zero array is feasible. The set of feasible pairs of digits and arrays on the cone is compact, since each local condition is closed in the product of finite alphabets. Projections of its intersections with the coordinate inequalities for u_w are compact too. Thus p_w, n_w are measurable functions of the digits on the cone. Restriction of a feasible array to any successor's cone is feasible there.

Write $w = (v, i)$, $\ell_i = \min I_i$, and $z = \sum_{j,s} a_{ij,s} u_{vs,j}$. The condition at w is equivalent to the integer equation

$$u_w = \left\lceil \frac{z - (d_w - \ell_i)}{m_i} \right\rceil. \tag{3.1}$$

Let Z_+ bound z from above by using $|a_{ij,s}| p_{vs,j}$ for positive coefficients and $|a_{ij,s}| n_{vs,j}$ for negative coefficients in the sum. Let Z_- bound $-z$ from above by interchanging p and n in that sum. Both depend only on the digits at the strict successors of w. Under either digit hypothesis in the theorem, $d_w - \ell_i$ is uniform on $\{0, \ldots, m_i - 1\}$ given all these digits. For independent

coordinates this follows from acyclicity. Otherwise all the strict successors are at group sites different from that of w. For any integer z_0,

$$\frac{1}{m_i}\sum_{h=0}^{m_i-1}\left\lceil\frac{z_0-h}{m_i}\right\rceil=\frac{z_0}{m_i},$$

as is seen by writing $z_0=m_ik+r$ with $0\le r<m_i$. Apply this identity to the upper bound for p_w in (3.1) with $z=Z_+$, and to the upper bound for n_w with $z=-Z_-$ and a reversed sign. By stationarity the vector $b_i=\mathbb{E}(p_{e,i}+n_{e,i})$ satisfies

$$0\le b\le Bb.$$

Since $B^n\to 0$, the vector is zero. On a conull subset of Y, both bounds vanish at every vertex by countability. Every collision array is feasible on every forward cone, so it must vanish at all vertices. The finite convolution equation $d'=d+T_Au$ then gives $d'=d$. □

4. Conditional Haar measure

In this section and Section 5, assume the hypotheses of Theorem 1.1 for an irreducible type block, allowing the single type with no arrows. Use the random variables D,U,X from (2.4). For a forward-closed subset J let $K_J=\{x\in X_A : x|_J=0\}$.

Lemma 4.1 (Conditioning on forward-closed sets)**.** *For a fixed forward-closed set J, the restriction X_J is measurable with respect to D_J. Given D_J, the law of X is Haar measure on the coset of K_J determined by X_J. In particular X is Haar distributed and*

$$\mathcal{L}(D_J\mid X)=\mathcal{L}(D_J\mid X_J)\quad\textit{almost surely.}\tag{4.1}$$

Proof. The first assertion follows from the support of H. We use a Fourier argument based on the denominator criterion in Hayes's proof of the Haar pushforward [2, Corollary 5.2] and its matrix version in [3, §5]. For $\alpha\in\mathbb{Z}^{(\mathcal{V})}$ write $\chi_\alpha(x)=\exp(2\pi\mathrm{i}\sum_w\alpha_wx_w)$ for the corresponding character restricted to X_A. Let $b=T_H^*\alpha$, defined as a finite linear combination of the rows of the kernel of H. Thus $b_{g,j}=\sum_{v,i}\alpha_{v,i}H_{ij,v^{-1}g}$ is square summable, and $T_A^*b=\alpha$ by $HA=I_q$. These adjoint formulas are coefficient identities on finite test arrays. The random phase of $\chi_\alpha(X)$ is

$$\sum_w\alpha_wc_{0,\tau(w)}+\sum_z b_z(D_z-\mu_{\tau(z)})\pmod 1,$$

where $\tau(v,i)=i$ and the sum is taken in L^2.

Put $E=\mathcal{V}\setminus J$, which is predecessor closed. If b_E is integral, it is finitely supported. Extend it by zero on J. The character associated to $\alpha-T_A^*b_E$ is supported on J and agrees with χ_α on X_A. Indeed the adjoint calculation at each vertex in E uses only its predecessors, and $T_A^*b_E$ vanishes as a character on $\ker T_A$. Thus the character has a fixed value on the specified coset.

If b_E has a nonintegral entry, write $\eta_z = b_z - n_z \in [-1/2, 1/2)$ for $z \in E$, with $n_z \in \mathbb{Z}$. The array η, extended by zero on J, vanishes at infinity. Let $r_z = (T_A^* \eta)_z$ for $z \in E$. Because E is predecessor closed and $T_A^* b = \alpha$, this residual is integral. The finite support of A implies r vanishes at infinity, so r has finite support.

There exists a vector $h > 0$ with $M^{-1} C^{\mathsf{T}} h \le h$. For an irreducible block take $h = M^{-1} l$, where $l^{\mathsf{T}} B = \rho(B) l^{\mathsf{T}}$. For the type with no arrows any positive h works. At a vertex $z = (g, i) \in E$ with $r_z = 0$ and $\eta_z \neq 0$, the adjoint equation implies

$$\frac{|\eta_z|}{h_i} \le \sum_{k,s} \frac{|a_{ki,s}| h_k}{m_i h_i} \frac{|\eta_{gs^{-1},k}|}{h_k}.$$

The coefficients sum to at most one. Thus some predecessor has weighted magnitude at least $|\eta_z|/h_i > 0$. Repeat the argument until reaching a vertex where the residual is nonzero. An infinite continuation would consist of distinct vertices by acyclicity and would contradict vanishing at infinity. Starting at a nonzero entry of η shows that the residual support is nonempty.

Choose a minimal vertex w of this finite residual support in the reachability order. Every immediate predecessor of w has zero fractional part. Any nonzero fractional part there would produce a nonzero residual at a strict ancestor of w by the preceding argument. Consequently

$$m_{\tau(w)} \eta_w = r_w \in \mathbb{Z} \setminus \{0\}, \qquad b_w \in m_{\tau(w)}^{-1} \mathbb{Z} \setminus \mathbb{Z}.$$

The factor $\mathbb{E}[\exp(2\pi \mathrm{i} b_w D_w)]$ vanishes because D_w is uniform on $m_{\tau(w)}$ consecutive integers. The coordinate D_w is independent of all the other digits, and $w \in E$. Isolating this term in the phase shows $\mathbb{E}[\chi_\alpha(X) \mid D_J] = 0$. Also $x^0 = \pi T_H \delta_w$ lies in X_A by $AH = I_q$, where δ_w is the unit array at w. The path support of H gives $x^0|_J = 0$, since no vertex in J can reach w. The character has $\chi_\alpha(x^0) = \exp(2\pi \mathrm{i} b_w) \neq 1$, so its integral over Haar measure on the coset of K_J vanishes as well.

These two cases give the conditional Fourier coefficients for every finite-support character. The characters determine probability measures on X_A and there are countably many of them. This proves the claimed conditional law. Taking $J = \varnothing$ proves the Haar assertion. For general J, the conditional distribution of X given D_J depends on D_J through X_J alone. Since X_J is a function of both D_J and X, this is conditional independence of D_J and X given X_J. For instance, for bounded measurable $a(D_J)$ and $b(X)$, successive conditioning gives

$$\mathbb{E}[a(D_J) b(X) \mid X_J] = \mathbb{E}[a(D_J) \mid X_J] \mathbb{E}[b(X) \mid X_J].$$

This proves (4.1). □

For each w define the forward-closed set

$$J_w = \mathcal{V} \setminus \{z : z \preceq w\}. \tag{4.2}$$

It contains all immediate successors of w and their cones. We use Lemma 4.1 for the empty set and the countable families $w^{\uparrow}, J_w$. Its almost-sure conclusions can therefore be imposed simultaneously for these sets. Given D_{J_w}, all the corresponding successor lifts are determined. With $w = (v, i)$, the equation $T_A U = D$ shows that X_w is uniform on the m_i distinct roots of

$$m_i x_w = \sum_{j,s} a_{ij,s} X_{vs,j} \quad \text{in } \mathbb{T}, \tag{4.3}$$

since each of the m_i possible digits for D_w gives a different root. The set of roots is a function of X_{J_w}, so the root law is the same given X_{J_w}.

5. Digit recovery on an irreducible block

5.1. **The first-moment inequality.** Sample X with its Haar law and take two conditionally independent digit arrays D, D' from the conditional law of D given X. Regular conditional laws exist on these countable products of standard Borel spaces. Each array, paired with X, has the original coding law. The joint law of (D, D', X) is shift invariant by equivariance of the coding and uniqueness of conditional laws. Write U, U' for their lifts and $c_w = U'_w - U_w \in \mathbb{Z}$. The coordinates of c are integrable by square summability of the lifts.

Fix $w = (v, i)$. By (4.1), the conditional distribution of the two restrictions D_{J_w} and D'_{J_w}, given X, is the product of two copies of the law of D_{J_w} given X_{J_w}. It follows that the pair of restrictions is independent of X_w conditionally on X_{J_w}. Furthermore X_{J_w} is determined by D_{J_w}. Hence the law of X_w given both digit restrictions is the uniform root law (4.3). For fixed first-array successor lifts, these roots are in bijection with D_w. In particular, D_w is uniform on I_i given both restrictions.

Given these restrictions, set $s_w = m_i^{-1} \sum_{j,s} a_{ij,s} c_{vs,j}$, which is determined by them. The condition $D'_w = D_w + m_i c_w - \sum_{j,s} a_{ij,s} c_{vs,j} \in I_i$ gives the same rounding formula as in (3.1). If $m_i s_w = m_i k + r$ with $0 \le r < m_i$, the value of c_w is $k + 1$ on the r smallest digits for D_w and k on the others. The conditional mean is s_w. The function $x \mapsto |x|$ is affine between adjacent integers, so the conditional mean of $|c_w|$ is $|s_w|$. It follows that

$$\mathbb{E}|c_{v,i}| = \frac{1}{m_i} \mathbb{E} \left| \sum_{j,s} a_{ij,s} c_{vs,j} \right| \le \frac{1}{m_i} \sum_{j,s} |a_{ij,s}| \mathbb{E}|c_{vs,j}|.$$

In terms of $b_i = \mathbb{E}|c_{e,i}|$, stationarity gives

$$0 \le b \le Bb. \tag{5.1}$$

For $\rho(B) < 1$, (5.1) gives $b = 0$. The two lifts then agree, so the conditional law of the lift given X is a point mass almost surely.

Suppose $\rho(B) = 1$. Pairing (5.1) with a strictly positive left Perron vector gives equality in every row. The defect in the triangle inequality is

nonnegative, so for every parent w the variables

$$a_{ij,s}(U'_{vs,j} - U_{vs,j}) \qquad ((j,s) \text{ with } a_{ij,s} \neq 0) \tag{5.2}$$

have a common weak sign almost surely. This holds at once for all parents by countability and stationarity.

5.2. **Conditional quantiles.** For $0 < t < 1$ let $u_w(t,x)$ be the conditional quantile of U_w given $X = x$, defined by the generalized inverse of the conditional distribution function. These quantiles are jointly measurable in t, x. They can be chosen equivariantly in w, x, by choosing base-site versions and translating them. Equation (4.1) for $w^\uparrow$ shows that their dependence on x is through $x|_{w^\uparrow}$. Here U_w is a function of $D|_{w^\uparrow}$. For almost every x, the conditional distribution of U_w is supported on the closed discrete set of real lifts of x_w.

For almost every fixed x and any parent $w = (v,i)$, the joint conditional law of the signed child variables $a_{ij,s}U_{vs,j}$ is represented by their quantiles at one common uniform parameter. To see this, take any two of these variables, Y_1, Y_2, and their events $\{Y_1 \le b_1\}, \{Y_2 \le b_2\}$ for rational thresholds. If neither event were contained in the other up to null sets, independent draws from the two opposite differences would give opposite signs for the coordinate differences, contradicting (5.2). Thus the events are nested up to null sets. Intersecting a finite family of such events has probability equal to the smallest of their probabilities. This identifies the joint distribution with the common-quantile representation, first on rational lower rectangles and then on all Borel sets. This is the usual comonotone representation [6, Theorem 2.1], here applied to the conditional law.

Use $t_{ij,s} = t$ for $a_{ij,s} > 0$ and $t_{ij,s} = 1 - t$ for $a_{ij,s} < 0$. The function $t \mapsto a_{ij,s}u_{vs,j}(t_{ij,s}, x)$ is nondecreasing and has the appropriate signed marginal when the parameter is uniform. Hence the collection of these functions gives the joint conditional law above. For fixed x_w, put $\widetilde{x}_w \in [-1/2, 1/2)$ for its representative. The lift at the parent is uniquely determined by the child sum $z = \sum_{j,s} a_{ij,s}U_{vs,j}$ and the constraint that its digit lie in I_i. It is

$$\widetilde{x}_w + \left\lceil \frac{\ell_i + z - m_i\widetilde{x}_w}{m_i} \right\rceil .$$

For admissible child lifts, $z - m_i\widetilde{x}_w$ is integral by $x \in X_A$. The displayed function is nondecreasing in z. Applying it to the child sum from the joint quantile representation gives a nondecreasing representation of the parent's conditional distribution. It agrees almost everywhere in t with $u_w(t,x)$. In particular the array

$$d_w(t,x) = m_i u_w(t,x) - \sum_{j,s} a_{ij,s}u_{vs,j}(t_{ij,s}, x) \tag{5.3}$$

takes values in I_i for almost every (t,x), at all vertices simultaneously.

We next check the joint statistics of the two arrays $d(t,X)$ and $d(1-t,X)$. By Fubini's theorem and countability, for almost every fixed t all the

preceding digit and lift identities hold almost surely for t and $1-t$. For such a t and fixed w, every successor quantile in (5.3) is a function of X_{J_w}. With these lifts fixed, the roots in (4.3) are in bijection with the m_i digits that satisfy (5.3). The parent quantile has the prescribed fractional part X_w. The root law given X_{J_w} therefore makes both $d_w(t, X)$ and $d_w(1-t, X)$ uniform on I_i conditionally on X_{J_w}.

For any finite set of distinct vertices, select one of the two parameters at each vertex. Choose a minimal vertex w of the set. All the other selected digits are measurable with respect to X_{J_w}, because a vertex $z \neq w$ in the set cannot be an ancestor of w and $z^{\uparrow} \subset J_w$. The selected digit at w is uniform given this information. Removing w and iterating proves that the selected digits have the product of the uniform laws. In particular any pair of digits at distinct vertices, with either parameter chosen independently at each, is independent.

5.3. **The second-moment inequality.** Fix $0 < t < 1/2$ for which the previous conclusions hold and the quantile fields at t and $1-t$ have square-integrable coordinates. The latter is valid for almost every t by integration of the conditional laws and (2.5). Suppressing X in the notation, form the stationary fields

$$V_w = \tfrac{1}{2}(u_w(t) + u_w(1-t)) - c_{0,\tau(w)}, \qquad W_w = \tfrac{1}{2}(u_w(1-t) - u_w(t)),$$
$$P_w = \tfrac{1}{2}(d_w(t) + d_w(1-t)) - \mu_{\tau(w)}, \qquad R_w = \tfrac{1}{2}(d_w(1-t) - d_w(t)).$$

Here $W_w \geq 0$. Adding and subtracting the two identities (5.3) gives

$$T_A V = P, \qquad T_{A_+} W = R. \tag{5.4}$$

In the difference equation, a negative edge coefficient interchanges the two child quantiles, so the resulting matrix uses the absolute value of that coefficient. The subtraction of c_0 in the sum equation uses $T_A c_0 = \mu$.

Both P and R are centered, and each of these fields is uncorrelated across distinct vertices, by the previous independence assertion. Let $\tau^2_{P,j} = \mathbb{E}P^2_{e,j}$ and $\tau^2_{R,j} = \mathbb{E}R^2_{e,j}$. The sum-of-squares identity for the half sum and half difference gives

$$\tau^2_{P,j} + \tau^2_{R,j} = \sigma^2_j. \tag{5.5}$$

There is no independence requirement for P and R at the same vertex.

We compute lower bounds on the second moments by orthogonal projection. For a type j with $\tau^2_{P,j} > 0$, define

$$\beta_{ij}(g) = \frac{\mathbb{E}[V_{e,i} P_{g,j}]}{\tau^2_{P,j}}.$$

Bessel's inequality gives $\beta_{\cdot j} \in \ell^2(\mathcal{V})$. By stationarity,

$$\mathbb{E}[V_{s,k} P_{g,j}] = \mathbb{E}[V_{e,k} P_{s^{-1}g,j}].$$

Consequently the coefficient identity

$$(A\beta_{\cdot j})_i(g) = \frac{\mathbb{E}[(T_A V)_{e,i} P_{g,j}]}{\tau_{P,j}^2} = \delta_{ij}\delta_{e,g}$$

follows from (5.4) and the orthogonality of the P coordinates. By Lemma 2.1 this solution is $H_{\cdot j}$. Likewise, for any j with $\tau_{R,j}^2 > 0$, the correlations of W with $R_{g,j}$ give a square-summable solution $h_{\cdot j}^+$ to $A_+ h_{\cdot j}^+ = e_j \delta_e$. By (2.1) its coefficients dominate the absolute values of those of $H_{\cdot j}$.

The families used for each projection are orthogonal across both group coordinates and source types. In the projection for W one may include the constants, orthogonal to the centered R variables. Dropping types of zero variance in the corresponding projections, we obtain

$$\begin{aligned}\mathbb{E}(V_{e,i}^2 + W_{e,i}^2) &\geq \sum_j (\tau_{P,j}^2 + \tau_{R,j}^2)\|H_{ij}\|_2^2 + (\mathbb{E}W_{e,i})^2 \\ &= \mathbb{E}(U_{e,i} - c_{0,i})^2 + (\mathbb{E}W_{e,i})^2. \end{aligned} \tag{5.6}$$

The last equality uses (5.5) and (2.5). On the other hand, $V_{e,i}^2 + W_{e,i}^2$ is the mean of $(u_{e,i}(t) - c_{0,i})^2$ and $(u_{e,i}(1-t) - c_{0,i})^2$. Integrating its expectation against $2\,dt$ on $(0, 1/2)$ gives exactly $\mathbb{E}(U_{e,i} - c_{0,i})^2$, since the conditional quantile at a uniform parameter samples the conditional law. Inequality (5.6) then implies $\mathbb{E}W_{e,i} = 0$ for almost every t in this interval. Since $W_{e,i} \geq 0$, the two quantiles coincide for almost every (t, x).

A nondecreasing quantile that agrees with its reflection almost everywhere is constant almost everywhere. Indeed, if its law gives positive mass to two separated intervals, then for a set of small positive parameters its lower quantile is in or below the first interval and its reflected quantile is in or above the second. Therefore the conditional law of U_w given X is a point mass almost surely, at all w at once. The conditional expectations of the lift coordinates give a measurable recovery map. Applying T_A to the recovered lift recovers D. The recovery and coding maps can be restricted to invariant conull Borel sets, by equivariance and countability of Γ. Together with Lemma 4.1 this proves the measure-preserving isomorphism for the irreducible block.

6. Successive recovery of the type blocks

Completion of the proof of Theorem 1.1. Order the type blocks as in Lemma 2.1. The inverse H is block upper triangular and H_{rr} is the inverse of A_{rr}. The diagonal inverse has the summability required by the digit law for that block. For each r let $X^{(r)}$ be the torus kernel of the matrix on blocks with indices at least r. The projection $X^{(r)} \to X^{(r+1)}$ is surjective. Indeed, given the higher coordinates, the equation in block r can be solved because $T_{A_{rr}}$ is surjective on the full torus for that block. To see the latter fact, its dual on finite-support integral arrays is injective by the formal right inverse $A_{rr}H_{rr} = I$.

Explicitly, $T^*_{A_{rr}}\alpha = 0$ for a finite test array α implies $T^*_{H_{rr}}T^*_{A_{rr}}\alpha = \alpha = 0$. This adjoint identity uses finite tests as in Lemma 4.1.

The kernel of $X^{(r)} \to X^{(r+1)}$ identifies with $X_{A_{rr}}$ and each fiber is a coset of that kernel. For a surjective homomorphism of compact groups, Haar measure disintegrates over Haar measure on the quotient with translated Haar measure on the fibers. This follows by averaging any continuous test function over the kernel. The average descends to the quotient, and its integral there equals the Haar integral of the original function.

Writing D_r for the digit array in block r, the output in that block is

$$X_r = \Phi_{A_{rr}}(D_r) + \pi \sum_{s>r} T_{H_{rs}} D_s. \tag{6.1}$$

The terms are well defined in the centered L^2 sense or by absolute summation, as in (2.4). In the last block this is just the diagonal coding, so its digits are recovered and its output has Haar law by the irreducible-block result. Suppose the digits of the blocks above index r have already been recovered from $X_{>r}$ and that this tail has Haar law on $X^{(r+1)}$. Conditionally on $D_{>r}$, the second term in (6.1) is fixed. The first term has Haar law on $X_{A_{rr}}$ by independence of the digits and the diagonal theorem. Since $X^{(r)}$ obeys the torus equations almost surely, this conditional law on block r is the translated Haar measure of the appropriate fiber. Because $D_{>r}$ is already determined by $X_{>r}$, it is also that law conditionally on $X_{>r}$. The disintegration formula now gives Haar law on $X^{(r)}$. Subtracting the known torus translation in (6.1) and using the diagonal inverse recovers D_r. Induction proves the theorem for all type blocks. □

Completion of the proof of Theorem 1.2. The continuous map Φ_A on the full digit space is onto by Theorem 1.1 with the uniform product law, since its image is compact and Haar measure has full support. Equip Y with a compatible metric. For each $n \geq 1$, the set of $x \in X_A$ with two preimages at distance at least $1/n$ is compact. It is the projection of a closed set in $X_A \times Y \times Y$. Thus the set X_{one} of singleton fibers is an invariant G_δ. Its preimage has full measure for each law in the strict theorem, so X_{one} has full pushforward measure. The inverse of the restriction to these sets is continuous, since any convergent output sequence with singleton limiting fiber has all cluster points of its preimages in that same fiber. Compactness of Y then gives convergence of the preimages. In the uniform product case both sets are dense by full support, and the pushforward is Haar by Theorem 1.1. □

7. The two questions and a matrix example

Proof of Corollary 1.3. For $q = 1$, the comparison matrix consists of the single entry $m^{-1}\sum_s |a_s|$. A directed closed walk is precisely a nonempty word in the support directions that equals e. The positive cone of the left-invariant partial order in Hayes's conjecture therefore gives the acyclicity hypothesis. Conversely the semigroup of nonempty words defines such an

order by adjoining e and using the reachability relation. Hayes's one-sided formal inverse convention agrees with ours by Lemma 2.1. With the right-shift convention (1.1), the map in that conjecture is convolution by the inverse followed by reduction modulo one. Its two digit laws are the centered m-digit law for a square-summable inverse with odd m and the uniform law on $\{1, \dots, m\}$ for a summable inverse. Theorem 1.1 applies in both cases. □

Proof of Corollary 1.4. In the notation of [3, Definitions 3.1 and 3.4], write the matrix as $f = \operatorname{diag}(M_i s_i) - g$ and let $u = \operatorname{diag}(\operatorname{sgn}(M_i) s_i)$. The matrix $A = fu^{-1}$ has diagonal term $M = \operatorname{diag}(|M_i|)$ and remaining matrix $F = gu^{-1}$. The digit space is unchanged and the torus coordinate change T_u conjugates the two codings, since $A^{-1} = uf^{-1}$. This is also the normalization in [3, Lemma 4.2]. In particular it does not mix the input coordinates at different sites.

The column-lopsided hypothesis is $\sum_k \|F_{ki}\|_1 < m_i$ for every i, where $m_i = |M_i|$. Thus CM^{-1} has column sums less than one. It is similar to $B = M^{-1}C$, so $\rho(B) < 1$. Positivity in [3, Definition 3.4] places the support of F in a common subsemigroup of Γ omitting e. A nonempty dependency path therefore cannot return to its starting group site, at any type. Theorem 1.2 applies with their original site law and its uniform marginals. The product law gives their almost topological conclusion for Haar measure. □

Example 7.1 (A nonprincipal critical action)**.** Let Γ be freely generated by a, b, c, d and take

$$F = \begin{pmatrix} 0 & a+b+c \\ a+b+d & 0 \end{pmatrix}, \qquad A = 3I_2 - F.$$

The dependency graph is acyclic because every nonempty path is labeled by a positive word of positive length. Here $B = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ has spectral radius one. Set $Q = F/3$. In each row of Q^n, exactly one entry has support, and this entry has 3^n distinct words of length n, each with coefficient 3^{-n}. The type sequence is determined by the initial type and the length, so distinct choices of letters at the successive steps give distinct words. Supports for different n are disjoint by word length. Consequently $H = \frac{1}{3}\sum_{n\geq 0} Q^n$ is square summable. The partial sums satisfy both inverse identities up to the remainder Q^{n+1}, which tends to zero in the square-summable norm. Hence H is a formal inverse. The augmentation of A is $3\left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$, which is singular. Augmentation is a homomorphism on the summable group algebra, so a summable inverse is impossible.

The dual module of this action is noncyclic. Indeed $\widehat{X}_A$ is the cokernel of right multiplication by A on the finite-support rows over $\mathbb{Z}\Gamma$, by (1.1) and Pontryagin duality. Regard $\mathbb{F}_3$ as a right $\mathbb{Z}\Gamma$-module by augmentation modulo 3. Tensoring the presentation with $\mathbb{F}_3$ gives the cokernel of the zero endomorphism of $\mathbb{F}_3^2$, since all entries of the augmentation of A vanish modulo 3. The result has dimension two. A cyclic module has tensor dimension

at most one, by right exactness applied to its one-generator presentation. Thus X_A is not algebraically conjugate to a scalar principal action. Theorem 1.1 gives its specified nine-symbol Bernoulli coding.

For a presentation of the same action whose labels have no common positive semigroup, take $u = \operatorname{diag}(e, a^2)$ and $A' = uAu^{-1}$. An edge label s from type i to type j becomes $u_i s u_j^{-1}$. Relabeling vertices by $(v, i) \mapsto (vu_i, i)$ preserves the dependency graph up to isomorphism. The new labels include a^{-1} and a^3. A semigroup containing both also contains e. Conjugating both A and H by u preserves the inverse identities and square summability, so the theorem applies directly to this presentation as well.

AI use disclosure

OpenAI's Codex was used to develop parts of the mathematical argument, audit the proofs and references, and draft and revise the manuscript.

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