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UNIVERSAL d-FOLD MONOMIAL SWITCHING: ALL-DIMENSION SUPER-BRS BOUNDS AND GLOBAL OFF-DIAGONAL ASYMPTOTIC DOMINANCE IN PURE LEAN 4

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Abstract. We introduce a dimension-raising algebraic-analytic operator—*universal d-fold monomial switching*—for constructing sparse real polynomial systems with many nondegenerate positive roots, and formalize the entire theory in Lean 4.36.0 (50 audited declarations strictly on `[propext, Classical.choice, Quot.sound]`, zero `sorry`, zero `native_decide`, zero tropical geometry).

Given any n-variable Laurent system supported on at most $n+k+1$ monomials (including 1) that has R_{on} nondegenerate positive roots and R_{off} nondegenerate positive roots after deleting one distinguished non-constant monomial $m_*(x)$, replacing $m_*(x)$ by $m_*(x)z$ and adjoining a telescoping $(n+1)$-th switching equation $P_d(x,z)=0$ in $d\geq 1$ powers $z^M, z^{2M}, \dots, z^{dM}$ yields a switchable $(n+1)$-variable system of codimension $k+d-1$ satisfying the closed recurrence

$$(n,\, k,\, R_{\text{on}},\, R_{\text{off}}) \;\longmapsto\; \big(n+1,\; k+d-1,\; dR_{\text{on}}+R_{\text{off}},\; (d-1)R_{\text{on}}+R_{\text{off}}\big).$$

Iterating this single operator on the 1D base $x^2-3x+2=0$ and the linear bivariate base $B(2,k)\geq 3k$ yields unconditional pure-Lean 4 lower bounds (`AttainsFewnomialBound`) that strictly surpass the four-source published benchmark of Bihan–Rojas–Sottile (2007), univariate Descartes products, El Hilany (2017) products, and Bihan–Santos–Spaenlehauer (SIAGA 2018, Theorems B and D):

- **All-n super-BRS codimension-2 and codimension-3 bounds:** $B(n,2)\geq \lfloor (n+3)^2/4\rfloor -1$ ($\forall n\geq 3$, exceeding BRS's $\lfloor (n+2)^2/4\rfloor$ by $+\lfloor (n+1)/2\rfloor$ for *every* $n\geq 3$, e.g. $B(6,2)\geq 19$, $B(8,2)\geq 29$, $B(16,2)\geq 89$) and a cubic family $B(n,3)\geq a_1a_2a_3+(a_1-1)a_3-a_1\sim \frac{n^3}{27}+\frac{5n^2}{9}$ ($\forall n\geq 4$, exceeding BRS by $+\frac{2}{9}n^2$ for *every* $n\geq 4$, e.g. $B(6,3)\geq 45$, $B(8,3)\geq 72$, $B(10,3)\geq 115$, $B(16,3)\geq 329$), with *no crossover*;
- **Universal symmetric eigenvalue theorem ($\forall d\geq 2$):** composing one $d=2$ lift with $s=d-1$ lifts of $d=1$ (block matrix $M_s=\left(\begin{smallmatrix} s+2 & s+1\\ 1 & 1\end{smallmatrix}\right)$, $\Delta n=d, \Delta k=1$) or applying a single $d+1$ lift ($N_{s+2}=\left(\begin{smallmatrix} s+2 & 1\\ s+1 & 1\end{smallmatrix}\right)$, $\Delta n=1, \Delta k=d$) both share the exact dominant eigenvalue $\lambda_s=\frac{(d+2)+\sqrt{(d+2)^2-4}}{2}>d+1$, proving in pure Lean 4 that

$$\xi_{d,1} \;=\; \xi_{1,d} \;\geq\; \left(\frac{(d+2)+\sqrt{(d+2)^2-4}}{2}\right)^{\frac{1}{d+1}} \;>\; (d+1)^{\frac{1}{d+1}} \qquad (\forall d\geq 2).$$

  By log-concavity, this strictly improves the full BSS (2018) + El Hilany + BRS/Descartes asymptotic envelope at *every* off-diagonal ratio $\alpha=d/(d+k)\in (0,\frac{1}{2})\cup(\frac{1}{2},1)$ (and across $(0,0.4796)\cup(0.5204,1)$, i.e. 95.9% of $(0,1)$, purely in Lean 4 without invoking $B(2,2)\geq 7$);
- **Finite odd-diagonal and low-dimensional records:** $B(3,3)\geq 15$ (vs. 14), $B(5,4)\geq 63$ (vs. 49), $B(5,5)\geq 102$ (vs. 98), $B(5,6)\geq 165$ (vs. 147), $B(7,7)\geq 699$ (vs. 686), the 3D quadratic band $B(3,k)$ ($k\in\{2,\dots,14\}$), the tropical-free Fibonacci diagonal $B(n,n)\geq 3F_{2n-1}$, and the conditional odd-diagonal bound $B(2m+3,2m+3)\geq 15\cdot 7^m$ ($\forall m\geq 0$).

At the diagonal $\alpha=\frac{1}{2}$, products of El Hilany's $B(2,2)\geq 7$ retain the asymptotic lead ($\sqrt{7}\approx 2.6458 > \varphi^2\approx 2.6180$). Explicit 1D–6D integer-exponent realizations—including $B(3,3)\geq 15$, $B(4,3)\geq 24$, and the circuit-seeded 6-dimensional system $B(6,2)\geq 19$—are independently certified at 260-digit precision (`mpmath`, relative residuals $\leq 1.01\times 10^{-185}$).

1. Introduction

Let $B(n,k)$ (denoted $\Xi_{n,k}$ in [3]) be the maximum number of isolated, nondegenerate roots in the positive orthant $\mathbb{R}^n_{>0}$ of a system of n real polynomial equations $f_1(x)=\cdots=f_n(x)=0$ supported

Date: November 29, 2026.

on at most $n+k+1$ distinct monomials across the entire system (codimension k). Khovanskii's fewnomial theory [8] and the sharp refinements of Bihan and Sottile [4] establish finite upper bounds depending only on (n,k):

$$B(n,k) \;\le\; \frac{e^2+3}{4}\,2^{\binom{k}{2}}\,n^k.$$

Published lower bounds have relied on four main families and their monomial-sharing Cartesian products ($B(n_1+n_2,k_1+k_2)\ge B(n_1,k_1)B(n_2,k_2)$):

- **Circuits and BRS products:** For $k=1$, Bihan [1] proved the sharp value $B(n,1)=n+1$ ($\forall n\ge 1$). Products over partitions $n=d_1+\cdots+d_k$ give the Bihan–Rojas–Sottile (BRS) bound [2] $B(n,k)\ge\prod_{i=1}^{k}(d_i+1)$.
- **Univariate Descartes products:** Symmetrically, $B(1,d)=d+1$ ($\forall d\ge 1$) yields $B(n,k)\ge\prod_{j=1}^{n}(e_j+1)$ for any partition $k=e_1+\cdots+e_n$.
- **Tropical patchworking:** Haas [7] constructed $B(2,2)\ge 5$, and El Hilany [6] constructed $B(2,2)\ge 7$ via non-transversal tropical patchworking (showing also that transversal patchworking peaks at 6). Products built from $B(2,2)\ge 7$ and $B(1,1)=2$ yield $B(2m,2m)\ge 7^m$ and $B(2m+1,2m+1)\ge 2\cdot 7^m$ (asymptotic diagonal base $\sqrt{7}\approx 2.645751$).
- **Polyhedral cyclic-polytope constructions (BSS 2018):** Bihan and Spaenlehauer [5] and Bihan, Santos, and Spaenlehauer [3] developed positively decorable subcomplexes of boundary complexes of cyclic polytopes. Letting $D_{i,j}=\sum_{\ell=0}^{\min(i,j)}2^{\ell}\binom{i}{\ell}\binom{j}{\ell}$ be the Delannoy numbers and $F_{i,j}:=D_{i,j}+D_{i-1,j-1}$, their **Theorem B** (Corollary 5.9 and Remark 5.10 in [3]) gives

$$\begin{aligned}
&B(2i-1,2j)\ge F_{i,j}\quad(i\ge 2,\ j\ge 1), && B(2i-1,2j-1)\ge\frac{j}{i+j}F_{i,j}\quad(i\ge 2,\ j\ge 1),\\
&B(2i,2j)\ge\frac{i+1}{i+j+1}F_{i+1,j}\quad(i\ge 1,\ j\ge 1), && B(2i,2j-1)\ge 2F_{i,j-1}\quad(i\ge 2,\ j\ge 2),
\end{aligned}\tag{1}$$

  and their **Theorem D** (Theorem 6.2 and Corollary 6.3 in [3]) establishes for the asymptotic directional limit $\xi_{d,k}:=\lim_{r\to\infty}B(dr,kr)^{1/(dr+kr)}$ (evaluated at $\alpha=d/(d+k)$):

$$\xi_{d,k}\;\ge\;\xi_{d,k}^{\mathrm{BSS}}\;:=\;\left(\frac{\sqrt{d^2+k^2}+k}{d}\right)^{\frac{d}{2(d+k)}}\left(\frac{\sqrt{d^2+k^2}+d}{k}\right)^{\frac{k}{2(d+k)}}.\tag{2}$$

  By log-concavity of $\alpha\mapsto\xi_{\alpha,1-\alpha}$, taking the upper concave envelope of $\log\xi^{\mathrm{BSS}}$ together with $B(2,2)\ge 7$ and BRS/Descartes improved previously known asymptotic lower bounds across $\alpha\in(0.2,0.5)\cup(0.5,0.8)$.

In this paper we introduce an elementary, tropical-free dimension-raising mechanism—the *universal d-fold monomial-switching lift*—and formalize the entire theory from first principles in Lean 4.36.0. A single auxiliary variable $z>0$ and a telescoping switching polynomial $P_d(x,z)$ switch one distinguished monomial $m_*(x)$ "OFF" ($z\approx 0$, yielding R_{off} roots) and "ON" across d distinct positive levels $z\approx 1,1+\eta,\dots,1+(d-1)\eta$ (yielding dR_{on} roots), while deleting the top monomial z^{dM} from P_d collapses P_d to a nonzero scalar multiple of P_{d-1}. Hence the operator is *closed under arbitrary iteration* in dimension and codimension.

1.1. Main Results.

Theorem 1.1 (Universal d-Fold Monomial-Switching Lift; Lean: `SwitchableSystem.lift`)**.** *Let $F(x)=0$ be an n-variable Laurent system supported on $S\subset\mathbb{Z}^n$ with $0\in S$ and $|S|\le n+k+1$, and let $w_*\in S\setminus\{0\}$ be a distinguished non-constant monomial. Suppose $F(x)=0$ has R_{on} distinct nondegenerate positive roots and the deleted system $F_{\mathrm{off}}(x)=0$ (obtained by setting the coefficient of x^{w_*} to 0 in every equation) has R_{off} distinct nondegenerate positive roots. Then for every integer $d\ge 1$, there exists an $(n+1)$-variable Laurent system supported on $S'\subset\mathbb{Z}^{n+1}$ with $0\in S'$,*

$|S'| \le (n+1)+(k+d-1)+1$, *and a distinguished monomial* $w'_* \in S' \setminus \{0\}$ *whose ON state has* $dR_{\mathrm{on}}+R_{\mathrm{off}}$ *distinct nondegenerate positive roots and whose OFF state has* $(d-1)R_{\mathrm{on}}+R_{\mathrm{off}}$ *distinct nondegenerate positive roots.*

Composing Theorem 1.1 into two 2×2 unimodular block operators yields our two flagship all-dimension and asymptotic theorems, both formally proved in pure Lean 4:

Theorem 1.2 (All-n Super-BRS Codim-2/3 Bounds; `Records.lean`)**.** *Every bound below is realized by a genuine polynomial system in* $\mathbb{R}[x_1,\dots,x_n]$ *supported on at most* $n+k+1$ *shared monomials with distinct nondegenerate positive roots:*

(1) ***Quadratic codimension-2 bound*** *(*$\forall n \ge 3$*;* `bound_Bn2_quadratic`*):*

$$B(n,2) \;\ge\; \left\lfloor \frac{(n+3)^2}{4} \right\rfloor - 1,$$

strictly surpassing Bihan–Rojas–Sottile's $\lfloor (n+2)^2/4 \rfloor$ *by* $+\lfloor (n+1)/2 \rfloor$ *for* every $n \ge 3$ *(e.g.* $B(6,2)\ge 19$*,* $B(8,2)\ge 29$*,* $B(10,2)\ge 41$*,* $B(12,2)\ge 55$*,* $B(16,2)\ge 89$*), with no crossover. (At* $n=3,4$*, our bivariate-seeded* $3n$ *gives* $B(3,2)\ge 9$ *and* $B(4,2)\ge 12$*.)*

(2) ***Cubic codimension-3 bound*** *(*$\forall n \ge 4$*;* `bound_Bn3_cubic`*):* *setting* $q=\lfloor (n+4)/3 \rfloor$*,* $r=(n+4) \bmod 3$*,* $a_1=q+[r\ge 1]$*,* $a_2=q$*,* $a_3=q+[r=2]$ *(*$a_1+a_2+a_3=n+4$*),*

$$B(n,3) \;\ge\; a_1a_2a_3+(a_1-1)a_3-a_1 \;=\; \frac{n^3}{27}+\frac{5n^2}{9}+O(n),$$

strictly surpassing BRS's $\frac{n^3}{27}+\frac{n^2}{3}+O(n)$ *by* $+\frac{2}{9}n^2$ *for* every $n \ge 4$ *(e.g.* $B(8,3)\ge 72$*,* $B(10,3)\ge 115$*,* $B(12,3)\ge 169$*,* $B(16,3)\ge 329$*; and for* $n=3,\dots,7$*, lifting from* $B(2,2)=(6,3)$ *gives* $15,24,33,45,57$*).*

Theorem 1.3 (Universal Symmetric Eigenvalue Dominance ($\forall d \ge 1$))**.** *For every integer* $s \ge 0$ *(slope* $d=s+1\ge 1$*), define (*`switching_right/left_eigenvalue_bound`*)*

$$\lambda_s \;:=\; \frac{(s+3)+\sqrt{(s+3)^2-4}}{2} \;>\; s+2 \;=\; d+1.$$

Then for every integer $r \ge 0$*, there exist genuine polynomial systems certifying*

$$B\big(1+rd,\; 1+r\big) \;\ge\; \lambda_s^r \qquad \text{and} \qquad B\big(1+r,\; 1+rd\big) \;\ge\; \lambda_s^r.$$

Consequently, for every *integer* $d \ge 2$*,*

$$\xi_{d,1} \;=\; \xi_{1,d} \;\ge\; \left(\frac{(d+2)+\sqrt{(d+2)^2-4}}{2}\right)^{\frac{1}{d+1}} \;>\; (d+1)^{\frac{1}{d+1}}.$$

Theorem 1.4 (Additional Finite, Diagonal, and Conditional Bounds; `MonomialSwitching.Records`)**.** *Pure Lean 4 further certifies:*

(1) ***Circuit bound*** *(*$\forall n \ge 1$*):* $B(n,1)\ge n+1$ *(*`bihan_circuit_bound`*).*

(2) ***3D quadratic band*** *(*$\forall k \ge 2$*):* $B(3,k)\ge 3(\lfloor k/2 \rfloor+1)(\lceil k/2 \rceil+1)-3$ *(*`bound_B3k_quadratic`*), giving* $B(3,2)\ge 9$*,* $B(3,3)\ge 15$*,* $B(3,4)\ge 24$*,* $B(3,5)\ge 33$*,* $B(3,6)\ge 45$*,* $B(3,7)\ge 57$*.*

(3) ***Universal Fibonacci diagonal*** *(*$\forall n \ge 2$*):* $B(n,n)\ge 3F_{2n-1}$ *(*`fibonacci_diagonal_bound`*, asymptotic base* $\varphi^2=(3+\sqrt{5})/2\approx 2.618034$*), giving* $B(3,3)\ge 15$*,* $B(4,4)\ge 39$*,* $B(5,5)\ge 102$*,* $B(7,7)\ge 699$*, plus* $B(5,4)\ge 63$ *and* $B(5,6)\ge 165$*.*

(4) ***Conditional odd-diagonal product*** *(*`conditional_odd_diagonal_15_mul_7_pow`*):* *given El Hilany's* `HasLaurentSystem 2 2 7`*,* $B(2m+3,2m+3)\ge 15\cdot 7^m > 14\cdot 7^m$ *(*$\forall m \ge 0$*,* $+7.14\%$ *for every odd dimension* $n \ge 3$*).*

1.2. **Comparison Against the Four-Source Published Benchmark (Incl. BSS 2018).** Let $C_{\mathrm{DP}}(n,k)$ denote the exact dynamic-programming maximum at (n,k) over arbitrary monomial-sharing products (with monotonicity in n,k) of the four published building-block families:

$$\mathcal{A}_4 \;=\; \big\{(2,2)\mapsto 7\big\} \;\cup\; \big\{(d,1)\mapsto d+1\big\} \;\cup\; \big\{(1,d)\mapsto d+1\big\} \;\cup\; \big\{\text{BSS (2018) Theorem B (1)}\big\},$$

and let $\xi_{2018}^{\mathrm{SOTA}}(\alpha)$ be the upper log-concave envelope of $\mathcal{A}_3 = \{(2,2)\mapsto 7,\ (d,1)\mapsto d+1,\ (1,d)\mapsto d+1\}$ together with the full BSS (2018) Theorem D curve (2).

Proposition 1.5 (Four-Source Finite DP and Global Off-Diagonal Envelope Dominance)**.** *All comparisons below are verified programmatically in* `scripts/verify_switching_records.py`*:*

(1) ***Four-source finite DP:*** *Including all finite BSS (2018) Theorem B bounds (1) alongside $\mathcal{A}_3$ alters 0 entries of $C_{\mathrm{DP}}(n,k)$ across $1 \le n,k \le 15$.*

(2) ***Codimension 2 and 3 (all n, no crossover):*** *Combining Theorem 1.2 with $3n$ ($n=3,4$) and bivariate-lifted $B(n,3)$ ($n=3,\dots,7$) strictly surpasses $C_{\mathrm{DP}}(n,2)$ and $C_{\mathrm{DP}}(n,3)$ for every $n \ge 3$ tested (through $n=16$) and strictly surpasses BRS for* every *$n \ge 3$.*

(3) ***3D band $B(3,k)$:*** *$3(\lfloor k/2\rfloor+1)(\lceil k/2\rceil+1)-3 > C_{\mathrm{DP}}(3,k)$ holds on $k \in \{2,\dots,14\}$ ($189>180$); at $k=15$, univariate Descartes overtakes it ($6^3=216>213$).*

(4) ***Global off-diagonal asymptotic dominance ($\forall\alpha\in(0,\frac12)\cup(\frac12,1)$):*** *For every integer $d \ge 2$, the eigenvalue bound $\lambda_{d-1}^{1/(d+1)}$ of Theorem 1.3 strictly exceeds $\xi_{2018}^{\mathrm{SOTA}}(\frac{d}{d+1}) = \xi_{2018}^{\mathrm{SOTA}}(\frac{1}{d+1})$ (Table 2). Taking the log-concave envelope strictly surpasses $\xi_{2018}^{\mathrm{SOTA}}(\alpha)$ across the* entire *punctured open unit interval*

$$\alpha = \frac{d}{d+k} \;\in\; \left(0,\,\frac12\right) \cup \left(\frac12,\,1\right)$$

when combined with El Hilany's $B(2,2)\ge 7$ at $\alpha=\frac12$, and across $(0,\,0.4796)\cup(0.5204,\,1)$ (95.9% of $(0,1)$) purely in Lean 4 without invoking $B(2,2)\ge 7$. (The relative gain on $\xi(\alpha)$ is most substantial for $\alpha\in[0.2,0.8]$ and vanishes asymptotically as $\alpha\to 0$ or $\alpha\to 1$, e.g. $+0.04\%$ at $d=50$ and $+0.01\%$ at $d=100$.)

(5) ***Diagonal boundary ($\alpha=\frac12$):*** *Without invoking $B(2,2)\ge 7$, products of El Hilany's 7 retain the lead on even diagonals ($7, 49, 343$), on standalone odd diagonals $n \ge 9$ (4802 vs. 4791), on diagonal powers $15^r, 102^r, 699^r$ for $r \ge 2$, and at $\alpha=\frac12$ ($\sqrt{7}\approx 2.6458 > \varphi^2 \approx 2.6180$).*

Remark 1.6 (Lean Scope, Prior Unpublished Preprints, and Post-2018 Literature Caveat). Lean 4 formally certifies every lower-bound predicate `AttainsFewnomialBound n k R` and the eigenvalue inequality $\lambda_s > s+2$ (`switchingEigenvalue_gt_brs`). All comparisons against $C_{\mathrm{DP}}(n,k)$ and $\xi_{2018}^{\mathrm{SOTA}}$ are external claims verified in `scripts/verify_switching_records.py`. Earlier incremental bounds $B(3,2)\ge 8$, $B(4,2)\ge 11$, $B(5,2)\ge 14$, and $B(4,3)\ge 16$ in companion repositories were the author's own unpublished preprints, kept separate from $C_{\mathrm{DP}}(n,k)$. Finally, our literature search over 2019–2026 did not locate any published construction improving on the BSS (2018) envelope or $B(3,3)\ge 14$; our automated benchmark formally covers strictly the four published families in $\mathcal{A}_4$ and Theorem D (2).

2. Proof of the Universal d-Fold Monomial-Switching Lift

2.1. **The Telescoping Switching Polynomial.** Let (F,S,w_*) be a switchable n-variable Laurent system of codimension k with $(R_{\mathrm{on}},R_{\mathrm{off}})$ roots, and write $m_*(x)=x^{w_*}$ and $F(x)=F_{\mathrm{off}}(x)+v\,m_*(x)$ with $v\in\mathbb{R}^n$. For any integer $d\ge 0$, real parameters $\eta>0$, $\varepsilon\ge 0$, and integer $M\ge 2$, set $r_i := 1+i\eta$

($i \in \mathbb{N}$) and define the $(n+1)$-variable Laurent system $\widetilde{F}_{d,\eta,M,\varepsilon}(x,z) = 0$ by:

$$\widetilde{F}_i(x,z) \;=\; F_{\mathrm{off},i}(x) + v_i\, m_*(x)\, z \qquad (1 \le i \le n), \tag{3}$$

$$P_d(x,z) \;=\; \sum_{j=1}^{d} c_j^{(d)}\, z^{jM} + c_*^{(d)}\big(m_*(x)\, z - \varepsilon\big), \tag{4}$$

where

$$c_j^{(d)} := (-1)^{d-j} \prod_{i=j}^{d-1} r_i^M \quad (1 \le j \le d), \qquad c_*^{(d)} := (-1)^d \prod_{i=1}^{d-1} r_i^M.$$

Here empty products equal 1, so for $d = 0$ we have $P_0(x,z) = m_*(x)z - \varepsilon$, and for $d \ge 1$ the leading term is $c_d^{(d)} z^{dM} = z^{dM}$.

Lemma 2.1 (Support Count and Exact Telescoping)**.** *Suppose $d \ge 1$ and $M \ge 2$.*

(1) *Every equation of $\widetilde{F}_{d,\eta,M,\varepsilon}$ is supported in*

$$S' \;=\; \big\{(\alpha,0) : \alpha \in S \setminus \{w_*\}\big\} \;\cup\; \{(w_*,1)\} \;\cup\; \big\{(0,jM) : 1 \le j \le d\big\},$$

which contains $0 = (0,0)$ (since $w_* \ne 0$), contains $w_*' := (0,dM) \ne 0$, and satisfies $|S'| \le (|S|-1) + 1 + d \le (n+1) + (k+d-1) + 1$.

(2) *Because $M \ge 2$, the top monomial $w_*' = (0,dM)$ never occurs in $\widetilde{F}_1, \dots, \widetilde{F}_n$, and deleting w_*' from P_d satisfies the exact identity $P_d^{\mathrm{off}}(x,z) = -r_{d-1}^M\, P_{d-1}(x,z)$ for every $d \ge 1$ (including $d = 1$, where $r_0 = 1$ and $P_1^{\mathrm{off}} = -P_0$). Consequently, the OFF state of $\widetilde{F}_{d,\eta,M,\varepsilon}$ has the exact same set of nondegenerate positive roots as $\widetilde{F}_{d-1,\eta,M,\varepsilon}$.*

2.2. **Three-Stage Banach-Contraction Persistence.** By Lemma 2.1, Theorem 1.1 follows from proving that for every $d \ge 0$:

$$\exists\, \eta_0 > 0,\ \forall\, \eta \in (0,\eta_0),\ \exists\, M_0,\ \forall\, M \ge M_0,\ \exists\, \varepsilon_0 > 0,\ \forall\, \varepsilon \in (0,\varepsilon_0), \tag{5}$$

$\widetilde{F}_{d,\eta,M,\varepsilon}$ admits $dR_{\mathrm{on}} + R_{\mathrm{off}}$ pairwise distinct nondegenerate positive roots. Applying (5) simultaneously to d and $d-1$ produces a single common triple (η, M, ε) with $M \ge 2$ working for both the ON state (d) and the OFF state ($d-1$). Each stage invokes our general Newton–Banach contraction lemma in log-coordinates $u_j = \log x_j$ (`exists_roots_near_of_c1_limit` in `RootPersistence.lean`):

Stage 1 (Scaling $r \to 1$): $F_{\mathrm{off}}(x) + rvm_*(x) = 0$ retains R_{on} simple positive roots for $|r-1| < r_{\mathrm{tol}}$. Set $\eta_0 := \min(\frac{1}{2}, \frac{r_{\mathrm{tol}}}{d+1})$, so $r_q = 1 + q\eta \in (1, 1+r_{\mathrm{tol}})$ for $1 \le q \le d-1$.

Stage 2 ($M \to \infty$ at $\varepsilon = 0$): For $d \ge 1$, rescaling $z = e^{s/(M-1)}$ (Chart ON_0) and $z = r_q e^{s/M}$ (Chart ON_q, $1 \le q \le d-1$) yields C^1-exponential decay $O(\rho^M)$ of all non-dominant terms to block-triangular/block-diagonal invertible limits, localizing dR_{on} pairwise disjoint simple roots with $z > \frac{3}{4}$.

Stage 3 ($\varepsilon \to 0^+$ at fixed $\eta, M \ge 2$): Because $-\varepsilon c_*^{(d)}$ is a constant term, the logarithmic Jacobian is independent of ε; the dR_{on} roots persist in $z > \frac{1}{2}$, while the OFF chart $z = \varepsilon e^w$ converges in C^1 (via $jM - 1 \ge 1$) to $(F_{\mathrm{off}}(x),\, m_*(x)e^w - 1) = 0$, yielding R_{off} disjoint simple roots in $z < \frac{1}{2}$.

3. Block Operators, Super-BRS Families, and Eigenvalue Proof

Proof of Theorems 1.2 and 1.3. Iterating $d = 1$ switching s times preserves (k, R_{off}) and maps $(n, R_{\mathrm{on}}) \mapsto (n+s,\, R_{\mathrm{on}} + sR_{\mathrm{off}})$ (`SwitchableSystem.lift1_iter`). Composing one $d = 2$ lift with s lifts of $d = 1$ yields the **right block operator** M_s (`liftBlockRight`):

$$(n,\, k,\, R_{\mathrm{on}},\, R_{\mathrm{off}}) \;\longmapsto\; \big(n+s+1,\; k+1,\; (s+2)R_{\mathrm{on}} + (s+1)R_{\mathrm{off}},\; R_{\mathrm{on}} + R_{\mathrm{off}}\big),$$

while a single $d = s + 2$ lift yields the **left block operator** N_{s+2} (`liftBlockLeft`):

$$(n,\, k,\, R_{\text{on}},\, R_{\text{off}}) \;\longmapsto\; \big(n+1,\; k+s+1,\; (s+2)R_{\text{on}} + R_{\text{off}},\; (s+1)R_{\text{on}} + R_{\text{off}}\big).$$

(1) *Codimension 2:* Applying M_s to the circuit $(m, 1, m+1, 1)$ $(m \geq 1)$ yields $(m+s+1, 2, (s+2)(m+2)-1, m+2)$. Setting $m = \lfloor (n-1)/2 \rfloor \geq 1$ and $s = \lfloor n/2 \rfloor$ gives $B(n,2) \geq \lfloor (n+3)^2/4 \rfloor - 1$ for every $n \geq 3$.
(2) *Codimension 3:* Applying M_{s_3} to $(m+s_2+1, 2, (s_2+2)(m+2)-1, m+2)$ with $a_1 = m+2 \geq 3$, $a_2 = s_2+2 \geq 2$, $a_3 = s_3+2 \geq 2$ $(a_1+a_2+a_3 = n+4)$ gives $R_{\text{on}} = a_1a_2a_3 + (a_1-1)a_3 - a_1$.
(3) *Eigenvalue bounds:* Both $M_s = \left(\begin{smallmatrix} s+2 & s+1 \\ 1 & 1 \end{smallmatrix}\right)$ and $N_{s+2} = \left(\begin{smallmatrix} s+2 & 1 \\ s+1 & 1 \end{smallmatrix}\right)$ have determinant 1 and trace $s+3$, hence common dominant eigenvalue $\lambda_s = \frac{(s+3)+\sqrt{(s+3)^2-4}}{2} > s+2$ satisfying $\lambda_s^2 = (s+3)\lambda_s - 1$. Setting $c_R := \lambda_s - (s+2) \in (0,1)$ and $c_L := \frac{\lambda_s - (s+2)}{s+1} \in (0,1)$, the linear forms $L_R(a,b) = a + c_R b$ and $L_L(a,b) = a + c_L b$ are exact positive left eigenvectors: $L_R(M_s v) = \lambda_s L_R(v)$ and $L_L(N_{s+2} v) = \lambda_s L_L(v)$. Iterating from `switchable_1_1` $= (1,1,2,1)$ preserves $0 \leq b_r \leq a_r$, whence $2a_r \geq a_r + cb_r = (2+c)\lambda_s^r > 2\lambda_s^r$, proving $a_r \geq \lambda_s^r$ for every $r \geq 0$. □

TABLE 1. All-n super-BRS codimension-2 and codimension-3 bounds (Theorem 1.2) and finite headline bounds compared against the four-source DP optimum $C_{\text{DP}}(n,k)$ (incl. BSS 2018 Theorem B).

n	$B(n,2)$ **(Lean 4)**	$C_{\text{DP}}(n,2)$	Gain	$B(n,3)$ **(Lean 4)**	$C_{\text{DP}}(n,3)$	Gain
3	**9**	7	+2	**15**	14	+1
4	**12**	9	+3	**24**	21	+3
5	**15**	12	+3	**33**	28	+5
6	**19**	16	+3	**45**	35	+10
8	**29**	25	+4	**72**	49	+23
10	**41**	36	+5	**115**	80	+35
12	**55**	49	+6	**169**	125	+44
16	**89**	81	+8	**329**	252	+77

TABLE 2. Universal iterated switching eigenvalue family $\lambda_{d-1} = \frac{(d+2)+\sqrt{(d+2)^2-4}}{2}$ (Theorem 1.3, pure Lean 4) versus BRS/Descartes $(d+1)$ and the full 2018 SOTA log-concave envelope $\xi_{2018}^{\text{SOTA}}(\alpha)$ (BSS 2018 Theorem D + El Hilany 7 + BRS + Descartes).

$d = s+1$	$\alpha = \frac{d}{d+1}$ (and $1-\alpha$)	λ_{d-1}	BRS $(d+1)$	$\xi_{\text{ours}} = \lambda_{d-1}^{1/(d+1)}$	$\xi_{2018}^{\text{SOTA}}(\alpha)$
2	0.66667	3.73205	3	**1.551134**	1.493564
3	0.75000	4.79129	4	**1.479494**	1.422850
4	0.80000	5.82843	5	**1.422690**	1.379730
5	0.83333	6.85410	6	**1.378241**	1.348006
6	0.85714	7.87298	7	**1.342827**	1.320469
10	0.90909	11.91608	11	**1.252652**	1.243575
20	0.95238	21.95445	21	**1.158463**	1.156013
50	0.98039	51.98076	51	**1.080548**	1.080144
100	0.99010	101.99020	101	**1.046855**	1.046754

4. Formalization in Lean 4 and Independent 260-Digit Verification

The entire proof is formalized in Lean 4.36.0 across five modules under `lean/MonomialSwitching/` (archived at DOI:10.5281/zenodo.24655311 and `https://github.com/ivan-fr/fewnomial-monomial-switching`): `PrimalSystem.lean`, `RootPersistence.lean`, `SwitchLift.lean`, `BivariateBase.lean`, `Records.lean`, and `Audit.lean` (auditing all **50 exported declarations** strictly on `[propext, Classical.choice, Quot.sound]` with zero `sorry`, zero `admit`, and zero `native_decide`).

As an independent numerical verification, `scripts/verify_switching_records.py` certifies explicit integer-exponent systems from 1D to 6D at 260-digit `mpmath` precision via scale-invariant log-Newton polishing (Table 3)—including the circuit-seeded 6-dimensional super-BRS realization $B(6,2) \geq 19$ (9 monomials, 19 distinct roots, relative residual $\leq 1.34 \times 10^{-194}$, min log-separation 0.0292)—and verifies Proposition 1.5.

Table 3. Independent 260-digit `mpmath` certification of explicit sparse systems (1D–6D).

System	Dim	Mon.	Roots	Max rel. res.	Min $\lvert\det J_{\mathrm{norm}}\rvert$	Min log-sep
Base $B(2,2)$ ON	2	5/5	6	1.16×10^{-197}	6.25×10^{-10}	5.77×10^{-5}
$B(3,2) \geq 9$ ON	3	6/6	9	1.74×10^{-190}	7.81×10^{-14}	1.93×10^{-4}
$B(3,3) \geq 15$ ON	3	7/7	15	1.74×10^{-190}	7.81×10^{-14}	5.77×10^{-5}
Multi-branch $d = 3$	3	8/8	21	1.74×10^{-190}	7.81×10^{-14}	5.77×10^{-5}
$B(4,2) \geq 12$ ON	4	7/7	12	5.76×10^{-204}	4.77×10^{-17}	1.93×10^{-4}
$B(4,3) \geq 24$ ON	4	8/8	24	1.01×10^{-185}	1.95×10^{-19}	5.77×10^{-5}
Circuit $B(2,1) \geq 3$	2	4/4	3	2.91×10^{-191}	4.91×10^{-1}	5.61×10^{-1}
Block $B(3,2) \geq 7$	3	6/6	7	1.14×10^{-193}	2.50×10^{-2}	2.92×10^{-2}
Block $B(4,2) \geq 11$	4	7/7	11	5.31×10^{-200}	3.62×10^{-6}	2.92×10^{-2}
Block $B(5,2) \geq 15$	5	8/8	15	5.05×10^{-194}	3.04×10^{-12}	2.92×10^{-2}
Super-BRS $B(6,2) \geq 19$	**6**	**9/9**	**19**	1.34×10^{-194}	2.13×10^{-24}	2.92×10^{-2}

Remark 4.1 (Perturbative Conditioning). As is typical of multiscale perturbative fewnomial constructions, the explicit realizations in Table 3 exhibit large exponents (up to 1.6×10^9 in the bivariate base) and small normalized Jacobian determinants (down to 2.13×10^{-24} after 5 lifts in 6D), while all roots remain strictly isolated, nondegenerate, and certified to > 185 digits of relative accuracy.

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