The Lih–Wang inequality in order five

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Abstract

We prove the Lih–Wang inequality for doubly stochastic matrices of order five. An exact rational certificate establishes the midpoint inequality. Two coefficient bounds then extend the result to the full parameter interval through the Bernstein representation of a cubic polynomial.

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1 Introduction

Let Ω_5 denote the set of nonnegative real 5×5 matrices whose row and column sums are one, and let J be the matrix whose entries are all $1/5$. For a matrix $X = (x_{ij})$, write

$$\operatorname{per}(X) = \sum_{\sigma \in S_5} \prod_{i=1}^{5} x_{i,\sigma(i)}.$$

We establish the following form of the Lih–Wang inequality.

Theorem 1.1. For every $A \in \Omega_5$ and $t \in [1/2, 1]$,

$$\begin{aligned} &\operatorname{per}(tJ + (1-t)A) \\ &\quad \le t\operatorname{per}(J) + (1-t)\operatorname{per}(A). \end{aligned}$$

The proof is given in Section 4. Section 2 establishes the coefficient bounds, and Section 3 proves the midpoint inequality by a finite weighted Gram representation.

2 Coefficient inequalities

Fix $A \in \Omega_5$, and put $B = A - J$ and $d = \operatorname{per}(J) = 24/625$. The row and column sums of B vanish. Hence

$$\begin{aligned} p(s) &= \operatorname{per}(J + sB) \\ &= d + c_2 s^2 + c_3 s^3 + c_4 s^4 + c_5 s^5. \end{aligned}$$

Let $\sigma_k(B)$ denote the sum of all permanents of $k \times k$ submatrices of B. For $2 \le k \le 5$, expansion in the entries of J gives

$$c_k = \frac{(5-k)!}{5^{5-k}} \sigma_k(B).$$

For $R = \sum_{ij} b_{ij}^2$ and $T = \sum_{ij} b_{ij}^3$, the zero row and column sums yield

$$\sigma_2(B) = \frac{R}{2}, \qquad \sigma_3(B) = \frac{2T}{3}.$$

These identities follow by inclusion–exclusion on repeated row and column indices; terms containing a singleton index vanish. Consequently,

$$c_2 = \frac{3}{125}R, \qquad c_3 = \frac{4}{75}T.$$

Lemma 2.1. The coefficients satisfy $c_2 \geq 0$ and $c_3 \geq -c_2/3$.

Proof. The first inequality follows from the expression for c_2. For a probability vector $x \in \mathbb{R}^5$, define

$$\varphi(x) = 20\sum_i x_i^3 - 9\sum_i x_i^2 + 1.$$

We first show that $\varphi(x) \geq 0$. Set $a = \min_i x_i$ and $\delta = 1 - 5a$. If $\delta = 0$, then x is uniform and $\varphi(x) = 0$. Otherwise, $x = a\mathbf{1} + \delta q$, where q is a probability vector with at most four nonzero coordinates. Write $q_2 = \sum_i q_i^2$ and $q_3 = \sum_i q_i^3$. The Cauchy–Schwarz inequality gives $q_2 \geq 1/4$ and $q_3 \geq q_2^2$. Thus

$$\begin{aligned}\varphi(q) &\geq 20q_2^2 - 9q_2 + 1\\ &= (5q_2 - 1)(4q_2 - 1) \geq 0.\end{aligned}$$

Substitution of $x = a\mathbf{1} + \delta q$ now gives

$$\begin{aligned}\varphi(x) = \delta^2\big[&(1-\delta)(3q_2 - 3/5)\\ &+ \delta\varphi(q)\big] \geq 0,\end{aligned}$$

since $0 \leq \delta \leq 1$ and $q_2 \geq 1/4$. Applying this inequality to the five rows of A and summing, we obtain

$$20T + 3R \geq 0.$$

The formulas for c_2 and c_3 give the second assertion. □

Lemma 2.2. The coefficients satisfy

$$c_4 \geq -\frac{3}{2}(c_2 + c_3).$$

Proof. Kopotun's Holens–Djoković inequality [1], specialized to order five and submatrix order four, states that

$$\sigma_4(A) \geq \frac{1}{5}\sigma_3(A),$$

where $\sigma_k(A)$ is the sum of all $k \times k$ subpermanents. The factor is $(5-4+1)^2/(5\cdot 4) = 1/5$. Comparing the coefficients of z^2 and z in

$$\operatorname{per}(A + zJ) = (1+z)^5 p\Big(\frac{1}{1+z}\Big)$$

gives

$$\begin{aligned}\sigma_3(A) &= \frac{24}{5} + \frac{75}{2}c_2 + \frac{25}{2}c_3,\\ \sigma_4(A) &= \frac{24}{25} + 15c_2 + 10c_3 + 5c_4.\end{aligned}$$

Substitution in Kopotun's inequality proves the claim. □

3 The midpoint inequality

Proposition 3.1. For every $A \in \Omega_5$,

$$\Delta(A) = \frac{\operatorname{per}(A) + d}{2} - \operatorname{per}\left(\frac{A+J}{2}\right) \geq 0.$$

Proof. We construct a nonnegative representation of Δ modulo the equal-line-sum relations. The rational coefficients, group operators and polynomial seeds used below are given in certificate_data.json.

3.1 Polynomial lift

For $X = (x_{ij})$, put $r_i = \sum_j x_{ij}$ and $h = (\sum_i r_i)/5$. Let $\mathcal{R}$ denote averaging over row permutations, column permutations and transposition. Define

$$H(X) = 50000\operatorname{per}(X) + 1920 \prod_i r_i - \sum_{\sigma \in S_5} \prod_i (5x_{i,\sigma(i)} + r_i),$$

$$\mathcal{T}(X) = \mathcal{R}\left[\frac{h}{100000} H(X)\right].$$

The polynomial $\mathcal{T}$ is homogeneous of degree six. At $A \in \Omega_5$, all r_i and h equal one, and the definition reduces to $\mathcal{T}(A) = \Delta(A)$.

The coefficients of H are obtained by expanding over the 5^5 row assignments and the 120 permutations. For a monomial $\prod_i x_{i,j_i}$, its coefficient is

$$1920 + 500001_{\{j_1,\ldots,j_5\}=\{1,\ldots,5\}} - \sum_{\sigma \in S_5} 6^{|\{i : j_i = \sigma(i)\}|}.$$

Multiplication by $\sum_{ij} x_{ij}$ and division by 500000, followed by group averaging, reconstruct $\mathcal{T}$ coefficient by coefficient.

3.2 Weighted Gram representation

Work in the coordinate ring in which all row and column sums have the common value h, and put $W_{ij} = 5x_{ij} - h$. The certificate consists of 66 weighted Gram blocks. Write S for their sum.

For each of the first 56 blocks, the specified group-algebra operator E satisfies

$$E = E^*, \qquad E^2 = \kappa E, \qquad \kappa > 0.$$

Every permutation in its support fixes the monomial weight w. If $z_i = E(\mathrm{seed}_i)$, the block is

$$\frac{1}{\kappa g} \mathcal{R}\left[w\, z^{\mathsf{T}} Q z\right],$$

where g is the positive integer normalization recorded in the data. Its coefficient map follows from

$$\mathcal{R}(w\,\mathrm{seed}_i\,E(\mathrm{seed}_j))$$
$$= \frac{1}{\kappa}\mathcal{R}(w\,E(\mathrm{seed}_i)E(\mathrm{seed}_j)).$$

Indeed, averaging is invariant under the permutations occurring in E; self-adjointness and $E^2 = \kappa E$ then give the identity. The operators are reconstructed from their YY^* descriptions, and their algebraic and stabilizer relations are checked over the integers.

Of these blocks, 24 have cubic seeds and weight one, 17 have quadratic seeds and weight $x_{11}x_{12}$, and 15 have quadratic seeds and weight $x_{11}x_{22}$. The other ten blocks have the form

$$\mathcal{R}\left[w\ell^{\mathsf{T}}Q\ell\right],$$

where w is a product of four distinct entries and ℓ is a specified vector of linear forms. Each of the 66 rational matrices Q has an exact LDL factorization with positive diagonal entries. There are 336 positive pivots in total. Since the weights are nonnegative on Ω_5, every block, and hence S, is nonnegative there.

3.3 Coefficient identity

Degree-six monomials form 156 orbits under $\mathcal{R}$. For an invariant polynomial f of degree six, let $[f]$ denote its vector of orbit coefficients. Let E_0 be the 156×315 integer matrix obtained by averaging products of the specified degree-five monomials with the first-row-sum minus the second-row-sum. Every column represents a polynomial vanishing on Ω_5.

The certificate supplies an integer matrix Π for which exact elimination gives

$$\Pi E_0 = 0,$$
$$\mathrm{rank}(E_0) = 130,$$
$$\mathrm{rank}(\Pi) = 26.$$

It follows that $\ker \Pi = \mathrm{col}(E_0)$. Reconstructing the coefficient map of each Gram block yields the 26 rational identities

$$\Pi[S] = \Pi[\mathcal{T}].$$

Thus $[S] - [\mathcal{T}]$ belongs to the column space of E_0, so $S - \mathcal{T}$ vanishes on Ω_5. Consequently,

$$\Delta(A) = \mathcal{T}(A) = S(A) \geq 0$$

for every $A \in \Omega_5$. This proves Proposition 3.1. $\square$

4 Proof of Theorem 1.1

Put $s = 1 - t$ and $P = \mathrm{per}(A) - d = c_2 + c_3 + c_4 + c_5$. The required difference factors as

$$(1-s)d + s\mathrm{per}(A) - p(s) = s(1-s)Q(s),$$

where

$$Q(s) = c_2 + (1+s)c_3$$
$$+ (1+s+s^2)c_4$$
$$+ (1+s+s^2+s^3)c_5.$$

It suffices to prove $Q(s) \geq 0$ for $0 \leq s \leq 1/2$. In the Bernstein basis with $u = 2s$,

$$Q(s) = \sum_{j=0}^{3} \beta_j \binom{3}{j} u^j (1-u)^{3-j},$$

where

$$\begin{aligned}
\beta_0 &= P, \\
\beta_1 &= \frac{7P - c_2}{6}, \\
\beta_2 &= \frac{17P - 5c_2 - c_3}{12}, \\
\beta_3 &= \frac{15P - 7c_2 - 3c_3 - c_4}{8}.
\end{aligned}$$

Define

$$\begin{aligned}
\rho &= c_3 + \frac{c_2}{3}, \\
\eta &= c_4 + \frac{3}{2}(c_2 + c_3), \\
\mu &= 15P - 7c_2 - 3c_3 - c_4.
\end{aligned}$$

Lemmas 2.1 and 2.2 give $\rho \geq 0$ and $\eta \geq 0$. Since $\Delta(A) = Q(1/2)/4$, Proposition 3.1 gives $\mu = 32\Delta(A) \geq 0$. The Bernstein coefficients can now be written as

$$\begin{aligned}
\beta_0 &= \frac{c_2}{3} + \frac{\mu + \eta}{15} + \frac{\rho}{10}, \\
\beta_1 &= \frac{2c_2}{9} + \frac{7(\mu + \eta)}{90} + \frac{7\rho}{60}, \\
\beta_2 &= \frac{c_2}{12} + \frac{17(\mu + \eta)}{180} + \frac{7\rho}{120}, \\
\beta_3 &= \frac{\mu}{8}.
\end{aligned}$$

Every term on the right is nonnegative. The Bernstein basis functions are nonnegative for $0 \leq u \leq 1$, so $Q(s) \geq 0$ throughout $[0, 1/2]$. Multiplication by $s(1-s)$ proves Theorem 1.1. □

5 Certificate verification

The data file specifies 1,939 upper-triangular rational Gram entries. The programs verify.py and verify_alternate.py reconstruct the polynomial target, the group operators, the relation matrix and the Gram coefficient maps, and check the coefficient identities and positive pivots. The program verify_bridge.py checks the polynomial identities and rational decompositions in Sections 2 and 4.

All three programs use the Python standard library. From the repository directory, run

```
python -I verify.py
python -I verify_alternate.py
python -I verify_bridge.py
```

The complete verification outputs and recorded running times are in verification/.

References

[1] K. A. Kopotun, *On some permanental conjectures*, Linear and Multilinear Algebra **36** (1994), 205–216. DOI: 10.1080/03081089408818290. Author's copy.