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HYPERSURFACES WITH LARGE AUTOMORPHISM GROUPS AT FIXED DEGREE AND SUFFICIENTLY MANY VARIABLES

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Abstract. For fixed $d \geq 3$, we show that when n is sufficiently large, the largest order of the automorphism group of a smooth degree-d hypersurface in $\mathbb{P}^{n-1}_{\mathbb{C}}$ whose defining polynomial has no one-variable summand is

$$\begin{cases} (n+1)!, & d \text{ prime and } \gcd(n+1, d-1) = 1, \\ n!\beta^{n-1}, & \text{otherwise,} \end{cases}$$

where β is the largest proper divisor of d. We also describe the defining forms with maximum group order and show that all are indecomposable.

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1. Introduction

Let $n, d \geq 3$ and let $X_F \subseteq \mathbb{P}^{n-1}_{\mathbb{C}}$ be a smooth hypersurface defined by a homogeneous polynomial $F \in \mathbb{C}[x_1, \ldots, x_n]$ of degree d. Its group of linear automorphisms $\mathrm{Lin}(X_F)$ is finite. It equals $\mathrm{Aut}(X_F)$ when $(n, d) \notin \{(3, 3), (4, 4)\}$. For $n \geq 4$, these are the theorems of Matsumura and Monsky [MM63, Theorems 1 and 2]. For plane curves, see [Poo05, Theorems 1.1 and 1.3]. Write

$$\mathrm{Aut}(F) = \{g \in \mathrm{GL}(n, \mathbb{C}) : F \circ g = F\}.$$

There is an exact sequence

$$1 \longrightarrow \mu_d \longrightarrow \mathrm{Aut}(F) \longrightarrow \mathrm{Lin}(X_F) \longrightarrow 1,$$

where μ_d acts on $\mathbb{C}^n$ by scalars.

Yang, Yu, and Zhu [YYZ25, Theorem 1.1] and Esser and Li [EL25, Theorem 2.3] independently proved that, for all but finitely many pairs (n, d), the Fermat hypersurface $x_1^d + \cdots + x_n^d = 0$ is the unique maximizer, up to isomorphism, of $|\mathrm{Lin}(X_F)|$ among smooth

degree-d hypersurfaces in $\mathbb{P}^{n-1}_{\mathbb{C}}$. Its group has order $d^{n-1}n!$. The cited theorems also describe the exceptional maximizers.

An *additive decomposition* of F is $F = F_1 + \cdots + F_t$ in suitable linear coordinates, where F_i are nonzero homogeneous polynomials in disjoint groups of variables. A decomposition is *maximal* if no summand can be split further in this way. The maximal additive decomposition of F is unique up to reordering and linear equivalence of its summands [Kle05, Theorem 3.7], [HLYZ22, Proposition 2.1(3)]. See Lemma 2.1.

The one-variable summands in the maximal decomposition are the *Fermat summands*, which are of the form x_i^d after rescaling. The form F is *indecomposable* if its maximal decomposition has one summand. Thus an indecomposable F has no Fermat summands. Two forms F and F' in n variables are *linearly equivalent* if $F' = F \circ g$ for some $g \in \mathrm{GL}(n, \mathbb{C})$.

We study the maximum of $|\operatorname{Lin}(X_F)|$ over smooth hypersurfaces whose defining forms have no Fermat summands.

Let $M_{n,d}$ be the maximum of $|\operatorname{Aut}(F)|$ over smooth degree-d forms in n variables without Fermat summands, and let $M^{\mathrm{ind}}_{n,d}$ be the corresponding maximum for indecomposable forms.

Theorem 1.1. *Fix $d \geq 3$. Let $q = q(d)$ be the smallest prime divisor of d and write $\beta = d/q$. Then there exist $C_d > 0$ and $N_d \geq 3$ such that*

(i) *For all $n \geq 3$,*

$$q\beta^n n! \leq M^{\mathrm{ind}}_{n,d} \leq M_{n,d} \leq C_d(n+1)!\beta^n.$$

(ii) *For all $n \geq N_d$,*

$$M^{\mathrm{ind}}_{n,d} = M_{n,d} = \begin{cases} d(n+1)!, & d \text{ prime and } \gcd(n+1, d-1) = 1, \\ q\beta^n n!, & \text{otherwise.} \end{cases}$$

To prove the upper bounds, we use primitive block decompositions of $\operatorname{Aut}(F)$ as in [YYZ25, Set-up 3.2] and [EL25, Lemma 3.1], together with Collins's bounds for finite primitive linear groups [Col08, Theorem A]. The order of the subgroup acting by scalars on the blocks is the index of the lattice generated by the block multidegrees of F. Hadamard's inequality gives an upper bound for this index.

When the permutation group on the one-dimensional blocks has small index, we use Pago's lemma [Pag23, Lemma 4.3.8] to find alternating subgroups on disjoint collections of these blocks. These subgroups impose divisibility conditions on the multidegrees. Smoothness and the absence of Fermat summands force the resulting divisors to be proper divisors of d.

Propositions A.2 and A.3 give the lower bounds in Theorem 1.1. Theorems A.5 and A.8 describe the forms with $|\operatorname{Aut}(F)| = M_{n,d}$ and determine their automorphism groups.

Dividing the maxima in Theorem 1.1 by d gives the corresponding maxima of $|\operatorname{Lin}(X_F)|$. Theorem 2.3 gives a bound sharper than d^n for abelian subgroups of $\operatorname{Aut}(F)$ when F has no Fermat summands.

For $r \geq 1$, let $\mathcal{O}_{r,d}$ consist of the orders $|\operatorname{Aut}(F_0)|$ for smooth degree-d forms F_0 in r variables without Fermat summands. Each $\mathcal{O}_{r,d}$ is finite by Theorem 2.3 and is nonempty for $r \geq 3$ by Proposition A.2. Write $\mathcal{O}_{0,d} = \{1\}$. Order the pairs (r, a) with $a \in \mathcal{O}_{r,d}$ by increasing r and then decreasing a.

Corollary 1.2. *Fix $d \geq 3$ and an integer $M \geq 1$. Write $(r_1, a_1), \ldots, (r_M, a_M)$ for the first M pairs in this order. For every sufficiently large n and each $1 \leq i \leq M$, the ith largest*

distinct value of $|\operatorname{Aut}(F)|$ *among smooth degree-d forms F in n variables is*

$$d^{n-r_i}(n-r_i)!\,a_i.$$

The paper is organized as follows. Section 2 treats additive decompositions and gives a uniform bound for abelian subgroups. Sections 3 and 4 develop the estimates for part (i). Sections 5 and 6 give the sharp bounds in composite and prime degree, respectively. Section 7 proves Corollary 1.2. Appendix A contains the constructions and the classification.

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2. Additive decompositions and a uniform bound

In this section we introduce additive decompositions and give a uniform bound for abelian subgroups of $\operatorname{Aut}(F)$ when F is smooth and has no Fermat summands. For fixed d, let q be the smallest prime divisor of d and $\beta = d/q$.

2.1. Centres and additive decompositions.

Let F be a form of degree d on a complex vector space V of dimension $n \geq 1$. A decomposition $V = \bigoplus_i V_i$ gives an *additive decomposition* of F if each $F_i = F|_{V_i}$ is nonzero and, for all $v_i \in V_i$,

$$F\Big(\sum_i v_i\Big) = \sum_i F_i(v_i).$$

The form F has a *dummy variable* if it is independent of one coordinate in some linear coordinate system. If $d \geq 2$, then $\mathrm{d}F$ vanishes along the corresponding axis, so a smooth form has no dummy variables.

Let H_F be the Hessian matrix of F in a basis of V. The *centre* of F [Har75] is

$$\mathcal{C}_F = \{A \in \operatorname{End}(V^\vee) : AH_F = H_F A^{\mathsf{t}}\}.$$

See [HLYZ22, Eq. (3)] for the corresponding formula on V.

Lemma 2.1. *Let $d \geq 3$ and let F be a degree-d form without dummy variables on a finite-dimensional complex vector space. Then $\mathcal{C}_F$ is a commutative algebra. The additive decompositions of F correspond to decompositions of the identity into nonzero orthogonal idempotents in $\mathcal{C}_F$. In particular, the subspaces of the maximal additive decomposition are unique up to reordering.*

Proof. Commutativity and the correspondence via dual projections follow from [Kle05, Proposition 2.21 and Theorem 3.7]. See also [HLYZ22, Proposition 2.1]. The identity of $\mathcal{C}_F$ is a sum of pairwise orthogonal primitive idempotents, unique up to reordering. The images of their transposes are the summand subspaces of the maximal additive decomposition. □

By uniqueness, $\operatorname{Aut}(F)$ permutes the summand subspaces of the maximal additive decomposition and preserves their linear equivalence classes. Write $F = \sum_{i=1}^{u} \sum_{j=1}^{k_i} F_{ij}$, where i

indexes the linear equivalence classes. For each i, j, choose a linear equivalence between F_{ij} and F_{i1}. Then

$$\operatorname{Aut}(F) \simeq \prod_{i=1}^{u} \big(\operatorname{Aut}(F_{i1})^{k_i} \rtimes S_{k_i}\big). \tag{2.1}$$

Here S_{k_i} permutes the factors of $\operatorname{Aut}(F_{i1})^{k_i}$.

Corollary 2.2. *Let F be a smooth form of degree $d \geq 3$ in $n \geq 1$ variables. In suitable coordinates, for some $0 \leq r \leq n$ we have*

$$F = x_1^d + \cdots + x_{n-r}^d + F_0.$$

The form F_0 is in the last r variables, and r is determined by F. When $r > 0$, F_0 is smooth, has no Fermat summands, and is unique up to linear equivalence. When $r = 0$, take $F_0 = 0$ and $\operatorname{Aut}(F_0) = 1$. Then

$$\operatorname{Aut}(F) \simeq \big(\mu_d^{n-r} \rtimes S_{n-r}\big) \times \operatorname{Aut}(F_0),$$

and

$$|\operatorname{Aut}(F)| = d^{n-r}(n-r)!\,|\operatorname{Aut}(F_0)|. \tag{2.2}$$

Proof. In the maximal decomposition $F = \sum_i F_i$, rescale the one-dimensional summands to x_i^d. For $r > 0$, the remaining summands give the maximal decomposition of F_0, which is smooth since $\mathrm{d}F = \bigoplus_i \mathrm{d}F_i$. Lemma 2.1 gives uniqueness, and (2.1) gives the group formulas. □

2.2. **Diagonal automorphisms.** Let D be the diagonal subgroup of $\operatorname{Aut}(F)$ in fixed coordinates. Write $F = \sum_{\alpha \in E} c_\alpha x^\alpha$, where E is the *support* and $c_\alpha \neq 0$ for $\alpha \in E$. Let $L \subset \mathbb{Z}^n$ be the lattice generated by E. A diagonal matrix $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ preserves F precisely when $\prod_i \lambda_i^{\alpha_i} = 1$ for every $\alpha \in E$. Thus

$$D \simeq \operatorname{Hom}(\mathbb{Z}^n/L, \mathbb{C}^\times). \tag{2.3}$$

If D is finite, then L has full rank and

$$|D| = [\mathbb{Z}^n : L]. \tag{2.4}$$

2.3. **Bounds for abelian subgroups.** The exact sequence in the introduction also holds for smooth binary forms. When $d \geq 3$, $\operatorname{Lin}(X_F)$ acts faithfully on the d points of X_F, so $\operatorname{Aut}(F)$ is finite in this case as well.

In coordinates diagonalizing an abelian subgroup $A \leq \operatorname{Aut}(F)$, we have $|A| \leq [\mathbb{Z}^n : L]$ by (2.4).

For $d \geq 3$, write

$$\alpha_d = \frac{\sqrt{(d-1)^2 + 1}}{d}, \qquad \gamma_d = \sqrt{\alpha_d}. \tag{2.5}$$

Both constants are less than one. By [YYZ25, Theorem 3.5], every abelian subgroup of $\operatorname{Aut}(F)$ has order at most d^n when F is smooth and $n \geq 2$.

Theorem 2.3. *Let F be a smooth form of degree $d \geq 3$ in n variables with no Fermat summands. Every abelian subgroup $A \leq \operatorname{Aut}(F)$ satisfies $|A| \leq d^n \gamma_d^n$. Moreover, $|\operatorname{Aut}(F)| \leq J(n) d^n \gamma_d^n$, where $J(n)$ is the complex Jordan constant. In particular, for $n \geq 71$,*

$$|\operatorname{Aut}(F)| \leq d^n (n+1)! \gamma_d^n. \tag{2.6}$$

Proof. Choose coordinates in which A is diagonal and let D be the diagonal subgroup of $\mathrm{Aut}(F)$. Every variable occurs in a mixed monomial, since otherwise it would be a dummy variable or give a Fermat summand.

Choose exponent vectors $\alpha_1, \dots, \alpha_t$ of mixed monomials that form a basis of their rational span. Since L has full rank by (2.4), extend this to a basis $\alpha_1, \dots, \alpha_n$ of $\mathbb{Q}^n$ with vectors from E. The supports of $\alpha_1, \dots, \alpha_t$ cover all coordinates, since every mixed exponent vector is in their span. Thus $s_i = |\operatorname{supp}(\alpha_i)| \geq 2$ for $i \leq t$ and $\sum_{i \leq t} s_i \geq n$.

For a nonnegative integral vector of coordinate sum d and support size s, the Euclidean norm is largest when one entry is $d - s + 1$ and the other nonzero entries are 1. Indeed, moving one unit from another entry greater than one to a largest entry increases the norm without changing the support. For $2 \leq s \leq d$,

$$(d - s + 1)^2 + s - 1 \leq d^2 - s(d - 1) \leq d^2 \alpha_d^s.$$

The middle expression minus the first is $(s - 2)(d - s)$. The second inequality follows from Bernoulli's inequality applied to $\alpha_d^s = (1 - 2(d-1)/d^2)^{s/2}$. Hence $\|\alpha_i\|_2 \leq d\gamma_d^{s_i}$ for $i \leq t$, while $\|\alpha_i\|_2 \leq d$ for the remaining vectors.

Since the lattice generated by the α_i lies in L, the index $[\mathbb{Z}^n : L]$ is at most the absolute determinant of the matrix with rows α_i. By (2.4) and Hadamard's inequality,

$$|A| \leq |D| \leq \prod_{i=1}^{n} \|\alpha_i\|_2 \leq d^n \gamma_d^{\sum_{i \leq t} s_i} \leq d^n \gamma_d^n.$$

Jordan's theorem gives a normal abelian subgroup $B \leq \mathrm{Aut}(F)$ of index at most $J(n)$. Thus $|\mathrm{Aut}(F)| \leq J(n)|B|$, and $J(n) = (n+1)!$ for $n \geq 71$ by [Col07, Theorem A]. □

3. Primitive blocks

To improve the bound in Theorem 2.3 for large n, we use the dimensions of the primitive blocks and the permutation group on the one-dimensional blocks.

A finite group $G \leq \mathrm{GL}(V)$ is *primitive* if it acts irreducibly and there is no decomposition $V = W_1 \oplus \cdots \oplus W_s$ with $s \geq 2$ and nonzero W_i permuted by G. For $r \geq 2$, write $\Xi(r)$ for the maximum of $|\Gamma/Z(\Gamma)|$ over finite primitive subgroups $\Gamma \leq \mathrm{GL}(r, \mathbb{C})$, as in [EL25, Definition 1.6].

3.1. Decomposition and kernels.

Proposition 3.1. *Let F be a smooth form of degree $d \geq 3$ on V and write $G = \mathrm{Aut}(F)$.*

(i) *There is a decomposition*

$$V = W_1 \oplus \cdots \oplus W_s$$

into nonzero subspaces permuted by G such that, for every a, the subgroup of G preserving W_a has primitive image $\Gamma_a \leq \mathrm{GL}(W_a)$. Call any such decomposition a primitive block decomposition*, and its summands* primitive blocks*.*

(ii) *Choose a primitive block decomposition. Let ℓ be the number of one-dimensional blocks, $b = s - \ell$ the number of higher-dimensional blocks, and $R = \dim V - \ell$ their total dimension.*

Let H be the kernel of the action on the blocks and let $D \leq H$ be the subgroup acting by scalars on each block. Write $Q = G/H$ and let $Q_1 \leq S_\ell$ be its image on the

one-dimensional blocks. For each a, write $r_a = \dim W_a$ and let $K_a \triangleleft \Gamma_a$ be the image of H on W_a.

A higher-dimensional block is active if the image of K_a in $\mathrm{PGL}(W_a)$ is nontrivial, and passive otherwise. Then

$$|G| \le |D|\,|Q_1|\,b! \prod_{a \text{ active}} \Xi(r_a), \qquad |D| \le d^s. \tag{3.1}$$

(iii) *There is an absolute constant $A \ge 1$ such that*

$$\Xi(r) \le A^r r!, \qquad \Xi(r) \le (r+1)!. \tag{3.2}$$

The first inequality holds for $r \ge 2$, and the second for $r > 12$.

Proof. The existence in (i) follows from [YYZ25, Set-up 3.2] or [EL25, Lemma 3.1].

If $\dim V = 1$, then $G = \mu_d$ and (3.1) holds. For $\dim V \ge 2$, we use the estimates for $[G : H]$, $[H : D]$, and $|D|$ as in [YYZ25, Lemma 3.7] and [EL25, Theorem 3.4]. The permutations of the two collections of blocks give $Q \le Q_1 \times S_b$. The action of H on every line and passive block is scalar, so the map from H to the product of the projective images of K_a on the active blocks has kernel D. By Schur's lemma, the scalar subgroup of Γ_a is its centre, so each projective image has order at most $\Xi(r_a)$. The bound $|D| \le d^s$ is [YYZ25, Theorem 3.5], also proved in [EL25, Theorem 3.3].

Part (iii) follows from [Col08, Theorem A] with A enlarged for $2 \le r \le 12$. □

Remark 3.2. When $d \ge 3$, every decomposition into primitive blocks for $\mathrm{Aut}(F)$ refines the maximal additive decomposition of F. Theorem A.5 gives indecomposable forms in n variables with n one-dimensional primitive blocks.

Write R_0 for the total dimension of the passive blocks.

Lemma 3.3. *In the notation of Proposition 3.1, the group H' acts trivially on the one-dimensional and passive blocks. On each active block W_a it has image K'_a, and $(W_a^\vee)^{K'_a} = 0$.*

Proof. The image of H' on W_a is K'_a. On a line or a passive block, K_a consists of scalars, so $K'_a = 1$. If W_a is active, then K_a is not abelian, because every normal abelian subgroup of Γ_a is scalar [Col08, Lemma 1]. Since K'_a is normal in Γ_a, its fixed covectors form a Γ_a-invariant subspace of $W_a^\vee$. Irreducibility of $W_a^\vee$ and $K'_a \ne 1$ give $(W_a^\vee)^{K'_a} = 0$. □

3.2. **Factorial estimates for higher-dimensional blocks.** With $|D| \le d^{\ell+b}$ in (3.1), the higher-dimensional blocks contribute the factor $d^b b! \prod_{a \text{ active}} \Xi(r_a)$. We estimate this factor in terms of R.

Lemma 3.4. *Fix $d \ge 3$. For integers $r_1, \dots, r_b \ge 2$ and $I \subseteq \{1, \dots, b\}$, write $R = \sum_i r_i$. There is a constant B_d such that*

$$d^b b! \prod_{i \in I} \Xi(r_i) \le B_d (R+1)!. \tag{3.3}$$

The left side is 1 if $b = 0$ and $O_d((R-1)!)$ if $b \ge 2$.

Proof. For nonnegative integers x, y,

$$\frac{(x+3)!(y+3)!}{(x+y+3)!} = 6 \prod_{k=4}^{x+3} \frac{k}{y+k} \le 6.$$

Iterating this inequality gives $\prod_i (r_i+1)! \le 6^{b-1}(R-2b+3)!$. For $b \ge 2$, the ratio $(R-2b+3)!/(R-1)!$ is nonincreasing in $R \ge 2b$. Thus, for each $C > 0$,

$$\frac{C^b b! \prod_i (r_i+1)!}{(R-1)!} \le a_b(C) := \frac{(6C)^b b!}{(2b-1)!}.$$

Since $(2b-1)!/b! \ge b^{b-1}$, we have $a_b(C) \le b(6C/b)^b \to 0$. With $B(C) = \sup_{b \ge 2} a_b(C)$, we obtain, for $b \ge 2$,

$$C^b b! \prod_{i=1}^{b} (r_i+1)! \le B(C)(R-1)!. \tag{3.4}$$

Choose $K \ge 1$ so that $\Xi(r) \le K(r+1)!$ for all $r \ge 2$, using (3.2). For $b \ge 2$, the left side of (3.3) is at most $B(dK)(R-1)!$ by (3.4) with $C = dK$. For $b = 1$ the left side is at most $dK(R+1)!$, so (3.3) holds with $B_d = \max\{1, dK, B(dK)\}$. □

Applying Lemma 3.4 in (3.1), with $n = \ell + R$, gives

$$|G| \le B_d d^\ell |Q_1|(R+1)! \le B_d d^\ell \ell!(R+1)!. \tag{3.5}$$

In particular,

$$\frac{|G|}{(n+1)!} \le B_d \frac{d^\ell}{\binom{n+1}{\ell}}. \tag{3.6}$$

3.3. **Consequences of a factorial lower bound.** Suppose $|G| \ge cn!$ for some $c > 0$. If $\ell \ge 1$, then $\ell!(R+1)! \le n!$, so (3.5) gives

$$[S_\ell : Q_1] \le (B_d/c) d^\ell \le B^\ell, \qquad B = d \max\{1, B_d/c\}. \tag{3.7}$$

Lemma 3.5. *Fix $d \ge 3$. Consider smooth degree-d forms F without Fermat summands in n variables along a sequence with $n \to \infty$. If $|\operatorname{Aut}(F)| \ge cn!$ for a fixed $c > 0$, then $b = o(n)$ and $R_0 = o(n)$ in the notation of Proposition 3.1.*

Proof. By (3.1) and (3.2),

$$|G| \le A_0^n \ell! b! \prod_{a \text{ active}} r_a!$$

for a constant A_0 depending only on d.

Suppose $b \ge \varepsilon n$ along a subsequence for some fixed $\varepsilon > 0$. Choose $K \ge 1$ with $\Xi(r) \le K(r+1)!$ for $r \ge 2$. Using the estimate defining $a_b(C)$ with $C = dK$ in (3.1) gives

$$\frac{|G|}{n!} \le d^\ell a_b(dK) \le d^n b(6dK/b)^b \longrightarrow 0.$$

Here $\ell!(R-1)! \le n!$. This contradicts the lower bound, so $b = o(n)$.

If $R_0 \ge \varepsilon n$ along a subsequence, then $\ell + b + \sum_{a \text{ active}} r_a = n + b - R_0 \le (1 - \varepsilon/2)n$ for large n. Thus

$$\frac{|G|}{n!} \le A_0^n \frac{\lfloor (1-\varepsilon/2)n \rfloor !}{n!} \longrightarrow 0, \tag{3.8}$$

since the denominator after cancellation has at least $\varepsilon n/2$ factors of size at least $(1-\varepsilon/2)n$. This contradicts the lower bound, so $R_0 = o(n)$. □

4. Scalar and permutation bounds

In this section we use the alternating supporting partition of Q_1 to estimate $|D|$ and $|Q_1|$ when $[S_\ell : Q_1] \le B^\ell$ as in (3.7).

4.1. **Alternating supporting partitions.** For an integer $m \ge 0$, write $[m] = \{1, \dots, m\}$. For $\Omega \subseteq [m]$, let S_Ω and A_Ω be the symmetric and alternating groups supported on Ω. A partition $\mathcal{P}$ of $[m]$ is an *alternating supporting partition* of $Q \le S_m$ if the product of the S_Ω for $|\Omega| < 5$ and the A_Ω for $|\Omega| \ge 5$ is contained in Q. Pago showed that there is a unique coarsest such partition $\mathrm{SPA}(Q)$, in the sense that every alternating supporting partition of Q refines it. Pago also shows that Q preserves $\mathrm{SPA}(Q)$, so

$$\prod_{\substack{\Omega \in \mathrm{SPA}(Q) \\ |\Omega| < 5}} S_\Omega \times \prod_{\substack{\Omega \in \mathrm{SPA}(Q) \\ |\Omega| \ge 5}} A_\Omega \le Q \le \mathrm{Stab}(\mathrm{SPA}(Q)). \tag{4.1}$$

Here $\mathrm{Stab}(\mathrm{SPA}(Q))$ is the subgroup of S_m preserving the partition. See [Pag23, Definition 4.2.7 and Lemmas 4.2.8, 4.2.10].

Lemma 4.1. *Let $m_j \to \infty$ and let $Q_j \le S_{m_j}$ satisfy $[S_{m_j} : Q_j] \le B^{m_j}$ for some fixed $B \ge 1$. Write $\mathcal{P}_j = \mathrm{SPA}(Q_j)$. For any integers $u_j \ge 2$ with $u_j = o(m_j)$, the parts of size at most u_j contain $o(m_j)$ points. In particular, $|\mathcal{P}_j| = o(m_j)$.*

Proof. Fix such a sequence u_j, and let Δ_j be the union of the parts of size at most u_j. Suppose that $t_j = |\Delta_j| \ge c m_j$ along a subsequence for some $c > 0$. Pass to a further subsequence on which t_j is strictly increasing. Let $H_j \le S_{\Delta_j}$ be the group induced on Δ_j by the elements of Q_j fixing $[m_j] \setminus \Delta_j$ pointwise. Since Δ_j is Q_j-invariant, $|Q_j| \le |H_j|(m_j - t_j)!$. Thus

$$[S_{\Delta_j} : H_j] \le \frac{[S_{m_j} : Q_j]}{\binom{m_j}{t_j}} \le (B^{1/c})^{t_j}.$$

Identify S_{Δ_j} with S_{t_j} and index the groups H_j by t_j, taking S_t for the missing indices t. With $k = \max\{1, \lceil c^{-1} \log_2 B \rceil\}$ in [Pag23, Lemma 4.3.8], there are $c' > 0$ and, for all sufficiently large j, subsets $\Lambda_j \subseteq \Delta_j$ with $|\Lambda_j| \ge c' t_j$ and $A_{\Lambda_j} \le H_j$. Extending by the identity outside Δ_j gives $A_{\Lambda_j} \le Q_j$.

Since $u_j = o(m_j)$, eventually $|\Lambda_j| > u_j$ and $|\Lambda_j| \ge 5$. The partition consisting of Λ_j and the singletons outside it is an alternating supporting partition of Q_j. Since this partition refines $\mathcal{P}_j$, the part of $\mathcal{P}_j$ containing Λ_j lies in Δ_j, contradicting $|\Lambda_j| > u_j$.

Take $u_j = \max\{2, \lfloor \sqrt{m_j} \rfloor\}$. Then there are $o(m_j)$ parts of size at most u_j and at most $m_j/u_j = o(m_j)$ larger parts. □

4.2. **Large parts of the supporting partition.** Fix $d \ge 3$. Let F_j be smooth degree-d forms without Fermat summands in n_j variables, with $n_j \to \infty$. For $G_j = \mathrm{Aut}(F_j)$, use the notation of Proposition 3.1 with subscript j. Suppose that for fixed $\delta > 0$ and $B \ge 1$,

$$b_j = o(n_j), \quad R_{0,j} = o(n_j), \quad \ell_j \ge \delta n_j, \quad [S_{\ell_j} : Q_{1,j}] \le B^{\ell_j}. \tag{4.2}$$

Number the blocks so that $W_1, \dots, W_{\ell_j}$ are the one-dimensional blocks. We suppress the subscript j in the local arguments below.

In SPA(Q_1), let s_0 be the number of points in parts of size less than $d+2$, and let a be the number of parts of size at least $d+2$. Lemma 4.1 gives $s_0 = o(\ell)$ and $a = o(\ell)$. Write

$$u = a + s_0 + R_0 + b + d + 2. \tag{4.3}$$

Call a part Ω *large* if $|\Omega| > u$, and write U_Ω for the sum of the lines labelled by Ω. By (4.2), $u = o(\ell)$, so Lemma 4.1 gives

$$\sum_{\Omega \text{ not large}} |\Omega| = o(\ell). \tag{4.4}$$

Write

$$M = \sum_{\Omega \text{ large}} |\Omega|, \qquad r = \ell - M, \qquad g = \#\{\Omega : \Omega \text{ is large}\}. \tag{4.5}$$

By (4.4) and $g \le a$,

$$r = o(\ell), \qquad g = o(\ell). \tag{4.6}$$

Lemma 4.2. *For every large part Ω, the restriction $F|_{U_\Omega}$ is nonzero.*

Proof. Suppose $F|_{U_\Omega} = 0$. For every $x \in U_\Omega$, $\mathrm{d}F_x$ vanishes on U_Ω. Thus we have a polynomial map

$$U_\Omega \longrightarrow (V/U_\Omega)^\vee, \qquad x \longmapsto \mathrm{d}F_x.$$

Since H' fixes U_Ω pointwise, Lemma 3.3 gives $\mathrm{d}F_x|_{W_a} = 0$ on every active block. The sum of the passive blocks and the lines in parts of size less than $d+2$ has dimension $R_0 + s_0$.

Let $\Pi \neq \Omega$ be a part of size at least $d+2$, and let $G(\Pi)$ be the preimage of A_Π under $G \to Q_1$. The group $G(\Pi)$ acts by scalars on each one-dimensional block in U_Ω, so $G(\Pi)'$ fixes U_Ω pointwise. Since A_Π is perfect, $G(\Pi)'$ maps onto A_Π and permutes the one-dimensional blocks in U_Π transitively. An invariant covector on U_Π is determined by its restriction to one of these blocks, so $\dim(U_\Pi^\vee)^{G(\Pi)'} \le 1$. For $x \in U_\Omega$, the restriction $\mathrm{d}F_x|_{U_\Pi}$ lies in this space because $G(\Pi)'$ fixes x.

The map $x \mapsto \mathrm{d}F_x$ therefore takes values in a subspace of dimension at most $R_0 + s_0 + a - 1 < |\Omega|$. In a basis of that subspace its components are homogeneous of degree $d-1$. By the projective dimension theorem these components have a common zero in $\mathbb{P}(U_\Omega)$, where all derivatives of F vanish. □

Lemma 4.3. *For every large part Ω,*

$$A_\Omega \times 1 \le Q \le Q_1 \times S_b,$$

where the second component acts on the higher-dimensional blocks.

Proof. The subgroup $Q \cap (A_\Omega \times S_b)$ projects onto A_Ω by (4.1). Its projection to S_b has a nontrivial kernel, since $|\Omega| > u \ge b + d + 2$ gives $|A_\Omega| = |\Omega|!/2 > b!$. The first projection of the kernel is a nontrivial normal subgroup of the simple group A_Ω. Thus the kernel is $A_\Omega \times 1$. □

4.3. **The lattice of block multidegrees.** Decompose F into components homogeneous in each block. Let $E \subseteq \mathbb{Z}^s_{\ge 0}$ consist of the *block multidegrees* of the nonzero components, and write $L = \langle E \rangle_\mathbb{Z}$. Each $\alpha \in E$ has coordinate sum d.

A tuple of block scalars (λ_a) multiplies the component of multidegree α by $\prod_a \lambda_a^{\alpha_a}$. As in (2.3), D is isomorphic to the character group of $\mathbb{Z}^s/L$. Since D is finite, L has full rank and

$$|D| = [\mathbb{Z}^s : L]. \tag{4.7}$$

For a finite coordinate set Ω, let

$$\mathsf{A}_\Omega = \left\{ z \in \mathbb{Z}^\Omega : \sum_{i\in\Omega} z_i = 0 \right\}.$$

We embed this lattice in a larger coordinate lattice by extending vectors by zero outside Ω.

Lemma 4.4. *For each large part Ω, write*

$$h_\Omega = \gcd\{\alpha_i - \alpha_k : \alpha \in E,\ i,k \in \Omega\}.$$

Then

$$h_\Omega \mathsf{A}_\Omega \subseteq L, \qquad 1 \le h_\Omega < d, \qquad h_\Omega \mid d, \qquad h_\Omega \le \beta. \tag{4.8}$$

Proof. By Lemma 4.3, each $\sigma \in A_\Omega$ lifts to an element of G whose permutation of the blocks is σ on Ω and trivial elsewhere. Hence E is A_Ω-invariant. Each $\alpha \in E$ has at least two zero coordinates on Ω because $|\Omega| \ge d+2$. Thus h_Ω is the gcd of the entries α_i for $\alpha \in E$ and $i \in \Omega$. If $\alpha_i \neq 0$, choose distinct $j,k \in \Omega$ with $\alpha_j = \alpha_k = 0$. The cycle $(i\ j\ k)$ gives $\alpha_i(e_j - e_i) \in L$. Since L is A_Ω-invariant and A_Ω is doubly transitive, $\alpha_i \mathsf{A}_\Omega \subseteq L$. Taking integer linear combinations gives $h_\Omega \mathsf{A}_\Omega \subseteq L$.

By Lemma 4.2, some $\alpha \in E$ is supported on Ω. Its entries sum to d, so $h_\Omega > 0$ and $h_\Omega \mid d$. If $h_\Omega = d$, every monomial involving x_i with $i \in \Omega$ is x_i^d. Since $F|_{U_\Omega} \neq 0$, some x_i^d with $i \in \Omega$ occurs and gives a Fermat summand. Therefore h_Ω is a proper divisor of d and $h_\Omega \le \beta = d/q$. □

Lemma 4.5. *Let $L \subseteq \mathbb{Z}^s$ be a full-rank lattice generated by nonnegative vectors with coordinate sum $d \ge 1$. Let $\Omega_1, \dots, \Omega_g$ be disjoint nonempty subsets of $[s]$, and suppose $h_i \mathsf{A}_{\Omega_i} \subseteq L$ for positive integers h_i. If $t = s - \sum_i (|\Omega_i| - 1)$, then*

$$[\mathbb{Z}^s : L] \le d^t \prod_i h_i^{|\Omega_i|-1}.$$

Proof. Write $K = \bigoplus_i h_i \mathsf{A}_{\Omega_i}$. Summing the coordinates on each Ω_i and keeping the coordinates outside their union defines a surjection $\pi : \mathbb{Z}^s \to \mathbb{Z}^t$ with kernel $\bigoplus_i \mathsf{A}_{\Omega_i}$. Since $K \subseteq L \cap \ker \pi$,

$$[\mathbb{Z}^s : L] = [\mathbb{Z}^t : \pi(L)]\,[\ker\pi : L \cap \ker\pi] \le [\mathbb{Z}^t : \pi(L)] \prod_i h_i^{|\Omega_i|-1}.$$

The images of the generators of L have nonnegative entries summing to d, so their Euclidean norms are at most d. Since $\pi(L)$ has full rank, choose t linearly independent images. The absolute determinant of the matrix with these rows is at least $[\mathbb{Z}^t : \pi(L)]$ and at most d^t by Hadamard's inequality. □

Apply Lemma 4.5 to the large parts, with $t = s - (M - g) = b + r + g$. Equations (4.7) and (4.8) give

$$|D| \le d^{b+r+g} \beta^{M-g} = d^b q^{r+g} \beta^\ell. \tag{4.9}$$

4.4. **Permutation index and group order.** To combine (4.9) with (3.1), we compare q^{r+g} with $[S_\ell : Q_1]$.

Lemma 4.6. *Under (4.2), for all sufficiently large j,*

$$q^{r_j+g_j} |Q_{1,j}| \le q\ell_j!. \tag{4.10}$$

Proof. Let $m_1, \dots, m_g$ be the sizes of the large parts and let ν_k count those of size k. Since Q_1 preserves the partition and part sizes, the union of the large parts is Q_1-invariant. Hence

$$[S_\ell : Q_1] \geq \binom{\ell}{r} \frac{M!}{\prod_i m_i! \prod_k \nu_k!}. \tag{4.11}$$

Since $M/\ell \to 1$, we have $g \geq 1$ for large j. Iterating $x!y! \leq (x+y-1)!$ gives $\prod_i m_i! \leq (M-g+1)!$, and $\prod_k \nu_k! \leq g!$. The factorial quotient in (4.11) is therefore at least $\binom{M}{g-1}/g$. Using $\binom{n}{k} \geq (n/(k+1))^k$ and $g \leq 2^{g-1}$ gives

$$\frac{[S_\ell : Q_1]}{q^{r+g-1}} \geq \left(\frac{\ell}{q(r+1)}\right)^r \left(\frac{M}{2qg}\right)^{g-1}. \tag{4.12}$$

Both bases tend to infinity by $r = o(\ell)$ and $g = o(M)$, so the product is eventually at least one. □

Equations (3.1), (4.9), and (4.10) give, for all sufficiently large j,

$$\begin{aligned} |G_j| &\leq q\beta^{\ell_j}\ell_j! \left(d^{b_j} b_j! \prod_{a \text{ active}} \Xi(r_{a,j})\right) \\ &\leq qB_d\beta^{\ell_j}\ell_j!(R_j+1)!. \end{aligned} \tag{4.13}$$

If every block is one-dimensional, the factor in parentheses is 1.

4.5. **Bounds in all degrees.** For $n \geq 3$, Proposition A.2 gives a smooth indecomposable form $\Phi_{n,d}$ with a monomial subgroup of order $q\beta^n n!$. Hence

$$M_{n,d}^{\text{ind}} \geq |\operatorname{Aut}(\Phi_{n,d})| \geq q\beta^n n!. \tag{4.14}$$

Proposition 4.7. *For each $d \geq 3$, there is a constant C_d such that $M_{n,d} \leq C_d(n+1)!\beta^n$ for every $n \geq 3$.*

Proof. Suppose there are smooth forms F_j without Fermat summands in n_j variables, with $G_j = \operatorname{Aut}(F_j)$, such that

$$\eta_j = \frac{|G_j|}{(n_j+1)!\beta^{n_j}} \longrightarrow \infty.$$

By (3.5), $\eta_j \leq B_d q^{n_j}$, so $n_j \to \infty$. For large j, we have $|G_j| \geq n_j!$. Lemma 3.5 then gives $b_j = o(n_j)$ and $R_{0,j} = o(n_j)$. If $1 \leq \ell_j \leq (n_j+1)/d$, then

$$\binom{n_j+1}{\ell_j} \geq \left(\frac{n_j+1}{\ell_j}\right)^{\ell_j} \geq d^{\ell_j}.$$

The inequality $\binom{n_j+1}{\ell_j} \geq d^{\ell_j}$ also holds for $\ell_j = 0$. Equation (3.6) gives $\eta_j \leq B_d$ in both cases. Consequently $\ell_j > (n_j+1)/d$ for all sufficiently large j.

By (3.7), $[S_{\ell_j} : Q_{1,j}] \leq B^{\ell_j}$ for a fixed B. Thus (4.2) holds, and (4.13) gives $\eta_j \leq qB_d$, contradicting $\eta_j \to \infty$. □

5. Composite degree

In this section we show that the lower bound (4.14) is sharp in composite degree.

Proposition 5.1. *Fix a composite degree d. For all sufficiently large n, let F be a smooth degree-d form in n variables without Fermat summands. Suppose that $|\operatorname{Aut}(F)| \geq q\beta^n n!$. Then*

$$|\operatorname{Aut}(F)| = q\beta^n n!.$$

Moreover, every primitive block decomposition of $\operatorname{Aut}(F)$ consists of one-dimensional blocks.

Proof. Let F_j satisfy the hypotheses in n_j variables, where $n_j \to \infty$. Choose an arbitrary primitive block decomposition for each $G_j = \operatorname{Aut}(F_j)$. Then

$$|G_j| \geq q\beta^{n_j} n_j!. \tag{5.1}$$

By Lemma 3.5, $b_j = o(n_j)$ and $R_{0,j} = o(n_j)$. Since $\ell_j!(R_j+1)!/n_j! \leq R_j + 1 \leq n_j + 1$, equations (3.5) and (5.1) give

$$q\beta^{n_j} \leq B_d(n_j+1)d^{\ell_j}. \tag{5.2}$$

Taking logarithms gives $\ell_j \geq \delta n_j$ for all sufficiently large j, where $\delta = (\log\beta)/(2\log d) > 0$.

Since $|G_j| \geq n_j!$, (3.7) gives the index bound in (4.2). If $R_j > 0$, then $1 \leq R_j \leq n_j - 1$ and (4.13) gives

$$\frac{|G_j|}{n_j!\beta^{n_j}} \leq qB_d \frac{R_j+1}{\binom{n_j}{R_j}\beta^{R_j}} \leq \frac{qB_d}{n_j}. \tag{5.3}$$

This contradicts (5.1) for large j. Hence $R_j = 0$ eventually, and the first line of (4.13) gives $|G_j| \leq q\beta^{n_j} n_j!$. Equation (5.1) gives the reverse inequality. □

6. Prime degree

In this section we relate the maximum order in prime degree to the existence of a smooth S_{n+1}-invariant form on the standard hyperplane.

6.1. **Invariant forms on the standard hyperplane.** For $n \geq 2$, let $V_n = \{y \in \mathbb{C}^{n+1} : \sum_i y_i = 0\}$. Permuting coordinates gives the faithful standard representation of S_{n+1} on V_n. Its image intersects $\mathbb{C}^\times I_{V_n}$ in $\{I_{V_n}\}$ because S_{n+1} has trivial centre. Thus a smooth S_{n+1}-invariant form F of degree d on V_n satisfies $|\operatorname{Aut}(F)| \geq d(n+1)!$.

Lemma 6.1. *Let $n \geq 4$ and $d \geq 3$. Every smooth degree-d form on V_n preserved by A_{n+1} in the standard representation is indecomposable.*

Proof. The group A_{n+1} permutes the t summands of the maximal additive decomposition. Since $t \leq n$, we have $|A_{n+1}| > t!$. By simplicity, A_{n+1} preserves each summand subspace. The irreducibility of V_n as an A_{n+1}-representation gives $t = 1$. □

Lemma 6.2. *Let $n \geq 2$. If $d < n(n+1)/2$, every A_{n+1}-invariant degree-d form on V_n is S_{n+1}-invariant.*

Proof. Let $\widetilde{F}$ be the average over A_{n+1} of a homogeneous extension of F to $\mathbb{C}^{n+1}$. Fix a transposition τ. The difference $\widetilde{F} - \widetilde{F}\circ\tau$ is alternating, so it vanishes on each hyperplane $y_i = y_j$ and is divisible by $\prod_{i<j}(y_i - y_j)$. The product has degree $n(n+1)/2 > d$, so the difference is zero. Thus $\widetilde{F}$ is symmetric, as is its restriction to V_n. □

Proposition 6.3. *Let $n \geq 2$ and $d \geq 2$. There is a smooth S_{n+1}-invariant form of degree d on V_n if and only if* $\gcd(n+1, d-1) = 1$.

Chênevert states this criterion for $n \geq 3$ [Chê11, Theorem 2.6].

Proof. Write $N = n+1$ and $m = d-1$. Suppose a prime ℓ divides both N and m. Write $N = a\ell$ and choose a primitive ℓ-th root of unity ζ. Let v have a coordinates equal to ζ^j for each $0 \leq j < \ell$. Then v lies in V_n.

For an S_{n+1}-invariant form F on V_n, the differential $\mathrm{d}F_v$ is fixed by the group $(S_a)^\ell$ of permutations among equal coordinates. Choose a permutation τ that cycles the coordinate classes and satisfies $\tau v = \zeta v$. Since $\zeta^m = 1$, homogeneity and invariance give $\mathrm{d}F_v \circ \tau = \mathrm{d}F_{\tau v} \circ \tau = \mathrm{d}F_v$. The group generated by $(S_a)^\ell$ and τ is transitive on the coordinates and has no nonzero fixed covector on V_n. Hence $\mathrm{d}F_v = 0$. Euler's identity gives $F(v) = 0$, so $[v]$ is singular on X_F.

The converse follows from Proposition A.3. □

6.2. **Nonprimitive automorphism groups.** We take $d \geq 3$ prime in this and the following subsection, so $q = d$ and $\beta = 1$.

Lemma 6.4. *Let F be a smooth form of prime degree $d \geq 3$ without Fermat summands. If a primitive block decomposition of* $\mathrm{Aut}(F)$ *consists of a line and an active block, then $D = \mu_d$.*

Proof. Use coordinates (x, w) on the line and the active block W. Since H' fixes x and preserves F, the coefficient of x^{d-1} lies in $(W^\vee)^{H'} = 0$ by Lemma 3.3. If the coefficient of x^d also vanished, $[1:0]$ would be singular.

If $F|_W = 0$, write $F = xP$. Since $\dim W > 1$ and P is homogeneous of degree $d-1$, the restriction $P|_W$ has a zero in $\mathbb{P}(W)$. At that point $x = P = 0$, so $\mathrm{d}F = 0$, contradicting smoothness. Since F has no Fermat summands, it has a nonzero component of bidegree $(k, d-k)$ with $0 < k < d$. An element of D with scalars (λ, μ) on the two blocks therefore satisfies

$$\lambda^d = \mu^d = 1, \qquad (\lambda/\mu)^k = 1.$$

Since d is prime, $\lambda = \mu$. Thus $D = \mu_d$, as the scalars in μ_d preserve F. □

Proposition 6.5. *Fix a prime $d \geq 3$. For all sufficiently large n, let F be a smooth degree-d form in n variables without Fermat summands, with $G = \mathrm{Aut}(F)$ nonprimitive. Then $|G| \leq dn!$. If $|G| = dn!$, every primitive block decomposition of G is of one of the following types.*

(i) *All blocks are one-dimensional.*
(ii) *There are two blocks, a line and an active block of dimension $n-1$.*

Proof. Suppose the assertion fails along a sequence with $n \to \infty$. Then $|G| \geq dn!$. Use the notation of Proposition 3.1 for a decomposition with more than one block. If $|G| = dn!$, choose a decomposition of neither type in the statement. By Lemma 3.5, $b = o(n)$ and $R_0 = o(n)$. Pass to a subsequence on which either $\ell = o(n)$ or $\ell \geq \delta n$ for some fixed $\delta > 0$.

Suppose first that $\ell = o(n)$. For $\ell \geq 2$, (3.5) gives

$$\frac{|G|}{n!} \leq B_d\, a_n(\ell), \qquad a_n(k) = d^k \frac{k!(n-k+1)!}{n!}.$$

The ratio $a_n(k+1)/a_n(k) = d(k+1)/(n-k+1)$ is less than one for $2 \leq k < \ell$ when n is large. It follows that $a_n(\ell) \leq a_n(2) = 2d^2/n$, contrary to $|G| \geq dn!$.

If $\ell \le 1$ and $b \ge 2$, (3.1) and the $b \ge 2$ bound in Lemma 3.4 give $|G| = O_d((R-1)!) = O_d((n-1)!)$, again contrary to $|G| \ge dn!$.

Since the decomposition has more than one block, $\ell = b = 1$. A passive block would give $|G| \le d^2$, contradicting $|G| \ge dn!$ for large n. The block of dimension $n-1$ is therefore active, and $D = \mu_d$ by Lemma 6.4. Equations (3.1) and (3.2) give $|G| \le d\,\Xi(n-1) \le dn!$ for $n - 1 > 12$.

Suppose instead that $\ell \ge \delta n$. By (3.7), the sequence satisfies the index bound in (4.2). If $R > 0$, then $R \ge 2$ and $\ell \ge 2$ for large n. The second line of (4.13) gives

$$\frac{|G|}{dn!} \le B_d \frac{R+1}{\binom{n}{\ell}} \le B_d \frac{n+1}{\binom{n}{2}} \longrightarrow 0.$$

Thus $R = 0$ for large n, and the first line of (4.13) gives $|G| \le dn!$. Both cases contradict the choice of the sequence. □

6.3. **Primitive automorphism groups.** If $\mathrm{Aut}(F)$ is primitive, Schur's lemma gives $Z(\mathrm{Aut}(F)) = \mu_d$. Equation (3.2) gives $|\mathrm{Aut}(F)| \le d(n+1)!$ when $n > 12$. Together with Proposition 6.5, this gives, for all sufficiently large n,

$$M_{n,d} \le d(n+1)!. \tag{6.1}$$

Lemma 6.6. *There is an absolute integer r_0 such that if $r \ge r_0$ and $\Gamma \le \mathrm{GL}(r, \mathbb{C})$ is finite and primitive with $|\Gamma/Z(\Gamma)| \ge r!$, then Γ contains a normal subgroup isomorphic to A_{r+1} acting by the standard representation.*

Proof. Let E be the central product of the components and quasicomponents of Γ in [Col08, Theorem 5 and p. 765]. Each component is quasisimple. For a prime p, the largest normal p-subgroup of Γ is a *quasicomponent* when it is noncyclic. By primitivity, $\mathbb{C}^r$ is isomorphic as an E-module to $W^{\oplus(r/m)}$, where W is irreducible of dimension m and $m \mid r$. Since E acts faithfully on $\mathbb{C}^r$, it acts faithfully on W.

Let $E_1, \dots, E_s$ be the factors of this central product. If $s = 0$, then the generalized Fitting subgroup of Γ is $Z(\Gamma)$. The self-centralizing property of the generalized Fitting subgroup would give $\Gamma = Z(\Gamma)$, contrary to primitivity for $r > 1$. Thus $s \ge 1$. The representation on W is a tensor product of irreducible representations of the E_i. Their degrees a_i satisfy $m = \prod_i a_i$. Each factor representation is faithful, since its kernel would act trivially on W, so $a_i \ge 2$.

For a component, write

$$c_i = |E_i/Z(E_i)|\,|\,\mathrm{Out}_c(E_i)|,$$

where $\mathrm{Out}_c(E_i)$ is the subgroup of outer automorphisms represented by automorphisms acting trivially on $Z(E_i)$. For a quasicomponent, write $|E_i/Z(E_i)| = p^{2t}$. Then $a_i \ge p^t$ and we take $c_i = p^{2t}|\,\mathrm{Sp}_{2t}(p)|$. By [Col08, Theorem 5],

$$|\Gamma/Z(\Gamma)| \le s! \prod_{i=1}^{s} c_i. \tag{6.2}$$

Collins's bounds [Col08, Theorems 6 and 9] give $c_i \le A(a_i+1)!$ for an absolute constant A chosen to cover the finitely many exceptions in small degrees.

If $s \geq 2$, replace pairs of integers a_i by their products until two integers $a, b \geq 2$ remain. The sum cannot decrease, so $\sum_i a_i \leq a + b \leq ab/2 + 2 = m/2 + 2$. Also $s \leq \log_2 m$. Hence

$$\log |\Gamma/Z(\Gamma)| \leq \log(s!) + s \log A + \sum_i \log((a_i + 1)!) \leq \left(\frac{r}{2} + O(\log r)\right) \log(r + 1).$$

The implied constant is absolute, so the right-hand side is less than $\log(r!)$ for all sufficiently large r. If $s = 1$ and $m < r$, then $m \leq r/2$, and $A(m + 1)! < r!$ for large r. We conclude that $s = 1$ and $a_1 = m = r$.

For a quasicomponent, the symplectic order formula gives $c_1 \leq p^{2t^2+3t}$. For a component with simple quotient S of Lie type of rank k over a field of size u, the bounds in [Col08, pp. 769–770] give $k \log u = O(\log r)$ and $\log |\operatorname{Aut}(S)| = O(k^2 \log u)$ with absolute constants. The faithful representation of the component induces a faithful projective representation of S, so the degree bounds apply. In both cases $\log c_1 = O((\log r)^2)$, using $\log p, \log u \geq \log 2$. If the simple quotient of E_1 is sporadic or among the small exceptions, $c_1 = O(1)$. In each of these cases, $c_1 < r!$ for sufficiently large r.

Thus E_1 is a component with simple quotient A_t. Since $c_1 \geq r!$, the bounded cases $t < 12$ are excluded for sufficiently large r. Then $c_1 \leq |\operatorname{Aut}(A_t)| = t!$, so $t \geq r$. The component is a quotient of the Schur double cover of A_t.

The representation of E_1 pulls back to a nontrivial irreducible representation of this cover of degree r. By [BNOTV15, Lemma 3.5(1)–(2)], this degree is $t - 1$ or greater than $2t$. Hence $r = t - 1$. The representation of degree $t - 1$ is unique and factors through the standard representation of A_t [EJM25, Lemma 2.5 in the arXiv version]. Since E_1 acts faithfully, it is isomorphic to A_{r+1} and acts by the standard representation. Conjugation by Γ permutes the components, so the sole component E_1 is normal in Γ. □

We now return to arbitrary degree $d \geq 3$ and combine the upper bounds with the constructions to prove Theorem 1.1.

Proof of Theorem 1.1. Equation (4.14) and Proposition 4.7 give the lower and upper bounds in (i). An indecomposable form in $n \geq 3$ variables has no Fermat summands, so $M^{\mathrm{ind}}_{n,d} \leq M_{n,d}$.

To prove (ii), take n sufficiently large. If d is composite, Proposition 5.1 gives $M_{n,d} \leq q\beta^n n!$, so (4.14) gives equality for both maxima.

Suppose d is prime. Let $G = \operatorname{Aut}(F)$ for a smooth degree-d form F in n variables without Fermat summands. If G is primitive and $|G| \geq dn!$, then $Z(G) = \mu_d$ and $|G/Z(G)| \geq n!$. By Lemma 6.6, in suitable coordinates G contains A_{n+1} acting on V_n by its standard representation. Since $d < \binom{n+1}{2}$, Lemma 6.2 gives S_{n+1}-invariance of F.

If $\gcd(n + 1, d - 1) = 1$, Proposition 6.3 gives a smooth S_{n+1}-invariant form on V_n. By Lemma 6.1 it is indecomposable, and its group contains $\mu_d \times S_{n+1}$. The bound (6.1) gives $M^{\mathrm{ind}}_{n,d} = M_{n,d} = d(n + 1)!$.

If $\gcd(n + 1, d - 1) > 1$, the preceding argument and Proposition 6.3 give $|\operatorname{Aut}(F)| < dn!$ when $\operatorname{Aut}(F)$ is primitive. Otherwise Proposition 6.5 gives $|\operatorname{Aut}(F)| \leq dn!$. By (4.14), $M^{\mathrm{ind}}_{n,d} = M_{n,d} = dn!$, since $q = d$ and $\beta = 1$. □

7. Largest orders of automorphism groups

In this section we prove Corollary 1.2. We first compare $|\operatorname{Aut}(F)|$ for forms without Fermat summands with $d^{n-\kappa}(n - \kappa)!$ for fixed κ.

Corollary 7.1. *Fix $d \geq 3$ and an integer $\kappa \geq 0$. As $n \to \infty$,*

$$\sup_F \frac{|\operatorname{Aut}(F)|}{d^{n-\kappa}(n-\kappa)!} = O_{d,\kappa}\big(n^{\kappa+1}q^{-n}\big),$$

where F runs over smooth degree-d forms in n variables without Fermat summands.

Proof. By Theorem 1.1(i),

$$\frac{|\operatorname{Aut}(F)|}{d^{n-\kappa}(n-\kappa)!} \leq C_d d^{\kappa} q^{-n} \frac{(n+1)!}{(n-\kappa)!} = O_{d,\kappa}(n^{\kappa+1}q^{-n}).$$

□

With this estimate, we can prove Corollary 1.2.

Proof of Corollary 1.2. By Corollary 2.2, each $|\operatorname{Aut}(F)|$ has the form $d^{n-r}(n-r)!\,a$ with $0 \leq r \leq n$ and $a \in \mathcal{O}_{r,d}$. Conversely, for each $1 \leq r \leq n$ and $a \in \mathcal{O}_{r,d}$, choose F_0 as in the definition of $\mathcal{O}_{r,d}$ with $|\operatorname{Aut}(F_0)| = a$ and adjoin $n-r$ Fermat summands. The pair $(0,1)$ gives the Fermat form.

Fix (T,a) and write $K = T+1$. Corollary 7.1 with $\kappa = K$ and the finiteness of each $\mathcal{O}_{r,d}$ give a constant $B_{d,T}$ such that, for every $r \geq K$ and $b \in \mathcal{O}_{r,d}$,

$$b \leq B_{d,T} d^{r-K}(r-K)!.$$

For $K \leq r \leq n$ we have $(n-r)!(r-K)! \leq (n-K)!$, so

$$\frac{d^{n-r}(n-r)!\,b}{d^{n-T}(n-T)!\,a} \leq \frac{B_{d,T}}{ad(n-T)}.$$

For all sufficiently large n, the bound is less than one for every $T < r \leq n$ and every $b \in \mathcal{O}_{r,d}$. When $r = T$ and $b < a$, the ratio is $b/a < 1$.

Take $n \geq \max_i r_i$ large enough for these comparisons to hold for each of the first M pairs $(T,a) = (r_i, a_i)$. The corresponding orders are strictly decreasing, and each exceeds all orders from later pairs with $r \leq n$. □

Remark 7.2. The Fermat hypersurface and the finite list of exceptions in [YYZ25, Theorem 1.1] and [EL25, Theorem 2.3] give a simple and complete answer to the maximum-order problem. Determining the second largest order for each pair (n,d) would require more calculations and separate arguments for exceptional cases.

The contributor's original question was how the largest few distinct orders compare for fixed d and large n. The estimate in Theorem 2.3 suffices for Corollary 1.2. The contributor finds the arithmetic dependence in Theorem 1.1 more interesting and was pleased to obtain an exact answer for all sufficiently large n. The formula depends on whether d is prime and, when it is, on $\gcd(n+1, d-1)$.

Appendix A. Constructions and rigidity

In this appendix we give the constructions for the lower bounds and describe the forms with $|\operatorname{Aut}(F)| = M_{n,d}$.

A.1. **Forms with monomial subgroups.**

Lemma A.1. *Let d be composite and $n \geq 3$. If f is a symmetric form of degree q with a mixed monomial and $F = f(x_1^\beta, \dots, x_n^\beta)$ is smooth, then F is indecomposable.*

Proof. Write $H_{ij} = (H_F)_{ij}$ and let e_i be the ith coordinate vector. The monomial exponents in H_{ij} are congruent to $-e_i - e_j$ modulo β. If $i \neq j$, then $H_{ij} \neq 0$ by symmetry and the presence of a mixed monomial in f. For $A = (a_{ij}) \in \mathcal{C}_F$ and $i \neq j$, comparing the terms of residue $-e_i - e_j$ modulo β in $AH_F = H_F A^{\mathsf{t}}$ and cancelling H_{ij} gives

$$A(E_{ij} + E_{ji}) = (E_{ij} + E_{ji})A^{\mathsf{t}},$$

where E_{ij} is the (i,j) matrix unit. Thus A preserves each coordinate plane $\langle e_i, e_j \rangle$. Since $n \geq 3$, their intersections give the coordinate lines, so A is diagonal. The same identity gives $a_{ii} = a_{jj}$. Hence $\mathcal{C}_F = \mathbb{C}I_n$, and Lemma 2.1 gives indecomposability. □

Let q be the smallest prime divisor of d and let $\beta = d/q$. For $r \geq 3$, let

$$\Phi_{r,d}(x_1, \dots, x_r) = \sum_{i=1}^{r} x_i^d + \left(\sum_{i=1}^{r} x_i^\beta \right)^q. \tag{A.1}$$

Proposition A.2. *For $d \geq 3$ and $r \geq 3$, the form $\Phi_{r,d}$ is smooth and indecomposable. The group $\mathrm{Aut}(\Phi_{r,d})$ contains a monomial subgroup of order $q\beta^r r!$.*

Proof. Write $\Phi = \Phi_{r,d}$ and $S = \sum_i x_i^\beta$. If $\lambda_i^\beta = \omega$ for all i, with $\omega \in \mu_q$, then $\mathrm{diag}(\lambda_1, \dots, \lambda_r)$ preserves $\sum_i x_i^d$ and sends S to ωS, so it preserves Φ. The coordinate permutations normalize this group of $q\beta^r$ diagonal matrices and intersect it trivially. Together they generate a subgroup of $\mathrm{Aut}(\Phi)$ of order $q\beta^r r!$.

The partial derivatives of Φ are

$$\frac{\partial \Phi}{\partial x_i} = d x_i^{\beta-1} \big((x_i^\beta)^{q-1} + S^{q-1} \big).$$

Suppose x is a nonzero critical point. Then $S \neq 0$, since for $S = 0$ the derivatives reduce to $d x_i^{d-1}$. Write $z_i = x_i^\beta / S$ for $x_i \neq 0$. We have $z_i^{q-1} = -1$ and $\sum_{x_i \neq 0} z_i = 1$. If $q = 2$, each z_i is -1, which contradicts $\sum_{x_i \neq 0} z_i = 1$.

If q is odd, write $m = q - 1$ and choose a primitive $2m$-th root ζ_{2m}. With $\zeta_m = \zeta_{2m}^2$, each z_i belongs to $\zeta_{2m}\mathbb{Q}(\zeta_m)$. Their sum is one, so $\zeta_{2m} \in \mathbb{Q}(\zeta_m)$. But $[\mathbb{Q}(\zeta_{2m}) : \mathbb{Q}(\zeta_m)] = 2$ for even m. Thus Φ is smooth.

If $\beta \geq 2$, the form $\sum_i y_i^q + (\sum_i y_i)^q$ has a mixed monomial, so Lemma A.1 gives indecomposability.

Suppose $\beta = 1$ and let $A = (a_{ij}) \in \mathcal{C}_\Phi$. Write $m = d-2$. The Hessian is $d(d-1)\big(\mathrm{diag}(x_i^m) + S^m J\big)$, where J is the matrix of ones. Write $b_i = \sum_k a_{ik}$. For $i \neq j$, the (i,j)-entry of the centre identity gives

$$a_{ij} x_j^m - a_{ji} x_i^m + (b_i - b_j) S^m = 0.$$

For distinct i, j, choose $k \notin \{i, j\}$, which is possible since $r \geq 3$. The coefficient of x_k^m gives $b_i = b_j$. Comparing the remaining coefficients gives $a_{ij} = a_{ji} = 0$. Now A is diagonal, so $b_i = b_j$ gives $a_{ii} = a_{jj}$. Hence $\mathcal{C}_\Phi = \mathbb{C}I_r$ and Φ is indecomposable by Lemma 2.1. □

A.2. **Smooth forms on the standard hyperplane.** The coefficient formula below gives the converse in Proposition 6.3.

Proposition A.3. *Let $n \geq 2$ and $d \geq 2$ with $\gcd(n+1, d-1) = 1$. Then $\Theta_{n,d} = \widetilde{F}|_{V_n}$ is a smooth S_{n+1}-invariant form of degree d, where*

$$\widetilde{F}(y_1, \dots, y_{n+1}) = [z^d] \prod_{i=1}^{n+1} (1 - z y_i)^{(d-1)/(n+1)}. \tag{A.2}$$

Here $[z^d]$ extracts the coefficient of z^d in the formal expansion at $z = 0$.

Proof. Write $N = n + 1$ and $m = d - 1$. The polynomial $\widetilde{F}$ in (A.2) is symmetric and homogeneous of degree d. At $y \in V_n$, write $\prod_{i=1}^{N} (1 - z y_i)^{m/N} = \sum_{j \geq 0} c_j z^j$. Differentiate the product with respect to y_i and take the coefficient of z^{m+1} to obtain

$$\frac{\partial \widetilde{F}}{\partial y_i}(y) = -\frac{m}{N} \sum_{k=0}^{m} c_{m-k} y_i^k.$$

At a critical point y of $\widetilde{F}|_{V_n}$, the partial derivatives of $\widetilde{F}$ are equal. Let λ be the common value of $\sum_{k=0}^{m} c_{m-k} y_i^k$. Then $A(t) = \sum_{k=0}^{m} c_{m-k} t^k - \lambda$ is monic of degree m since $c_0 = 1$, and $A(y_i) = 0$ for every i.

Let $T(t) = \prod_{i=1}^{N} (t - y_i)$ and consider the Laurent series

$$P(t) = t^m \prod_{i=1}^{N} (1 - y_i/t)^{m/N} \in \mathbb{C}((t^{-1})).$$

By the definitions of T and P, we have $P^N = T^m$. The coefficients c_j give $P - A - \lambda \in t^{-1}\mathbb{C}[[t^{-1}]]$. Logarithmic differentiation gives $P' = (m/N)(T'/T)P$.

The poles of T'/T are simple, and A vanishes at each of them, so $(T'/T)A$ is a polynomial. Since $P' - A' = O(t^{-2})$ and $(T'/T)(P - A) = O(t^{-1})$, the polynomial $A' - (m/N)(T'/T)A$ is $O(t^{-1})$ and hence zero.

Hence $(A^N/T^m)' = 0$, and $A^N = T^m$ by monicity. Since $\gcd(N, m) = 1$, every root multiplicity of T is divisible by $N = \deg T$. Thus all y_i are equal, and $\sum_i y_i = 0$ gives $y = 0$. This proves that $\Theta_{n,d}$ is smooth. □

A.3. **Decomposable forms.**

Proposition A.4. *Fix a prime $d \geq 3$. Let F be a smooth degree-d form in n variables without Fermat summands. If F is decomposable, then*

$$|\operatorname{Aut}(F)| = O_d\big((n-1)!\big).$$

In particular, if n is sufficiently large and $|\operatorname{Aut}(F)| = M_{n,d}$, then F is indecomposable.

Proof. Theorem 1.1(i) gives $|\operatorname{Aut}(F)| \leq A_d(m+1)!$ for smooth degree-d forms in $m \geq 3$ variables without Fermat summands. For $m = 2$, $|\operatorname{Lin}(X_F)| \leq d!$, so $|\operatorname{Aut}(F)| \leq d\, d!$. Increasing A_d extends the bound to $m = 2$.

Write $m_1, \dots, m_t$ for the dimensions of the $t \geq 2$ summands in the maximal additive decomposition of F. Each summand is smooth and has no Fermat summands, so $m_i \geq 2$. By

(2.1) and (3.4),

$$|\operatorname{Aut}(F)| \le A_d^t t! \prod_{i=1}^{t} (m_i+1)! \le B(A_d)(n-1)!.$$

By (4.14), $M_{n,d}^{\mathrm{ind}} \ge dn!$. For sufficiently large n, the bound for decomposable forms is below $dn!$, so every form with $|\operatorname{Aut}(F)| = M_{n,d}$ is indecomposable. □

A.4. **Forms with maximum group order in composite degree.** Under (4.2), the right side of (4.12) tends to infinity along any subsequence with $r_j + g_j > 1$. Hence

$$\frac{q^{r_j+g_j}|Q_{1,j}|}{q\ell_j!} \longrightarrow 0. \tag{A.3}$$

Let F be a smooth degree-d form in n variables without Fermat summands, with $d \ge 3$. Suppose a primitive block decomposition of $G = \operatorname{Aut}(F)$ consists of lines and $|G| = q\beta^n n!$. Since $|D| \le d^n$, we have $[S_n : Q_1] = n!|D|/|G| \le q^{n-1}$. Along any sequence of such forms with $n \to \infty$, (4.2) holds. By (4.9) and (4.10),

$$q\beta^n n! = |D|\,|Q_1| \le q^{r+g}\beta^n |Q_1| \le q\beta^n n!.$$

By (A.3), for sufficiently large n we have $r = 0$ and $g = 1$, so $[n]$ is the only large part. Lemmas 4.4 and 4.5 give $|D| \le dh_{[n]}^{n-1} \le q\beta^n$. Since $|D|\,|Q_1| = q\beta^n n!$, we obtain

$$Q_1 = S_n, \qquad |D| = q\beta^n, \qquad h_{[n]} = \beta. \tag{A.4}$$

Theorem A.5. *Fix a composite degree d. For all sufficiently large n, a smooth degree-d form F in n variables without Fermat summands satisfies $|\operatorname{Aut}(F)| = M_{n,d}$ if and only if in some linear coordinates it is of the form*

$$F(x_1, \dots, x_n) = f(x_1^\beta, \dots, x_n^\beta),$$

where f is symmetric of degree q and is not proportional to $\sum_i y_i^q$. These forms are all indecomposable.

In these coordinates $\operatorname{Aut}(F) = D_{n,d} \rtimes S_n$, with S_n acting by coordinate permutations and

$$D_{n,d} = \{\operatorname{diag}(\lambda_1, \dots, \lambda_n) : \lambda_1^\beta = \cdots = \lambda_n^\beta \in \mu_q\}.$$

Two forms in the stated coordinates are linearly equivalent if and only if their degree-q polynomials f are proportional.

Proof. Suppose $|\operatorname{Aut}(F)| = M_{n,d} = q\beta^n n!$, and write $G = \operatorname{Aut}(F)$. By Proposition 5.1, every primitive block is one-dimensional for sufficiently large n. Choose a basis vector on each line. By (A.4), the coordinates of each support vector are congruent modulo β. Since $n > d$, each vector has a zero coordinate, so $F = f(x_1^\beta, \dots, x_n^\beta)$ with $\deg f = q$. The group $D_{n,d}$ preserves this form and has order $q\beta^n = |D|$, so $D = D_{n,d}$.

On the monomial group, the homomorphism $\rho\colon \operatorname{diag}(\lambda_i)\sigma \mapsto \operatorname{diag}(\lambda_i^\beta)\sigma$ has kernel μ_β^n. The surjective map $y_i = x_i^\beta$ intertwines the action of G on x with that of $\rho(G)$ on y. Thus $\rho(G)$ preserves f. The projection $\rho(G) \to S_n$ is surjective with kernel μ_q. Since $\beta \ge 2$, any coordinate point on X_F would be singular. Thus every pure power y_i^q occurs in f. Rescale the y_i so that all coefficients of y_i^q are one. In the resulting coordinates $\rho(G) \subseteq \mu_q^n \rtimes S_n$.

For $n \ge 5$, two lifts in $\rho(G)$ of disjoint transpositions preserve a common coordinate line. Their commutator lies in μ_q and acts trivially on that line, so the lifts commute. A lift of

$(i\ j)$ commutes with lifts of all transpositions on $[n] \setminus \{i, j\}$ and hence acts by a common scalar on the other coordinate lines. Divide by this scalar to obtain a lift that fixes those lines pointwise. The square of the lift lies in μ_q and fixes those lines pointwise, so it is the identity.

Take such a lift t_i for each adjacent transposition, and write $t_i e_i = a_i e_{i+1}$. Then $t_i e_{i+1} = a_i^{-1} e_i$. In the basis $(b_i e_i)$ with $b_1 = 1$ and $b_{i+1} = a_i b_i$, the t_i are the adjacent coordinate transpositions. Thus $\mu_q \times S_n \le \rho(G)$. Both groups have order $qn!$, so $\rho(G) = \mu_q \times S_n$ and f is symmetric.

These diagonal changes of coordinates in y lift to diagonal changes in x. Since $\mu_\beta^n \subseteq G$, we have $G = \rho^{-1}(\mu_q \times S_n) = D_{n,d} \rtimes S_n$. If f were proportional to $\sum_i y_i^q$, then F would be a Fermat form, contrary to the hypothesis.

Conversely, let f be symmetric of degree q and not proportional to $\sum_i y_i^q$, so f has a mixed monomial. If $F = f(x_1^\beta, \dots, x_n^\beta)$ is smooth, it is indecomposable by Lemma A.1. The group $D_{n,d} \rtimes S_n$ preserves F and has order $q\beta^n n!$. By Theorem 1.1, it equals $\mathrm{Aut}(F)$ for sufficiently large n.

Let $G = D_{n,d} \rtimes S_n$ in these coordinates. Every abelian normal subgroup of G has trivial image in S_n for $n \ge 5$, so $D_{n,d}$ is characteristic in G. The common eigenspaces of $D_{n,d}$ are the coordinate lines because $\mu_\beta^n \subseteq D_{n,d}$. A normalizing matrix is therefore monomial. After multiplying by a permutation, write it as $\Lambda = \mathrm{diag}(\lambda_i)$. The product $\Lambda(i\ j)\Lambda^{-1}(i\ j)$ lies in $D_{n,d}$ and fixes every other coordinate line pointwise. This gives $\lambda_i^\beta = \lambda_j^\beta$ for all i, j. Thus $\Lambda \in \mathbb{C}^\times \mu_\beta^n$ and $N_{\mathrm{GL}(n,\mathbb{C})}(G) = \mathbb{C}^\times G$.

A linear equivalence between two such forms normalizes G and hence belongs to $\mathbb{C}^\times G$, so the forms are proportional. The map $x \mapsto (x_1^\beta, \dots, x_n^\beta)$ is surjective, so the polynomials f are proportional as well. Conversely, rescaling x by a dth root of a proportionality factor gives a linear equivalence. □

For sufficiently large n, consider the open subset of $\mathbb{P}\big((\mathrm{Sym}^q(\mathbb{C}^n)^\vee)^{S_n}\big)$ where $f(x_1^\beta, \dots, x_n^\beta)$ is smooth, with the point $[\sum_i y_i^q]$ removed. By Theorem A.5, this subset parametrizes the linear equivalence classes. It is nonempty by Proposition A.2.

A.5. **Forms with maximum group order in prime degree.** Fix a prime $d \ge 3$ in this subsection. For primitive groups we use the standard representation from the proof of Theorem 1.1. If $\mathrm{Aut}(F)$ is nonprimitive and has order $dn!$, Proposition 6.5 gives two possibilities for the block dimensions.

Corollary A.6. *Fix a prime $d \ge 3$. For all sufficiently large n, let F be a smooth degree-d form in n variables without Fermat summands. If $\mathrm{Aut}(F)$ is primitive and $|\mathrm{Aut}(F)| \ge dn!$, then F is linearly equivalent to an S_{n+1}-invariant form on V_n, with $\mathrm{Aut}(F) = \mu_d \times S_{n+1}$ in these coordinates.*

Proof. The proof of Theorem 1.1 shows that F is S_{n+1}-invariant in suitable coordinates on V_n. Thus $\mu_d \times S_{n+1} \le \mathrm{Aut}(F)$, and equality follows from (6.1). □

Proposition A.7. *Fix a prime $d \ge 3$. For all sufficiently large n, let F be a smooth degree-d form in n variables without Fermat summands. If $G = \mathrm{Aut}(F)$ is nonprimitive and $|G| = dn!$, then in suitable linear coordinates $G = \mu_d \times S_n$, where S_n acts by coordinate permutations.*

Proof. Use the two possibilities for the block dimensions in Proposition 6.5. Suppose first that all blocks are one-dimensional. By (A.4) with $q = d$ and $\beta = 1$, we have $Q_1 = S_n$ and $|D| = d$. Thus $D = \mu_d$.

Each adjacent transposition has a unique involutory lift t_i because the central kernel μ_d has odd order. For $i \neq k$, let m be the order of $(i\ i+1)(k\ k+1)$. Conjugation by t_i inverts the central element $(t_i t_k)^m \in \mu_d$. Since μ_d has odd order, $(t_i t_k)^m = 1$. The Coxeter presentation gives a complement S_n in G.

Let S_{n-1} be the subgroup of the complement preserving a chosen coordinate line. The representation of the complement on $\mathbb{C}^n$ is induced from the character of S_{n-1} on that line. The sign character would give the sign twist of the permutation representation. Since d is odd, F would be alternating and divisible by the Vandermonde product of degree $\binom{n}{2} > d$. Hence the character is trivial and the complement acts by coordinate permutations.

Now suppose the blocks have dimensions 1 and $n-1$, and write W for the active block. Their dimensions differ, so G preserves both. By Lemma 6.4, $D = \mu_d$. The restriction $G \to \mathrm{GL}(W)$ is faithful because its kernel consists of elements of D acting trivially on W. Its image Γ is primitive, so by Schur's lemma its centre consists of the scalar matrices in Γ. These are exactly the images of elements of D, so $|\Gamma/Z(\Gamma)| = n!$, and Lemma 6.6 gives a subgroup A_n acting on W by its standard representation. The inverse image of A_n in G fixes the one-dimensional block pointwise because A_n has no nontrivial one-dimensional character for $n \geq 5$.

Expand F in the coordinate on the one-dimensional block. Each coefficient is an A_n-invariant form on W of degree at most d. The argument of Lemma 6.2 gives S_n-invariance of these coefficients when $d < n(n-1)/2$. In the permutation representation of S_n, the trivial and standard summands give $\mathbb{C}^n = \mathbb{C}(1,\dots,1) \oplus V_{n-1}$. Thus $\mu_d \times S_n \leq G$ in these coordinates, and both groups have order $dn!$. □

Theorem A.8. *Fix a prime $d \geq 3$. For all sufficiently large n, the smooth degree-d forms F in n variables without Fermat summands for which $|\mathrm{Aut}(F)| = M_{n,d}$ are, up to linear equivalence, as follows.*

(i) *If $\gcd(n+1, d-1) = 1$, they are the smooth S_{n+1}-invariant forms on V_n, with $\mathrm{Aut}(F) = \mu_d \times S_{n+1}$.*

(ii) *If $\gcd(n+1, d-1) > 1$, they are the smooth symmetric forms on $\mathbb{C}^n$ without Fermat summands, with $\mathrm{Aut}(F) = \mu_d \times S_n$.*

These forms are all indecomposable.

Proof. Suppose $|\mathrm{Aut}(F)| = M_{n,d}$. In (i), $M_{n,d} = d(n+1)!$, so $\mathrm{Aut}(F)$ is primitive by Proposition 6.5 and Corollary A.6 applies. In (ii), $M_{n,d} = dn!$, and Corollary A.6 and Proposition 6.3 imply that $\mathrm{Aut}(F)$ is nonprimitive. Proposition A.7 gives the stated group.

Conversely, Lemma 6.1 gives indecomposability in (i). The indicated groups preserve the forms and have order $M_{n,d}$. In (ii), indecomposability then follows from Proposition A.4. □

A.6. **Linear equivalence in prime degree.**

Proposition A.9. *Fix a prime $d \geq 3$ and take n sufficiently large. If $\gcd(n+1, d-1) = 1$, the smooth locus of $\mathbb{P}\big((\mathrm{Sym}^d V_n^\vee)^{S_{n+1}}\big)$ parametrizes the linear equivalence classes in Theorem A.8(i).*

If $\gcd(n+1, d-1) > 1$, the classes are the orbits of

$$\{uI_n + vJ_n : u(u+nv) \neq 0\}/\mathbb{C}^\times$$

on the projective classes of smooth indecomposable forms in $(\mathrm{Sym}^d(\mathbb{C}^n)^\vee)^{S_n}$. Here J_n is the $n \times n$ matrix all of whose entries are one, and S_n acts by coordinate permutations.

Proof. By Theorem A.8, $\mathrm{Aut}(F) = \mu_d \times S_N$, with $N = n+1$ in the first case and $N = n$ in the second. A linear equivalence therefore normalizes the derived group A_N.

Every automorphism of A_N for $N \geq 7$ is induced by conjugation by a permutation [Wil09, Theorem 2.3]. The centralizer of A_N on V_{N-1} consists of scalars by Schur's lemma. On $\mathbb{C}^N$, the centralizer consists of invertible matrices with constant diagonal entries and constant off-diagonal entries by double transitivity. Thus

$$N_{\mathrm{GL}(V_{N-1})}(A_N) = \mathbb{C}^\times S_N,$$
$$N_{\mathrm{GL}(N,\mathbb{C})}(A_N) = \{uI_N + vJ_N : u(u+Nv) \neq 0\}\, S_N.$$

The S_N factors fix the forms, and scalar matrices give all nonzero rescalings. □

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