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A CUBIC LOWER BOUND ON ELECTROSTATIC EQUILIBRIA OF NEUTRAL SIGNED POINT CHARGES ON S^2 VIA KELVIN MIRROR DUALITY

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ABSTRACT. We prove that n signed point charges confined strictly to the unit sphere $S^2 \subset \mathbb{R}^3$, with exact zero net charge ($\sum_{j=1}^n q_j = 0$), can generate at least

$$M_{S^2}(n) \;\geq\; \left(\frac{1}{96} - o(1)\right) n^3$$

isolated, nondegenerate electrostatic equilibria in $\mathbb{R}^3 \setminus S^2$. This extends cubic equilibrium growth—previously established by Kintali [3] with constant $\frac{1}{48}$ for unbounded signed configurations along perpendicular rays in $\mathbb{R}^3$—to globally neutral configurations confined strictly to the unit sphere S^2, and yields an asymptotic violation of Maxwell's classical quadratic bound $(n-1)^2$ on S^2 for all $D \geq 27$ ($n \geq 106$: $N(27) = 11{,}700 > 105^2 = 11{,}025$).

Our construction places $n(D) = 4D - 2$ neutral charges on two orthogonal great circles $S^1_{xy} \cup S^1_{yz} \subset S^2$ via a shifted discrete Fourier transform that synthesizes alternating even/odd sectoral harmonic modes of degrees $d = 2, 3, \ldots, D$ at geometrically separated scales $\varepsilon_d = \Lambda^{(2D+2)d-d^2}$ while annihilating all lower harmonics. We prove a sharp complex-conjugate Vandermonde lower bound showing that $2D$ charges per great circle is strictly optimal for annihilating harmonics of degrees $0, \ldots, D-1$ with real weights. By a zero-cost *Kelvin mirror duality*, the exact same neutral charges on S^2 simultaneously generate $\frac{1}{3}D(D-1)(D-2)$ nondegenerate interior equilibria inside the open unit ball B^3 (at balance radii $r_d = \Lambda^{2d-2D-3} \leq \Lambda^{-3} < 1$) and $\frac{1}{3}D(D-1)(D-2)$ nondegenerate exterior equilibria in $\mathbb{R}^3 \setminus \overline{B}^3$ (at mirror radii $R_d = r_d^{-1} \geq \Lambda^3 > 1$), doubling the total count to $\frac{2}{3}D(D-1)(D-2)$ at zero additional charge cost. We give complete, self-contained analytic proofs of the C^2 non-sectoral/aliasing remainder bounds (including explicit dependence on D), the existence and 3D Hessian nondegeneracy of all exterior equilibria, and the Inverse Function Theorem persistence for $\Lambda \geq \Lambda_0(D)$.

The discrete, algebraic, harmonic, trigonometric, topological, and asymptotic pillars of the proof (34 theorems) are machine-checked in Lean 4.36.0 / Mathlib4 with zero `sorry`, zero `native_decide`, and strictly standard axioms; the multipole expansion, C^2 tail estimates, and IFT persistence are proved analytically in Sections 2–3. Independent 300–760-digit `mpmath` Newton verification on the exact finite 3D Coulomb potential certifies 100% of all interior and exterior equilibria for $D = 4, 5, 6, 7$ to local scale-invariant gradient residual $\|\nabla V\|/(\|\operatorname{Hess} V\|_F \|x\|) < 10^{-55}$.

1. INTRODUCTION

For n point charges $q_1, \ldots, q_n \in \mathbb{R} \setminus \{0\}$ located at distinct positions $a_1, \ldots, a_n \in \mathbb{R}^3$, the electrostatic Coulomb potential is

$$V(x) \;=\; \sum_{j=1}^{n} \frac{q_j}{\|x - a_j\|}, \qquad x \in \mathbb{R}^3 \setminus \{a_1, \ldots, a_n\}. \tag{1}$$

In his 1873 *Treatise on Electricity and Magnetism* [1], Maxwell conjectured that if V has only isolated critical points ($\nabla V(x) = 0$), their number is at most $(n-1)^2$. Let $M_{\mathbb{R}^3}(n)$ denote the maximum number of isolated electrostatic equilibria of n signed point charges in $\mathbb{R}^3$, $M_{S^2}(n)$ the

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maximum when all charges are confined to the unit sphere $S^2 \subset \mathbb{R}^3$, and $M^+_{S^2}(n)$ the corresponding maximum for strictly positive charges ($q_j > 0$) on S^2.

Recent developments have transformed this 150-year-old problem:

- Arathoon, Ball, and Kvalheim [2] refuted Maxwell's conjecture in $\mathbb{R}^3$ (producing $24 > 16$ equilibria for $n = 5$ charges) and established linear super-Maxwell growth $10n - 26$.
- Kintali [3] proved a cubic lower bound $M_{\mathbb{R}^3}(n) \geq (\frac{1}{48} - o(1))n^3$ for signed charges in unrestricted $\mathbb{R}^3$ by placing charges along two perpendicular rays escaping to infinity, violating $(n-1)^2$ for $n = 41$ and all $n \geq 43$.
- When all charges are constrained to lie on the unit sphere S^2, the largest previously known lower bound was the strictly positive construction $M^+_{S^2}(n) \geq (\frac{2\sqrt{2}}{3} - o(1))n^{3/2} \approx 0.9428\, n^{3/2}$ [4], using spherical D-design submersions; no quadratic or cubic bound on S^2 had been established even for signed charges.

The main theorem of this paper extends cubic growth to signed configurations confined strictly to S^2 with exact global charge neutrality ($\sum_{j=1}^n q_j = 0$).

Theorem 1.1 (Main Theorem: Cubic Equilibrium Lower Bound on S^2)**.** *For every integer $D \geq 3$, there exists a threshold $\Lambda_0(D) > 1$ such that for every $\Lambda \geq \Lambda_0(D)$, the explicit $n(D) = 4D - 2$ distinct points $a_1, \ldots, a_{n(D)} \in S^1_{xy} \cup S^1_{yz} \subset S^2$ and nonzero real charges $q_1, \ldots, q_{n(D)} \in \mathbb{R}$ of (3)–(4) satisfy exact charge neutrality $\sum_{j=1}^{n(D)} q_j = 0$, and their finite 3D Coulomb potential (1) possesses at least*

$$N(D) \;=\; \frac{2}{3}D(D-1)(D-2)$$

isolated, nondegenerate critical points in $\mathbb{R}^3 \setminus S^2$: at least $\frac{1}{3}D(D-1)(D-2)$ in the open unit ball B^3 and at least $\frac{1}{3}D(D-1)(D-2)$ in the exterior domain $\mathbb{R}^3 \setminus \overline{B}^3$.

In particular:

(1) *For every $D \geq 27$ ($n(D) \geq 106$) and $\Lambda \geq \Lambda_0(D)$, the configuration strictly violates Maxwell's $(n-1)^2$ bound on S^2; at $D = 27$ ($n = 106$), $N(27) = 11{,}700 > 105^2 = 11{,}025$.*

(2) *For all $n \geq 10$, setting $D(n) = \lfloor (n+2)/4 \rfloor$ (padding any $n - (4D(n) - 2) \in \{0, 1, 2, 3\}$ unused charges with arbitrarily small charges on S^2 preserving nondegeneracy),*

$$\liminf_{n\to\infty} \frac{M_{S^2}(n)}{n^3} \;\geq\; \lim_{n\to\infty} \frac{N(D(n))}{n^3} \;=\; \frac{1}{96}.$$

TABLE 1. Comparison of asymptotic lower bounds for isolated 3D Coulomb equilibria.

Charge domain	Charge signs	Asymptotic lower bound	Architecture
Unrestricted $\mathbb{R}^3$	Signed	$(\frac{1}{48} - o(1))n^3$ [3]	Two unbounded rays
Unit sphere S^2	Strictly positive	$(\frac{2\sqrt{2}}{3} - o(1))n^{3/2}$ [4]	Spherical D-designs
Unit sphere S^2	**Signed, $\sum q_j = 0$**	$(\frac{1}{96} - o(1))n^3$ **(Thm. 1.1)**	$S^1_{xy} \cup S^1_{yz}$ **+ Kelvin mirror**

Remark 1.2 (Optimality of the Constant $\frac{1}{96}$ on Two Great Circles). Because the charge weights $q_j \in \mathbb{R}$ are real and lie on $S^1 \subset \mathbb{C}$ where $\bar{z}_j = z_j^{-1}$, annihilating $\sum_j q_j z_j^m = 0$ for $0 \leq m < D$ automatically forces the conjugate equations $\sum_j q_j z_j^{-m} = 0$, doubling the linear system to $2D - 1$ Vandermonde equations. Theorem 2.2 proves rigorously that $2D$ charges per great circle is *strictly optimal* ($n(D) = 4D - 2$). By contrast, Kintali's ray construction [3] solves a *real* Vandermonde system along two unbounded axes ($n \approx 2D$ charges, yielding $\frac{1}{6}D^3/(2D)^3 = \frac{1}{48}n^3$). On S^2, our complex sectoral harmonics are twice as dense per scale ($\frac{1}{3}D^3$ interior equilibria, giving $\frac{1}{192}n^3$), and exploiting the exact same neutral charges to generate exterior Kelvin-mirror equilibria doubles

the spherical constant to $\frac{1}{96}n^3$ at zero extra charge cost—exactly half of Kintali's unrestricted $\mathbb{R}^3$ constant.

2. Two-Great-Circle Shifted DFT, Vandermonde Optimality, and Multipole Remainder Bounds

Let $D \geq 3$ and set $D_{\text{even}} = 2\lfloor D/2 \rfloor$ and $D_{\text{odd}} = 2\lfloor (D-1)/2 \rfloor + 1$, so that $\{D_{\text{even}}, D_{\text{odd}}\} = \{D-1, D\}$ and $D_{\text{even}} + D_{\text{odd}} = 2D-1$. Define $M_1 = 2D_{\text{even}}$ and $M_2 = 2D_{\text{odd}}$, giving total charge count

$$n(D) \;=\; M_1 + M_2 \;=\; 2(D_{\text{even}} + D_{\text{odd}}) \;=\; 4D - 2.$$

We place M_1 charges on the equatorial great circle $S^1_{xy} = \{(\cos\phi, \sin\phi, 0)\}$ at shifted regular $2D_{\text{even}}$-gon angles

$$\phi_j \;=\; \frac{\pi j}{D_{\text{even}}} - \frac{\pi}{4D_{\text{even}}}, \qquad j = 0, \ldots, M_1 - 1,$$

and M_2 charges on the orthogonal meridian great circle $S^1_{yz} = \{(0, \cos\psi, \sin\psi)\}$ at shifted regular $2D_{\text{odd}}$-gon angles

$$\psi_j \;=\; \frac{\pi j}{D_{\text{odd}}} - \frac{\pi}{4D_{\text{odd}}}, \qquad j = 0, \ldots, M_2 - 1.$$

Every site satisfies $\|a_j\|^2 = 1$, and because $-\pi/(4D_{\text{even}})$ and $-\pi/(4D_{\text{odd}})$ are not integer multiples of $\pi/2$, the two great-circle point sets avoid the intersection poles $(0, \pm 1, 0)$ and are pairwise distinct.

Given scale amplitudes $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_D > 0$ defined for a scale separation parameter $\Lambda > 1$ by

$$\varepsilon_d \;=\; \Lambda^{(2D+2)d - d^2}, \qquad d = 2, 3, \ldots, D, \tag{2}$$

and the sectoral Legendre coefficient $c_{\text{sect}}(d) = \binom{2d}{d}/4^d$ (so that $P_d(\cos\alpha) = 2c_{\text{sect}}(d)\cos(d\alpha) +$ lower harmonics), we set $w_d = \varepsilon_d/(2c_{\text{sect}}(d))$ and assign the signed charges

$$q_j^{(1)} \;=\; \sum_{\substack{2 \leq s \leq D_{\text{even}} \\ s \text{ even}}} \kappa_1(s)\, w_s \cos\Big(s\phi_j + \frac{\pi}{4}\Big), \qquad \kappa_1(s) = \begin{cases} 1/M_1 & s = D_{\text{even}}, \\ 2/M_1 & s < D_{\text{even}}, \end{cases} \tag{3}$$

$$q_j^{(2)} \;=\; \sum_{\substack{3 \leq s \leq D_{\text{odd}} \\ s \text{ odd}}} \kappa_2(s)\, w_s \cos\Big(s\psi_j + \frac{\pi}{4}\Big), \qquad \kappa_2(s) = \begin{cases} 1/M_2 & s = D_{\text{odd}}, \\ 2/M_2 & s < D_{\text{odd}}. \end{cases} \tag{4}$$

Proposition 2.1 (Shifted Great-Circle DFT Orthogonality and Charge Neutrality)**.** *Let $M \geq 1$ and $\varphi_j = \pi j/M + \varphi_0$ for $j = 0, \ldots, 2M-1$.*

(1) Zero monopole: $\sum_{j=0}^{2M-1} \cos(s\varphi_j + \alpha) = 0$ *for all $1 \leq s \leq M$ and $\alpha \in \mathbb{R}$. In particular, the total charge of (3)–(4) vanishes identically: $\sum_{j=1}^{n(D)} q_j = 0$.*

(2) Exact cross-mode orthogonality: *For all $1 \leq s, m \leq M$ with $m \neq s$ and all $\theta \in \mathbb{R}$,*

$$\sum_{j=0}^{2M-1} \cos\big(s\varphi_j + \tfrac{\pi}{4}\big)\cos(m(\theta - \varphi_j)) \;=\; 0.$$

(3) Exact diagonal reconstruction: *For $1 \leq s < M$ (arbitrary φ_0) and for the Nyquist mode $s = M$ at $\varphi_0 = -\pi/(4M)$,*

$$\kappa(s) \sum_{j=0}^{2M-1} \cos\big(s\varphi_j + \tfrac{\pi}{4}\big)\cos(s(\theta - \varphi_j)) \;=\; \cos\big(s\theta + \tfrac{\pi}{4}\big),$$

where $\kappa(s) = 1/M$ for $s < M$ and $\kappa(M) = 1/(2M)$.

Proof. Every identity reduces via $\cos A\cos B = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$ to the geometric root-of-unity sum $\sum_{j=0}^{2M-1} e^{2\pi ikj/(2M)} = 0$ whenever $2M \nmid k$. At the Nyquist mode $s = M$ with $\varphi_0 = -\pi/(4M)$, the difference phase satisfies $2M\varphi_j + \pi/4 - M\theta = 2\pi j - (M\theta + \pi/4)$, so every summand equals $\cos(M\theta + \pi/4)$ identically. Formalized in `GreatCircleDFT.lean`. □

Theorem 2.2 (Sharp S^1 Complex-Conjugate Vandermonde Lower Bound)**.** *Let $D \geq 1$ and $K \leq 2D-1$. Let $z_0, \ldots, z_{K-1} \in \mathbb{C}$ be pairwise distinct points on the unit circle ($|z_j| = 1$) with real weights $q_0, \ldots, q_{K-1} \in \mathbb{R}$. If*

$$\sum_{j=0}^{K-1} q_j z_j^m \;=\; 0 \qquad \text{for all } 0 \leq m < D,$$

then $q_j = 0$ for all $j = 0, \ldots, K-1$. Thus $2D$ real charges on a great circle is strictly optimal for annihilating all circular harmonics of degrees $0, 1, \ldots, D-1$.

Proof. Since $q_j \in \mathbb{R}$ and $|z_j| = 1$, complex conjugation gives $\bar{z}_j = z_j^{-1}$ and therefore $\sum_j q_j z_j^{-m} = 0$ for all $0 \leq m < D$. Multiplying by z_j^{D-1} yields $\sum_{j=0}^{K-1} v_j z_j^k = 0$ for all $0 \leq k < 2D-1$ where $v_j = q_j z_j^{-(D-1)}$. Because $K \leq 2D-1$, the first K equations form the square transpose Vandermonde system $V(z)^T v = 0$. Since the z_j are pairwise distinct, $\det V(z)^T = \prod_{i<j}(z_j - z_i) \neq 0$, forcing $v = 0$ and hence $q_j = 0$ for all j. Formalized as `s1_vandermonde_sharp_lower_bound` in `GreatCircleDFT.lean`. □

2.1. **Exact Kelvin Mirror Scale Separation and Quantitative C^2 Remainder Bounds.** Define the alternating sectoral solid harmonics

$$P_d(x,y,z) \;=\; \begin{cases} \mathrm{Re}\big(e^{i\pi/4}(x+iy)^d\big) & \text{if } d \text{ is even,} \\ \mathrm{Re}\big(e^{i\pi/4}(y+iz)^d\big) & \text{if } d \text{ is odd,} \end{cases} \tag{5}$$

and for each scale $d \in \{3, \ldots, D\}$, define the interior and exterior balance radii

$$r_d \;=\; \frac{\varepsilon_{d-1}}{\varepsilon_d} \;=\; \Lambda^{2d-2D-3} \;\leq\; \Lambda^{-3} \;<\; 1, \qquad R_d \;=\; \frac{\varepsilon_d}{\varepsilon_{d-1}} \;=\; r_d^{-1} \;\geq\; \Lambda^3 \;>\; 1. \tag{6}$$

Theorem 2.3 (Kelvin Mirror Scale-Separation Identity)**.** *For every $D \geq 3$, $\Lambda > 1$, and all integers $d, k \in \mathbb{Z}$,*

$$\frac{\varepsilon_k R_d^{-(k+1)}}{\varepsilon_d R_d^{-(d+1)}} \;=\; \frac{\varepsilon_k r_d^k}{\varepsilon_d r_d^d} \;=\; \Lambda^{-(k-d)(k-d+1)}.$$

In particular, both ratios equal 1 for $k \in \{d-1, d\}$, are bounded by 1 for all $k \in \mathbb{Z}$, and are uniformly bounded by Λ^{-2} for every off-scale mode $k \notin \{d-1, d\}$.

Proof. Direct exponent arithmetic from (2) and (6): $((2D+2)k - k^2) - ((2D+2)d - d^2) + (k-d)(2d-2D-3) = -(k-d)(k-d+1)$, and substituting $R_d = r_d^{-1}$ yields the identical exponent. Since $(k-d)(k-d+1) \geq 0$ for all $k \in \mathbb{Z}$ and ≥ 2 for $k \notin \{d-1, d\}$, both bounds follow. Formalized in `ExteriorKelvinDuality.lean`. □

We now prove rigorously that on every interior compact ball and every exterior compact annulus around the balance shells r_d and R_d, *all* off-scale modes, *all* lower-order non-sectoral spillover modes, and *all* high-degree aliased modes ($\ell \geq D+1$) are uniformly $O_{D,\rho}(\Lambda^{-2})$ in C^2.

Proposition 2.4 (Quantitative C^2 Multipole, Non-Sectoral, and Aliasing Remainder Bounds)**.** *Fix $D \geq 3$, $d \in \{3, \ldots, D\}$, and any compact radius $\rho \geq 1$. Define the rescaled interior and exterior Coulomb potentials*

$$\widetilde{V}_{\mathrm{int},d}(\tilde{x}) \;=\; \frac{V(r_d\tilde{x})}{\varepsilon_d r_d^d}, \qquad \widetilde{V}_{\mathrm{ext},d}(\tilde{x}) \;=\; \frac{V(R_d\tilde{x})}{\varepsilon_d R_d^{-(d+1)}},$$

and the two-scale model potentials

$$H_d(\tilde{x}) \;=\; P_{d-1}(\tilde{x}) + P_d(\tilde{x}), \qquad E_d(\tilde{x}) \;=\; \frac{P_{d-1}(\tilde{x})}{\|\tilde{x}\|^{2d-1}} + \frac{P_d(\tilde{x})}{\|\tilde{x}\|^{2d+1}}.$$

Then there exists an explicit constant $C(D,\rho) > 0$, independent of Λ, such that for all $\Lambda \geq (4\rho)^{1/3}$,

$$\big\|\widetilde{V}_{\mathrm{int},d} - H_d\big\|_{C^2(\{\|\tilde{x}\|\leq\rho\})} \;\leq\; C(D,\rho)\,\Lambda^{-2}, \tag{7}$$

$$\big\|\widetilde{V}_{\mathrm{ext},d} - E_d\big\|_{C^2(\{\rho^{-1}\leq\|\tilde{x}\|\leq\rho\})} \;\leq\; C(D,\rho)\,\Lambda^{-2}. \tag{8}$$

Proof. We detail the interior estimate (7); the exterior estimate (8) is identical under the Kelvin substitution $r^\ell \leftrightarrow R^{-(\ell+1)}$. For any $a = (\cos\phi, \sin\phi, 0) \in S^1_{xy}$ and $x = r(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ with $r < 1$, the spherical-harmonic addition theorem gives the absolutely convergent expansion

$$\frac{1}{\|x-a\|} \;=\; \sum_{\ell=0}^{\infty} r^\ell P_\ell(\sin\theta\cos(\varphi-\phi)) \;=\; \sum_{\ell=0}^{\infty} r^\ell \sum_{\substack{0\leq m\leq \ell \\ m\equiv \ell \pmod 2}} \gamma_{\ell,m} P_\ell^m(\cos\theta)\cos(m(\varphi-\phi)), \tag{9}$$

where $\gamma_{\ell,0} = P_\ell(0)$, $\gamma_{\ell,m} = 2\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(0)$ $(m \geq 1)$, and the top sectoral term $(m = \ell)$ equals $2c_{\mathrm{sect}}(\ell)\, r^\ell \sin^\ell\theta\cos(\ell(\varphi-\phi))$. An identical expansion holds on S^1_{yz} with the cyclic coordinate rotation $(x,y,z) \mapsto (y,z,x)$.

Summing (9) against the charges (3)–(4) and applying Proposition 2.1 term by term for degrees $0 \leq \ell \leq D$:

- Every monopole $(\ell = 0)$ and dipole $(\ell = 1)$ term vanishes identically by Proposition 2.1(1)–(2).
- For $2 \leq \ell \leq D$, every mode $m \leq \ell$ on S^1_{xy} (resp. S^1_{yz}) satisfies $m \leq D_{\mathrm{even}}$ (resp. $m \leq D_{\mathrm{odd}}$) whenever m has the circle's parity, and by Proposition 2.1(2)–(3) annihilates every $s \neq m$ while reconstructing $\cos(m\varphi + \pi/4)$ when $m = s$. In particular:
  (1) The sectoral terms $(m = s = \ell)$ sum to $\sum_{\ell=2}^{D} \varepsilon_\ell P_\ell(x)$ exactly.
  (2) The only remaining terms of degree $\ell \leq D$ are *non-sectoral spillover harmonics* $Q_{\ell,s}(x)$ with $2 \leq s \leq \ell - 2$ $(s \equiv \ell \pmod 2)$, homogeneous of degree ℓ, carrying amplitude $w_s = \varepsilon_s/(2c_{\mathrm{sect}}(s))$.

Setting $x = r_d\tilde{x}$ and dividing by $\varepsilon_d r_d^d$, the remainder $\widetilde{V}_{\mathrm{int},d}(\tilde{x}) - H_d(\tilde{x})$ splits into three exhaustive parts:

1. Off-scale sectoral modes ($\ell \in \{2,\dots,D\}\backslash\{d-1,d\}$). By Theorem 2.3, each term $(\varepsilon_\ell r_d^\ell/(\varepsilon_d r_d^d))P_\ell(\tilde{x})$ carries the factor $\Lambda^{-(\ell-d)(\ell-d+1)} \leq \Lambda^{-2}$, so its C^2 norm on $\{\|\tilde{x}\| \leq \rho\}$ is at most $C_{\mathrm{off}}(D,\rho)\,\Lambda^{-2}$.

2. Non-sectoral spillover modes ($2 \leq s \leq \ell - 2$, $\ell \leq D$). The rescaled term has amplitude ratio

$$\frac{\varepsilon_s r_d^\ell}{\varepsilon_d r_d^d} \;=\; \left(\frac{\varepsilon_s r_d^s}{\varepsilon_d r_d^d}\right) r_d^{\ell-s} \;\leq\; 1\cdot r_d^2 \;\leq\; \Lambda^{-6} \;\leq\; \Lambda^{-2},$$

since $\varepsilon_s r_d^s/(\varepsilon_d r_d^d) = \Lambda^{-(s-d)(s-d+1)} \leq 1$ by Theorem 2.3, $\ell - s \geq 2$, and $r_d \leq \Lambda^{-3}$. Summing over the finitely many pairs (s,ℓ) yields a $C^2(\{\|\tilde{x}\| \leq \rho\})$ bound $C_{\mathrm{spill}}(D,\rho)\,\Lambda^{-6}$.

3. High-degree tail and aliased modes ($\ell \geq D+1$). Since $\varepsilon_2 \leq \varepsilon_3 \leq \cdots \leq \varepsilon_D$ and $c_{\mathrm{sect}}(s) \geq 4^{-D}$, the total ℓ^1 charge obeys

$$\sum_{j=1}^{4D-2} |q_j| \;\leq\; K_1(D)\,\varepsilon_D$$

for an explicit constant $K_1(D)$ independent of Λ. For each point charge $a_j \in S^2$, the degree-ℓ term $r^\ell P_\ell(\hat{x}\cdot a_j)$ is a homogeneous harmonic polynomial bounded in magnitude by r^ℓ on the closed ball of radius r. Applying the standard Cauchy/interior gradient estimates for harmonic functions on the concentric ball of radius $2\rho r_d$ (valid since $2\rho r_d \leq 2\rho\Lambda^{-3} \leq \frac{1}{2}$ for $\Lambda \geq (4\rho)^{1/3}$), the C^2 norm of

$\tilde{x} \mapsto r_d^\ell P_\ell(\widehat{\tilde{x}} \cdot a_j)$ on $\{\|\tilde{x}\| \le \rho\}$ is bounded by $K_2\,(2\rho r_d)^\ell$ for an absolute constant K_2. Summing over $\ell \ge D+1$ and dividing by $\varepsilon_d r_d^d$:

$$\frac{K_1(D)\,\varepsilon_D}{\varepsilon_d r_d^d} \sum_{\ell=D+1}^{\infty} K_2(2\rho r_d)^\ell \;=\; \underbrace{\left(\frac{\varepsilon_D r_d^D}{\varepsilon_d r_d^d}\right)}_{\le 1} K_1(D) K_2\,(2\rho)^{D+1} \frac{r_d}{1-2\rho r_d} \;\le\; 2K_1(D) K_2 (2\rho)^{D+1}\,\Lambda^{-3},$$

because $r_d \le \Lambda^{-3}$ and $2\rho r_d \le \frac{1}{2}$. Notice that every aliased mode on S^1_{xy} or S^1_{yz} has degree $\ell \ge 2D_{\text{odd}} - D_{\text{odd}} \ge D-1$, and by Proposition 2.1 no aliasing occurs for $\ell \le D$, so *all* aliased modes are contained strictly within this $\ell \ge D+1$ tail!

For the exterior estimate (8) on $\{\rho^{-1} \le \|\tilde{x}\| \le \rho\}$, we have $R_d\|\tilde{x}\| \ge R_d\rho^{-1} \ge \Lambda^3\rho^{-1} \ge 4$, so: (1) off-scale sectoral modes carry $\Lambda^{-(\ell-d)(\ell-d+1)} \le \Lambda^{-2}$; (2) non-sectoral spillover modes carry $(\varepsilon_s R_d^{-(s+1)}/(\varepsilon_d R_d^{-(d+1)}))R_d^{-(\ell-s)} \le R_d^{-2} \le \Lambda^{-6}$; and (3) the $\ell \ge D+1$ tail (containing all exterior aliased modes) is bounded in $C^2(\{\rho^{-1} \le \|\tilde{x}\| \le \rho\})$ by

$$\left(\frac{\varepsilon_D R_d^{-(D+1)}}{\varepsilon_d R_d^{-(d+1)}}\right) K_1(D) K_2'(2\rho)^{D+2} \frac{R_d^{-1}}{1-2\rho R_d^{-1}} \;\le\; 2K_1(D) K_2'(2\rho)^{D+2}\,\Lambda^{-3}.$$

Combining the three estimates yields (7) and (8) with

$$C(D,\rho) \;=\; C_{\text{off}}(D,\rho) + C_{\text{spill}}(D,\rho) + 2K_1(D)\max(K_2,K_2')(2\rho)^{D+2}. \qquad \square$$

Remark 2.5 (Growth of the Perturbation Threshold $\Lambda_0(D)$). As D increases, the outermost critical points of H_d and E_d lie at dimensionless radius $\rho_d \sim (d/\pi)^d$ (arising from the smallest nodal angle $\pi/(4(d-1))$). Consequently, the constant $C(D,\rho_D)$ in Proposition 2.4 grows rapidly with D, so the persistence threshold $\Lambda_0(D)$ genuinely depends on D. This theoretical dependence is directly observed in Section 5: $\Lambda = 400$ suffices for $D = 4, 5$, while $D = 6$ requires $\Lambda \approx 4{,}000$ and $D = 7$ requires $\Lambda \approx 10^5$.

3. Interior Lattice, Exterior Kelvin Mirror Theorem, and Proof of Theorem 1.1

3.1. **Interior Equilibria in B^3.** By the closed-form complex-plane harmonic lattice of [4] (re-verified in `InteriorLattice.lean`), for every $d \ge 3$, the two-scale interior harmonic polynomial $H_d(\tilde{x}) = P_{d-1}(\tilde{x}) + P_d(\tilde{x})$ is harmonic ($\Delta H_d \equiv 0$) and possesses $(d-1)(d-2)$ explicit pairwise distinct critical points $p_{j,k}^{(d)} \in \mathbb{R}^3 \setminus \{0\}$ with strictly nonzero 3×3 Hessian determinant $\det \operatorname{Hess} H_d(p_{j,k}^{(d)}) \ne 0$.

3.2. **Exterior Kelvin-Mirror Equilibria in $\mathbb{R}^3 \setminus \overline{B}^3$.** Because spherical inversion $x \mapsto x/\|x\|^2$ is not gradient-preserving (the Kelvin transform multiplies by $\|x\|^{-1}$, shifting radial powers from r^{d-1}, r^d to $R^{-d}, R^{-(d+1)}$), the existence and nondegeneracy of exterior critical points of E_d require an independent proof.

Consider odd $d \ge 3$ (even $d \ge 4$ is identical upon interchanging the roles of the (x,y) and (y,z) sectoral factors). In the open affine chart $y > 0$, parametrize $\mathbb{R}^3$ by spherical-affine coordinates $(\rho, t_1, t_2) \in (0,\infty) \times \mathbb{R}^2$:

$$\Psi(\rho,t_1,t_2) \;=\; \rho\,\hat{u}(t_1,t_2) \;=\; \frac{\rho}{\sqrt{W}}\,(t_1,1,t_2), \qquad W = 1 + t_1^2 + t_2^2. \tag{10}$$

A direct computation (`exterior_chart_jacobian_det_ne_zero`) shows that Ψ is a smooth diffeomorphism onto $\{y > 0\}$ with Jacobian determinant

$$\det D\Psi(\rho,t_1,t_2) \;=\; -\frac{\rho^2}{W^{3/2}} \;\ne\; 0. \tag{11}$$

Setting $m = d-1 \geq 2$ and $\phi_1(t_1) = \frac{\pi}{2} - \arctan t_1$, we write $E_d(\rho\hat{u}) = \rho^{-d}A(t_1,t_2) + \rho^{-(d+1)}B(t_1,t_2)$, where

$$A(t_1,t_2) = P_{d-1}(\hat{u}) = \frac{(1+t_1^2)^{m/2}\cos(m\phi_1(t_1)+\pi/4)}{W^{m/2}},$$
$$B(t_1,t_2) = P_d(\hat{u}) = \frac{(1+t_2^2)^{d/2}\cos(d\arctan t_2+\pi/4)}{W^{d/2}}.$$

Whenever $AB \neq 0$, because $P_{d-1}(-\hat{u}) = (-1)^{d-1}P_{d-1}(\hat{u})$ and $P_d(-\hat{u}) = (-1)^d P_d(\hat{u})$ have opposite parities, exactly one of $\pm\hat{u}$ satisfies $AB < 0$. On that ray, radial stationarity $\partial_\rho(\rho^{-d}A+\rho^{-(d+1)}B) = 0$ has the unique positive solution

$$\rho_*(t_1,t_2) \;=\; -\frac{d+1}{d}\frac{B(t_1,t_2)}{A(t_1,t_2)} \;>\; 0, \tag{12}$$

where the exact radial second derivative is strictly nonzero:

$$E_{\rho\rho\rho}(\rho_*,t_1,t_2) \;=\; -d\,\rho_*^{-(d+2)}A \;\neq\; 0. \tag{13}$$

Evaluating E_d on the radial-balance surface $\rho = \rho_*(t_1,t_2)$ yields

$$\Phi(t_1,t_2) \;=\; E_d(\rho_*(t_1,t_2),t_1,t_2) \;=\; \frac{1}{d+1}\rho_*^{-d}A \;=\; \frac{\operatorname{sgn}(A)}{d+1}\left(\frac{d}{d+1}\right)^d \exp\big(-L_{\mathrm{ext}}(t_1,t_2)\big), \tag{14}$$

where the affine-chart log-potential is

$$L_{\mathrm{ext}}(t_1,t_2) \;=\; d\ln|B| - (d+1)\ln|A| \;=\; d\,g_2(t_2) - (d+1)\,g_1(t_1) - \frac{1}{2}\ln(1+t_1^2+t_2^2), \tag{15}$$

with $g_1(t_1) = \frac{m}{2}\ln(1+t_1^2) + \ln|\cos(m\phi_1(t_1)+\frac{\pi}{4})|$ and $g_2(t_2) = \frac{d}{2}\ln(1+t_2^2) + \ln|\cos(d\arctan t_2+\frac{\pi}{4})|$.

Theorem 3.1 (Exterior Strict-Saddle Existence and 3D Nondegeneracy)**.** *For every $d \geq 3$, let $s_0 < s_1 < \cdots < s_{d-2}$ be the $d-1$ roots of $\cos((d-1)\phi_1(t_1)+\pi/4) = 0$ and $r_0 < r_1 < \cdots < r_{d-1}$ be the d roots of $\cos(d\arctan t_2 + \pi/4) = 0$.*

(1) Strict curvature separation: *At every point with $\partial_{t_1}L_{\mathrm{ext}} = 0$, $\partial_{t_1}^2 L_{\mathrm{ext}} > \frac{5}{1+t_1^2} > 0$; at every point with $\partial_{t_2}L_{\mathrm{ext}} = 0$, $\partial_{t_2}^2 L_{\mathrm{ext}} < -\frac{13}{1+t_2^2} < 0$. Hence at every critical point of L_{ext},*

$$\det \operatorname{Hess} L_{\mathrm{ext}}(t_1,t_2) \;=\; L_{11}L_{22} - L_{12}^2 \;<\; 0.$$

(2) Existence in every bounded nodal rectangle: *Every one of the $(d-2)(d-1)$ pairwise disjoint open rectangles $(s_k,s_{k+1})\times(r_j,r_{j+1})$ $(0 \leq k < d-2,\ 0 \leq j < d-1)$ contains a critical point (t_1^*,t_2^*) of L_{ext}.*

(3) 3D nondegeneracy: *At the corresponding 3D point $q_{k,j}^{(d)} = \pm\rho_*(t_1^*,t_2^*)\,\hat{u}(t_1^*,t_2^*) \in \mathbb{R}^3 \setminus \{0\}$, we have $\nabla E_d(q_{k,j}^{(d)}) = 0$ and $\det \operatorname{Hess}_{x,y,z} E_d(q_{k,j}^{(d)}) \neq 0$.*

Consequently, E_d possesses at least $(d-1)(d-2)$ pairwise distinct, nondegenerate critical points in $\mathbb{R}^3 \setminus \{0\}$.

Proof. Step 1 (Curvature separation). Differentiating g_1 and g_2 using $\phi_1'(t_1) = -(1+t_1^2)^{-1}$ and $\sec^2\theta = 1+\tan^2\theta$ gives, with $u = \tan(m\phi_1(t_1)+\frac{\pi}{4})$ and $v = \tan(d\arctan t_2 + \frac{\pi}{4})$,

$$g_1'(t_1) = \frac{m(t_1+u)}{1+t_1^2}, \qquad g_1''(t_1) = -\frac{m\big[(m-1)(1+u^2)+(t_1+u)^2\big]}{(1+t_1^2)^2},$$
$$g_2'(t_2) = \frac{d(t_2-v)}{1+t_2^2}, \qquad g_2''(t_2) = -\frac{d\big[(d-1)(1+v^2)+(t_2-v)^2\big]}{(1+t_2^2)^2}.$$

Hence $\partial_{t_1}L_{\mathrm{ext}} = -(d+1)g_1'(t_1) - t_1/W$ and $\partial_{t_2}L_{\mathrm{ext}} = d\,g_2'(t_2) - t_2/W$. If $\partial_{t_1}L_{\mathrm{ext}} = 0$, then $(d+1)m(t_1+u) = -t_1(1+t_1^2)/W$. Since $1+t_1^2 \leq W$ and $(d+1)m = (d+1)(d-1) \geq 8$ for $d \geq 3$,

we have $|t_1+u| \le |t_1|/8$, so $|u| \ge \frac{7}{8}|t_1|$ and $1+u^2 \ge \frac{49}{64}(1+t_1^2)$. As $(d+1)m(m-1) \ge 8$ and $(1+t_2^2-t_1^2)/W^2 \le 1/W \le 1/(1+t_1^2)$, we deduce

$$L_{11} \;=\; -(d+1)g_1''(t_1) - \frac{1+t_2^2-t_1^2}{W^2} \;\ge\; \frac{8\cdot\frac{49}{64}(1+t_1^2)}{(1+t_1^2)^2} - \frac{1}{1+t_1^2} \;=\; \frac{41/8}{1+t_1^2} \;>\; \frac{5}{1+t_1^2} \;>\; 0.$$

Symmetrically, if $\partial_{t_2} L_{\mathrm{ext}} = 0$, then $d^2(t_2-v) = t_2(1+t_2^2)/W$, so $|t_2-v| \le |t_2|/9$, $|v| \ge \frac{8}{9}|t_2|$, $1+v^2 \ge \frac{64}{81}(1+t_2^2)$, and $d^2(d-1) \ge 18$, yielding

$$L_{22} \;=\; d\,g_2''(t_2) - \frac{1+t_1^2-t_2^2}{W^2} \;\le\; -\frac{18\cdot\frac{64}{81}(1+t_2^2)}{(1+t_2^2)^2} + \frac{1}{1+t_2^2} \;=\; -\frac{119/9}{1+t_2^2} \;<\; -\frac{13}{1+t_2^2} \;<\; 0.$$

Therefore $L_{11}L_{22} - L_{12}^2 < 0$ at every critical point of L_{ext}.

Step 2 (Uniqueness in t_1, IFT-free continuity, and cell existence). Fix $0 \le k < d-2$ and $0 \le j < d-1$. For any fixed $t_2 \in \mathbb{R}$, suppose $s_k < a < b < s_{k+1}$ were two distinct roots of $f(t_1) := \partial_{t_1} L_{\mathrm{ext}}(t_1,t_2) = 0$. Since $f'(a) = L_{11}(a,t_2) > 0$, we have $f(t_1) > 0$ on $(a, a+\delta)$ for some $\delta > 0$. Setting $x_0 = \inf\{x \in [a+\delta/2, b] : f(x) \le 0\} \in (a,b]$, continuity forces $f(x_0) = 0$ and $f > 0$ on (a, x_0); yet $f'(x_0) = L_{11}(x_0,t_2) > 0$ implies $f(x) < 0$ just left of x_0, a contradiction. Thus $f(\cdot) = 0$ has at most one root in (s_k, s_{k+1}).

Next, because $|t_1/W| \le 1$ uniformly for all $t_2 \in \mathbb{R}$, while $u(t_1) \to +\infty$ as $t_1 \to s_k^+$ and $u(t_1) \to -\infty$ as $t_1 \to s_{k+1}^-$, choosing explicit boundary offsets $\delta = \min(\frac{\pi}{4}, \frac{1}{2(M+2+M^2)})$ (where $M = \max(|s_k|, |s_{k+1}|)+1$) produces fixed endpoints $s_k < a_1 < b_1 < s_{k+1}$ such that $\partial_{t_1} L_{\mathrm{ext}}(a_1,t_2) < 0 < \partial_{t_1} L_{\mathrm{ext}}(b_1,t_2)$ for all $t_2 \in \mathbb{R}$. By the Intermediate Value Theorem and uniqueness, there is a unique function $\varphi : \mathbb{R} \to (a_1,b_1)$ satisfying $\partial_{t_1} L_{\mathrm{ext}}(\varphi(t_2),t_2) = 0$. To see that φ is continuous at any $t_{2,0}$ without invoking the Implicit Function Theorem: for any $\varepsilon > 0$, set $x_L = \max(a_1, \varphi(t_{2,0}) - \varepsilon/2)$ and $x_R = \min(b_1, \varphi(t_{2,0}) + \varepsilon/2)$. By uniqueness and IVT, $\partial_{t_1} L_{\mathrm{ext}}(x_L, t_{2,0}) < 0 < \partial_{t_1} L_{\mathrm{ext}}(x_R, t_{2,0})$; by continuity in t_2, both strict inequalities persist on a δ-neighborhood of $t_{2,0}$, forcing $\varphi(t_2) \in (x_L, x_R)$ there.

Finally, since $|t_2/W| \le 1$ uniformly in t_1 while $v(t_2) \to -\infty$ as $t_2 \to r_j^+$ and $v(t_2) \to +\infty$ as $t_2 \to r_{j+1}^-$, the same explicit offset yields $r_j < a_2 < b_2 < r_{j+1}$ such that $\partial_{t_2} L_{\mathrm{ext}}(t_1,a_2) > 0 > \partial_{t_2} L_{\mathrm{ext}}(t_1,b_2)$ for all $t_1 \in \mathbb{R}$. Applying the Intermediate Value Theorem to the continuous function $t_2 \mapsto \partial_{t_2} L_{\mathrm{ext}}(\varphi(t_2),t_2)$ on $[a_2,b_2]$ produces $t_2^* \in (r_j, r_{j+1})$ with $t_1^* = \varphi(t_2^*) \in (s_k, s_{k+1})$ and $\nabla L_{\mathrm{ext}}(t_1^*, t_2^*) = 0$.

Step 3 (3D Hessian factorization). At (ρ_*, t_1^*, t_2^*), radial balance $E_\rho \equiv 0$ along $\rho = \rho_*(t_1,t_2)$ implies $\partial_i \rho_* = -E_{\rho i}/E_{\rho\rho}$, so the Hessian of $\Phi(t_1,t_2) = E_d(\rho_*(t_1,t_2),t_1,t_2)$ equals the exact 2×2 Schur complement $\Phi_{ij} = E_{ij} - E_{\rho i}E_{\rho j}/E_{\rho\rho}$ of the 3×3 coordinate Hessian $H_{\mathrm{coord}} = \mathrm{Hess}_{\rho,t_1,t_2} E_d$. By the 3×3 Schur-complement determinant identity (`exterior_schur_3x3_det`),

$$\det H_{\mathrm{coord}} \;=\; E_{\rho\rho} \cdot \det \mathrm{Hess}\, \Phi(t_1^*, t_2^*).$$

Differentiating (14) at $\nabla L_{\mathrm{ext}}(t_1^*,t_2^*) = 0$ gives $\Phi_{ij} = -\Phi(t_1^*,t_2^*)\, L_{ij}$, so $\det \mathrm{Hess}\,\Phi = \Phi(t_1^*,t_2^*)^2 \det \mathrm{Hess}\, L_{\mathrm{ext}} < 0$. By (11) and the change-of-variables formula at a critical point ($\nabla E_d = 0$),

$$\det \mathrm{Hess}_{x,y,z}\, E_d(q_{k,j}^{(d)}) \;=\; \frac{E_{\rho\rho}\cdot \Phi(t_1^*,t_2^*)^2 \cdot \det \mathrm{Hess}\, L_{\mathrm{ext}}(t_1^*,t_2^*)}{(\det D\Psi(\rho_*,t_1^*,t_2^*))^2} \;\neq\; 0,$$

as formalized in `ExteriorKelvinDuality.lean` (see `exterior_3d_hess_det_ne_zero`). Since the $(d-2)(d-1)$ open rectangles are pairwise disjoint (see `exterior_cells_pairwise_disjoint`), the rays $\pm\hat{u}(t_1^*,t_2^*)$ are pairwise distinct across all (k,j). □

3.3. **Proof of Theorem 1.1.** Fix $D \ge 3$. For each $d \in \{3,\dots,D\}$, choose $\rho_d \ge 2$ large enough that all $(d-1)(d-2)$ interior critical points $p_{j,k}^{(d)}$ of H_d lie strictly inside the open ball $\{\|\tilde{x}\| < \rho_d/2\}$ and all $(d-1)(d-2)$ exterior critical points $q_{k,j}^{(d)}$ of E_d lie strictly inside the open annulus $\{2\rho_d^{-1} < \|\tilde{x}\| < \rho_d/2\}$.

Because every critical point $p_{j,k}^{(d)}$ and $q_{k,j}^{(d)}$ is nondegenerate, we can surround each of them by pairwise disjoint closed balls $B(p_{j,k}^{(d)}, \eta_d)$ and $B(q_{k,j}^{(d)}, \eta_d)$ on which the Jacobian of ∇H_d (resp. ∇E_d) is uniformly invertible.

By Proposition 2.4, taking $\Lambda \geq \Lambda_0(D)$ sufficiently large makes $\|\widetilde{V}_{\mathrm{int},d} - H_d\|_{C^2(\{\|\tilde{x}\| \leq \rho_d\})}$ and $\|\widetilde{V}_{\mathrm{ext},d} - E_d\|_{C^2(\{\rho_d^{-1} \leq \|\tilde{x}\| \leq \rho_d\})}$ arbitrarily small simultaneously for all $d \in \{3, \ldots, D\}$, while ensuring that the rescaled shells $r_d \cdot \{\|\tilde{x}\| \leq \rho_d\} \subset B^3$ and $R_d \cdot \{\rho_d^{-1} \leq \|\tilde{x}\| \leq \rho_d\} \subset \mathbb{R}^3 \setminus \overline{B}^3$ are pairwise disjoint across $d = 3, \ldots, D$ (since $r_d/r_{d-1} = \Lambda^2 \gg \rho_d \rho_{d-1}$). By the quantitative Inverse Function Theorem (Newton–Kantorovich), each critical point of H_d and E_d persists to a unique nondegenerate critical point of the exact finite 3D Coulomb potential V inside its isolating ball. Summing over $d = 3, \ldots, D$ yields

$$N(D) \;=\; \sum_{d=3}^{D} 2(d-1)(d-2) \;=\; \frac{2}{3} D(D-1)(D-2)$$

distinct nondegenerate equilibria from $n(D) = 4D - 2$ neutral charges on S^2. For $D \geq 27$ ($n \geq 106$), $N(D) > (n(D)-1)^2$, and $\lim_{n\to\infty} N(\lfloor (n+2)/4 \rfloor)/n^3 = \frac{1}{96}$ (`AsymptoticBound.lean`). □

4. Remark on the Strictly Positive Charge Barrier

A natural question is whether an axisymmetric multi-shell variant can push the strictly positive-charge bound $M_{S^2}^{+}(n) = \Omega(n^{3/2})$ past the $n^{3/2}$ exponent. In `AsymptoticBound.lean` (see Table 2), we record a sharp topological obstruction:

Proposition 4.1 (Positive Axisymmetric Poincaré–Hopf Ceiling)**.** *In any axisymmetric configuration in $\mathbb{R}^3$ consisting of L continuous strictly positive coaxial rings and $K \geq 1$ strictly positive point charges on the z-axis with isolated axial equilibria (n_{ax}) and isolated circular equilibrium rings (c_{circ}) in the open meridian half-plane $\rho > 0$, the Intermediate Value Theorem forces $n_{\mathrm{ax}} \geq K - 1$ and the 2D meridian Poincaré–Hopf index balance yields $K + 2L - n_{\mathrm{ax}} - 2c_{\mathrm{circ}} \geq 1$, whence $c_{\mathrm{circ}} \leq L$.*

5. Machine Verification Scope and High-Precision 3D Coulomb Certification

5.1. **Lean 4.36.0 / Mathlib4 Audit and Scope.** We formalize **34 core theorems** in Lean 4.36.0 with Mathlib4 across four self-contained modules, with zero `sorry`, zero `admit`, zero `native_decide`, and strictly the three standard axioms (`propext`, `Classical.choice`, `Quot.sound`), verified automatically in GitHub Actions CI (Table 2).

Scope boundary: The Lean 4 formalization machine-checks the interior harmonic lattice (`InteriorLattice.lean`), the shifted great-circle DFT orthogonality, charge neutrality, S^2-confinement, and sharp S^1 Vandermonde lower bound (`GreatCircleDFT.lean`), the Kelvin mirror scale-separation identity, the complete exterior affine-chart proof and 3D Hessian determinant factorization (`ExteriorKelvinDuality.lean`), and the exact double-scale count, $D \geq 27$ violation, $\frac{1}{96}$ limits, and positive axisymmetric ceiling (`AsymptoticBound.lean`). The analytic bridge from the finite 3D Coulomb sum (1) to the two-scale model potentials H_d and E_d—namely the spherical-harmonic addition theorem (9), the quantitative C^2 non-sectoral/aliasing remainder bounds (Proposition 2.4), and the Inverse Function Theorem persistence for $\Lambda \geq \Lambda_0(D)$—is proved analytically in Sections 2–3 and independently tested on the exact finite 3D Coulomb potential in Section 5.2.

5.2. **Exact Finite 3D Coulomb Certification with Strict Local Convergence (`mpmath`).** To verify persistence on the *actual* finite-sum 3D Coulomb potential (1) (with all non-sectoral spillover and aliased modes included verbatim), our standalone verifier (`verify_s2_cubic.py`) polishes every

TABLE 2. Lean 4.36.0 formalization modules and key machine-checked theorems (34 audited theorems).

Module	Key verified theorems (zero `sorry`, standard axioms only)
`InteriorLattice.lean`	`laplacian_Hd_zero`, `critPt_injective`, `critPt_grad_zero`, `critPt_hess_det_ne_zero` (odd and even scales), `nestedH_rescale`
`GreatCircleDFT.lean`	`shifted_dft_zero_monopole`, `shifted_dft_orthogonality`, `shifted_dft_diag_strict`, `shifted_dft_nyquist`, `great_circle_superposition_neutral`, `s1_vandermonde_sharp_lower_bound` (Theorem 2.2)
`ExteriorKelvinDuality.lean`	`kelvin_mirror_scale_identity`, `kelvin_mirror_off_scale_suppression`, `exterior_d11L_pos`, `exterior_d22L_neg`, `exterior_L_strict_saddle`, `exterior_t1_stationary_unique`, `exterior_exists_strict_saddle_in_cell`, `exterior_cells_pairwise_disjoint`, `exterior_radial_balance_and_curvature`, `exterior_schur_3x3_det`, `exterior_chart_jacobian_det_ne_zero`, `exterior_3d_hess_det_ne_zero`
`AsymptoticBound.lean`	`sum_double_scale_count`, `beats_maxwell_quadratic_on_S2` ($D \geq 27$), `tendsto_subseq_1_96`, `tendsto_all_n_1_96`, `positive_axisymmetric_poincare_hopf_ceiling` (Proposition 4.1)

interior and exterior equilibrium until **both** scale-invariant local criteria are satisfied:

$$r_{\mathrm{loc}}(x) \;=\; \frac{\|\nabla V(x)\|_2}{\|\operatorname{Hess} V(x)\|_F\,\|x\|_2} \;<\; 10^{-55} \qquad \text{and} \qquad \frac{\|(\operatorname{Hess} V(x))^{-1}\nabla V(x)\|_2}{\|x\|_2} \;<\; 10^{-55}.$$

Using the local Hessian scale $\|\operatorname{Hess} V(x)\|_F\|x\|_2$ (rather than the multipole-cancellation-inflated ℓ^1 sum $\sum_j |q_j|\|x-a_j\|^{-2}$) guarantees genuine convergence even on deep shells where charge cancellation exceeds 10^{200}. We also check strict nondegeneracy (all three Hessian eigenvalues nonzero, Morse indices $c_1, c_2 \in \{1, 2\}$), bounded relative displacement from the model seed (< 0.1), strict pairwise relative separation (> 0.3), unit-sphere confinement ($\max_j |\|a_j\|^2 - 1| < 10^{-290}$), and exact zero net charge ($|\sum q_j|/\|q\|_1 = 0$). Table 3 records the certified counts for $D = 4, 5, 6, 7$.

TABLE 3. Strict local-scale `mpmath` certification on the exact finite 3D Coulomb potential (1).

D	n	Λ	dps	Interior (B^3)	Exterior ($\mathbb{R}^3 \setminus \overline{B}^3$)	Total	Max r_{loc}	Min rel. sep.
4	14	400	300	8/8 (c_1=4, c_2=4)	8/8 (c_1=4, c_2=4)	16/16	1.05×10^{-61}	1.01
5	18	400	360	20/20 (c_1=12, c_2=8)	20/20 (c_1=12, c_2=8)	40/40	1.38×10^{-57}	0.672
6	22	4,000	500	40/40 (c_1=20, c_2=20)	40/40 (c_1=20, c_2=20)	80/80	2.19×10^{-57}	0.510
7	26	10^5	760	70/70 (c_1=38, c_2=32)	70/70 (c_1=38, c_2=32)	140/140	8.23×10^{-57}	0.413

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