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PRIME CADENCES: IHARA ZETA, GRAPHENE, AND VOICE-LEADING HOMOLOGY ON THE TONNETZ

ERGENTICS

Abstract. The 24 major and minor triads, joined by the neo-Riemannian moves P, L, R, form a 3-regular graph X. We call the primes of its Ihara zeta function *prime cadences* and prove a number of exact statements about them. X is the honeycomb lattice on a 12-cell torus; its zeta zeros are the tight-binding band energies, equal to the moduli of the triad's discrete Fourier coefficients, and X is Ramanujan. Over $\mathbb{Q}$, ζ_X^{-1} factors into Weil polynomials over $\mathbb{F}_2$. The sequence of moving notes alone determines a cadence (*tune theorem*), and in every equal-tempered system with any step set the tune determines the cadence, apart from one explicit family. The voices' net winding is a torus homology coordinate, asymptotically Gaussian with variance $m/24$; its fourth cumulant is isotropic and hexagonal anisotropy first appears at order six. Twisting by it preserves the Riemann hypothesis on an explicit window. Three involutions of $PGL_2(\mathbb{Q}_2)$, one per move, generate a Schottky group whose Mumford curve C has genus 13, reduction graph X and automorphism group the transposition/inversion group. We compute its component group and gonality, find its order-12 factor in an explicit genus-5 curve over $\mathbb{F}_2$ (genus 5 is minimal), and show via a trace invariant and 2-adic edge-length obstructions that C admits no Čerednik–Drinfeld uniformisation except possibly by a group with torsion at a ramified place ($e_v = 2, f_v = 1$ forces Eichler mass ≥ 36) or an inert place (where inversion-free uniformisation at even $f_v \geq 2$ forces $\mathbb{Q}(\zeta_{2f_v+1})^+ \subset E$, sharp via an explicit non-arithmetic group $\Gamma_5 \subset PGL_2(\mathbb{Q}_4)$ with edge inversions). No Shimura curve with torsion-free uniformisation has reduction graph X over a totally real field of degree at most 5, or of degree at least 9 (sharpened unconditionally from $n \leq 14$ to $n \leq 8$ via a three-pincer Hilbert class field obstruction, which also eliminates sextic narrow-class ratios $u \in \{16, 32\}$). We separate known results from new ones.

1. Introduction

Euler drew a lattice of fifths and thirds in 1739 [20]; Riemann's harmonic dualism and its neo-Riemannian revival [30, 12, 13, 14] turned that lattice

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into a space of chords linked by parsimonious moves. Douthett and Steinbach [17] identified the chord graph as the hexagonal "chicken-wire torus", and Crans, Fiore and Satyendra [15] showed that the PLR-group is dihedral of order 24 and acts simply transitively. Tymoczko and collaborators [52, 9, 53] placed these structures inside continuous voice-leading orbifolds.

On the number-theoretic side, Ihara [24] attached a zeta function to discrete subgroups of PGL_2 over p-adic fields; Serre observed that it is the zeta function of a finite regular graph, and Sunada, Hashimoto and Bass [47, 22, 6] developed the general theory (see also [46, 49]). Ramanujan graphs [32], named after Ramanujan's conjecture on $\tau(n)$ [42] (proved by Deligne [16]), are those whose zeta functions satisfy the analogue of the Riemann hypothesis.

This note joins these lines. Throughout, a *cadence* is a closed walk in the chord graph, and a *prime cadence* is an Ihara prime. The results are of three kinds: identifications that are classical once stated (graphene, Coxeter groups); theorems that we believe are new (tune theorem, winding law, Riemann window, the arithmetic of the spectrum); and one refuted conjecture with its correction (Eisenstein ideals). All numerical claims were checked in exact or 50-digit arithmetic; Section 8 lists what is known.

2. Set-up

2.1. **Triads, moves and the chord graph.** Pitch classes are $\mathbb{Z}_{12}$. A major triad with root r is $\{r, r+4, r+7\}$ and a minor triad with root r is $\{r, r+3, r+7\}$. The moves are

$$\begin{aligned} P:&\ \{r, r+4, r+7\} \leftrightarrow \{r, r+3, r+7\},\\ L:&\ \{r, r+4, r+7\} \leftrightarrow \{r+4, r+7, r+11\},\\ R:&\ \{r, r+4, r+7\} \leftrightarrow \{r+9, r, r+4\}. \end{aligned}$$

Each move keeps two notes and moves the third by a semitone (P, L) or a whole tone (R). Let X be the graph with the 24 triads as vertices and an edge for each move; it is 3-regular with $V = 24$, $E = 36$, and bipartite (major/minor).

Proposition 2.1 ([15])**.** [lit.] *The group generated by P, L, R acting on triads is dihedral of order 24 and acts simply transitively. Hence X is the Cayley graph*

$$X \cong \mathrm{Cay}(D_{12}, \{P, L, R\}), \qquad P = L(LR)^4, \quad R = L(LR).$$

Let $W_3 = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 = \langle p, l, r \rangle$ be the universal Coxeter group of rank 3; its Cayley graph is the 3-regular tree T_3, which is the Bruhat–Tits tree of $PGL_2(\mathbb{Q}_2)$.

Corollary 2.2. [GB] *The kernel K of $W_3 \to D_{12}$ equals $\pi_1(X)$ and is free of rank $E - V + 1 = 13$. Non-backtracking walks in X are reduced words in W_3, and prime cadences are in bijection with the primitive conjugacy classes of*

K. Thus ζ_X is the Ihara–Selberg zeta function of the lattice K acting on T_3, Ihara's original setting with $q = 2$.

Proof. $X = K\backslash T_3$ since D_{12} acts simply transitively. A free action on a tree gives a free group of rank $E - V + 1$. Closed non-backtracking, tailless walks up to rotation correspond to cyclically reduced words up to cyclic permutation, i.e. conjugacy classes in K. □

2.2. **The Ihara zeta function.** For a finite graph, $\zeta_X(u) = \prod_{[\gamma]}(1-u^{\ell(\gamma)})^{-1}$ over prime cycles. For X $(q+1)$-regular,

$$\zeta_X(u)^{-1} = (1-u^2)^{E-V}\det(I - uA + qu^2 I) = \det(I - uB), \tag{2.1}$$

where A is the adjacency matrix and B is the $2E \times 2E$ Hashimoto (non-backtracking) matrix [24, 22, 6, 28]. X is *Ramanujan* if every eigenvalue $\lambda \neq \pm(q+1)$ of A satisfies $|\lambda| \leq 2\sqrt{q}$; equivalently all nontrivial poles lie on $|u| = q^{-1/2}$.

2.3. **Tonnetz coordinates.** Let $\mathbb{Z}^2 \ni (a, b)$ map to the pitch class $7a + 4b$ (fifths and major thirds). The kernel is the lattice Λ with basis $P_u = (4, -4)$, $P_v = (0, 3)$; $\mathbb{Z}^2/\Lambda \cong \mathbb{Z}_{12}$. With unit edges at 60° the Gram matrix of (P_u, P_v) is $G = \left(\begin{smallmatrix} 16 & -6 \\ -6 & 9 \end{smallmatrix}\right)$, $\det G = 108$. Triads are the triangles of the triangulated plane modulo Λ (the Tonnetz torus), and P, L, R are flips across the three edge directions. Dually, X is the honeycomb lattice modulo Λ (Figure 1).

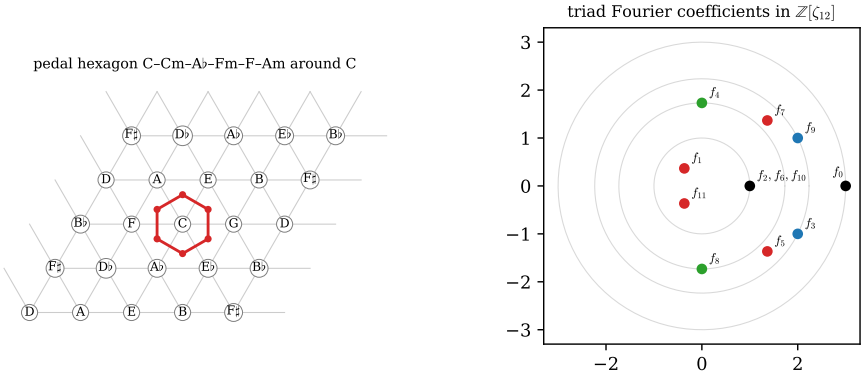

FIGURE 1. Left: the Tonnetz with the six triads containing C; the length-6 prime cadence around them (a held-note hexagon) is a hexagonal face of the honeycomb. Right: the twelve Fourier coefficients f_k of the major triad: units (red), the Gaussian primes $2 \mp i$ (blue), $\pm i\sqrt{3}$ (green).

3. Spectrum, graphene and arithmetic

Write $\zeta = e^{2\pi i/12}$, $\omega = \zeta^4$, and $f_k = \zeta^0 + \zeta^{4k} + \zeta^{7k}$, the k-th Fourier coefficient of the indicator of $\{0,4,7\}$.

Theorem 3.1 (spectrum)**.** [E→GB] *The adjacency eigenvalues of X are $\pm|f_k|$, $k \in \mathbb{Z}_{12}$. Explicitly: ± 3 ($k=0$); ± 1 three times each ($k = 2,6,10$); $\pm\sqrt{5}$ twice ($k = 3,9$); $\pm\sqrt{3}$ twice ($k = 4,8$); $\pm 2\cos\frac{\pi}{12}$ and $\pm 2\cos\frac{5\pi}{12}$ twice each ($k = 1,5,7,11$). In particular X is Ramanujan, with largest nontrivial eigenvalue $\sqrt{5} < 2\sqrt{2}$.*

Proof. Index triads by (type, root). From the major triad of root r, P, L, R lead to the minor triads of roots r, $r+4$, $r+9$. Thus $A = \left(\begin{smallmatrix} 0 & C \\ C^* & 0 \end{smallmatrix}\right)$ with C the circulant $I + S^4 + S^9$ (S the shift). Fourier diagonalisation gives $C \mapsto g_k = 1 + \zeta^{4k} + \zeta^{9k}$, and the eigenvalues of A are $\pm|g_k|$. Now $\{0,4,9\} = 9 + \{0,3,7\}$ and $\{0,3,7\} = 7 - \{0,4,7\}$; translation multiplies a Fourier coefficient by a root of unity and reflection conjugates it, so $|g_k| = |f_k|$. The values follow by direct evaluation. (This is an instance of the spectrum of a dihedral Cayley graph; cf. the discussion in [35].) The 1-dimensional representations of D_{12} give $k = 0, 6$ and the 2-dimensional ones ρ_j give $k = 7j$. □

Remark 3.2 (graphene). C is the Bloch matrix of nearest-neighbour hopping on the honeycomb with bonds from cell r to cells $r, r+4, r+9$; the spectrum is $\pm|\Delta(\mathbf{k})|$ (Wallace [56]) sampled at the 12 momenta of $\Lambda^*/\mathbb{Z}^{2*}$. The identification of these numbers with the triad Fourier magnitudes is in Marín Muñoz [35]. The Ramanujan property means that no allowed momentum lies in the cap $|\Delta| > 2\sqrt{2}$ around Γ (4.8% of the Brillouin zone). The note Tonnetz is the "square" of the honeycomb: A^2 restricted to major triads is $3I + \mathrm{Cay}(\mathbb{Z}_{12}, \{\pm 3, \pm 4, \pm 5\})$, a square-root-lattice relation in the sense of [2].

Theorem 3.3 (counting)**.** [GB] *Let $N_m = \operatorname{tr} B^m$ and $\pi(m)$ the number of prime cadences of length m. Then $N_m = \sum_{j=1}^{48} u_j^{-m} + 12(1 + (-1)^m)$ where u_j are the zeros of $\det(I - uA + 2u^2)$, $\pi(m) = 0$ for m odd, $\pi(m) = \frac{1}{m}\sum_{d\mid m} \mu(m/d) N_d$, and $\pi(m) \sim 2^{m+1}/m$ for even $m \to \infty$.*

Proof. Take logarithmic derivatives in (2.1); the factor $(1-u^2)^{12}$ contributes $12(1+(-1)^m)$. Möbius inversion of $N_m = \sum_{d\mid m} d\,\pi(d)$ gives $\pi(m)$. Bipartiteness kills odd m, and the poles $u = \pm\frac{1}{2}$ dominate, the others having $|u| \geq 2^{-1/2}$. □

Chebotarev for graph coverings [46, 22]: X is a $\mathbb{Z}_{12}$ (transposition) cover of the theta graph, so long cadences return to their starting chord with density $1/12$; we observed maximal deviation below 10^{-6} at $m = 40$.

3.1. The arithmetic of the spectrum.

Theorem 3.4. [E→GB] *The coefficients f_k lie in $\mathbb{Z}[\zeta]$, and*

(1) $f_3 = 2 - i$, $f_9 = 2 + i$: the split Gaussian primes above 5;

m	6	8	10	12	14	16	20	24	40
$\pi(m)$	32	78	240	576	2400	8196	105096	1394120	54975717792
$\pi(m)\, m/2^{m+1}$	1.500	1.219	1.172	0.844	1.025	1.000	1.002	0.997	1.000

TABLE 1. Prime cadences by length (exact, by transfer matrix; verified against (2.1)).

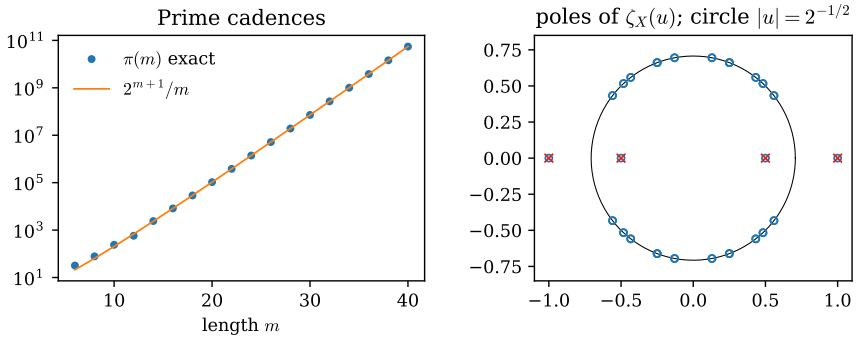

FIGURE 2. Left: prime cadence counts against the prime number theorem $2^{m+1}/m$. Right: poles of ζ_X; the 44 non-trivial ones lie on $|u| = 2^{-1/2}$ (Riemann hypothesis); crosses mark $u = \pm 1, \pm\frac{1}{2}$.

(2) *$f_4 = i\sqrt{3} = 1 + 2\omega$, $f_8 = -i\sqrt{3}$: the ramified prime of $\mathbb{Z}[\omega]$ above 3;*
(3) *for $k \in \{1, 5, 7, 11\}$, f_k is a unit of $\mathbb{Z}[\zeta]$, with $|f_5|^2 = |f_7|^2 = 2+\sqrt{3} = \varepsilon$, the fundamental unit of $\mathbb{Q}(\sqrt{3})$, and $|f_1|^2 = |f_{11}|^2 = \varepsilon^{-1}$; the minimal polynomial of f_1 is $x^4 - 2x^3 + 2x^2 + 2x + 1$.*

The Galois group $(\mathbb{Z}/12)^\times$ acts by $\sigma_a f_k = f_{ak}$. The element σ_{11} is induced by inversion $x \mapsto -x$, an automorphism of X exchanging major and minor. The elements σ_5, σ_7 are induced by the multiplications M_5, M_7; they are not automorphisms of X, but M_7 maps the 24 consonant triads bijectively onto the 24 transpositions and inversions of the set class $[0, 1, 4]$ and maps X isomorphically onto the graph of single-note parsimonious moves among them.

Proof. (1)–(2): $f_3 = 1 + \zeta^{12} + \zeta^{21} = 2 + \zeta^9 = 2 - i$, and $f_4 = 1 + \zeta^{16} + \zeta^{28} = 1 + 2\omega$ with $(1 + 2\omega)^2 = -3$. (3): the minimal polynomial of f_1 (computed exactly) has constant term 1, so $N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(f_1) = 1$ and f_1 is a unit; its conjugates are the f_k, $k \in (\mathbb{Z}/12)^\times$. Further $f_5 = 1 + \zeta^8 + \zeta^{11}$, so $|f_5|^2 = 3+2(\cos 240^\circ+\cos 330^\circ+\cos 90^\circ) = 2+\sqrt{3}$, and $|f_1|^2|f_5|^2 = N(f_1) = 1$. Galois: $\sigma_a(\zeta^{kx}) = \zeta^{akx}$. $M_7\{0, 4, 7\} = \{0, 4, 1\}$; the edge correspondence was checked on all 36 edges. □

So, read arithmetically, the eigenvalue $\sqrt{5}$ (the augmented-triad / diminished-seventh character of order 4) is the modulus of a Gaussian prime,

$\sqrt{3}$ (the order-3 character) the modulus of the Eisenstein prime, and the primitive characters give square roots of $\varepsilon^{\pm 1}$.

Theorem 3.5 (ζ_X over $\mathbb{Q}$)**.** [GB] *The characteristic polynomial of A is $(x-3)(x+3)(x-1)^3(x+1)^3(x^2-5)^2(x^2-3)^2(x^4-4x^2+1)^2$, and*

$$\zeta_X(u)^{-1} = (1-u^2)^{12}(1-u)(1-2u)(1+u)(1+2u)\,L_1^3\,L_1'^3\,L_2^2\,L_2'^2\,L_4^2,$$

with $L_1 = 1-u+2u^2$, $L_1' = 1+u+2u^2$, $L_2 = 1-u^2+4u^4$, $L_2' = 1+u^2+4u^4$, $L_4 = 1+4u^2+9u^4+16u^6+16u^8$. Each of $L_1, L_1', L_2, L_2', L_4$ is the reversed characteristic polynomial of Frobenius of a simple isogeny class of abelian varieties over $\mathbb{F}_2$ (Honda–Tate [48, 23]*); in LMFDB* [50] *labels: L_1: `1.2.ab`, the curve $y^2+xy = x^3+x^2+1$, Frobenius field $\mathbb{Q}(\sqrt{-7})$, which is the Euler factor at 2 of the CM newform `49.2.a.a`; L_1': `1.2.b`; L_2: `2.2.a_ab`, a Jacobian with endomorphism algebra $\mathbb{Q}(\sqrt{-3},\sqrt{5})$; L_2': `2.2.a_b`, endomorphism algebra $\mathbb{Q}(\sqrt{3},\sqrt{-5})$; L_4: `4.2.a_e_a_j`, a simple fourfold that is not a Jacobian. The factor $(1-u)(1-2u)$ is the zeta denominator of $\mathbb{P}^1_{\mathbb{F}_2}$.*

Proof. For each eigenvalue λ the Ihara factor is $1-\lambda u+2u^2$; multiply over Galois orbits. A polynomial $1-\lambda u+2u^2$ with $|\lambda| \le 2\sqrt{2}$ has reciprocal roots of modulus $\sqrt{2}$, so each orbit product is a Weil 2-polynomial; irreducibility over $\mathbb{Q}$ was checked, and the labels and endomorphism data were read from LMFDB. □

Remark 3.6. The Ramanujan property of X is here literally the Hasse–Weil bound for these isogeny classes; the total dimension $3{\cdot}1+3{\cdot}1+2{\cdot}2+2{\cdot}2+2{\cdot}4 = 22$ equals the number of nontrivial eigenvalues. This is a factor-by-factor identification; we do not claim a variety over $\mathbb{F}_2$ whose zeta function is ζ_X. Over $\mathbb{F}_4$ both surfaces become squares of elliptic curves with CM by $\mathbb{Q}(\sqrt{-15})$, and $\mathbb{Q}(\sqrt{-3},\sqrt{5})$ is the Hilbert class field of $\mathbb{Q}(\sqrt{-15})$.

3.2. **Eisenstein ideals: refuted and corrected.** It is tempting to view the Tonnetz as $\mathbb{Z}[\omega]$ modulo an ideal. This fails.

Proposition 3.7. [E→GB] *No additive surjection $\mathbb{Z}[\omega] \to \mathbb{Z}_{12}$ sending 1 and $1+\omega$ to a fifth and a major third (any of the 12 assignments of the Tonnetz axes) has an ideal as kernel; the multiplier ring of each kernel is $\mathbb{Z}+12\mathbb{Z}[\omega]$. Moreover $\mathbb{Z}_{12}$ is not a quotient ring of $\mathbb{Z}[\omega]$, $\mathbb{Z}[i]$ or $\mathbb{Z}[\zeta]$.*

Proof. A ring quotient $\mathbb{Z}[\omega] \to \mathbb{Z}_{12}$ needs a root of x^2+x+1 modulo 4, and there is none since 2 is inert; similarly x^2+1 has no root mod 4 and $\Phi_{12} = x^4-x^2+1$ has no root mod 3 or mod 4. The multiplier rings were computed exactly for all 12 maps. For example, with $\psi(a+b\omega) = 7a+9b$, $\psi(-12-8\omega) = 0$ but $\psi(\omega(-12-8\omega)) = 8$. □

Theorem 3.8. [E→GB] *Let $\mathcal{O} = \mathbb{Z}[\sqrt{-3}]$, the order of conductor 2 in $\mathbb{Q}(\sqrt{-3})$. The ring map $\psi(x+y\sqrt{-3}) = x+3y$ induces*

$$\mathbb{Z}_{12} \;\cong\; \mathcal{O}/(\sqrt{-3}\,(1+\sqrt{-3})),$$

a principal ideal of norm 12, under which the minor third is $\sqrt{-3}$*, the major third* $1+\sqrt{-3}$*, the fifth* $1+2\sqrt{-3}$ *and the semitone* 1*. Multiplication by* $\sqrt{-3}$ *acts on pitch classes as* $x \mapsto 3x$*. Among the orders* $\mathbb{Z}[f\omega]$*, exactly those with* f *even admit* $\mathbb{Z}_{12}$ *as a quotient ring.*

Proof. ψ is a ring map since $\psi(\sqrt{-3})^2 = 9 \equiv -3$. Its kernel contains $3-\sqrt{-3} = -\sqrt{-3}(1+\sqrt{-3})$, of norm 12, and ψ is onto, so this is the kernel. The basis $\{\sqrt{-3}, 1+\sqrt{-3}\}$ is unimodular, so the triangulation by minor and major thirds is preserved. For $\mathbb{Z}[f\omega]$ one needs a root of x^2+fx+f^2 (the minimal polynomial of $f\omega$) mod 4 and mod 3; mod 4 this exists iff f is even. □

Remark 3.9. The torus $\mathbb{C}/\mathcal{O}$ has $j = 54000$. It is not the equilateral (musical) torus: there the triad sides are equal, the period ratio is $\tau = -\frac{3}{8} + \frac{3\sqrt{3}}{8}i$ with minimal form $(9, -6, 13)$ of discriminant -432 and class number 6, and the multiplier ring is the order of conductor 12. So the *geometry* of harmony has conductor 12 while its *arithmetic* lives in conductor 2. The obstruction at 2 vanishes because $\mathcal{O}/(2) \cong \mathbb{F}_2[\epsilon]/(\epsilon^2)$ is non-reduced; the major third $1+\sqrt{-3}$ is the factor in $4 = (1+\sqrt{-3})(1-\sqrt{-3})$.

4. Melody: the tune theorem

Along a non-backtracking cadence $c = (T_0, T_1, \ldots, T_m = T_0)$ exactly one note changes at each step; let n_i be the note that arrives at step i. The *tune* of c is the cyclic word $(n_1, \ldots, n_m)$.

Lemma 4.1. [E→GB] *(L1) At each triad the three moves change three different notes. (L2) Only the backtracking move changes the note that has just arrived. (L3) No major or minor triad contains two notes 1 or 2 semitones apart, while every move changes a note by 1 or 2 semitones.*

Proof. Exhaustive check over the 72 directed edges; (L3) since the intervals of $\{0,4,7\}$ and $\{0,3,7\}$ are $3,4,5$ up to inversion. □

Theorem 4.2 (tune theorem)**.** [E→GB] *The tune of a non-backtracking cadence determines the cadence. Consequently the map from prime cadences to tunes is injective and preserves primitivity: no prime cadence has a periodic tune.*

Proof. By (L2), $T_i \ni n_{i-1}, n_i$. If the voices moved at i and $i-2$ differ, then $T_i = \{n_{i-2}, n_{i-1}, n_i\}$ is a consonant triad. If they coincide, then $n_{i-2} \to n_i$ is a single move of one voice and by (L3) $\{n_{i-2}, n_{i-1}, n_i\}$ is not consonant. Hence the tune determines which steps move the same voice, hence the partition of steps into voices, hence each $T_i = \{n_i, n_{i-1}, \text{latest arrival of the third voice}\}$. If only two voices ever move (the 24 held-note hexagons), the held note is the unique note completing every consecutive pair to a triad. The decoding is local and commutes with rotation, so the period of the tune equals the period of the cadence. □

We verified the decoder on all 174,864 rooted primes of length ≤ 16 (131,136 of them have length 16; the earlier version quoted only this last figure, a slip found during independent verification [G4N]).

4.1. **Other tuning systems.** Let $N \geq 5$, let $S \subset \{1, \ldots, \lfloor N/2 \rfloor\}$ be a set of *step* sizes, and let $\mathcal{C}$ be a transposition (or transposition/inversion) class of three-note chords in $\mathbb{Z}_N$ such that no two notes of a chord are a step apart. Join two chords when they differ in one note and that note moves by a step. The 12-tone case is $S = \{1, 2\}$ with the consonant triads.

Theorem 4.3 (general reduction)**.** [E→GB] *In any such system, the tune of a closed non-backtracking walk determines the full path of every voice that moves at least once: the note n_j arriving at step j leaves at the first later step i for which $n_i - n_j \in \pm S$. In particular the tune determines the walk whenever all three voices move, and the theorem can only fail through the held notes of walks in which one voice never moves.*

Proof. At step i the departing note d lies in T_{i-1} and $n_i - d \in \pm S$. Any other $x \in T_{i-1} \setminus \{d\}$ lies in T_i together with n_i, so $n_i - x \notin \pm S$. Thus d is the unique note of T_{i-1} at step distance from n_i, and a note present in the chord stays until the first arrival at step distance from it. □

It remains to identify held notes. The *link* $\mathrm{Lk}(b)$ of a note b is the graph on notes in which $u \sim w$ when $\{b, u, w\} \in \mathcal{C}$.

Theorem 4.4 (held notes)**.** [E→GB; independently E→G4N & E; X] *Let N, S and $\mathcal{C}$ be as in Theorem 4.3.*

- *(a) If exactly two voices move, a single step determines the held note. Hence the tune determines the walk.*
- *(b) If exactly one voice moves, the tune determines the walk unless $8 \mid N$, $\mathcal{C}$ is the transposition/inversion class of $\{0, \frac{N}{8}, \frac{N}{2}\}$, and $\frac{N}{4} \in S$. In that case take the held pairs $\{0, \frac{N}{2}\}$ and $\{\frac{N}{4}, \frac{3N}{4}\}$, with the moving voice running through $\frac{N}{8}, \frac{7N}{8}, \frac{5N}{8}, \frac{3N}{8}$. They give two different closed non-backtracking walks with the same tune.*
- *(c) Hence the tune of a closed non-backtracking walk determines the walk for every N and every S outside the family in (b). For $S \subseteq \{1, 2\}$ the only exception is $N = 8$, $S = \{2\}$ with the class of $\{0, 1, 4\}$. For $S = \{1\}$ and $S = \{1, 2\}$ there is none.*

Proof. By Theorem 4.3, two walks with the same tune move the same voices along the same paths, so they can differ only in what is held. Let the chord have intervals x, y, z with $x + y + z = N$. The chords through b are at most three transpositions and three inversions. If x, y, z are distinct, the six edges of $\mathrm{Lk}(b)$ form the hexagon $b{+}x$, $b{-}z$, $b{+}y$, $b{-}x$, $b{+}z$, $b{-}y$. If one interval is $N/2$, two opposite vertices coincide and the hexagon becomes two triangles sharing a vertex. If two intervals are equal, $\mathrm{Lk}(b)$ has at most three edges. In no case does $\mathrm{Lk}(b)$ contain a 4-cycle. (a) Suppose held notes $h \neq h'$ both fit a step that moves a to a' and keeps b. Then $\{h, a, b\}$, $\{h, a', b\}$, $\{h', a, b\}$ and

$\{h', a', b\}$ are chords, so h, a, h', a' is a 4-cycle in $\mathrm{Lk}(b)$, which is impossible. (b) The moving voice visits $m \geq 3$ distinct notes, and each of them completes the held pair. A pair with difference d fills an oriented interval $\pm x, \pm y, \pm z$ of the chord. At most one of x, y, z is $\equiv d$ and at most one is $\equiv -d$, so a pair has at most two completions unless $2d \equiv 0$. Hence some interval, say y, equals $N/2$. The held pair is then $Q = \{h, h + \frac{N}{2}\}$ with completions $W_Q = h + \{\pm x\} + H$, where $H = \{0, \frac{N}{2}\}$ and $x + z = \frac{N}{2}$. For a second such pair $Q' = Q + t$, the intersection $W_Q \cap W_{Q'}$ is a union of H-cosets with at least three points, so $W_{Q'} = W_Q$. Since $t \notin H$, this gives $t \equiv \pm 2x$ and $4x \equiv 0 \pmod{N/2}$, so $x \in \{\frac{N}{8}, \frac{3N}{8}\}$ ($x = \frac{N}{4}$ would give $x = z$). Differences in H are not steps, because $N/2$ is an interval. So the walk uses the four edges between $x + H$ and $-x + H$, whose differences are $\pm\frac{N}{4}$, and therefore $\frac{N}{4} \in S$. Conversely, the walks in (b) are valid. (c) This follows from (a) and (b): $\frac{N}{4} \in S \subseteq \{1, 2\}$ with $\frac{N}{8} \notin S$ forces $N = 8$ and $S = \{2\}$. □

The earlier exact checks agree with Theorem 4.4: 6028 systems with $N \leq 72$ and $S = \{1\}$ or $\{1, 2\}$, by two implementations, and brute force for $N \leq 24$. We also confirmed the link lemma for all $N \leq 60$, and part (b) against a brute-force search over all step sets for $N \leq 26$. Finally, we checked the whole statement against all closed non-backtracking walks of length ≤ 8, for all step sets with $N \leq 16$; every tune collision found lies in the family of (b). Non-backtracking is needed: $C \to Cm \to C$ and $E \to A\flat m \to E$ both have the tune $(E\flat, E)$ up to rotation.

Corollary 4.5. [G4N; X] *In the setting of Theorem 4.3, the tune determines the walk whenever $\mathcal{C}$ is a transposition class, or $8 \nmid N$, or S is closed under halving ($2d \in S \Rightarrow d \in S$). In particular it does so for every $S = \{1, \ldots, k\}$.*

Proof. In the exceptional family of Theorem 4.4(b), halving $\frac{N}{4} \in S$ would put $\frac{N}{8}$ in S, but $\frac{N}{8}$ is the distance between two notes of the chord $\{0, \frac{N}{8}, \frac{N}{2}\}$, which a step may not be. □

Independent confirmation [E→G4N & E; X]. Working independently from the `v0.1.0` baseline under Ergentics' direction, Gemini 4 Neon reached the same classification: it proves that two notes share at most two common link edges (disposing of walks with two moving voices for every $N \geq 3$) and obtains the family of Theorem 4.4(b) in the form $N = 8m$, chord class $\{0, m, 4m\}$, $2m \in S$, with minimal counterexample $N = 8$, $S = \{2\}$, chord $\{0, 1, 4\}$ (walks holding $\{0, 4\}$ and $\{2, 6\}$ share the cyclic tune $(3, 5, 7, 1)$). These statements are machine-checked in Lean 4 (without Mathlib), and an exhaustive sweep over the maximal step set of every chord class with $N \leq 72$ finds exactly nine collisions, one for each $N = 8, 16, \ldots, 72$. Grok Bot's independent re-check (`code/xcheck/`) reproduces the link bound, the nine collisions, the absence of any other collision for all step sets with $N \leq 24$, and the $N = 8$ example. A third independent clean-room derivation by Ergentics (`cleanroom/prime_cadences_resolution/paper.typ`) obtains the identical criterion, verifies

all 13,382 valid systems for $N \le 18$ (17 predicted, 17 observed, 0 mismatches), and records the Euclidean chord-length defect $D(a) = 2 - 2\cos(\pi a) = 4\sin^2(\pi a/2) > 0$ on $(0, \frac{1}{4})$ separating non-diameter reflection chords $\{0, a, \frac{1}{2} - a\}$ from diameter chords on $\mathbb{R}/\mathbb{Z}$.

5. Voice threads and the winding law

Track each voice continuously (in $\mathbb{Z}$, not $\mathbb{Z}_{12}$). Around a closed cadence the total displacement of the three voices is $12W$ for an integer W, the *winding*. A cadence also has a homology class $(u, v) \in H_1(\text{torus}) \cong \mathbb{Z}^2$ with respect to P_u, P_v.

Theorem 5.1 ($W = -u$)**.** [GB] *Every flip moves one vertex of the triangle by*

$$\delta \in \{\pm(1,-2), \pm(1,1), \pm(-2,1)\},$$

all in the coset lattice $\{2a + b \equiv 0 \bmod 3\}$*, and the voice moves by* $-\delta_a$ *semitones. Hence each voice stays in a fixed coset and its continuous pitch is* $-a + const$*, and* $W = -u$*. The u-period permutes the three voices cyclically, the v-period fixes them. In particular the voices return home exactly when* $u = 0$*, and every voice moves by an octave exactly when* $u = \pm 3$*.*

Proof. For a flip, the moving vertex is reflected across the opposite edge; in Tonnetz coordinates the six possible displacements are as listed (checked on all 72 directed moves), and the pitch change $7\delta_a + 4\delta_b$ reduces to $-\delta_a$ in $\{\pm 1, \pm 2\}$, e.g. $\delta = (1, -2)$ gives $7 - 8 = -1$. Summing along a closed cadence the a-displacements total the a-component of $uP_u + vP_v$, and since the triangle returns to itself translated by $uP_u + vP_v$, the three vertex tracks together have a-displacement $3 \cdot 4u = 12u$; so the total voice displacement is $-12u$. The u-period $(4, -4)$ changes $2a + b$ by $4 \equiv 1 \pmod 3$, cycling the cosets; $(0, 3)$ changes it by $3 \equiv 0$. □

The first prime cadences with $u = \pm 3$ occur at length 24, for example the alternating cadence $(RL)^{12}$ through all 24 triads, along which each voice rises by exactly one octave. The voice colouring of honeycomb edges is the Kekulé ($\sqrt{3} \times \sqrt{3}$) colouring; it lives on the triple cover W mod 3, not on the torus.

5.1. **Flux.** For $\varphi \in \mathbb{R}$, let A_φ be the adjacency operator twisted by $e^{i\theta s}$ on a move that shifts a voice by s semitones, $\theta = \varphi/12$. On closed walks the total phase is $e^{i\varphi W}$, so A_φ is the twist by the unitary character $\chi_\varphi : \pi_1(X) \to U(1)$, $\gamma \mapsto e^{i\varphi W(\gamma)}$; for $\varphi = 2\pi j/n$ it is an Artin–Ihara L-function of the $\mathbb{Z}_n$ winding cover. The Bass identity holds for unitary twists, $\det(I - uB_\varphi) = (1 - u^2)^{12} \det(I - uA_\varphi + 2u^2)$.

Theorem 5.2 (twisted spectrum)**.** [GB] *The eigenvalues of* A_φ *are* $\pm\lambda_k(\varphi)$ *with* $\lambda_k(\varphi) = |1 + \zeta^{4k} + e^{i\varphi/4}\zeta^{9k}|$*,* $k \in \mathbb{Z}_{12}$*.*

Proof. From the major triad of root r, P and L shift a voice by -1 and R by $+2$. So the twisted block is $C_\theta = e^{-i\theta}(I + S^4) + e^{2i\theta}S^9$, with Fourier symbol $e^{-i\theta}(1 + \zeta^{4k} + e^{3i\theta}\zeta^{9k})$ and $3\theta = \varphi/4$. □

Theorem 5.3 (Riemann window)**.** [E→GB] *For $\varphi \in [0, 2\pi)$, all eigenvalues of A_φ satisfy $|\lambda| \le 2\sqrt{2}$ (the Riemann hypothesis for $L(u, \chi_\varphi)$) if and only if*

$$\varphi \in \left[\, 4\arccos\tfrac{3}{4},\ 2\pi - 4\arccos\tfrac{3}{4} \,\right] \approx [2.8909,\ 3.3922].$$

Consequently, among the cyclic winding covers X_n ($\mathbb{Z}_n$-covers through W mod n), X_n is Ramanujan iff $n \le 2$. The double cover X_2 (48 triads, voice parity) is a Ramanujan 2-lift in the sense of Bilu–Linial [8]*, with top new eigenvalue $\sqrt{5 + 2\sqrt{2}} \approx 2.798$.*

Proof. If $4k \not\equiv 0 \pmod{12}$ then $|1 + \zeta^{4k}| = 1$ and $\lambda_k \le 2 < 2\sqrt{2}$. If $k \in \{0, 3, 6, 9\}$ then $\zeta^{4k} = 1$ and $\lambda_k^2 = |2 + e^{i(\varphi/4 + j\pi/2)}|^2 = 5 + 4\cos(\varphi/4 + j\pi/2)$, $j = 0, 3, 2, 1$. The condition $\lambda^2 \le 8$ reads $\cos(\cdot) \le \frac{3}{4}$. For $\varphi/4 \in [0, \pi/2)$ the largest of the four cosines is $\max(\cos\frac{\varphi}{4}, \sin\frac{\varphi}{4})$, so the condition is $\frac{\varphi}{4} \in [\arccos\frac{3}{4}, \arcsin\frac{3}{4}]$, and $4\arcsin\frac{3}{4} = 2\pi - 4\arccos\frac{3}{4}$. The cover X_n is Ramanujan iff all characters $\varphi = 2\pi j/n$, $1 \le j < n$, lie in the window; for $n \ge 3$, $j = 1$ gives $\varphi \le 2\pi/3 < 2.89$. For $n = 2$, $\varphi = \pi$ gives $\max \lambda^2 = 5 + 2\sqrt{2}$. □

Remark 5.4. At $\varphi = 2\pi/3$ the flux lands on a Dirac point ($\lambda_k = 0$ for some k), which the untwisted torus lacks. Ramanujan signings of X are not rare: of the 8191 nontrivial classes of signings (2^{13} characters of π_1 to ± 1), 7675 are Ramanujan, the best with top eigenvalue $1 + \sqrt{2}$; the voice-parity signing ranks 7372 of 8192. Thus the musical double cover is canonical but not extremal, and the existence theorem of Marcus–Spielman–Srivastava [34] adds nothing specific here.

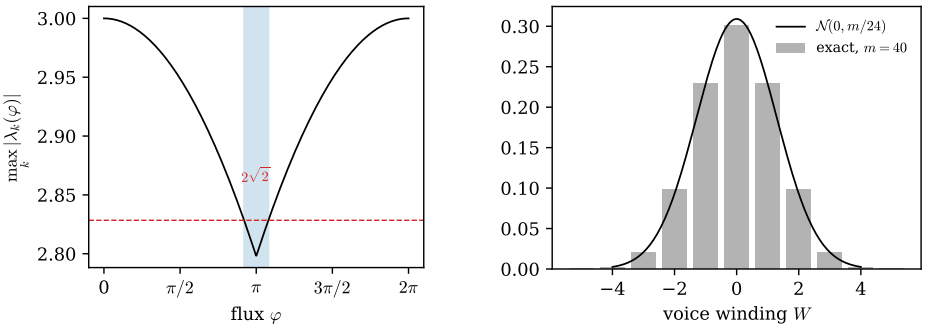

FIGURE 3. Left: the largest twisted eigenvalue against flux; the shaded band is the Riemann window of Theorem 5.3. Right: distribution of the voice winding W over all 5.5×10^{10} prime cadences of length 40, against $\mathcal{N}(0, 40/24)$.

6. The homology law

Theorem 6.1 (winding law)**.** [GB] *Let $\pi(m, w)$ be the number of prime cadences of length m (even) with winding $W = w$. Then, uniformly for $w = O(\sqrt{m})$,*

$$\pi(m, w) \sim \frac{2^{m+1}}{m}\sqrt{\frac{12}{\pi m}}\, e^{-12w^2/m},$$

so W is asymptotically Gaussian with variance $m/24$. More generally the homology class (u, v) has covariance $\frac{m}{2}G^{-1} = m\left(\begin{smallmatrix} 1/24 & 1/36 \\ 1/36 & 2/27 \end{smallmatrix}\right)$: isotropic Brownian motion on the Tonnetz plane with unit mean-square displacement per step.

Proof sketch. By Theorems 5.2 and (2.1) for B_φ, $\operatorname{tr} B_\varphi^m = \sum_k (\alpha_k^m + \beta_k^m) + 12(1 + (-1)^m)$, where α, β solve $x^2 - \lambda x + 2 = 0$. The dominant branch near $\varphi = 0$ is $\lambda(\varphi) = |2 + e^{i\varphi/4}| = \sqrt{5 + 4\cos(\varphi/4)}$, with $\rho(\varphi) = \frac{1}{2}(\lambda + \sqrt{\lambda^2 - 8})$. Expanding, $\lambda \approx 3 - \varphi^2/48$, $d\rho/d\lambda = 2$ at $\lambda = 3$, so $\log \rho(\varphi) = \log 2 - \varphi^2/48 + O(\varphi^4)$. The number of closed non-backtracking walks with winding w is $\frac{1}{2\pi}\int \operatorname{tr} B_\varphi^m e^{-i\varphi w} d\varphi$; the Riemann window theorem shows that away from $\varphi = 0$ (and $\varphi = \pm 2\pi$-periodic images, handled by the bipartite branch) the integrand is exponentially smaller, so Laplace's method gives a Gaussian of variance $2m/48 = m/24$; Möbius inversion passes to primes. For the full class, repeat with a two-parameter twist; the Hessian of $-\log \rho$ equals $\frac{1}{2}G^{-1}$, and $\operatorname{tr}(\frac{1}{2}G^{-1}G) = 1$ is the mean-square displacement: non-backtracking honeycomb steps have length $1/\sqrt{3}$ and persistence factor 3. □

Exact check: at $m = 40$, $\operatorname{Var}(W)/m = 0.041668$ versus $1/24 = 0.041667$ (Figure 3). Theorem 6.1 is the graph instance of the prime geodesic theorems in homology classes [39, 25, 29, 41]; for graphs see [31, 33].

6.1. Higher cumulants and hexagonal anisotropy.

Theorem 6.1 is a statement about second moments. The next terms are also exact, and they show how the hexagonal lattice behind the Tonnetz enters the statistics of long cadences.

Theorem 6.2 (higher cumulants)**.** [E→G4N; X] *Let ℓ be a non-zero linear function of the homology class (u, v), and let κ_j be the cumulants of ℓ over the prime cadences of length m (even). The odd cumulants vanish, and up to exponentially small errors*

$$\frac{\kappa_4}{\kappa_2^2} = -\frac{17}{2m}, \qquad \frac{\kappa_6}{\kappa_2^3} = \frac{1470 + \cos 6\alpha}{4m^2},$$

where α is the angle between ℓ and $4u + 3v$ in the inner product given by the covariance of Theorem 6.1. Equivalently, in coordinates $\xi = r(\cos\alpha, \sin\alpha)$ in which the covariance per step is $\frac{1}{2}|\xi|^2$, the cumulant generating function per step is

$$-\frac{r^2}{4} - \frac{17\, r^4}{192} - \Big(\frac{49}{768} + \frac{\cos 6\alpha}{23040}\Big) r^6 + O(r^8).$$

For the winding $W = -u$ (where $\cos 6\alpha = -1$) the cumulant rates are $\kappa_2/m = \frac{1}{24}$, $\kappa_4/m = -\frac{17}{1152}$, $\kappa_6/m = \frac{1469}{55296}$.

So the fourth cumulant (excess kurtosis κ_4/κ_2^2) is rotationally isotropic, like the covariance, and the six-fold (C_6) symmetry of the honeycomb lattice first appears at order six. The extreme directions are the bond-perpendicular zigzag axis $4u + 3v$ ($\cos 6\alpha = 1$, ratio $1471/4m^2$) and the bond-parallel armchair axes W and v ($\cos 6\alpha = -1$, ratio $1469/4m^2$).

Proposition 6.3 (all cumulants of the winding)**.** [GB; X] *Up to exponentially small errors, $\log \mathbb{E}\, e^{i\varphi W} = m \log \frac{\rho(\varphi)}{2}$ over the prime cadences of length m, where $\rho = \frac{1}{2}(\lambda + \sqrt{\lambda^2 - 8})$ and $\lambda = \sqrt{5 + 4\cos(\varphi/4)}$. Expanding gives every cumulant of W; for example $\kappa_8/m = -\frac{887099}{7962624}$.*

Proof sketch of Theorem 6.2 and Proposition 6.3. As in the proof of Theorem 6.1, twist the edges by a character $e^{i\langle \xi, \cdot \rangle}$ of the homology class. The twisted adjacency matrix has a simple top eigenvalue $\mu(\xi)$ near 3, and the twisted Hashimoto matrix has the eigenvalue $\rho(\xi)$ with $\rho^2 - \mu\rho + 2 = 0$, its bipartite partner $-\rho(\xi)$, and otherwise eigenvalues of modulus at most $\sqrt{2}$ near $\xi = 0$ (Theorem 3.1). Hence, over closed non-backtracking walks, the characteristic function of the class is $\rho(\xi)^m(1 + (-1)^m)/N_m$ up to a relative error $O(2^{-m/2})$, and Möbius inversion to primes changes it by a further exponentially small amount. Cumulants are Taylor coefficients at $\xi = 0$, so by Cauchy's estimates they equal m times those of $\log(\rho/2)$ up to exponentially small errors. The series of μ follows from perturbation theory around the constant eigenvector, and for W the top eigenvalue is $|2 + e^{i\varphi/4}|$ in closed form by Theorem 5.2. □

Verification. [G4N]: the exact hexagonal power sums and cumulant series are machine-checked in Lean 4, the predictions were tested in C++20, Swift and Rust against the exact distributions at $m = 40$, and Proposition 6.3 was independently verified in exact rational arithmetic through $\kappa_8/m = -887099/7962624$. [GB]: an independent exact computation of the perturbation series of μ, from a twisted adjacency matrix built from scratch, gives $\kappa_4/\kappa_2^2 = -17/2$ per step along six functionals (W, v, $4u + 3v$, $u + v$, $2u + v$, $u + 2v$) and $\kappa_6/\kappa_2^3 = (1470 + \cos 6\alpha)/4$ exactly along each. The exact distributions over the 54,975,717,792 prime cadences of length 40 give $m\kappa_4/\kappa_2^2 = -8.4952$, -8.4898, -8.4909 and $m^2\kappa_6/\kappa_2^3 = 366.3$, 368.6, 368.2 for W, v, $4u + 3v$.

Remark 6.4 (Edgeworth corrections). [E→G4N; X] At $m = 40$, Edgeworth expansions built from Theorem 6.2 reduce the total-variation distance between the exact law of W and its approximation from $1.09 \cdot 10^{-2}$ (Gaussian) to $9.95 \cdot 10^{-4}$ (fourth order) and $1.44 \cdot 10^{-4}$ (sixth order), with each approximant normalised over $|w| \le 24$. Truncated Edgeworth densities become negative far in the tail (at $w = \pm 5$ for $m = 40$). They must be clipped there, and the comparison should then use a bounded divergence such as Jensen–Shannon

rather than Kullback–Leibler, which is infinite once the clipped density vanishes where the exact one does not. Grok Bot independently reproduced these values to relative accuracy 10^{-12}.

7. The geometric object: a 2-adic Mumford curve

The Ihara zeta function was born in p-adic geometry [24, 45]: the Bruhat–Tits tree $\mathcal{T}$ of $PGL_2(\mathbb{Q}_2)$ is 3-regular, a finite 3-regular graph is a quotient $\Gamma\backslash\mathcal{T}$, and when Γ is a Schottky group Mumford's construction [36, 21] turns $\Gamma\backslash\mathcal{T}$ into the reduction graph of an algebraic curve. This section carries that out explicitly for the chord graph and asks how far the resulting curve is arithmetic.

7.1. **A Schottky group generated by P, L, R.** Vertices of $\mathcal{T}$ are homothety classes of $\mathbb{Z}_2$-lattices in $\mathbb{Q}_2^2$; the three neighbours of $v_0 = [\mathbb{Z}_2^2]$ are its index-2 sublattices, indexed by $\mathbb{P}^1(\mathbb{F}_2)$. Put

$$\sigma_P = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \qquad \sigma_L = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_R = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \in PGL_2(\mathbb{Q}_2).$$

Theorem 7.1 (Schottky realisation)**.** [GB]

(a) Each σ_s satisfies $\sigma_s^2 = 2I$, so it is an involution of $PGL_2(\mathbb{Q}_2)$; it swaps v_0 with one of its neighbours, and the three neighbours reached are distinct.

(b) $W = \langle \sigma_P, \sigma_L, \sigma_R \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$, and W acts simply transitively on the vertices of $\mathcal{T}$.

(c) Let K be the kernel of $W \to PLR \cong D_{12}$, $\sigma_s \mapsto s$. Then K is a Schottky group, free of rank 13, and $K\backslash\mathcal{T} \cong X$.

(d) The Mumford curve $C = K\backslash\Omega$ has genus 13. Its stable reduction is 24 copies of $\mathbb{P}^1_{\mathbb{F}_2}$, one per triad, glued at 36 $\mathbb{F}_2$-rational nodes, one per move, so the dual graph is X (Figure 4). Every $\mathbb{F}_2$-point of every component is a node, $\#C_0(\mathbb{F}_{2^n}) = 24 \cdot 2^n - 12$, and $J(C)$ has split toric reduction.

Proof. (a) $\sigma_P(x, y) = (y, 2x)$ maps $\mathbb{Z}_2^2$ onto $\mathbb{Z}_2 \oplus 2\mathbb{Z}_2$ and that lattice onto $2\mathbb{Z}_2^2 \sim \mathbb{Z}_2^2$. Up to scalars, $\sigma_L = g\sigma_P g^{-1}$ with $g = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\sigma_R = h\sigma_P h^{-1}$ with $h = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. Both $g, h \in GL_2(\mathbb{Z}_2)$ fix v_0 and send the line $\langle e_1 \rangle$ of $\mathbb{F}_2^2$ to $\langle e_2 \rangle$ and $\langle e_1 + e_2 \rangle$. (b) The orbit map $w \mapsto wv_0$ sends the Cayley tree of $\mathbb{Z}_2^{*3}$ to $\mathcal{T}$, and the neighbours of wv_0 are $w\sigma_s v_0$. So the map is a local bijection of 3-regular trees, hence an isomorphism. This forces both freeness of the product and simple transitivity. (c) K is normal of index 24 and lies in the even (bipartition-preserving) subgroup, so it acts without inversions. It meets no vertex stabiliser, so it acts freely and is free [45]. It is discrete and cocompact with no elliptic elements, so it is Schottky [24, 21]. $K\backslash\mathcal{T}$ is the coset graph of W/K, which is the Cayley graph X, and its rank is $E - V + 1 = 13$. (d) This is Mumford's theorem [36]: components correspond to vertices of $K\backslash\mathcal{T}$ and double points to its edges. The point count is $24(2^n + 1) - 36$. □

Verification. We computed the 13 Schreier generators of K from a spanning tree of X as integer matrices. Their determinants are 2^6, 2^8 and 2^{10}; each is hyperbolic ($2v_2(\mathrm{tr}) < v_2(\det)$), and its translation length on $\mathcal{T}$ equals the length (6 or 8) of its cycle in X. For example, the hexagon $(LP)^3$ at E is $\mathrm{diag}(64, 1)$, and the word $PLRLPRPL$ is $\left(\begin{smallmatrix} -5 & 12 \\ 12 & -80 \end{smallmatrix}\right)$, with determinant 2^8 and trace -85. We also checked simple transitivity on the ball of radius 7 (382 reduced words, all giving distinct vertices).

Proposition 7.2 (symmetries)**.** [E→GB] $\mathrm{Aut}(C) \cong D_{12}$. *It acts on the reduction graph as the group of transpositions and inversions, the centraliser of the PLR-group.*

Proof. $\mathrm{Aut}(C) = N(K)/K$ [36, 21], and this group injects into $\mathrm{Aut}(X)$. Indeed, if $n \in N(K)$ acts trivially on X, then after multiplying by an element of K it fixes v_0. Local bijectivity of $\mathcal{T} \to X$ then makes it fix every vertex, and $PGL_2(\mathbb{Q}_2)$ acts faithfully on $\mathcal{T}$. A direct computation gives $|\mathrm{Aut}(X)| = 24$. The group W/K already supplies 24 automorphisms, and they preserve the edge colours P, L, R. So they commute with the PLR-action, and they are Lewin's dual T/I group [30, 15]. □

In words: the moves P, L, R build the curve, and transposition and inversion are its symmetries.

Proposition 7.3 (not arithmetic)**.** [E→GB] *No finite-index subgroup of K is contained in an arithmetic lattice of $PGL_2(\mathbb{Q}_2)$ coming from a quaternion algebra over $\mathbb{Q}$.*

Proof. A cocompact arithmetic lattice of $PGL_2(\mathbb{Q}_2)$ defined over $\mathbb{Q}$ comes from the $\mathbb{Z}[\frac{1}{2}]$-units of an order in a quaternion algebra $B/\mathbb{Q}$ that is split at 2 and definite. Its non-central elements have reduced trace t and norm $n > 0$ with $t^2 < 4n$. The quantity $\mathrm{tr}^2 / \det$ is conjugation-invariant, and an element that is hyperbolic at the real place stays hyperbolic in all its powers. The generator above has $\mathrm{tr}^2 / \det = 85^2/2^8 > 4$. □

Proposition 7.3 concerns this particular K. Other Schottky groups with quotient X exist (move the fixed-point pairs of the three involutions). Whether one of them is arithmetic is the subject of §7.5.

7.2. **The tower over the two-line curve.**

Theorem 7.4 (the $\mathbb{Z}/12$ tower)**.** [E→GB] *Let $W^+ = \langle \sigma_P\sigma_L, \sigma_L\sigma_R \rangle$ be the even subgroup.*

(a) W^+ is a Whittaker group (the index-2 even subgroup of a group generated by three involutions), and $H = W^+\backslash\Omega$ is a hyperelliptic Mumford curve of genus 2 [21]*. Its stable reduction is two copies of $\mathbb{P}^1_{\mathbb{F}_2}$, "major" and "minor", glued pointwise along $\mathbb{P}^1(\mathbb{F}_2) = \{P, L, R\}$. The dual graph is the theta graph Θ (Figure 5).*

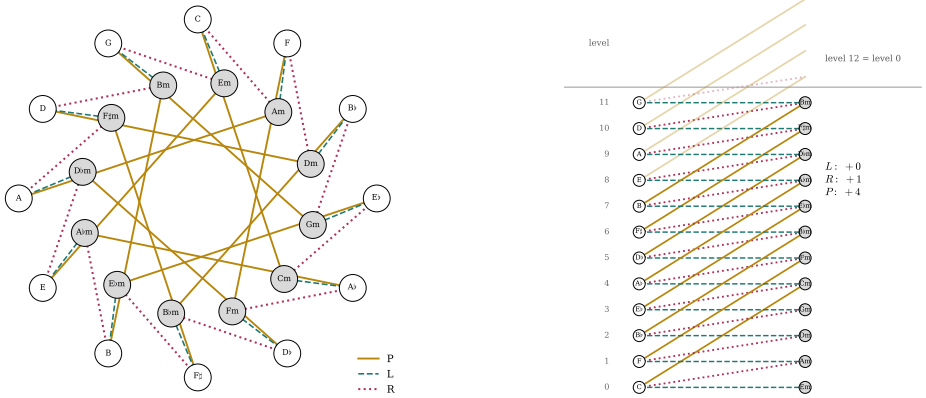

FIGURE 4. Left: the reduction graph of C. Vertices are the 24 components $\mathbb{P}^1_{\mathbb{F}_2}$, edges are the 36 nodes, and solid, dashed and dotted lines are P, L and R. Majors sit on the outer ring and minors on the inner ring, at their level ℓ (Theorem 7.4), which runs through the circle of fifths. Right: the same graph as the $\mathbb{Z}/12$ cover of Θ, drawn as twelve levels with ramps $L : +0$, $R : +1$, $P : +4$; faint ramps wrap from level 11 to level 0.

(b) *$C \to H$ is an unramified Galois cover with group $W^+/K \cong \mathbb{Z}/12$. On reduction graphs, X is the $\mathbb{Z}/12$ voltage cover of Θ with voltages $P : 4$, $L : 0$, $R : 1$, reading off $P = L\rho^4$, $R = L\rho$, $\rho = LR$. The level of a major triad with root r is $5r$, and that of a minor triad with root r is $5(r-4)$ mod 12: the levels run through the circle of fifths.*

(c) *A loop on Θ with net voltage v lifts to a cycle of length $2{\cdot}12/\gcd(v, 12)$. The loops PL, PR, LR have voltages 4, 3, ± 1, and they lift to the hexatonic 6-cycle, the octatonic 8-cycle and the Hamiltonian 24-cycle.*

(d) *$C \to \mathbb{P}^1 = W\backslash\Omega$ is a D_{12}-cover branched over six points, the two fixed points of each σ_s, each with monodromy an involution.*

Proof. (a) $W^+\backslash\mathcal{T}$ has two vertices (the bipartition classes) and three edges. (b) W^+/K is the rotation subgroup of D_{12}, cyclic of order 12 and generated by ρ. The voltage of an edge records the power of ρ picked up when the edge is crossed from major to minor, and the levels follow by solving these congruences. We checked all 36 edges. (c) A lift closes after $12/\gcd(v, 12)$ laps of two edges. (d) Riemann–Hurwitz: $24 = 24(-2) + 72$, and the 72 comes from 6×12 ramification points of index 2. □

Corollary 7.5 (Ihara factors are characters of the cover)**.** [E→GB] *With $\varphi_k = 1 + \zeta^k + \zeta^{4k}$ and $\zeta = e^{2\pi i/12}$ (the voltages $L, R, P \mapsto 0, 1, 4$; for each order d the values $|\varphi_k|$ are those of the $|f_k|$ of Section 3, possibly permuted*

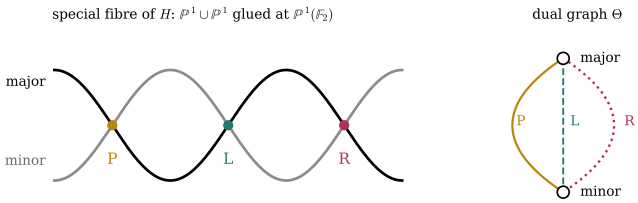

FIGURE 5. The genus-2 curve H underneath C: two lines meeting in the three points P, L, R, and its dual graph Θ. The chord graph is the $\mathbb{Z}/12$ cover of Θ (Figure 4, right).

within the class),

$$\zeta_X(u)^{-1} = \prod_{k=0}^{11} L_\Theta(u, \chi_k)^{-1}, \qquad L_\Theta(u, \chi_k)^{-1} = (1-u^2)\big((1+2u^2)^2 - |\varphi_k|^2 u^2\big).$$

Grouping the characters by their order d gives the rational factorisation of Theorem 3.5:

d	k	$\lvert\varphi_k\rvert$	*factor (without* $(1-u^2)$*)*
1	*0*	*3*	$(1-u)(1+u)(1-2u)(1+2u)$
2	*6*	*1*	$(1-u+2u^2)(1+u+2u^2)$
3	*4, 8*	$\sqrt{3}$	$(1+u^2+4u^4)^2$
4	*3, 9*	$\sqrt{5}$	$(1-u^2+4u^4)^2$
6	*2, 10*	*1*	$(1-u+2u^2)^2(1+u+2u^2)^2$
12	*1, 5, 7, 11*	$2\cos\frac{\pi}{12}$, $2\cos\frac{5\pi}{12}$	$(1+4u^2+9u^4+16u^6+16u^8)^2$

Correspondingly, Chevalley–Weil gives an isogeny $J(C) \sim J(H) \times A_2 \times A_3 \times A_4 \times A_6 \times A_{12}$, where A_d is the part on which $\mathbb{Z}/12$ acts through characters of exact order d. The dimensions are $2, 1, 2, 2, 2, 4$.

Proof. This is the Artin–Ihara factorisation for abelian covers [46, 49]. The twisted adjacency of Θ at χ_k is $\left(\begin{smallmatrix} 0 & \varphi_k \\ \bar{\varphi}_k & 0 \end{smallmatrix}\right)$. We checked the identity exactly in SymPy. □

Remark 7.6. The Ihara factors are *not* the Frobenius polynomials of the reduction of $J(C)$, which is a split torus. The bridge between the two is the cover $X \to \Theta$, not a congruence relation; compare §7.5.

7.3. **Tropical invariants: sandpile group and gonality.**

Theorem 7.7. [E→GB] *The critical (sandpile) group of X is*

$$K(X) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/736 \oplus \mathbb{Z}/6624,$$
$$|K(X)| = 156{,}008{,}448 = 2^{15} \cdot 3^2 \cdot 23^2.$$

Its order is $\frac{1}{24} \cdot 6 \prod_{k \neq 0} (9 - |\varphi_k|^2)$, and the factor 23^2 comes from the order-12 characters: $(9-(2-\sqrt{3}))(9-(2+\sqrt{3})) = 46$. By Raynaud [43]*, $K(X)$ is the*

group of components of the Néron model of $J(C)$. The cycle-pairing matrix on $H_1(X,\mathbb{Z})$, which gives the valuations of the multiplicative periods of $J(C)$, has determinant $|K(X)|$ and cokernel $K(X)$.

Proof. Smith normal form of $3I - A$ (PARI); Kirchhoff's matrix-tree theorem [26]; the eigenvalues of $3I - A$ are $3 \pm |\varphi_k|$. □

Theorem 7.8 (gonality)**.** [E→GB] *The divisorial gonality of X [4] is 6, with witness $2\,C + 2\,Am + D\flat + A\flat m$, which is invariant under the inversion exchanging C and Am. No involution of X has a tree quotient, so the metric graph is not hyperelliptic* [11]*. Hence C is not hyperelliptic* [3, Cor. 3.5]*, while H is.*

Proof. By vertex-transitivity, a rank-≥ 1 divisor class contains a divisor $D \geq (C)$. We searched all effective D of degree ≤ 6 of this form, using Dhar burning to compute q-reduced divisors. The search was done twice, with independent implementations (compiled and pure Python). We enumerated all 13 involutions of X. □

7.4. **Curves over $\mathbb{F}_2$.**

Proposition 7.9. [E→GB] *The degree-44 polynomial $P_{44}(u) = \prod_{d>1}$(factors above) is a 2-Weil polynomial, but it is not the zeta numerator of any curve over $\mathbb{F}_2$. Its point counts would be $N_n = 3, 39, 9, 75, 33, 27, \ldots$, and $N_6 < N_2$ is impossible. Every irreducible factor is a factor of the L-polynomial of an explicit curve over $\mathbb{F}_2$ (checked by point counting): $y^2 + xy = x^3 + x^2 + 1$ and $y^2 + xy = x^3 + 1$ $(1 \mp u + 2u^2)$; $y^2 + (x^2 + x)y = x^5 + x^3 + x^2 + x$ $(1 - u^2 + 4u^4)$; $y^2 + (x^2 + x + 1)y = x^5 + x^3 + x^2 + 1$ $(1 + u^2 + 4u^4)$; and, for the order-12 octic (isogeny class 4.2.a_e_a_j, which contains no Jacobian* [50]*), the genus-6 curve*

$$y^2 + (x^6 + x^5 + x^4 + x^3 + 1)\,y = x^{13} + x^3 + x,$$

$$L(u) = (1 - 3u^2 + 4u^4)(1 + 4u^2 + 9u^4 + 16u^6 + 16u^8).$$

We searched all smooth hyperelliptic models over $\mathbb{F}_2$ of genus 4, 5, 6 and 7, up to $y \mapsto y + t(x)$ (1536, 6144, 24576 and 98304 models). Among these hyperelliptic models the octic occurs only in genus 6, where 12 models have it. The minimal genus is smaller once non-hyperelliptic curves are allowed:

Proposition 7.10 (the octic in genus 5)**.** [E→GB] *No curve of genus ≤ 4 over $\mathbb{F}_2$ has the octic $1 + 4u^2 + 9u^4 + 16u^6 + 16u^8$ as a factor of its L-polynomial, and two curves of genus 5 do. They are the smooth complete intersections in*

$\mathbb{P}^4 = \{(T : U : X : Y : Z)\}$

$$
\begin{aligned}
A:\quad & TX + TZ + Y^2 + YZ + Z^2 = 0,\\
& T^2 + TU + TY + TZ + U^2 + UX + UZ + XZ + Y^2 = 0,\\
& TY + U^2 + UX + UZ + X^2 + XY + XZ + Y^2 + YZ + Z^2 = 0;\\
B:\quad & TX + TZ + Y^2 + YZ + Z^2 = 0, \qquad TX + UX + UY + UZ = 0,\\
& TU + U^2 + UX + UY + UZ + X^2 + XY + XZ + Y^2 + Z^2 = 0,
\end{aligned}
$$

with $L_A = (1-u+2u^2)\cdot$ octic and $L_B = (1+u+2u^2)\cdot$ octic, so their Jacobians are isogenous to $E \times A_{12}$ for the two elliptic curves of Proposition 7.9.

Proof. Genus 4 would need L = octic itself, i.e. $(N_1, N_2, N_3, N_4) = (3, 13, 9, 21)$, and this class contains no Jacobian [50]; we confirmed it by enumerating all hyperelliptic models and all quadric–cubic complete intersections in $\mathbb{P}^3$ over $\mathbb{F}_2$. Genus ≤ 3 is excluded by degree. In genus 5 the cofactor is $1 - au + 2u^2$ with $a \in \{0, \pm 1, \pm 2\}$; $a = 0$ would give a negative number of closed points of degree 4, and Dragutinović's complete census of genus-5 curves over $\mathbb{F}_2$ [18] has exactly the two classes above (none with $a = \pm 2$). We recomputed the point counts of A and B over $\mathbb{F}_{2^k}$, $k \le 5$, and verified smoothness by Gröbner bases. □

7.5. **No Shimura curve over $\mathbb{Q}$.** By Čerednik–Drinfeld [54, 19], in the form used by Ribet [44], the Shimura curve $X_0^D(N)$ with $2 \mid D$ has a 2-adic reduction graph. It is the bipartite double of the 2-Brandt graph of an Eichler order of level N in the definite algebra of discriminant $D/2$. Its adjacency spectrum is ± 3 together with $\pm a_2(f)$, for f running over the $D/2$-new part of $S_2(\Gamma_0(DN/2))$.

Theorem 7.11. [E→GB; corrected G4N; X] *No Shimura curve over $\mathbb{Q}$ of the form $X_0^D(N)$ with $2 \mid D$, nor any of its Γ_H-variants with 24 components, has reduction graph X.*

Proof. Genus 13 leaves exactly 19 pairs (D, N): $D = 2p$ for 16 pairs, plus three four-prime discriminants $D = 546$, $D = 570$, and $D = 690 = 2 \cdot 3 \cdot 5 \cdot 23$ with $N = 1$ (the last has $\phi(690) = 176$, $e_2 = 0$, $e_3 = 8$, $g = 1 + 176/12 - 8/3 = 13$; an earlier cutoff $\phi(D) \le 168$ in `v0.1.0`–`v0.2.0` omitted $X_0^{690}(1)$, caught in `NEG-E10`). It also leaves four Γ_H cases. For each pair, PARI's Hecke matrices give the multiset $\{|a_2(f)|\}$ (for $X_0^{690}(1)$, $\dim S_2^{345\text{-new}}(\Gamma_0(345)) = 15$ and $\mathrm{charpoly}(T_2) \bmod (t^4 - 4t^2 + 1) = 3t^3 + t^2 + 21t + 7 \neq 0$). In the 11 torsion-free cases it has exactly 11 elements, consistent with the Brandt class number 12. None equals the spectrum $\{1, 1, 1, \sqrt{5}^2, \sqrt{3}^2, (2\cos\tfrac{\pi}{12})^2, (2\cos\tfrac{5\pi}{12})^2\}$ of X (exponents are multiplicities). The nearest cases, $(D, N) = (26, 9)$ and $(570, 1)$, agree on $1, 1, 1, \sqrt{3}, \sqrt{3}$ but have $1 + \sqrt{2}$ in place of the rest. Independently, without calling PARI, a standalone Rust implementation of Manin modular symbols computes T_2 on $\mathcal{M}_2(\Gamma_0(M))^-$ over $\mathbb{F}_{1000003}$ for all 18 distinct levels $M = DN/2$ covering all

19 pairs and verifies full rank of $T_2^4 - 4T_2^2 + I$ on every level (at $M = 345$: $\dim = 45$, $\operatorname{tr}(T_2) \equiv -5$, rank $= 45/45$). □

Remark 7.12. The coincidence with $X_0(49)$ noted in Theorem 3.5 therefore reflects a single factor with $a_2 = 1$, not an arithmetic structure. Over larger totally real fields see §7.6 and §7.7.

7.6. **No arithmetic uniformisation of C at all.** We now remove the restriction to $\mathbb{Q}$ as far as the curve C itself is concerned. Call a subgroup $\Gamma \subset PGL_2(F_v)$, for a number field F and a finite place v, *arithmetic* if it is commensurable with the image of $G(\mathcal{O}_E[1/v])$ for an inner form G of PGL_2 over a subfield $E \subset F_v$ with $G(E_v) \cong PGL_2(F_v)$ and $G(E \otimes \mathbb{R})$ compact. Discreteness and cocompactness force exactly these conditions (a non-compact archimedean factor would make the image dense), so E is totally real. The groups of Čerednik–Drinfeld uniformisation of Shimura curves over totally real fields are of this kind [54, 19]: they come from the totally definite algebra $\bar{B}$ obtained by interchanging the invariants at v and at one real place.

Theorem 7.13 (trace obstruction)**.** [E→GB; X] *Let F be any number field, v any finite place, $L \supset F_v$ any complete extension, and $\Gamma \subset PGL_2(F_v)$ any torsion-free arithmetic group. Then $C_L \not\cong \Gamma\backslash\Omega_L$. More strongly, no conjugate of a finite-index subgroup of K by an element of $PGL_2(L)$ is commensurable with an arithmetic group.*

Proof. The function $\tau(g) = \operatorname{tr}(g)^2/\det(g)$ is well defined on PGL_2, invariant under conjugation, and is a regular function defined over the base on every inner form (on $B^\times/E^\times$ it is $\operatorname{trd}^2/\operatorname{nrd}$). If γ lies in an arithmetic Γ, some power γ^m lies in $G(\mathcal{O}_E[1/v])$, so $\tau(\gamma^m) \in E$ and every real conjugate of $\tau(\gamma^m)$ is the value of τ on an element of a compact form, i.e. of $\mathbb{H}^\times/\mathbb{R}^\times$, where $0 \le \tau \le 4$. For $\gamma_0 = PLRLPRPL = \pm\left(\begin{smallmatrix} -5 & 12 \\ 12 & -80 \end{smallmatrix}\right)$ we have $\tau(\gamma_0) = 7225/256$, and if μ is the ratio of the eigenvalues then $\tau(\gamma_0^m) = 2 + \mu^m + \mu^{-m}$ is a rational number > 4 for every $m \ge 1$, since $\mu > 0$, $\mu \ne 1$. It is fixed by every embedding of E and violates the bound. This proves the second statement, because $\gamma_0 \in K$ (Proposition 7.3). For the first, an isomorphism $C_L \cong \Gamma\backslash\Omega_L$ with Γ torsion-free is, by the uniqueness of the Schottky group of a Mumford curve [36, 21], a conjugacy $gKg^{-1} = \Gamma$ in $PGL_2(L)$. □

Theorem 7.13 is insensitive to the field, the place, the residue degree and passage to commensurable or conjugate groups. The only uniformisations it does not see are those by groups *with torsion*, whose quotient is C even though the group is not a Schottky group (Shimura curves with elliptic points). These are controlled by edge lengths. Normalise lengths on the Bruhat–Tits tree $\mathcal{T}_v$ of $PGL_2(F_v)$ so that $v(2) = 1$; an edge then has length $1/e_v$, where e_v is the ramification index of $F_v/\mathbb{Q}_2$. The skeleton of C is X with all 36 edges of length 1, every vertex of valence 3 and every component of genus 0.

Lemma 7.14 (edge lengths)**.** [E→GB; sharpened E; X] *Let $\Gamma \subset PGL_2(F_v)$ be discrete and cocompact and e an edge of $\mathcal{T}_v$ whose pointwise stabiliser Γ_e is non-trivial and not inverted. Then the image of e in the skeleton of $\Gamma\backslash\Omega$ has length $|\Gamma_e|/e_v \geq 2/e_v$.*

Proof. Γ_e is finite and acts on the open annulus A_e of points of Ω^{an} retracting to the interior of e, preserving both ends. Pick $\sigma \in \Gamma_e$ of prime order ℓ. If ℓ is odd, σ is conjugate to $z \mapsto \zeta z$ with $|\zeta - 1| = 1$, the circles of A_e are centred at a fixed point, and $w = z^\ell$ multiplies log-radii by ℓ. If $\ell = 2$, σ is conjugate to $z \mapsto -z$ or is unipotent, and on a disc $|z - c| \leq r$ fixed by σ one has $|z^2 - c^2| = |z - c|\,|z + c| = |z - c|^2$ whenever $|2c| < r$, so again the log-modulus is doubled (both the slope-2 doubling and its slope-1 control off the fixed region are formalised in Lean 4 and confirmed numerically). In residue characteristic 2 the action of Γ_e on A_e need not be free—as noted in Ergentics' clean-room monograph, the wild involution $\sigma(z) = 3/z \in PGL_2(\mathbb{Q}_2)$ fixes the oriented edge $e = (L_0, L_1)$ with $L_0 = \mathbb{Z}_2^2$, $L_1 = \langle (1,1), (2,0) \rangle$ while its fixed points $\pm\sqrt{3}$ satisfy $(1 + x)^2 = 3 \implies |x|_2 = 2^{-1/2} \in (\frac{1}{2}, 1)$ strictly inside $A_e = \{\frac{1}{2} < |x|_2 < 1\}$—yet the invariant coordinate $u(z) = z + 3/z$ obeys the exact cancellation-free identity $u(1 + x) - 4 = x(x - 2)/(1 + x)$, giving $|u(1 + x) - 4|_2 = |x|_2^2$ on A_e despite interior branching. In full generality, every interior ramification point lies off the Berkovich skeleton segment e on an open disc branch, so the finite end-preserving quotient $A_e \to A_e/\Gamma_e$ is an open annulus whose piecewise-linear slope along e equals the generic degree $|\Gamma_e|$ [5], yielding normalised length $|\Gamma_e|/e_v \geq 2/e_v$. □

Theorem 7.15 (no 2-adic uniformisation with $F_v = \mathbb{Q}_2$)**.** [E→GB; X] *Let F be a totally real field and $v \mid 2$ a place with $F_v \cong \mathbb{Q}_2$ (a prime of norm 2 that is unramified, or any degree-one place of a larger field). Let $\Gamma \subset PGL_2(F_v)$ be any arithmetic group, with or without torsion, for example the group attached to a connected component of a Shimura curve over F uniformised at v. Then $\Gamma\backslash\Omega$ is not isomorphic to C over any extension of $\mathbb{Q}_2$.*

Proof. Suppose $\Gamma\backslash\Omega_L \cong C_L$. The quotient $\Gamma\backslash\Omega^{\mathrm{an}}$ retracts onto $\Gamma\backslash\mathcal{T}_v$, the fibres being open discs because every type-2 point of $\Gamma\backslash\Omega^{\mathrm{an}}$ has genus 0 (it is dominated by a $\mathbb{P}^1$; Lüroth). Hence the minimal skeleton X is the core of the metric graph $\Gamma\backslash\mathcal{T}_v$, with valence-2 points smoothed [7, 5]. Here $e_v = 1$, so an edge of $\mathcal{T}_v$ has length 1 and an inverted edge folds to a leaf of length $\frac{1}{2}$, outside the core. By Lemma 7.14 an edge with non-trivial stabiliser has length ≥ 2. A skeleton edge of X has length exactly 1, so it is the image of a single edge of $\mathcal{T}_v$ with trivial stabiliser, and there is no valence-2 point on it. Every vertex of $\Gamma\backslash\mathcal{T}_v$ therefore has valence 3 in the core (valence 2 would create a skeleton edge of length ≥ 2), so its three neighbours lie in three different Γ_x-orbits; thus Γ_x fixes the three edges at x, and $\Gamma_x = 1$. With no inversions either, Γ acts freely on $\mathcal{T}_v$ and is torsion-free [45], contradicting Theorem 7.13. □

Remark 7.16 (what Theorems 7.13 and 7.15 leave). The curve C is not a Shimura curve, nor a connected component of one, for any 2-adic Čerednik–Drinfeld uniformisation by a torsion-free group (any F, any place), nor for any uniformisation at a place with $F_v \cong \mathbb{Q}_2$. The one case not covered is a group *with torsion* at a place $v \mid 2$ with $e_v \geq 2$ or residue degree $f_v \geq 2$. Propositions 7.17 and 7.18 sharply obstruct both regimes.

Proposition 7.17 (torsion at a place above 2)**.** [E→GB; sharpened E; X] *Let $\Gamma \subset PGL_2(F_v)$ be discrete and cocompact with $\Gamma\backslash\Omega_L \cong C_L$, and let $q = 2^{f_v}$.*

(a) If $e_v = 1$, the core of $\Gamma\backslash\mathcal{T}_v$ is X. Its 36 edges are single edges of $\mathcal{T}_v$ with trivial stabilisers. At each of its 24 vertices the stabiliser Γ_x permutes the three core directions freely, so $3|\Gamma_x| \leq q+1$. The other $q+1-3|\Gamma_x|$ directions lead into finite trees of groups, which become open discs in C. Hence $\Gamma \cong A_1 * \cdots * A_{24} * F_{13}$, where each A_i is generated by elements of finite order and F_{13} is a Schottky group with quotient graph X.

(b) If $f_v = 1$, Γ is torsion-free (Theorem 7.15). If $f_v = 2$, all 24 core stabilisers are trivial and each core vertex has exactly two contracted directions, so all torsion lies off the core. If $f_v \geq 3$, the valence count allows core stabilisers of order up to $(2^{f_v}+1)/3$. For example, all 56 elements of order 3 in $PGL_2(\mathbb{F}_8)$ permute the nine points of $\mathbb{P}^1(\mathbb{F}_8)$ in three free orbits.

(c) If $e_v \geq 2$, each edge $\bar{e}$ of X is a chain of quotient edges $e_1, \ldots, e_k$ of $\mathcal{T}_v$ obeying the exact length-conservation law $\sum_{i=1}^{k} |\Gamma_{e_i}| = e_v$. When $f_v = 1$ ($\mathcal{T}_v$ is 3-regular) and Γ is inversion-free, no unsubdivided stabilised edge ($k = 1, |\Gamma_e| = e_v$) can occur at all. In particular, for $e_v = 2, f_v = 1$, every one of the 36 edges of X must subdivide into two unstabilised half-edges meeting at 36 trivial-stabiliser midpoint vertices lying in a single bipartition class of $\mathcal{T}_v$; for a Shimura group $\Gamma^+ = \mathcal{O}^1/\{\pm 1\}$ this forces $M(\mathcal{O}) \geq 36$, where $M(\mathcal{O}) = \sum_y 1/|\Gamma_y^+|$ is the mass of either bipartition class of $\Gamma^+\backslash\mathcal{T}_v$ (a combinatorially sharp floor).

Proof. (a) As in the proof of Theorem 7.15, an edge of X is a single unstabilised edge of $\mathcal{T}_v$, and core vertices have valence 3. So the three core directions at x lie in different Γ_x-orbits, each of size $|\Gamma_x|$. The rest of the quotient is a forest of finite trees attached to the core. Every point of $\Gamma\backslash\Omega^{\mathrm{an}}$ has genus 0, so these trees retract onto the core through discs. In the graph of groups the core edges carry the trivial group. By Bass–Serre theory [45], Γ is therefore the free product of $\pi_1(X) \cong F_{13}$ with the fundamental groups A_x of the trees of groups hanging at the core vertices; each A_x is generated by its finite vertex groups. Lifting a spanning tree of X realises F_{13} as a group acting freely on a 3-regular subtree of $\mathcal{T}_v$ with quotient X; the map from the universal cover of X is locally injective, hence injective. A finitely generated, discrete, torsion-free subgroup is a Schottky group [24, 21]. (b) We have $q+1 = 3|\Gamma_x| + r$ with $r \geq 0$. For $q = 2$ this forces $|\Gamma_x| = 1$ and

$r = 0$; for $q = 4$ it forces $|\Gamma_x| = 1$ and $r = 2$. An element of order 3 lies in a non-split torus, because 3 divides $q+1 = 9$, so it has no fixed point on $\mathbb{P}^1(\mathbb{F}_8)$ (verified exhaustively across all 504 elements of $PGL_2(\mathbb{F}_8)$ and kernel-checked in Lean 4). (c) The conservation law $\sum_{i=1}^{k} |\Gamma_{e_i}| = e_v$ is Lemma 7.14 together with the smoothing of valence-2 vertices. Now suppose $f_v = 1$ and Γ is inversion-free. At any lift $x \in V(\mathcal{T}_v)$ of a Tonnetz vertex, the three core edges incident to $\bar{x}$ lie in three distinct Γ_x-orbits; since $\mathcal{T}_v$ is 3-regular, Γ_x must fix all three incident edges pointwise, so $\Gamma_x = \Gamma_{e_1} = \Gamma_{e_2} = \Gamma_{e_3}$. If $|\Gamma_{e_1}| \geq 2$ on an unsubdivided edge, then $|\Gamma_{e_2}| = |\Gamma_{e_3}| = |\Gamma_x| \geq 2$ as well; propagating across the connected 3-regular tree forces Γ_x to act trivially on $\mathcal{T}_v$, contradicting the faithfulness of $PGL_2(F_v)$ on $\mathcal{T}_v$. At $e_v = 2$ this eliminates $(0, 1)$ and forces every Tonnetz edge to subdivide as $(1, 1)$ at a midpoint $\bar{y}_e$. A nontrivial stabiliser at $\bar{y}_e$ cannot swap the two Tonnetz half-edges (which would fold $\bar{y}_e$ into a leaf) nor fix one of them (which would force $|\Gamma_{e_i}| \geq 2$), so $|\Gamma_{\bar{y}_e}| = 1$ for all 36 midpoints. Since every midpoint is adjacent only to Tonnetz vertices (and vice versa), subdividing every edge places all 36 midpoints in one bipartition class of $\mathcal{T}_v$ (and all 24 Tonnetz vertices in the opposite class), so the bipartition class of midpoints contributes at least $36 \cdot 1 = 36$ to $M(\mathcal{O})$. Combinatorially this floor is sharp: attaching 36 C_3-leaves to the third direction of each midpoint puts all 36 leaves in the opposite bipartition class alongside the 24 Tonnetz vertices (which already have valence 3 in the subdivided graph), balancing mass at $24 + 36/3 = 36$. □

Proposition 7.18 (unramified cyclotomic obstruction and inversion sharpness)**.** [E; X] *Let $F_v/\mathbb{Q}_2$ be unramified ($e_v = 1$) of residue degree $f_v \geq 2$, and suppose a discrete cocompact subgroup $\Gamma \subset PGL_2(F_v)$ has minimal Berkovich skeleton isometric to X.*

- *(a)* (Inversion-free cyclotomic obstruction.) *Suppose Γ is inversion-free (in particular, any Shimura group $\Gamma^+ = \mathcal{O}^1/\{\pm 1\}$ over a totally real field E with $E_v = F_v$; the conclusions about E refer to such arithmetic Γ). If f_v is even, then $3 \nmid (2^{f_v} + 1)$ forces a pruned leaf vertex y with $(2^{f_v} + 1) \mid |\Gamma_y|$ and hence $\mathbb{Q}(\zeta_{2^{f_v}+1})^+ \subset E$; in particular, at $f_v = 2$ ($\mathbb{Q}_4$), no Shimura curve over any totally real field $E \not\supset \mathbb{Q}(\sqrt{5})$ can uniformise C. If $f_v \geq 3$ is odd, then either $\mathbb{Q}(\zeta_{2^{f_v}+1})^+ \subset E$ (via a pruned leaf) or $\Gamma\backslash\mathcal{T}_v = X$ is leaf-free with $\mathbb{Q}(\zeta_{(2^{f_v}+1)/3})^+ \subset E$.*
- *(b)* (Sharpness with edge inversions.) *Over $F_v = \mathbb{Q}_4 = \mathbb{Q}_2(\omega)$ ($\omega^2 + \omega + 1 = 0$), take $g_1 = 1$, $g_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $g_3, g_4, g_5 = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ with $a = 1, \omega, \omega^2$, which map $0 \mapsto \{0, \infty, 1, \omega, \omega^2\} = \mathbb{P}^1(\mathbb{F}_4)$, and set $s_i = g_i \left(\begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix}\right) g_i^{-1}$. Then $G = \langle s_1, \ldots, s_5 \rangle \cong (\mathbb{Z}/2)^{*5} \subset PGL_2(\mathbb{Q}_4)$ is discrete and cocompact by Klein ping-pong, and $\Gamma_5 = \ker(\phi : G \to D_{12})$ for $\phi(s_1, s_2, s_3, s_4, s_5) = (P, L, R, 1, 1)$ acts freely on all 24 vertices and 36 Tonnetz edges while inverting the remaining 48 edges into capped discs ($u = z + 2/z$). Thus $\Gamma_5\backslash\Omega_{\mathbb{Q}_4}$ is a non-arithmetic Mumford curve with minimal skeleton strictly isometric to X and no dependence on $\sqrt{5}$.*

Proof. (a) Since $e_v = 1$, Proposition 7.17(a) forces every core edge of X to be unsubdivided with trivial stabiliser. If f_v is even, $2^{f_v}+1 \equiv 2 \pmod 3$, so the $3|\Gamma_x|$ core directions at a Tonnetz vertex cannot exhaust the $2^{f_v}+1$ incident edges of $\mathcal{T}_v$; therefore a non-empty finite pruned tree attaches to X. Because Γ is inversion-free, any finite pruned tree ends in at least one leaf vertex y (valence 1 in $\Gamma\backslash\mathcal{T}_v$, with unique quotient edge e_y). All $2^{f_v}+1$ edges of $\mathcal{T}_v$ at y form a single Γ_y-orbit, so by the orbit–stabiliser theorem $[\Gamma_y : \Gamma_{e_y}] = 2^{f_v}+1$, hence $(2^{f_v}+1) \mid |\Gamma_y|$. Every finite subgroup of $PGL_2(E)$ for totally real E is cyclic, dihedral, A_4, S_4 or A_5. Since $3 \nmid (2^{f_v}+1)$, the only odd divisor > 1 of $|A_4| = 12$, $|S_4| = 24$, $|A_5| = 60$ coprime to 3 is 5 (at $f_v = 2$, where A_5 itself contains an element of order 5); for every even $f_v \geq 4$, Γ_y must therefore be cyclic or dihedral and (since $2^{f_v}+1$ is odd) contains an element of exact order $2^{f_v}+1$, forcing $\zeta_{2^{f_v}+1} + \zeta_{2^{f_v}+1}^{-1} \in E$, i.e. $\mathbb{Q}(\zeta_{2^{f_v}+1})^+ \subset E$. For odd $f_v \geq 3$, either a pruned leaf exists ($2^{f_v}+1 \geq 9$ never divides 60 since $2^{f_v} \neq 14$, forcing $\mathbb{Q}(\zeta_{2^{f_v}+1})^+ \subset E$) or there are no pruned branches and $3|\Gamma_x| = 2^{f_v}+1$ at every Tonnetz vertex (where $(2^{f_v}+1)/3$ divides 60 only at $f_v = 3$, giving $|\Gamma_x| = 3$), forcing $\mathbb{Q}(\zeta_{(2^{f_v}+1)/3})^+ \subset E$. (b) Directly in $\mathbb{Z}_4 = \mathbb{Z}_2[\omega]$ (independently verified in `ergentics/open1_torsion_xcheck.py`), each s_i satisfies $s_i^2 = 2I \equiv I$ in $PGL_2(\mathbb{Q}_4)$, $v_2(\det s_i) = 1$, and maps every residue ball B_{α_j} ($j \neq i$) strictly into B_{α_i}. By Klein ping-pong, $G \cong (\mathbb{Z}/2)^{*5}$ acts freely on $V(\mathcal{T}_5)$ with five inverted edge orbits. Passing to the index-24 kernel $\Gamma_5 = \ker(\phi)$ unwinds (s_1, s_2, s_3) into the 24-vertex, 36-edge Tonnetz X with unit edge lengths, while each conjugate of s_4, s_5 in Γ_5 acts on its open annulus $\{\frac{1}{2} < |z|_2 < 1\}$ as $z \mapsto 2/z$; its invariant $u = z + 2/z$ maps the quotient isomorphically onto the open unit disc $\{|u|_2 < 1\}$, capping all 48 inverted edges with zero genus. Finally, $s_4 s_5 \in \Gamma_5$ and $\tau(s_4 s_5) = 49/4 > 4$, so by the argument of Theorem 7.13 no finite-index subgroup of Γ_5 is commensurable with an arithmetic group: Γ_5 is non-arithmetic. □

For $f_v = 2$ and an arithmetic group Γ with $E \supset \mathbb{Q}(\sqrt{5})$ (or an Atkin–Lehner normaliser admitting edge inversions), C and $F_{13}\backslash\Omega_{F_{13}}$ still agree after removing two residue discs at each core vertex; whether F_{13} must be conjugate to K remains the residual gap of Open Problem (1).

7.7. **Other Schottky groups with quotient X.** A different question is whether *some* Shimura curve has reduction graph X; its uniformising group would be torsion-free with quotient graph X, but its curve need not be C. For a totally real F of degree n with a prime v of norm 2, a totally definite algebra of finite discriminant D'' and an Eichler order of level N (with $v \nmid D''N$, $n + |D''|$ even), the dual graph is the bipartite double of the Brandt graph at v, which needs exactly 12 classes, all with trivial unit group modulo $\mathcal{O}_F^\times$. Eichler's mass formula then forces

$$2^{1-n}\,|\zeta_F(-1)|\,h_F \prod_{\mathfrak{p}\mid D''} (N\mathfrak{p}-1) \prod_{\mathfrak{p}^e \| N} N\mathfrak{p}^{e-1}(N\mathfrak{p}+1) = 12.$$

We consider torsion-free uniformising groups, the Schottky case of C itself. When $F_v \cong \mathbb{Q}_2$ and the reduction graph is X with unit edge lengths, the argument of Theorem 7.15 shows this is automatic. Torsion would appear as classes whose unit group is larger than $\mathcal{O}_F^\times$.

Proposition 7.19 (graph-level search)**.** [E→GB; independently E; X] *Let F be a totally real field of degree $n \le 4$ and narrow class number 1, with a prime v of norm 2. No Shimura curve over F that is 2-adically uniformised at v has reduction graph X. In more detail:*

(a) For real quadratic F, the only solutions of the mass equation are $\mathbb{Q}(\sqrt{41})$ with $N = v'\mathfrak{p}_5$ (two choices) and $\mathbb{Q}(\sqrt{137})$ with $N = v'$, where v' is the conjugate of v. All have torsion: the class numbers are 14 and 13.

(b) For cubic fields of discriminant ≤ 5200 and quartic fields of discriminant ≤ 60000 (by $|\zeta_F(-1)| \ge d_F^{3/2}/(2\pi^2)^n$ these contain every solution), the mass equation has 13 solutions. Six have torsion, with class numbers between 13 and 18: discriminant 564 ($D'' = \mathfrak{p}_{17}$), 13448 ($N = \mathfrak{p}_5$, two choices), 13676 ($N = \mathfrak{p}_5$), 6809 ($N = \mathfrak{p}_{17}$) and 10273 ($N = \mathfrak{p}_8$).

(c) The other seven are torsion-free, with 12 classes and trivial unit groups: discriminant 733 ($D'' = \mathfrak{p}_{13}$), 148 ($D'' = \mathfrak{p}_{25}$, $N = \mathfrak{p}_5$), 469 ($D'' = \mathfrak{p}_4$, $N = \mathfrak{p}_7$, two choices), 316 ($D'' = \mathfrak{p}_{37}$) and 2777 ($N = \mathfrak{p}_{71}$, or $N = \mathfrak{p}_8^2$). For every choice of v, none of their Brandt matrices $B(v)$ has the factor $t^4 - 4t^2 + 1$ in its characteristic polynomial. That factor is forced by the eigenvalues $\pm 2\cos\frac{\pi}{12}, \pm 2\cos\frac{5\pi}{12}$ of X, so none of these reduction graphs is X.

Proof. The fields, class groups and $\zeta_F(-1)$ were computed with PARI/GP [51]: `nflist`, `bnrclassno`, and Siegel's formula cross-checked by `lfun`. For each solution and each choice of D'', N and v (22 configurations in all), we computed the Brandt graph at v directly, with no external data. We built the definite algebra and its Eichler order with PARI's `alg` functions. We then enumerated the right ideal classes by v-neighbours, the three index-$N(v)^2$ sub-ideals of each ideal, in the spirit of [27]. Two ideals J, R with $\mathrm{nrd}(R)/\mathrm{nrd}(J) = (\nu)$, $\nu \gg 0$, are equivalent if and only if the positive definite form $\mathrm{Tr}_{F/\mathbb{Q}}(\mathrm{nrd}(x)/\nu)$ takes the value n on the transporter $\{x : xJ \subseteq R\}$, by the arithmetic–geometric mean inequality and $h^+ = 1$. Unit groups were counted in the same way.

Four independent checks support the result. First, the weighted class count $\sum_i 1/[\mathcal{O}_i^\times : \mathcal{O}_F^\times]$ equals the mass 12 in all 22 configurations. Second, recomputing $B(v)$ by counting elements of norm $\pi\nu$ (the theta coefficients) gives the same matrix. Third, the same code reproduces the Hecke eigenvalues of the LMFDB Hilbert newforms [50] in the six cases where they are available (733, 148, one choice for 469, both for 316, and 2777 at $\mathfrak{p}_{71}$), and a separate LMFDB Jacquet–Langlands query matches all seven configurations it covers including 2777 at $\mathfrak{p}_8^2$. Fourth, all eight cubic and quartic torsion-free

Brandt matrices (F_{733}, F_{148}, both $\mathfrak{p}_7$ over F_{469}, both $v \mid 2$ over F_{316}, and both levels over F_{2777}) were independently recomputed by two further zero-shared-data implementations—Ergentics' pure-Python monogenic Brandt engine (`cleanroom/prime_cadences_resolution/paper.typ`, printing explicit 12×12 matrices for the six cases over $F_{469}, F_{316}, F_{2777}$ and verifying all eight $\mathbb{Q}[x]$-irreducible factorisations, all satisfying $|\lambda|_{\mathrm{nt}} \le 2.7508 < 2\sqrt{2}$ and spectral distance $\Delta_X \ge 0.4371$) and Gemini 4 Neon's standalone PARI/GP verifier (`ergentics/shimura_brandt_xcheck.py`)—producing 8/8 identical characteristic polynomials. □

Theorem 7.20 (any narrow class number)**.** [E→GB; X] *Let F be a totally real field of degree n. Suppose a connected component of a Shimura curve over F is 2-adically uniformised at v by a torsion-free group and has reduction graph X. Write $u = h_F^+/h_F$.*

(a) $N(v) = 2$, and $2^{1-n}|\zeta_F(-1)| \prod_{\mathfrak{p}\mid D''}(N\mathfrak{p}-1) \prod_{\mathfrak{p}^e \| N} N\mathfrak{p}^{e-1}(N\mathfrak{p}+1) = 12u$.

(b) $d_F^{1/n} \le (8\pi^2)^{2/3}(9/4)^{2/(3n)}$ (about 24.1, 22.0, 21.1, 20.5, 20.1 for $n = 2, \dots, 6$), and $n \le 14$ (sharpened to $n \le 8$ in Theorem 7.21).

(c) There is no such curve for $n \le 5$, nor for $n = 6$ when $h_F^+ = h_F$ or $d_F \le 8{,}256{,}125$. The equation in (a) has 55 solutions with $h_F^+ > 1$ for $n \le 4$, four for $n = 5$ (one of them with $h_F^+ = 2$) and none for the sextic fields covered. None of them gives X.

Proof. (a) A torsion-free group acts freely on $\mathcal{T}_v$, so its quotient graph is $(N(v)+1)$-regular. It is X, so $N(v) = 2$ and the component has genus 13. Over $\mathbb{C}$ the component is $\Gamma\backslash\mathfrak{H}$, where Γ is the image of the units of totally positive norm in an Eichler order $\mathcal{O}$ of level N. Here $\mathcal{O}$ lies in the indefinite algebra ramified at $D'' \cup \{v\}$ and at $n-1$ real places. The group Γ has no elliptic points, because these correspond to fixed points in Ω of torsion elements of the 2-adic group. Its area is therefore $2\pi(2 \cdot 13 - 2) = 48\pi$. The volume formula [55] gives $\mathcal{O}^1/\{\pm 1\}$ the coarea $4\pi\, 2^{1-n}|\zeta_F(-1)| \prod (N\mathfrak{p} - 1) \prod N\mathfrak{p}^{e-1}(N\mathfrak{p}+1)$, with the first product over $D'' \cup \{v\}$ and $N(v) - 1 = 1$. Moreover Γ contains $\mathcal{O}^1/\{\pm 1\}$ with index $[\mathcal{O}_F^{\times,+} : \mathcal{O}_F^{\times 2}] = u$, by Eichler's norm theorem and the exact sequence $\mathcal{O}_F^\times \to \{\pm 1\}^n \to \mathrm{Cl}_F^+ \to \mathrm{Cl}_F \to 1$. (b) We have $|\zeta_F(-1)| = d_F^{3/2}\zeta_F(2)/(2\pi^2)^n$, and $\zeta_F(2) \ge \frac{4}{3}$ from the Euler factor at v. With $u \le 2^{n-1}$ and both products at least 1, this gives $d_F^{3/2} \le \frac{9}{4}(8\pi^2)^n$. For every $n \ge 15$, the unconditional Poitou–Odlyzko explicit formula [40, 37, 38], with the contribution of the prime v of norm 2, makes $d_F^{1/n}$ larger than this bound. (c) For $n \le 4$ we took all totally real fields within the bound from PARI's `nflist`: 2583 fields. For $n = 5$ we took the LMFDB list of totally real quintic fields [50], which is complete for $d_F \le 10^8$ and so covers the bound: 2507 fields. For $n = 6$ the LMFDB list is complete for $d_F \le 481{,}890{,}304$, but we could retrieve it only up to $d_F = 8{,}256{,}125$ (300 fields) before the server began to refuse automated queries. With $u = 1$ the bound in (b) becomes $d_F^{3/2} \le 288(2\pi^2)^6$, that is $d_F \le 6.63 \cdot 10^6$, so this covers every sextic field with

$h_F^+ = h_F$. For $n = 5, 6$ we first discarded fields for which the Euler product of $\zeta_F(2)$ over $p < 2000$, a lower bound, already gives $|\zeta_F(-1)| > 12u\,2^{n-1}$; this was confirmed on a random sample by computing $\zeta_F(-1)$ exactly. The values of $\zeta_F(-1)$ come from PARI's L-functions, rounded to rationals with error $< 10^{-35}$; for quadratic fields they agree with Siegel's formula. Restricted to $h_F^+ = 1$ and $n \le 4$, the scan reproduces exactly the fields of Proposition 7.19. We ran every other solution through the Brandt computation generalised to arbitrary narrow class number. It enumerates all right ideal classes, by v-neighbours together with neighbours at a few further primes. It tests equivalence over the totally positive units modulo squares, checks the class count against the mass and recounts by theta series. It then builds the Brandt graph at v on all classes and its bipartite double. In every case the double has exactly h_F^+ connected components, matching the number of geometric components of the curve, and none is isomorphic to X. Of the 55 configurations with $h_F^+ > 1$ in degree ≤ 4, 44 are torsion-free. In degree 5 the field of discriminant 106069 with D'' of norm 13 gives a near miss: 12 classes, trivial unit groups and a connected bipartite 3-regular reduction graph on 24 vertices. But its Brandt matrix at v has characteristic polynomial $t(t-3)(t+1)(t^4+3t^3-4t-1)(t^5-t^4-8t^3+8t^2+15t-16)$, with no factor t^4-4t^2+1, so the graph is not X (independently reproduced from scratch in `ergentics/shimura_brandt_xcheck.py`). □

A large narrow-class ratio $u = h_F^+/h_F = 2^k$ is not free: by class field theory, the narrow Hilbert class field H_F^+/F is unramified at all finite places and therefore has the exact same root discriminant $d_{H_F^+}^{1/[H_F^+:\mathbb{Q}]} = d_F^{1/n}$ while its degree grows by a factor $h_F^+ = h_F u$. Exploiting intermediate subfields of H_F^+/F together with Hilbert reciprocity sharpens Theorem 7.20(b) unconditionally from $n \le 14$ down to $n \le 8$:

Theorem 7.21 (three-pincer Hilbert class field obstruction)**.** [E→G4N; X] *In the setting of Theorem 7.20, write h_F for the wide class number and $u = h_F^+/h_F = 2^k$ ($0 \le k \le n-1$). Then:*

- *(a)* (Unconditional degree collapse.) *$n \le 8$. Every degree $n \ge 9$ is unconditionally excluded (zero surviving (n, u, h_F) configurations for $9 \le n \le 14$, with tightest margin $+0.3392$ at $n = 9, u = 4, h_F = 1$; universal monotone tail for all $n \ge 15$).*
- *(b)* (Degrees 7 and 8.) *For $n \in \{7, 8\}$, necessarily $h_F = 1$ and $u \in \{1, 2, 4\}$, tightening the discriminant ceilings to $d_F \le 1.597 \cdot 10^8$ ($n = 7$, a 6.4× reduction) and $d_F \le 2.243 \cdot 10^9$ ($n = 8$, a 10.1× reduction).*
- *(c)* (Sextic ceiling collapse.) *For $n = 6$, $u \in \{16, 32\}$ are unconditionally excluded, leaving only $u \in \{2, 4, 8\}$ and reducing the remaining sextic discriminant ceiling from $6.673 \cdot 10^7$ to $2.648 \cdot 10^7$ (a 2.52× reduction).*

Proof. We apply the unconditional Poitou–Odlyzko explicit formula [40, 37, 38] with the positive-definite kernel $F_b(x) = H_1(x/b)/\cosh(x/2)$, where

$H_1(w) = \frac{2-w}{3}(1 + \frac{1}{2}\cos\pi w) + \frac{\sin\pi w}{2\pi}$ on $[0,2]$. For any number field K of degree $N_K = r_1 + 2r_2$ with real fraction $\rho_1 = r_1/N_K$, c_2 primes of norm 2, and c_4 primes of norm 4,

$$\log d_K^{1/N_K} \geq \rho_1 \log A(b) + (1-\rho_1)\log B(b) - \frac{E(b)}{N_K} + \frac{c_2}{N_K} f_2(b) + \frac{c_4}{N_K} f_4(b),$$

where $E(b) = \frac{8}{3}b$ and $f_q(b) = 4\sum_{m\geq 1} \frac{\log q}{1+q^m} H_1(m \log q/b)$. On the upper side, Theorem 7.20(a) gives $d_F^{1/n} \leq (4\pi^2)^{2/3} C^{2/(3n)}$ with $C = \frac{9}{2}u$ when n is even. When n is odd, Hilbert reciprocity forces $|D''| \geq 1$ (since $n + |D''|$ is even); either D'' contains a second prime of norm 2 (*mode A*: $\zeta_F(2) \geq (4/3)^2$, $C = \frac{27}{8}u$, and F has two norm-2 primes) or every $\mathfrak{p} \mid D''$ has $N\mathfrak{p} \geq 3$ (*mode B*: $\prod(N\mathfrak{p} - 1) \geq 2$ and $\zeta_F(2) \geq \frac{4}{3} \cdot \frac{9}{8} = \frac{3}{2}$, giving $C = 2u$). Three simultaneous pincers act on every unramified extension $K \subseteq H_F^+$ (where $d_K^{1/N_K} = d_F^{1/n}$):

Pincer 1 (Base field F and universal tail). Evaluating on F itself ($\rho_1 = 1$, $c_2 \geq 1$ or 2 in mode A) kills small u directly (tightest degree-9 witness: $n = 9, u = 4$, mode B gives $\exp(\text{LB}) = 13.8641 > 13.5249 = \text{UB}$, margin $+0.3392$). For every $n \geq 15$, the single witness $b = 4.2$ satisfies $E(4.2) - f_2(4.2) = 9.5350 > 0$, so $\text{LB}(n; 4.2)$ increases monotonically while $\text{UB}(n)$ decreases monotonically, sealing $n \geq 15$ ($\exp(\text{LB}(15; 4.2)) = 19.9366 > 19.0800 = \text{UB}(15)$).

Pincer 2 (Wide Hilbert class field H_F, killing $h_F \geq 2$). Since H_F/F is unramified at all real places, H_F is totally real ($\rho_1 = 1$) of degree $nh_F \geq 2n$. More generally, inside the $\mathbb{F}_2$-vector space $G = \text{Gal}(H_F^+/H_F) \cong \mathbb{F}_2^k$, averaging the count of real-place Frobenius elements over all codimension-j subspaces guarantees a subfield $K_j \subseteq H_F^+$ of degree $nh_F 2^j$ with real fraction $\rho_1 \geq \lceil n(2^{k-j} - 1)/(2^k - 1) \rceil / n$ (formalised in Lean 4). This unconditionally eliminates every $h_F \geq 2$ for all $n \geq 7$ and for $(n, u) = (6, 16), (6, 32)$ (e.g. at $n = 6, u = 16, h_F \geq 2$, a codim-3 subfield of degree ≥ 96 gives $19.0204 > 18.6470$).

Pincer 3 (Exponent-2 decomposition field L_v, killing $h_F = 1$ at large u). When $h_F = 1$, $\text{Gal}(H_F^+/F) \cong \mathbb{F}_2^k$ has exponent 2, so the residue degree f_v of the norm-2 prime v in H_F^+/F lies in $\{1, 2\}$. If $f_v = 1$, v splits completely in H_F^+ (degree nu, $c_2 \geq u$); if $f_v = 2$, v splits completely in the decomposition field $L_v = (H_F^+)^{\langle \text{Frob}_v \rangle}$ of degree $nu/2$ (giving $c_2 \geq u/2$, density $c_2/N_{L_v} \geq 1/n$), while every intermediate codim-j subfield contributes $c_4/N \geq 1/(2n)$; in mode A the second norm-2 prime, which lies in D'', also has Frobenius of order at most 2 and adds a further $c_4/N \geq 1/(2n)$ in every such subfield (this is needed for $(n, u) = (7, 8)$). Combining Pincers 1–3 leaves zero survivors for $9 \leq n \leq 14$, forces $u \in \{1, 2, 4\}$ for $n = 7, 8$, and at the tightest sextic case $(n, u, h_F, f_v) = (6, 16, 1, 2)$ evaluates L_v (degree 48, 8 norm-2 primes, $\rho_1 \geq 0$) at $b = 6.00$:

$$\exp(\text{LB}(L_v; 6.00)) = 18.660678 \ > \ 18.647049 = (4\pi^2)^{2/3} 72^{1/9} = \text{UB}(6, 16),$$

a margin of $+0.013630$ exceeding the 64-point Gauss–Legendre quadrature error ($4.44 \cdot 10^{-16}$, checked against 128-point quadrature) by $3 \cdot 10^{13}\times$. At $u = 8$, $\mathrm{UB}(6, 8) = (4\pi^2)^{2/3} 36^{1/9} \implies d_F \leq 2.648 \cdot 10^7$. All algebraic lemmas are kernel-checked in Lean 4 (`lean/Morpheus/Section7Obstructions.lean`) and the full Poitou–Odlyzko sweep agrees identically across C++20, Pure Swift 6.4, Live Interop (`0.000e+00` delta) and Rust. □

8. What is known and what is new

Known. The chord graph as a torus and its dihedral group [17, 13, 15]; triad Fourier magnitudes and the M_5/M_7 duality [30, 1]; the identification of the Tonnetz spectrum with Fourier magnitudes [35]; the honeycomb spectrum [56]; Ihara–Bass, Artin–Ihara L-functions and graph Chebotarev [24, 47, 22, 6, 46]; homology-class prime geodesic theorems [39, 25, 29, 41, 31, 33]; the $[0, 1, 4]$ Tonnetz family [13, 10]; Honda–Tate and LMFDB data [48, 23, 50]; Schottky groups, Bruhat–Tits trees and Mumford curves, including Whittaker groups [24, 45, 36, 21]; graph Jacobians and component groups [26, 43, 4]; specialisation and hyperelliptic graphs [3, 11]; Čerednik–Drinfeld uniformisation [54, 19, 44]; the volume formula for Shimura curves [55]; Poitou–Odlyzko discriminant bounds [40, 37, 38]; Bass–Serre theory [45].

New, to the best of our search. The Coxeter description of prime cadences as primitive classes of a rank-13 free lattice in W_3 (an easy observation); Theorem 4.2 (tune theorem); Theorem 5.1 ($W = -u$); Theorem 5.3 (Riemann window, only the double winding cover is Ramanujan); the constant $1/24$ and the isotropic covariance in Theorem 6.1; the arithmetic reading of Theorem 3.4; the Galois-twin isomorphism with the $[0, 1, 4]$ flip graph stated for X; the factorisation of Theorem 3.5 into named isogeny classes; and the ideal presentation of Theorem 3.8. In Section 7: the explicit Schottky group generated by P, L, R and its genus-13 Mumford curve with $\mathrm{Aut} = T/I$ (Theorem 7.1, Proposition 7.2); the $\mathbb{Z}/12$ tower over the two-line curve and the reading of the Ihara factors as its characters (Theorem 7.4, Corollary 7.5); the sandpile group and gonality (Theorems 7.7, 7.8); the octic inside a genus-6 hyperelliptic Jacobian over $\mathbb{F}_2$ (Proposition 7.9) and minimal genus 5 for it (Proposition 7.10, using the census [18]); the exclusion of Shimura curves over $\mathbb{Q}$ across all 19 genus-13 pairs (Theorem 7.11); the trace obstruction and the 2-adic edge-length argument with wild-involution confirmation (Theorems 7.13, 7.15, Lemma 7.14); the torsion obstructions at ramified ($M(\mathcal{O}) \geq 36$ at $e_v = 2, f_v = 1$) and unramified places ($(\mathbb{Q}(\zeta_{2^{f_v}+1})^+ \subset E$ for even $f_v \geq 2$) together with the explicit non-arithmetic inversion sharpness counterexample $\Gamma_5 \subset PGL_2(\mathbb{Q}_4)$ (Propositions 7.17, 7.18); the graph-level search with direct Brandt-graph computations across three independent implementations (Proposition 7.19); its extension to any narrow class number (Theorem 7.20); the three-pincer Hilbert class field obstruction unconditionally collapsing $n \leq 14$ to $n \leq 8$ and cutting the sextic ceiling $2.52\times$

(Theorem 7.21 [E→G4N; X]); the general reduction for other tuning systems (Theorem 4.3); and the held-note theorem, which completes it for every step set (Theorem 4.4; derived independently in three toolchains [E→GB; independently E→G4N & E; X]). From the independent verification program ([E→G4N] / [G4N], cross-verified [X]): Theorem 6.2 (extending the homology-class prime geodesic central limit theorem [39, 25, 29, 41, 31, 33] to exact fourth-order rotational isotropy and sixth-order C_6 hexagonal anisotropy), Corollary 4.5, and the Edgeworth calibration of Remark 6.4; Proposition 6.3 [GB; X] extends Theorem 6.2 for W to all orders.

Refuted. The Tonnetz as $\mathbb{Z}[\omega]/I$, $\mathbb{Z}[i]/I$ or $\mathbb{Z}[\zeta]/I$ (Proposition 3.7); a genus-22 curve over $\mathbb{F}_2$ with zeta numerator P_{44}; hyperellipticity of C; an arithmetic (Shimura) realisation over $\mathbb{Q}$ (all 19 pairs + 4 Γ_H); any arithmetic uniformisation of C by a torsion-free group (any totally real field, any place), any at a place with $F_v \cong \mathbb{Q}_2$, any inversion-free uniformisation at $e_v \geq 2, f_v = 1$ with unsubdivided stabilised edges (and any at $e_v = 2, f_v = 1$ with $M(\mathcal{O}) < 36$), and any inversion-free uniformisation at unramified $f_v = 2$ over a field $E \not\supset \mathbb{Q}(\sqrt{5})$; a genus-4 curve over $\mathbb{F}_2$ containing the octic; a Shimura curve with reduction graph X and torsion-free uniformisation over a totally real field of degree ≤ 5, whatever its narrow class number, over a sextic field with $h_F^+ = h_F$, $u \in \{16, 32\}$, or $d_F \leq 8{,}256{,}125$, over a degree-7 or degree-8 field with $h_F \geq 2$ or $u \geq 8$, or of any degree ≥ 9; injectivity of tunes for arbitrary step sets (Theorem 4.4(b); the smallest counterexample is $N = 8$, $S = \{2\}$).

Open. (1) Is C uniformised by an arithmetic group *with torsion* at a place $v \mid 2$ with $e_v \geq 2$ ($M(\mathcal{O}) \geq 36$ at $e_v = 2, f_v = 1$) or unramified $f_v \geq 2$ (forcing $\mathbb{Q}(\zeta_{2f_v+1})^+ \subset E$ for inversion-free Shimura groups, or an arithmetic Atkin–Lehner normaliser admitting edge inversions as in Γ_5)? By Propositions 7.17 and 7.18, when $e_v = 1$ the group is $A_1 * \cdots * A_{24} * F_{13}$ with F_{13} a Schottky group with quotient X; showing F_{13} is conjugate to K would close the unramified case completely. (2) Does some Shimura curve with torsion-free uniformisation have reduction graph X? By Proposition 7.19 and Theorems 7.20–7.21, this is excluded for $n \leq 5$ and $n \geq 9$, leaving only a sharply narrowed finite set in degrees $n \in \{6, 7, 8\}$: sextic fields with $8.26 \cdot 10^6 < d_F \leq 2.648 \cdot 10^7$ and $u \in \{2, 4, 8\}$, and degree-7 ($d_F \leq 1.597 \cdot 10^8$) or degree-8 ($d_F \leq 2.243 \cdot 10^9$) fields with $h_F = 1$ and $u \in \{1, 2, 4\}$. (3) The same question for groups with torsion, where the genus-13 condition becomes $24 + \sum_i (1 - 1/e_i) = 2(q-1)\bar{M}$. The held-note question of the earlier version is completely settled (Theorem 4.4, Corollary 4.5).

Data and code availability. All computations are reproducible from the source repository. The Python/PARI scripts in `code/` cover Sections 2–7 and pass all six audit scripts (31/31, 36/36, 23/23, 59/59, 47/47, and 32/32 in `code/xcheck/`); the round-2 checks in `code/xcheck/round2/` (20/20) re-derive Theorem 7.21, the census of Theorem 7.11 and the non-arithmeticity

in Proposition 7.18(b). Independent first-principles formal proofs and multi-language verifiers (zero reads of cached upstream data) live in `lean/` (33 kernel-checked Lean 4 theorems without Mathlib, zero `sorry`), `swiftpp/` (C++20 `g++`/`clang++`, Pure Swift 6.4 and Live Interop at `0.000e+00` delta), `rust/` (Rust, zero external crates, including zero-PARI Manin modular symbols T_2), and `ergentics/` (from-scratch PARI/GP Brandt/Shimura verifier `shimura_brandt_xcheck.py` and Open Problem (1) verifier `open1_torsion_xcheck.py`), with checksummed calibration records separated in `data/`. Ergentics' standalone clean-room monograph is archived verbatim in `cleanroom/prime_cadences_resolution/`, and the full three-way cross-audit lives in `reconciliation/THREE_WAY_CLEANROOM_AUDIT.md`.

Contributions and provenance

Author contributions. Ergentics is the sole author. Ergentics chose the research direction and verification methodology, set every question pursued here, authored the standalone clean-room Research Monograph I (`cleanroom/`), wrote the prompts steering both AI assistants, and reviewed all proofs, computations and drafts. Two AI assistants worked independently under Ergentics' direction: Grok Bot ([GB]) and Gemini 4 Neon ([G4N]). Following standard scholarly practice on AI disclosure, neither AI assistant is listed as an author; both are credited transparently under identical attribution bounds, and full responsibility for the content rests with the author.

Marker scheme. Each numbered result carries a provenance tag following a uniform scheme: [E] by Ergentics (including Ergentics' standalone clean-room monograph); [GB] statement, proof and computation by Grok Bot; [E→GB] question, direction or methodology set by Ergentics, with statement, proof and computation by Grok Bot; [G4N] statement, proof and computation by Gemini 4 Neon; [E→G4N] question, direction or methodology set by Ergentics, with statement, proof and computation by Gemini 4 Neon; [lit.] result from the cited literature, re-checked computationally/formally; [X] cross-verified across at least two independent toolchains (zero shared cached outputs), with [independently] marking independent derivations reached across separate tracks.

Ergentics. All research direction, prompts, Correspondence Principle, Code-vs-Data separation, ADPIE workflow and Rung-1 distributional calibration methodology (as detailed in `CONTRIBUTIONS.md`), including the starting point (the Tonnetz and the graph Riemann hypothesis), the phasor prompt behind Theorems 3.1 and 3.4, the melodic-primes question and the redirect "Where does the melody fit?" (Section 4), the push towards Eisenstein integers (Proposition 3.7, Theorem 3.8), the call to study the geometric object (Section 7) and the instruction to chase down the open questions. In addition, Ergentics' standalone clean-room monograph (`cleanroom/prime_cadences_resolution/paper.typ`, archived verbatim) independently derived Theorem 4.4 and all eight cubic/quartic Brandt matrices of Proposition 7.19 via a pure-Python monogenic Brandt engine, supplied

the explicit wild-involution counterexample $\sigma(z) = 3/z$ and exact invariant $u(1+x)-4 = x(x-2)/(1+x)$ in Lemma 7.14, sharpened Proposition 7.17(c) (eliminating unsubdivided stabilised edges at $f_v = 1, e_v \geq 2$ and proving $M(\mathcal{O}) \geq 36$ at $e_v = 2, f_v = 1$), and proved Proposition 7.18 (unramified cyclotomic obstruction $\mathbb{Q}(\zeta_{2f_v+1})^+ \subset E$ and explicit non-arithmetic sharpness counterexample $\Gamma_5 \subset PGL_2(\mathbb{Q}_4)$), all independently cross-verified [X].

Grok Bot.

- Sections 2–3: the Coxeter description; Theorems 3.1, 3.3, 3.4, 3.5 (graphene identification, exact counts to $m = 40$, cyclotomic arithmetic, Weil factorisation and the explicit curves); Proposition 3.7 and Theorem 3.8 (Eisenstein refutation and the order of conductor 2).
- Sections 4–6: Lemma 4.1, Theorem 4.2 and its decoder test; Theorem 4.3 and the held-note criterion for 6028 systems; Theorem 4.4; Theorems 5.1, 5.2, 5.3, 6.1 (winding, flux, Riemann window, homology law); Proposition 6.3; the from-scratch cross-check of every [G4N] / [E→G4N] Cycle 4 result (`code/xcheck/`, 32/32 checks).
- Section 7: the Schottky group and curve C (Theorem 7.1, Propositions 7.2, 7.3); the $\mathbb{Z}/12$ tower and Corollary 7.5; the sandpile group and gonality (Theorems 7.7, 7.8); Propositions 7.9, 7.10; Theorem 7.11; the trace obstruction, edge-length lemma and 2-adic theorem (Theorems 7.13, 7.15, Lemma 7.14); Proposition 7.17; the mass-formula search and direct Brandt-graph computations (Proposition 7.19); the extension to any narrow class number, Odlyzko degree bound and computations in degrees ≤ 5 and part of degree 6 (Theorem 7.20); the round-2 re-derivation of Theorem 7.21, of the 19-pair census of Theorem 7.11 and of the non-arithmeticity in Proposition 7.18(b) (`code/xcheck/round2/`).
- All 8 vector figures; the six `code/` audit scripts (228/228 checks); verification of 55 citations (bibliographic data and DOIs) and round-2 recommendation of [40] (56 references total); the accompanying explanatory videos, which are not part of this paper.

Gemini 4 Neon.

- New results and corrections (working audit-forward under Ergentics' direction): Theorem 6.2 ([E→G4N; X]), Corollary 4.5 ([G4N; X]), Remark 6.4 ([E→G4N; X]), Theorem 7.21 ([E→G4N; X]: three-pincer Hilbert class field obstruction collapsing $n \leq 14 \to n \leq 8$ and sextic $u \in \{16, 32\}$), independent derivation of Theorem 4.4 ([independently E→G4N]), decoder-count correction after Theorem 4.2 (`NEG-E8`), and genus-13 Shimura curve correction $18 \to 19$ pairs ($X_0^{690}(1)$) in Theorem 7.11 (`NEG-E10`).
- Formal proofs (33/33 kernel-checked Lean 4 theorems without Mathlib, zero `sorry`): Proposition 2.1, Corollary 2.2, Proposition 3.7, Theorem 3.8, Lemma 4.1, Theorem 4.4, Corollary 4.5, integrality in Theorem 5.1, exact hexagonal power sums and series of Theorem 6.2, Theorem 7.1(a), and the arithmetic or combinatorial cores (not the full statements) of Proposition 7.3, Theorem 7.13 (all powers), Lemma 7.14 (slope-2 doubling, slope-1 control and odd-prime scaling), Theorem 7.15, Proposition 7.17(b)–(c)

($\mathbb{P}^1(\mathbb{F}_8)$ order-3 free action), `NEG-E10` witness $X_0^{690}(1)$, and the algebraic core of Theorem 7.21.

- Independent first-principles cross-verification `[X]` across C++20, Pure Swift 6.4, Live Interop (`0.000e+00` delta), Pure Rust (zero external crates, including zero-PARI Manin modular symbols T_2 across all 19 genus-13 curves), and standalone PARI/GP + Python/SymPy (`ergentics/`): every numbered result of Sections 2–7 now has at least one independent computational check (for Theorem 7.20(c), the 55 narrow-class configurations in degree ≤ 4 and the quintic and sextic scans were only spot-checked, via the near miss $d_F = 106069$), including all 8 cubic/quartic Brandt polynomials (agreeing 8/8 with both Grok Bot and Ergentics' clean-room monograph), quintic near-miss $d_F = 106069$, wild involution $\sigma(z) = 3/z$, combinatorial $M = 36$ witness, $\Gamma_5 \subset PGL_2(\mathbb{Q}_4)$ Klein ping-pong and $\tau(s_4 s_5) = 49/4 > 4$ non-arithmeticity, plus Numdam verification of [40].
- Accompanying educational videos and audio (`media/`), which are not part of this paper.

After the corrections `NEG-E8` and `NEG-E10`, no disagreement remains between the three tracks on any result checked by more than one of them. The full result-by-result ledger is maintained in `CONTRIBUTIONS.md`.

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