COUNTEREXAMPLES TO NEGATIVE CORRELATION IN THE RANDOM-CLUSTER MODEL FOR $0 \le q < 1$

ABSTRACT. We construct finite simple graphs in which two edges are positively correlated in the homogeneous random-cluster model for sufficiently small $q > 0$, as well as under the uniform measure on connected spanning subgraphs. This refutes the random-cluster negative-correlation conjecture in the regime $0 \le q < 1$, including conjectures of Grimmett and Winkler (2004) and Semple and Welsh (2008). By matroid duality, we also obtain regular basis-Rayleigh matroids whose uniform measures on independent sets violate negative correlation, refuting a conjecture of Wagner (2008). The corresponding negative-correlation conjecture for spanning forests of graphs remains open.

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1. INTRODUCTION

1.1. **The random-cluster model and negative correlation conjectures.** Let $G = (V, E)$ be a finite undirected multigraph with $n = |V|$ vertices and $m = |E|$ edges. For an edge subset $A \subseteq E$, let $\kappa(A)$ denote the number of connected components of the spanning subgraph (V, A), isolated vertices included. Given a *cluster weight* $q > 0$ and positive *edge weights* $\boldsymbol{\gamma} = (\gamma_g)_{g\in E} \in \mathbb{R}^E_{>0}$ with $\gamma(A) := \prod_{g\in A} \gamma_g$ (taking $\gamma(\varnothing) := 1$), the *random-cluster partition function* and *random-cluster measure* [FK72, Gri06] on G are defined by

$$Z_G(q, \boldsymbol{\gamma}) := \sum_{A\subseteq E} q^{\kappa(A)}\gamma(A) \quad \text{and} \quad \mu_{G,q,\boldsymbol{\gamma}}(A) := \frac{q^{\kappa(A)}\gamma(A)}{Z_G(q, \boldsymbol{\gamma})} \quad (A \subseteq E).$$

Letting $p_g := \gamma_g/(1 + \gamma_g) \in (0, 1)$ for each $g \in E$, so that $\gamma_g = p_g/(1 - p_g)$, yields the equivalent bond-percolation parameterisation $\mu_{G,q,\boldsymbol{\gamma}}(A) \propto q^{\kappa(A)} \prod_{g\in A} p_g \prod_{g\in E\setminus A}(1 - p_g)$. When every edge carries the same weight $\gamma_g = \gamma > 0$, we write $\mu_{G,q,\gamma}$ and call the model *homogeneous*.

At $q = 1$, the factor $q^{\kappa(A)}$ is constant and $\mu_{G,1,\boldsymbol{\gamma}}$ is independent bond percolation. For $q \ge 1$, the Fortuin–Kasteleyn–Ginibre (FKG) lattice condition holds [FKG71], so $\mu_{G,q,\boldsymbol{\gamma}}$ is positively associated and any two edges $e, f \in E$ satisfy $\mathrm{Cov}_{\mu_{G,q,\boldsymbol{\gamma}}}(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\}) \ge 0$. In the complementary regime $0 < q < 1$, given the configuration on $E \setminus \{g\}$, the conditional odds of opening an edge g are $\gamma_g/q > \gamma_g$ when its endpoints lie in distinct components and γ_g when they are already connected. Opening other edges can connect the endpoints of g and remove this $1/q > 1$ advantage, suggesting repulsive dependence among edges. A longstanding conjecture in probability and combinatorics [Pem00, GW04, Sok05, Gri06, SW08, KN10] asserts that for every finite graph G, every positive weight vector $\boldsymbol{\gamma} \in \mathbb{R}^E_{>0}$, every $q \in (0, 1)$, and every pair of distinct edges $e, f \in E$,

$$\mathrm{Cov}_{\mu_{G,q,\boldsymbol{\gamma}}}(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\}) \le 0, \tag{1.1}$$

with a stronger version [GW04, KN10] asserting full negative association for increasing events depending on disjoint edge sets.

As $q \downarrow 0$, the random-cluster measure admits three distinct combinatorial limits depending on how $\boldsymbol{\gamma}$ scales with q [GW04, Section 4] (see also [Sok05, Section 1.5] and [SW08, Section 3]):

(i) *Connected spanning subgraphs ($q \downarrow 0$ at fixed weights).* Assume G is connected, hold $\boldsymbol{\gamma} \in \mathbb{R}^E_{>0}$ fixed, and let $q \downarrow 0$. Only configurations with $\kappa(A) = 1$ survive in $q^{-1}Z_G(q, \boldsymbol{\gamma})$, yielding the

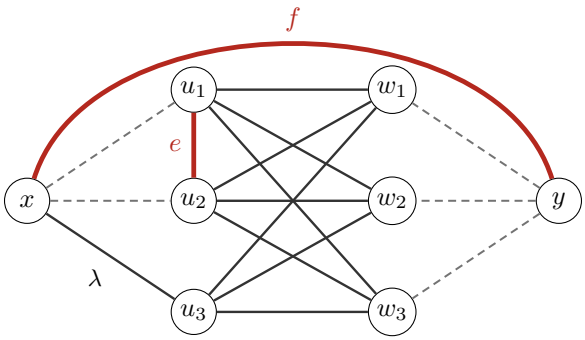

FIGURE 1. The eight-vertex, seventeen-edge simple graph G. Thin solid edges (the nine edges of $K_{U,W}$ and the single spoke xu_3) have weight λ, dashed edges (the five spokes $xu_1, xu_2, yw_1, yw_2, yw_3$) have weight λ^2, and the two thick red marked edges $e = u_1u_2$ (weight λ) and $f = xy$ (weight λ^2) are vertex-disjoint.

connected-spanning-subgraph partition function and measure

$$C_G(\boldsymbol{\gamma}) := \sum_{\substack{A \subseteq E \\ \kappa(A)=1}} \gamma(A) \quad \text{and} \quad \mu_{G,0,\boldsymbol{\gamma}}(A) := \frac{\gamma(A)\,\mathbf{1}\{\kappa(A)=1\}}{C_G(\boldsymbol{\gamma})}. \tag{1.2}$$

When $\gamma_g = 1$ for all $g \in E$, the measure $\mu_{G,0,1}$ is the uniform distribution on connected spanning subgraphs of G.

(ii) *Unrooted spanning forests ($q \downarrow 0$ with $\gamma_g = q\beta_g$).* Fix $\boldsymbol{\beta} = (\beta_g)_{g\in E} \in \mathbb{R}^E_{>0}$ and set $\gamma_g = q\beta_g$. Letting $\ell(A) := |A| - |V| + \kappa(A) \geq 0$ denote the cycle rank of (V, A), we have $q^{-|V|}Z_G(q, q\boldsymbol{\beta}) = \sum_{A\subseteq E} q^{\ell(A)}\beta(A)$, which converges as $q \downarrow 0$ to the *forest partition function* and *arboreal gas* (or weighted spanning-forest) measure

$$F_G(\boldsymbol{\beta}) := \sum_{\substack{A \subseteq E \\ \ell(A)=0}} \beta(A) \quad \text{and} \quad \nu_{G,\boldsymbol{\beta}}(A) := \frac{\beta(A)\,\mathbf{1}\{\ell(A)=0\}}{F_G(\boldsymbol{\beta})}. \tag{1.3}$$

(iii) *Spanning trees ($q \downarrow 0$ and $\gamma_g \downarrow 0$ with $q/\gamma_g \to 0$).* Scaling $\gamma_g = \tau\beta_g$ and letting $q \downarrow 0$ faster than $\tau^{|V|}$ concentrates the measure on spanning trees of a connected graph G, where pairwise negative correlation (and determinantal strong Rayleigh structure) has been known since Kirchhoff and Brooks, Smith, Stone, and Tutte [BSST40, BP93].

Grimmett and Winkler formulated pairwise edge negative correlation for the uniform forest measure $\nu_{G,1}$ on every finite graph G and for the uniform connected-spanning-subgraph measure $\mu_{G,0,1}$ on every finite connected graph G in [GW04, Conjecture 1.1], and discussed the stronger negative-association property immediately afterwards. Semple and Welsh [SW08, Conjectures 1.1 and 1.2] formulated the two assertions separately and extended them to matroids, asking which matroids satisfy pairwise negative correlation under weighted independent-set measures (*I-Rayleigh*) and weighted spanning-set measures (*S-Rayleigh*). In this hierarchy, Wagner [Wag08, Conjecture 5.11] conjectured that graphic matroids, regular matroids, sixth-root-of-unity matroids, matroids with the half-plane property, and basis-Rayleigh (B-Rayleigh) matroids are all I-Rayleigh.

1.2. **Main results.** We show that the universal negative-correlation conjecture (1.1) for $0 < q < 1$, its connected-spanning-subgraph counterpart at $q = 0$ (both weighted and uniform), and Wagner's conjecture for regular matroids are all false.

To state the construction explicitly, let $G = (V, E)$ be the simple graph with eight vertices and seventeen edges depicted in Figure 1. Specifically, let $U := \{u_1, u_2, u_3\}$ and $W := \{w_1, w_2, w_3\}$ be

disjoint three-vertex sets, let x and y be two additional apex vertices so that $V = U \cup W \cup \{x, y\}$, let $e := u_1u_2$ and $f := xy$ be two vertex-disjoint marked edges, and let the fifteen background edges be

$$E(H) := \{u_iw_j : i, j \in \{1, 2, 3\}\} \cup \{xu_1, xu_2, xu_3\} \cup \{yw_1, yw_2, yw_3\}, \tag{1.4}$$

where $H := (V, E(H)) = G \setminus \{e, f\}$. For any parameter $\lambda > 0$, define the two-level edge-weight vector $\boldsymbol{\gamma}(\lambda) = (\gamma_g(\lambda))_{g \in E}$ on G by

$$\gamma_g(\lambda) := \begin{cases} \lambda & \text{if } g \in \{u_iw_j : i, j \in \{1, 2, 3\}\} \cup \{xu_3, e\}, \\ \lambda^2 & \text{if } g \in \{xu_1, xu_2, yw_1, yw_2, yw_3, f\}, \end{cases} \tag{1.5}$$

assigning weight λ to eleven edges and weight λ^2 to six edges.

Theorem 1.1. *Let $G = (V, E)$ be the eight-vertex, seventeen-edge simple graph with vertex-disjoint edges $e = u_1u_2$ and $f = xy$ and edge weights $\boldsymbol{\gamma}(\lambda)$ defined in (1.5). For every $\lambda \in (0, 1/100]$, under the random-cluster measure $\mu_{G,\lambda^4,\boldsymbol{\gamma}(\lambda)}$ with cluster weight $q = \lambda^4 \in (0, 10^{-8}]$, we have*

$$\mathrm{Cov}_{\mu_{G,\lambda^4,\boldsymbol{\gamma}(\lambda)}}(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\}) > 0.$$

In particular, at $\lambda = 1/100$ (where $q = 10^{-8}$ and every edge weight equals 10^{-2} or 10^{-4}),

$$\mathrm{Cov}_{\mu_{G,10^{-8},\boldsymbol{\gamma}(1/100)}}(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\}) = 9.217468414162535\ldots \times 10^{-10} > 0.$$

Because the individual edge indicators $\mathbf{1}\{e \in A\}$ and $\mathbf{1}\{f \in A\}$ are increasing functions depending on disjoint coordinate sets $\{e\}$ and $\{f\}$, Theorem 1.1 also refutes universal negative association [GW04, KN10] for the $0 < q < 1$ random-cluster model. In Theorem 1.1, the cluster weight $q = \lambda^4$ varies together with $\boldsymbol{\gamma}(\lambda)$, so letting $q \downarrow 0$ along that curve also sends $\lambda \downarrow 0$. Holding $\lambda \in (0, 1/100]$ fixed instead and extracting the coefficient of q in each sector as $q \downarrow 0$ requires a separate Rayleigh-difference calculation (Section 3) and yields a counterexample for connected spanning subgraphs.

Theorem 1.2. *Let $G = (V, E)$ and $\boldsymbol{\gamma}(\lambda)$ be as in Theorem 1.1. For every $\lambda \in (0, 1/100]$, under the connected-spanning-subgraph measure $\mu_{G,0,\boldsymbol{\gamma}(\lambda)}$ defined in (1.2), we have*

$$\mathrm{Cov}_{\mu_{G,0,\boldsymbol{\gamma}(\lambda)}}(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\}) > 0.$$

In particular, at $\lambda = 1/100$,

$$\mathrm{Cov}_{\mu_{G,0,\boldsymbol{\gamma}(1/100)}}(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\}) = 9.438685544466864\ldots \times 10^{-10} > 0.$$

The sign of $\mathrm{Cov}(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\})$ depends only on the fifteen background weights on $H = G \setminus \{e, f\}$, and not on the positive weights $\gamma_e, \gamma_f > 0$ assigned to the two marked edges (Lemma 2.1). Replacing the fifteen background edges of H by series-parallel gadgets of common edge weight preserves the cluster weight q and converts both Theorem 1.1 and Theorem 1.2 into homogeneous (unweighted) simple-graph counterexamples.

Corollary 1.3. *The following hold for homogeneous random-cluster measures and uniform connected spanning subgraphs:*

(a) For every cluster weight $q \in (0, 10^{-8}]$ and every common edge weight $\gamma > 0$ (equivalently, every common bond probability $p = \gamma/(1 + \gamma) \in (0, 1)$, including $p = 1/2$), there exists a finite simple graph G' with two vertex-disjoint edges $e, f \in E(G')$ such that

$$\mathrm{Cov}_{\mu_{G',q,\gamma}}(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\}) > 0.$$

(b) Subdividing the ten weight-10^{-2} background edges of H into paths of length 100 and the five weight-10^{-4} background edges of H into paths of length 10,000 produces an explicit simple graph G_{unif} with $|V(G_{\mathrm{unif}})| = 50{,}993$ vertices and $|E(G_{\mathrm{unif}})| = 51{,}002$ edges such that, under the uniform measure $\mu_{G_{\mathrm{unif}},0,1}$ on connected spanning subgraphs of G_{unif},

$$\mathrm{Cov}_{\mu_{G_{\mathrm{unif}},0,1}}(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\}) > 0.$$

Part (b) of Corollary 1.3 refutes the connected-spanning-subgraph part of [GW04, Conjecture 1.1] and [SW08, Conjecture 1.2]. Passing from connected spanning subgraphs of G and G_{unif} to their edge complements gives independent sets of the dual cographic matroids $M(G)^*$ and $M(G_{\mathrm{unif}})^*$, which are regular matroids and therefore B-Rayleigh, balanced [FM92], and half-plane-property matroids [COSW04].

Corollary 1.4. *The graphic matroid $M(G)$ of rank 7 and corank 10 on 17 elements is neither S-Rayleigh nor Z-Rayleigh. Its dual cographic matroid $M(G)^*$ of rank 10 and corank 7 is a regular (hence B-Rayleigh) matroid that is not I-Rayleigh, and the cographic regular matroid $M(G_{\mathrm{unif}})^*$ violates pairwise negative correlation even under the uniform distribution on its independent sets.*

Corollary 1.4 refutes Wagner's conjecture [Wag08, Conjecture 5.11(c)–(f)] (restated for regular matroids in [FM26, Conjecture 14.15]) and answers affirmatively the question of Semple and Welsh [SW08, Section 5] on whether a B-Rayleigh matroid can fail to be I-Rayleigh or S-Rayleigh. Because both regular matroids and B-Rayleigh matroids contain all graphic and cographic matroids, both classes fail S-Rayleigh (via $M(G)$) as well as I-Rayleigh (via $M(G)^*$), whereas universal S-Rayleigh for cographic matroids is equivalent by duality (Lemma 4.2) to universal I-Rayleigh for graphic matroids, the open forest problem.

Several scope boundaries deserve emphasis right away. First, Theorem 1.1 and Corollary 1.3(a) establish positive covariance for small cluster weights $q \in (0, 10^{-8}]$. Whether pairwise negative correlation holds at a fixed moderate cluster weight such as $q = 1/2$ remains open, and we make no claim that $n = 8$ or $m = 17$ is minimal. Second, while Theorem 1.1 already operates on an eight-vertex graph, the parallel path-bundle reduction in Corollary 1.3(a) is an existence construction producing a much larger simple graph. Third, and most importantly, our construction does *not* refute the spanning-forest negative-correlation conjecture [Kah00, GW04, SW08, KN10]: under the forest scaling (1.3), the exact same graph G satisfies strict pairwise negative correlation between e and f across the entire four-parameter symmetric weight family containing (1.5) (Proposition 5.1), and because G contains $K_{3,3}$ as a subgraph, its cographic matroid $M(G)^*$ is non-graphic and cannot be realised as the forest matroid of another graph.

1.3. **Related work and recent partial progress.** To situate Theorems 1.1 and 1.2 and Corollaries 1.3 and 1.4, we survey both the classical correlation hierarchy for graphs and matroids and a recent wave of partial positive results on arXiv. Table 1 summarises the resulting status of the main conjectures.

Stronger polynomial properties and earlier distinct counterexamples. Several properties strictly stronger than (1.1) were already known to fail on small examples without violating positive-weight pairwise negative correlation on graphs. Borcea, Brändén, and Liggett [BBL09] developed the theory of real stable (strongly Rayleigh) polynomials, showing that basis generating polynomials of regular matroids (and hence spanning-tree polynomials [COSW04]) are real stable, whereas for $0 < q < 1$ the random-cluster polynomial of the four-cycle C_4 is Rayleigh on $\mathbb{R}^E_{>0}$ yet fails to be real stable because its Rayleigh difference takes negative values at mixed-sign real arguments [BBL09, Section 3.4]. Similarly, Royle and Sokal [RS04] disproved the Brown–Colbourn conjecture on complex roots of reliability ($q = 0$ connected-subgraph) polynomials, while K_4 already separates complex zero-freeness and coefficientwise Rayleigh positivity from positive-weight

TABLE 1. Status of pairwise negative correlation conjectures for edges or matroid elements under positive weights (and uniform specialisations where noted).

Model or matroid class	Status	Reference
Spanning trees of any graph ($q \downarrow 0$ tree limit)	Holds	[BSST40, BP93]
Matroids of rank ≤ 3 or corank ≤ 3 (Z-Rayleigh, $0 < q \leq 1$)	Holds	[ZZ26]
Series-parallel (K_4-minor-free) graphs ($0 < q \leq 1$)	Holds	[SW08, Man11, Eri12]
Complete graph K_n: uniform random forest, all $n \geq 2$	Holds	[Sta11, Gup26]
Complete graph K_n: connected subgraphs (homogeneous $\gamma \geq 1$), all n	Holds	[TZ26c, ÇR26]
Complete graph K_n: $0 < q < 1$, homogeneous $\gamma \geq 1$, all n	Holds	[ÇR26]
Regular expanders ($d_n \to \infty$, all sufficiently large n, fixed p, $0 \leq q < 1$, $\mathrm{dist}(e, f) = o(d_n / \log d_n)$)	Holds	[TZ26a]
Random-cluster $0 < q < 1$, arbitrary graph, positive weights	**False**	**Theorem 1.1**
Random-cluster $0 < q < 1$, arbitrary graph, homogeneous weights	**False**	**Corollary 1.3(a)**
Connected spanning subgraphs ($q = 0$, connected G), positive weights	**False**	**Theorem 1.2**
Uniform connected spanning subgraphs ($q = 0$, $\gamma = 1$, connected G)	**False**	**Corollary 1.3(b)**
Graphic S-Rayleigh and cographic I-Rayleigh matroids	**False**	**Corollary 1.4**
Regular and B-Rayleigh matroids: I- and S-Rayleigh	**False**	**Corollary 1.4**
Unrooted forests (graphic I-Rayleigh / cographic S-Rayleigh)	Open	[Kah00, GW04, SW08, ÇR26]
Random-cluster at fixed moderate q (e.g. $q = 1/2$)	Open	[GW04, Gri06, NP25]

Rayleigh inequalities [SW08]. For non-regular matroids, Seymour and Welsh [SW75] constructed the eight-element binary matroid S_8 whose basis measure violates pairwise negative correlation (see also [FM92, CW06]). Finally, an example of Seymour, Winkler, and Sudan [FM92], recorded in [HSW22, Example 2], shows that uniform *fixed-size* (k-edge) forests of a graph can have positively correlated edges (specifically, six-edge forests of $K_{2,5}$ augmented by the hub edge e and a pendant edge f have covariance $1/576 > 0$), whereas in the full unrooted forest measure $\nu_{G,\boldsymbol{\beta}}$ a pendant edge is independent of all other edges (Section 5).

On the positive side, minor closure holds for the I-Rayleigh and S-Rayleigh properties [SW08, Proposition 3.5] as well as the Z-Rayleigh property [Man11, Proposition 31(1)], and these properties are preserved under series-parallel extensions [SW08, Man11]. Further, Cocks [Coc08] showed that the universal weighted and unweighted formulations are equivalent within matroid classes closed under series and parallel extensions at $q = 0$. Mani [Man11] proved that series-parallel, Fano, and non-Fano matroids are Z-Rayleigh, and Erickson [Eri12] gave sum-of-squares certificates for forests of series-parallel graphs and proved that all regular matroids are I-Rayleigh if and only if all binary B-Rayleigh matroids are I-Rayleigh.

Lorentzian polynomials and universal multiplicative bounds. For every $q \in (0, 1]$, Brändén and Huh [BH20, Theorem 4.10] proved that the homogenised multivariate Tutte polynomial of any matroid (and hence the homogenisations of the random-cluster partition function and of its $q \downarrow 0$ connected and forest limits) is Lorentzian. Anari, Liu, Oveis Gharan, Vinzant, and Vuong [ALO+21]

established related log-concavity and approximate exchange properties for matroid independent-set and forest polynomials. Log-concavity of Lorentzian polynomials yields universal 2-negative correlation, $\Pr[e \in A, f \in A] \le 2 \Pr[e \in A] \Pr[f \in A]$, for all finite graphs, all positive weights, and all $q \in (0, 1]$, as well as at $q = 0$ under $\mu_{G,0,\boldsymbol{\gamma}}$ on connected finite graphs and under the forest measure $\nu_{G,\boldsymbol{\beta}}$. Very recently, Tang and Zhang [TZ26b] sharpened this universal bound for $q \in [0, 1]$ on arbitrary finite connected graphs by retaining explicit dependence on γ_e and γ_f and incorporating connectivity of $G\backslash\{e, f\}$. In particular, for the uniform connected-spanning-subgraph measure $\mu_{G,0,1}$ on any finite connected graph G, Tang and Zhang [TZ26b] proved $\Pr[e \in A, f \in A] \le \frac{10}{9} \Pr[e \in A] \Pr[f \in A]$. Theorems 1.1 and 1.2 and Corollary 1.3 complement [BH20, TZ26b] by showing that no universal multiplicative upper bound on arbitrary finite simple graphs can improve the constant all the way down to 1.

Recent progress on complete graphs, expanders, trees, and matroids. Several recent works establish genuine pairwise negative correlation (constant 1) under structural or parameter restrictions:

- *Behaviour near $q = 1$.* Nguyen and Pylyavskyy [NP25] derived an exact combinatorial formula for the first derivative in q of $\mu_{G,q,\boldsymbol{\gamma}}(e \in A, f \in A) - \mu_{G,q,\boldsymbol{\gamma}}(e \in A)\mu_{G,q,\boldsymbol{\gamma}}(f \in A)$ at the independent-percolation point $q = 1$, confirming negative correlation infinitesimally below $q = 1$ while restating the global $0 < q < 1$ conjecture [NP25, Conjectures 2.2 and 2.6].
- *Complete graphs.* For the uniform random forest $\nu_{K_n,1}$ on K_n, Stark [Sta11] proved pairwise negative correlation for sufficiently large n via asymptotic singularity analysis, and Gupta [Gup26] recently resolved all remaining orders by proving pairwise negative correlation on K_n for every $n \ge 2$ via moment inequalities on component counts and squared degrees. Tang and Zhang [TZ26c] proved pairwise negative correlation on K_n for sufficiently large n under the uniform measures on connected spanning subgraphs, k-component forests, and connected spanning subgraphs of fixed excess k. Çiçeksiz and Ravichandran [ÇR26] proved pairwise negative correlation for $\mu_{K_n,q,\boldsymbol{\gamma}}$ across all $q \in (0, 1)$ whenever $\gamma_g \ge 2$ for every edge, and in the homogeneous case whenever $\gamma_g = \gamma \ge 1$ for every n. Taking $q \downarrow 0$ yields pairwise negative correlation for connected spanning subgraphs of K_n at any common edge weight $\gamma \ge 1$ (including the uniform case $\gamma = 1$) for every n [ÇR26, Corollary 1.10].
- *High-degree regular graphs with expansion.* Tang and Zhang [TZ26a] proved two results for homogeneous random-cluster models. First, they showed that the universal homogeneous conjecture over all finite graphs is equivalent to its restriction to sequences of d_n-regular graphs with $d_n \to \infty$, even for edge pairs at distance $o(d_n/\log d_n)$. Second, for sequences of d_n-regular graphs ($d_n \to \infty$) satisfying a *two-scale edge-isoperimetric condition* at scale r_n (including uniformly expanding regular graphs [TZ26a, Corollary 6.2], random regular graphs, and hypercubes), they proved strict pairwise negative correlation for every fixed common bond probability $p \in (0, 1)$ on all sufficiently large graphs in the sequence, uniformly over $q \in [0, 1)$ and over distinct edges e, f with $\mathrm{dist}(e, f) \le r_n$ whenever $r_n \log d_n = o(d_n)$ [TZ26a, Theorem 1.2], and hence across all edge pairs when $\mathrm{diam}(G_n) \log d_n = o(d_n)$ and the condition holds at $r_n = \mathrm{diam}(G_n)$ [TZ26a, Corollary 6.7]. Corollary 1.3 does not conflict with [TZ26a] because subdividing edges introduces long paths of degree-two vertices that violate expansion, while homogeneity of the edge weights is essential even on K_n ($n \ge 8$): embedding $(G, \boldsymbol{\gamma}(1/100))$ into K_n and assigning sufficiently small weight $\varepsilon > 0$ to all remaining edges multiplies each marked-sector sum at $\varepsilon = 0$ by q^{n-8} and preserves positive covariance at $q = 10^{-8}$ by continuity. Conversely, combining Corollary 1.3 with the regularity reduction of [TZ26a] implies that even within diverging-degree regular graph sequences at fixed common p and sub-linear edge distance, some isoperimetric or expansion hypothesis in [TZ26a] is genuinely indispensable.

- *High-fugacity forests and wired trees.* For the homogeneous arboreal gas $\nu_{G,\beta}$ at common fugacity β on a connected finite graph with distinct non-bridge edges e and f, Çiçeksiz and Ravichandran [ÇR26, Theorems 1.4, 1.5, and 4.8] showed that at large β, the first leading coefficient of the Rayleigh difference is the classical transfer-current square and, when that vanishes, the second leading coefficient also admits an electrical sum-of-squares formula. Those top two coefficients both vanish precisely along equipotential interfaces, where the sign depends on higher-order terms (resolved in special families such as chains of K_4 blocks [ÇR26, Example 4.13]), while the general $0 < q < 1$ random-cluster conjecture was restated in [ÇR26, Conjecture 1.8]. For $0 < q < 1$ on wired trees, Park [Par26, Sections 5 and 6] proved uniqueness of the supercritical wired specification on infinite regular trees and established conditional negative association under external fields for branch-connection indicators on finite wired trees, deducing nonpositive covariance in infinite volume for bounded increasing cylinder observables supported on disjoint collections of branches at a common vertex.
- *Gårding polynomials and low-rank matroids.* Fang and Ma [FM26] introduced Gårding polynomials, showing that multiaffine Gårding polynomials with nonnegative coefficients extend multiaffine real stable polynomials with nonnegative coefficients while preserving the Rayleigh property on $\mathbb{R}^E_{>0}$ [FM26, Lemma 5.21], and derived new negative-dependence results for matroid and graph polynomials while recording the regular S-/I-Rayleigh and graphic I-Rayleigh assertions as open [FM26, Conjectures 14.15 and 14.16]. Most recently, Zhao and Zhou [ZZ26] proved that every matroid of rank at most 3 or corank at most 3 is Z-Rayleigh across all $q \in (0, 1]$ and all positive weightings, settling a conjecture of Mani [Man11]. In particular, every connected graph with at most 4 vertices (rank ≤ 3) or cycle rank at most 3 (corank ≤ 3) satisfies (1.1). Our counterexample G has rank 7 and corank 10, lying safely above both thresholds.

1.4. **Proof overview and main obstacles.** Let $H = G \setminus \{e, f\}$ be the graph with e and f removed, and expand $Z_G(q, \boldsymbol{\gamma}) = Z_0 + \gamma_e Z_e + \gamma_f Z_f + \gamma_e\gamma_f Z_{ef}$, where $Z_s := \sum_{A \subseteq E(H)} q^{\kappa(A \cup s)}\gamma(A)$ for $s \subseteq \{e, f\}$. Elementary algebra (Lemma 2.1) gives $\mathrm{Cov}(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\}) = -\gamma_e\gamma_f(Z_eZ_f - Z_0Z_{ef})/Z_G^2$, so positive covariance is equivalent to strict negativity of the *Rayleigh difference* $R_G := Z_eZ_f - Z_0Z_{ef}$.

To see what structural constraints any witness with $R_G < 0$ must navigate, consider first the leading spanning-tree regime on a connected graph. When edge weights are scaled toward spanning trees on a connected graph, Kirchhoff's matrix-tree theorem and Dodgson condensation (or the transfer-current theorem of Burton and Pemantle [BP93]) express the coefficient of $C_eC_f - C_0C_{ef}$ at the lowest possible (spanning-tree) weight order as a positive multiple of the squared signed transfer current between oriented copies of e and f, which is always nonnegative (and can vanish, leaving the first nonzero coefficient to higher-order terms). Specifically, a weight-preserving automorphism that reverses an orientation of e while preserving an orientation of f (for instance, by fixing both endpoints of f individually) negates the signed transfer current and forces the entire spanning-tree Rayleigh difference to vanish without constraining cyclic terms: on K_4 with opposite edges e and f and unit background weights, swapping the endpoints of e while fixing both endpoints of f forces the spanning-tree counts $(B_0, B_e, B_f, B_{ef}) = (4, 4, 4, 4)$ to have zero determinant, yet including cyclic subgraphs still gives $(C_0, C_e, C_f, C_{ef}) = (5, 9, 9, 15)$ and $C_eC_f - C_0C_{ef} = 6 > 0$. More generally, connected planar graphs cannot violate S-Rayleigh unless planar graphic matroids also violate I-Rayleigh via planar duality, and graphs of rank or corank at most 3 satisfy $R_G(q, \boldsymbol{\gamma}) \geq 0$ for all positive weights and all $q \in (0, 1]$ [ZZ26] (and hence $R_{G,\mathrm{conn}}(\boldsymbol{\gamma}) = \lim_{q \downarrow 0} q^{-2}R_G(q, \boldsymbol{\gamma}) \geq 0$ on connected graphs of rank or corank at most 3).

Our construction overcomes these barriers on the non-planar bipartite core $K_{U,W} \cong K_{3,3}$ by attaching the apexes x (to U) and y (to W) with *asymmetric* weak spoke weights (Figure 1): the nine core edges and the single spoke xu_3 receive weight λ, whereas the remaining spokes xu_1, xu_2 and yw_1, yw_2, yw_3 receive weight λ^2. Conceptually, when $f = xy$ is present, x and y can bridge a two-component spanning subgraph of $K_{U,W}$ whenever x attaches to one component and y attaches to the other, probing the quadratic form of two-component cuts of $K_{U,W}$. Under the two-level assignment (1.5), look at the minimum-weight connected configurations in each state $s \in \{\varnothing, e, f, ef\}$:

- Without f, connecting x and y to the core at minimum λ-degree forces x to use its unique heavy spoke xu_3 (weight λ, 1 choice), y to use one of its three spokes yw_j (weight λ^2, 3 choices), and $K_{U,W}$ to be spanned by a tree (81 trees of weight λ^5 without e, or 54 contracted trees of weight λ^4 with e). This gives leading connected terms $C_0 = 243\lambda^8 + O(\lambda^9)$ and $C_e = 162\lambda^7 + O(\lambda^8)$.
- With $f = xy$ present, y is already joined to x, so minimum λ-degree configurations omit all spokes at y and connect $\{x, y\}$ to the core solely through the heavy spoke xu_3, giving $C_f = 81\lambda^6 + O(\lambda^7)$ and $C_{ef} = 54\lambda^5 + O(\lambda^6)$.

When f is absent, minimum-degree configurations attach x via xu_3 and y separately via one of the three spokes yw_j (3 choices, giving $243 = 3 \cdot 81$ and $162 = 3 \cdot 54$), whereas when f is present, the joined pair $\{x, y\}$ attaches through xu_3 alone (1 choice, giving 81 and 54). Because these apex-attachment factors (3 without f and 1 with f) act proportionally whether or not e is contracted in the $K_{U,W}$ spanning tree, and because the weight-preserving involution $u_1 \leftrightarrow u_2$ reverses $e = u_1u_2$ while fixing both endpoints x and y of f, the spanning-tree Rayleigh difference vanishes identically across all powers of λ, beginning with $(3 \cdot 54) \cdot 81 - (3 \cdot 81) \cdot 54 = 0$ at λ^{13}. Remarkably, the cyclic contributions at λ^{14} and λ^{15} in $C_eC_f - C_0C_{ef}$ (and at λ^{22} and λ^{23} in $Z_eZ_f - Z_0Z_{ef}$ when $q = \lambda^4$) also cancel identically. Grouping the connected pairs contributing to $[\lambda^{16}](C_eC_f - C_0C_{ef})$ by the sum $\ell_1 + \ell_2$ of their cycle ranks ($\ell(A \cup s) = |A \cup s| - 7$) isolates what survives at the fourth order: total cycle rank 0 (spanning-tree pairs) contributes 0, total cycle rank 1 (a spanning tree paired with a connected spanning unicycle) contributes -1296, total cycle rank 2 contributes $+684$, and total cycle rank ≥ 3 contributes 0, summing to $-1296 + 684 = -612 < 0$.

To turn this mechanism into a short, human-verifiable proof without enumerating all $2^{17} = 131{,}072$ edge subsets, we carry out four steps:

(1) In Section 2, we prove an exact polynomial identity (Lemma 2.2) expressing Z_0, Z_e, Z_f, Z_{ef} for a four-parameter weight family on H as a sum over only *three* Potts colour-partition types $(1,1,1)$, $(2,1)$, and (3) of the right shore $W = \{w_1, w_2, w_3\}$.

(2) In Section 3, we verify combinatorially that the normalised polynomials A, B, C, D (for $q = \lambda^4$) and $A_{\mathrm{conn}}, B_{\mathrm{conn}}, C_{\mathrm{conn}}, D_{\mathrm{conn}}$ (for $q = 0$) have nonnegative integer coefficients summing to at most 2^{15} at $\lambda = 1$ (Lemma 3.1). We then compute their seven-term prefixes explicitly on the page from the three partition types (Lemma 3.2) and bound the degree-≥ 4 remainder $E(\lambda)$ of $R(\lambda)/\lambda^{24} = (BC - AD)/\lambda^3$ in absolute value by $2^{31}\lambda^4 \leq 21.48$ for all $\lambda \in (0, 1/100]$, which cannot overcome -612 (Theorems 1.1 and 1.2).

(3) In Section 4, we prove a unified series-parallel gadget lemma (Lemma 4.1) covering both $q > 0$ and $q = 0$, deduce Corollary 1.3, and apply spanning-to-independent matroid duality (Lemma 4.2) to deduce Corollary 1.4.

(4) In Section 5, we extract the forest scaling $q^{-|V|}Z_G(q, q\boldsymbol{\beta})|_{q=0}$ directly from Lemma 2.2 via finite differences across the three partition types, exhibiting an explicit positive factorisation of the forest Rayleigh polynomial (Proposition 5.1) because the forest limit discards all cyclic core subgraphs.

2. Three-class Potts identity for the four-parameter family

In this section, we express the two-edge covariance in terms of four background partition sums on $H = G \setminus \{e, f\}$ and evaluate all four sums closed-form via a three-class Potts colour-partition identity on the right shore $W = \{w_1, w_2, w_3\}$.

We begin with the standard reduction of two-edge covariance to the Rayleigh difference on any multigraph.

Lemma 2.1. *Let $\widetilde{G} = (V, E)$ be a finite multigraph, let $e, f \in E$ be distinct edges, and let $\widetilde{H} := \widetilde{G} \setminus \{e, f\}$. Given $q > 0$ and $\boldsymbol{\gamma} \in \mathbb{R}^E_{>0}$, for each $s \subseteq \{e, f\}$ let*

$$Z_s := \sum_{A \subseteq E(\widetilde{H})} q^{\kappa(A \cup s)} \prod_{g \in A} \gamma_g, \tag{2.1}$$

writing Z_0, Z_e, Z_f, Z_{ef} for $s = \varnothing, \{e\}, \{f\}, \{e, f\}$, and let $R_{\widetilde{G}}(q, \boldsymbol{\gamma}) := Z_e Z_f - Z_0 Z_{ef}$. Then $Z_{\widetilde{G}}(q, \boldsymbol{\gamma}) = Z_0 + \gamma_e Z_e + \gamma_f Z_f + \gamma_e \gamma_f Z_{ef}$, and under $\mu := \mu_{\widetilde{G}, q, \boldsymbol{\gamma}}$,

$$\operatorname{Cov}_\mu(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\}) = -\frac{\gamma_e \gamma_f \, R_{\widetilde{G}}(q, \boldsymbol{\gamma})}{Z_{\widetilde{G}}(q, \boldsymbol{\gamma})^2}. \tag{2.2}$$

When $\widetilde{G}$ is connected, the same identity holds under $\mu_{\widetilde{G}, 0, \boldsymbol{\gamma}}$ with $C_s := \sum_{A \subseteq E(\widetilde{H}) : \kappa(A \cup s) = 1} \prod_{g \in A} \gamma_g$ and $R_{\widetilde{G}, \mathrm{conn}}(\boldsymbol{\gamma}) := C_e C_f - C_0 C_{ef}$ in place of Z_s and $R_{\widetilde{G}}$.

Proof. Partitioning $A \subseteq E$ according to $A \cap \{e, f\} \in \{\varnothing, \{e\}, \{f\}, \{e, f\}\}$ decomposes $Z_{\widetilde{G}}(q, \boldsymbol{\gamma}) = Z_0 + \gamma_e Z_e + \gamma_f Z_f + \gamma_e \gamma_f Z_{ef}$, with the four joint probabilities of $(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\})$ equal to $Z_0 / Z_{\widetilde{G}}$, $\gamma_e Z_e / Z_{\widetilde{G}}$, $\gamma_f Z_f / Z_{\widetilde{G}}$, and $\gamma_e \gamma_f Z_{ef} / Z_{\widetilde{G}}$. Therefore,

$$\begin{aligned}
\operatorname{Cov}_\mu(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\}) &= \frac{\gamma_e \gamma_f Z_{ef}}{Z_{\widetilde{G}}} - \frac{\gamma_e Z_e + \gamma_e \gamma_f Z_{ef}}{Z_{\widetilde{G}}} \cdot \frac{\gamma_f Z_f + \gamma_e \gamma_f Z_{ef}}{Z_{\widetilde{G}}} \\
&= \frac{\gamma_e \gamma_f}{Z_{\widetilde{G}}^2} \Big(Z_{ef}(Z_0 + \gamma_e Z_e + \gamma_f Z_f + \gamma_e \gamma_f Z_{ef}) \\
&\qquad - (Z_e + \gamma_f Z_{ef})(Z_f + \gamma_e Z_{ef}) \Big) \\
&= \frac{\gamma_e \gamma_f (Z_0 Z_{ef} - Z_e Z_f)}{Z_{\widetilde{G}}^2} = -\frac{\gamma_e \gamma_f \, R_{\widetilde{G}}(q, \boldsymbol{\gamma})}{Z_{\widetilde{G}}^2}.
\end{aligned}$$

Multiplying Z_s by q^{-1} and letting $q \downarrow 0$ yields C_s and the identical formula for $\mu_{\widetilde{G}, 0, \boldsymbol{\gamma}}$. □

Notice from (2.1) and (2.2) that $R_{\widetilde{G}}(q, \boldsymbol{\gamma})$ depends only on q and the weights $(\gamma_g)_{g \in E(\widetilde{H})}$ of the background edges, so the strict sign of $\operatorname{Cov}_\mu(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\})$ is invariant under any choice of $\gamma_e, \gamma_f > 0$.

Now specialise $\widetilde{G}$ to our eight-vertex graph $G = (V, E)$ with fifteen-edge background $H = G \setminus \{e, f\}$ from (1.4). To encompass both the two-level assignment (1.5) and the broader weight family studied in Sections 3 and 5, consider the four-parameter background weight vector $\boldsymbol{\gamma}(x_0, a, b, c)$ on $E(H)$ that assigns:

- weight x_0 to each of the nine core edges $u_i w_j$ $(i, j \in \{1, 2, 3\})$,
- weight a to the two spokes $x u_1$ and $x u_2$, and weight b to the spoke $x u_3$,
- weight c to each of the three spokes $y w_1$, $y w_2$, and $y w_3$.

The two-level parameterisation (1.5) corresponds to $x_0 = b = \lambda$ and $a = c = \lambda^2$.

For a partition of the three-element set $W = \{w_1, w_2, w_3\}$ into $p \in \{1, 2, 3\}$ nonempty blocks with block-size multiset $h = (h_1, \dots, h_p)$, there are only three possible block-size types:

(2.3)

h	p	m_h	$\phi_h(q,\zeta) := q - p + \sum_{r=1}^{p} \zeta^{h_r}$
$(1,1,1)$	3	1	$q - 3 + 3\zeta$
$(2,1)$	2	3	$q - 2 + \zeta^2 + \zeta$
(3)	1	1	$q - 1 + \zeta^3$

where $m_h \in \{1, 3, 1\}$ denotes the number of set partitions of W having block-size type h (namely, one discrete partition $w_1|w_2|w_3$, three two-block partitions $w_1w_2|w_3$, $w_1w_3|w_2$, $w_2w_3|w_1$, and one single-block partition $w_1w_2w_3$). For $p \geq 1$, let $(q)_p := \prod_{j=0}^{p-1}(q - j)$ denote the falling factorial.

Lemma 2.2. *Let $H = G \setminus \{e, f\}$ carry the four-parameter background weights (x_0, a, b, c), and let $u := 1 + x_0$ and $v := 1 + c$. For each block-size type $h \in \{(1,1,1), (2,1), (3)\}$ in (2.3), let*

$$\begin{aligned} L_h &:= \phi_h(q, u), \quad M_h := \phi_h(q, u^2), \quad N_h := \phi_h(q, u^3), \quad S_h := \phi_h(q, v), \\ T_h &:= \phi_h(q, vu), \quad U_h := \phi_h(q, vu^2), \quad V_h := \phi_h(q, vu^3), \end{aligned} \tag{2.4}$$

and define the four polynomials

$$\begin{aligned} F_0(h) &:= S_h\big[(q + 2a + b)L_h^3 + (a^2 + 2ab)L_hM_h + a^2bN_h\big], \\ F_e(h) &:= S_h\big[(q + 2a + b + a^2)L_hM_h + b(2a + a^2)N_h\big], \\ F_f(h) &:= S_hL_h^3 + (2a + b)L_h^2T_h + (a^2 + 2ab)L_hU_h + a^2bV_h, \\ F_{ef}(h) &:= S_hL_hM_h + bM_hT_h + (2a + a^2)L_hU_h + b(2a + a^2)V_h. \end{aligned} \tag{2.5}$$

Then as polynomials in $\mathbb{Z}[q, x_0, a, b, c]$, the four background partition sums (2.1) satisfy

$$Z_s = \sum_{h \in \{(1,1,1),\,(2,1),\,(3)\}} m_h (q)_{p_h} F_s(h) \quad \text{for each } s \in \{0, e, f, ef\}. \tag{2.6}$$

Proof. We first prove (2.6) when $q \geq 3$ is an integer, and then conclude equality in $\mathbb{Z}[q, x_0, a, b, c]$ by polynomial interpolation.

Let $q \geq 3$ be an integer, and let $[q] := \{1, \dots, q\}$. For any edge subset $s \subseteq \{e, f\}$ and any spin configuration $\sigma : V \to [q]$, let $\mathbf{1}_s(\sigma) := \prod_{\alpha\beta \in s} \mathbf{1}\{\sigma(\alpha) = \sigma(\beta)\}$ denote the indicator that σ is constant across the endpoints of every edge in s (with $\mathbf{1}_\varnothing(\sigma) := 1$). Expanding the product over $E(H)$ gives the Fortuin–Kasteleyn identity [FK72]

$$\begin{aligned} \sum_{\sigma : V \to [q]} \mathbf{1}_s(\sigma) \prod_{\alpha\beta \in E(H)} \big(1 + \gamma_{\alpha\beta}\, \mathbf{1}\{\sigma(\alpha) = \sigma(\beta)\}\big) &= \sum_{A \subseteq E(H)} \left(\prod_{g \in A} \gamma_g\right) \sum_{\sigma : V \to [q]} \prod_{\alpha\beta \in A \cup s} \mathbf{1}\{\sigma(\alpha) = \sigma(\beta)\} \\ &= \sum_{A \subseteq E(H)} q^{\kappa(A \cup s)} \prod_{g \in A} \gamma_g = Z_s, \end{aligned} \tag{2.7}$$

because a colouring $\sigma : V \to [q]$ satisfies $\sigma(\alpha) = \sigma(\beta)$ for every edge $\alpha\beta \in A \cup s$ if and only if σ is constant on each of the $\kappa(A \cup s)$ connected components of $(V, A \cup s)$, which permits exactly $q^{\kappa(A \cup s)}$ colourings.

We evaluate the left-hand side of (2.7) by conditioning first on the restriction of σ to the right shore $W = \{w_1, w_2, w_3\}$, and next on the colour $\sigma(x) \in [q]$ of the left apex. Fix a set partition $\pi = \{B_1, \dots, B_p\}$ of W of block-size type $h = (h_1, \dots, h_p)$, where $h_r = |B_r|$. There are m_h such partitions π, and for each fixed π there are exactly $(q)_p = q(q-1)\cdots(q-p+1)$ injective assignments of distinct colours $k_1, \dots, k_p \in [q]$ to the blocks $B_1, \dots, B_p$ (well-defined since $q \geq 3 \geq p$).

Fix such a colouring of W with palette $K := \{k_1, \dots, k_p\} \subseteq [q]$. Notice first that for any parameter ζ, summing over a single colour $k \in [q]$ that contributes ζ^{h_r} when $k = k_r \in K$ and 1 when $k \in [q] \setminus K$ yields

$$\sum_{k \in [q] \setminus K} 1 + \sum_{r=1}^{p} \zeta^{h_r} = (q-p) + \sum_{r=1}^{p} \zeta^{h_r} = \phi_h(q, \zeta). \tag{2.8}$$

In particular, for any single left-shore vertex $u_i \in U$ (ignoring its spoke to x for a moment), if $\sigma(u_i) = k_r \in K$ then u_i agrees in colour with the h_r vertices of $B_r \subseteq W$ and contributes $(1 + x_0)^{h_r} = u^{h_r}$, whereas if $\sigma(u_i) \notin K$ it agrees with no vertex of W and contributes 1. Summing over $\sigma(u_i) \in [q]$ therefore gives $\phi_h(q, u) = L_h$. Similarly, if two left-shore vertices u_1, u_2 are constrained to share a common colour $k = \sigma(u_1) = \sigma(u_2)$, their combined interaction with W is u^{2h_r} when $k = k_r$ and 1 when $k \notin K$, summing to $\phi_h(q, u^2) = M_h$.

Now fix $\sigma(x) \in [q]$ as well, and define the quantity

$$\rho(\sigma(x)) := \begin{cases} u^{h_r} & \text{if } \sigma(x) = k_r \in K, \\ 1 & \text{if } \sigma(x) \in [q] \setminus K. \end{cases}$$

Once $\sigma|_W$ and $\sigma(x)$ are fixed, the vertices of $U = \{u_1, u_2, u_3\}$ interact only with $W \cup \{x\}$ (and with each other if $e = u_1u_2 \in s$), while y interacts only with W (and with x if $f = xy \in s$). We compute the conditionally independent sums over $\sigma|_U$ and $\sigma(y)$ separately:

- *Sum over $\sigma|_U$ when $e \notin s$.* The three vertices u_1, u_2, u_3 are conditionally independent given $\sigma|_W$ and $\sigma(x)$. For u_1, summing $(1 + a\,\mathbf{1}\{\sigma(u_1) = \sigma(x)\})$ times its W-interaction over $\sigma(u_1) \in [q]$ gives $L_h + a\rho(\sigma(x))$, because the 1 term sums to L_h and the $a\,\mathbf{1}\{\sigma(u_1) = \sigma(x)\}$ term evaluates the W-interaction at $\sigma(u_1) = \sigma(x)$, which is $\rho(\sigma(x))$. Likewise, u_2 sums to $L_h + a\rho(\sigma(x))$ and u_3 (whose spoke to x has weight b) sums to $L_h + b\rho(\sigma(x))$. Multiplying the three factors gives

$$\begin{aligned} \big(L_h + a\rho(\sigma(x))\big)^2 \big(L_h + b\rho(\sigma(x))\big) &= L_h^3 + (2a+b)L_h^2\rho(\sigma(x)) \\ &\quad + (a^2 + 2ab)L_h\rho(\sigma(x))^2 + a^2 b\rho(\sigma(x))^3. \end{aligned} \tag{2.9}$$

- *Sum over $\sigma|_U$ when $e \in s$.* The factor $\mathbf{1}\{\sigma(u_1) = \sigma(u_2)\}$ identifies $\sigma(u_1)$ with $\sigma(u_2)$, while u_3 remains conditionally independent with sum $L_h + b\rho(\sigma(x))$. For the merged pair $k = \sigma(u_1) = \sigma(u_2)$, its combined spoke factor to x is $(1 + a\,\mathbf{1}\{k = \sigma(x)\})^2 = 1 + (2a + a^2)\mathbf{1}\{k = \sigma(x)\}$ and its combined W-interaction is u^{2h_r} if $k = k_r$ and 1 if $k \notin K$. Summing over $k \in [q]$ yields $M_h + (2a + a^2)\rho(\sigma(x))^2$, and multiplying by $L_h + b\rho(\sigma(x))$ gives

$$\begin{aligned} \big(M_h + (2a + a^2)\rho(\sigma(x))^2\big)\big(L_h + b\rho(\sigma(x))\big) &= L_hM_h + bM_h\rho(\sigma(x)) \\ &\quad + (2a + a^2)L_h\rho(\sigma(x))^2 + b(2a + a^2)\rho(\sigma(x))^3. \end{aligned} \tag{2.10}$$

- *Sum over $\sigma(y)$.* For a given colour $\sigma(y) \in [q]$, the three weight-c spokes from y to W contribute $(1 + c)^{h_r} = v^{h_r}$ if $\sigma(y) = k_r \in K$, and 1 if $\sigma(y) \notin K$. When $f \notin s$, summing freely over $\sigma(y) \in [q]$ gives $\phi_h(q, v) = S_h$ by (2.8), independent of $\sigma(x)$. When $f \in s$, the factor $\mathbf{1}\{\sigma(x) = \sigma(y)\}$ forces $\sigma(y) = \sigma(x)$, contributing the weight

$$\tau(\sigma(x)) := \begin{cases} v^{h_r} & \text{if } \sigma(x) = k_r \in K, \\ 1 & \text{if } \sigma(x) \in [q] \setminus K. \end{cases}$$

Finally, we sum over $\sigma(x) \in [q]$ using (2.8): for each exponent $d \in \{0,1,2,3\}$, the unweighted sum $\sum_{\sigma(x)\in[q]} \rho(\sigma(x))^d$ applies (2.8) at $\zeta = u^d$, giving

$$\sum_{\sigma(x)\in[q]} 1 = q, \qquad \sum_{\sigma(x)\in[q]} \rho(\sigma(x)) = L_h,$$
$$\sum_{\sigma(x)\in[q]} \rho(\sigma(x))^2 = M_h, \qquad \sum_{\sigma(x)\in[q]} \rho(\sigma(x))^3 = N_h,$$

while the τ-weighted sum $\sum_{\sigma(x)\in[q]} \tau(\sigma(x))\rho(\sigma(x))^d$ applies (2.8) at $\zeta = vu^d$, giving

$$\sum_{\sigma(x)\in[q]} \tau(\sigma(x)) = S_h, \qquad \sum_{\sigma(x)\in[q]} \tau(\sigma(x))\rho(\sigma(x)) = T_h,$$
$$\sum_{\sigma(x)\in[q]} \tau(\sigma(x))\rho(\sigma(x))^2 = U_h, \qquad \sum_{\sigma(x)\in[q]} \tau(\sigma(x))\rho(\sigma(x))^3 = V_h.$$

Substituting these four moments into (2.9) and (2.10) (multiplied by S_h when $f \notin s$, or weighted by $\tau(\sigma(x))$ when $f \in s$) produces $S_h[qL_h^3 + (2a+b)L_h^3 + (a^2+2ab)L_hM_h + a^2bN_h] = F_0(h)$ for $s = \varnothing$, $S_h[qL_hM_h + bM_hL_h + (2a+a^2)L_hM_h + b(2a+a^2)N_h] = F_e(h)$ for $s = \{e\}$, and $F_f(h)$ and $F_{ef}(h)$ for $s = \{f\}$ and $s = \{e,f\}$. Multiplying by $m_h(q)_{p_h}$ and summing over the three types h establishes (2.6) for every integer $q \geq 3$.

For fixed $(x_0, a, b, c) \in \mathbb{R}^4$, both sides of (2.6) are univariate polynomials in q (of degree at most 8) that agree at infinitely many values $q \in \{3, 4, 5, \dots\}$, so they are identical polynomials in $\mathbb{Z}[q, x_0, a, b, c]$. □

Dividing (2.6) by q and sending $q \to 0$ immediately yields the corresponding three-class formula for the connected-spanning-subgraph sums C_s.

Corollary 2.3. *In the setting of Lemma 2.2, let $F_s^{(0)}(h)$ denote the polynomial obtained by setting $q = 0$ in (2.3), (2.4), and (2.5). Then as polynomials in $\mathbb{Z}[x_0, a, b, c]$, for each $s \in \{0, e, f, ef\}$,*

$$C_s = 2F_s^{(0)}(1,1,1) - 3F_s^{(0)}(2,1) + F_s^{(0)}(3). \tag{2.11}$$

Proof. Since $\kappa(A \cup s) \geq 1$ for every $A \subseteq E(H)$, we have $C_s = \lim_{q\to 0} q^{-1} Z_s$. In (2.6), each falling factorial $(q)_p = q(q-1)\cdots(q-p+1)$ satisfies $\lim_{q\to 0} q^{-1}(q)_p = (-1)^{p-1}(p-1)!$, so $\lim_{q\to 0} q^{-1} m_h(q)_{p_h}$ equals $1 \cdot 2 = 2$ for $h = (1,1,1)$, $3 \cdot (-1) = -3$ for $h = (2,1)$, and $1 \cdot 1 = 1$ for $h = (3)$. □

3. Exact finite-q and connected-limit certificates

In this section, we specialise the background weights to the two-level assignment $x_0 = b = \lambda$ and $a = c = \lambda^2$ from (1.5), compute the seven leading coefficients of the four partition polynomials by hand from Lemma 2.2 and Corollary 2.3, and bound the remaining higher-order terms to prove Theorems 1.1 and 1.2.

We first verify combinatorially the minimum λ-degrees, coefficient nonnegativity, and $\lambda = 1$ coefficient sums of Z_s (with $q = \lambda^4$) and C_s.

Lemma 3.1. *Under the two-level background assignment $x_0 = b = \lambda$ and $a = c = \lambda^2$ on $H = G \setminus \{e, f\}$ with $q = \lambda^4$, there exist polynomials $A, B, C, D \in \mathbb{Z}_{\geq 0}[\lambda]$ and $A_{\mathrm{conn}}, B_{\mathrm{conn}}, C_{\mathrm{conn}}, D_{\mathrm{conn}} \in \mathbb{Z}_{\geq 0}[\lambda]$ with nonnegative integer coefficients such that*

$$\begin{aligned} &Z_0(\lambda) = \lambda^{12}A(\lambda), && Z_e(\lambda) = \lambda^{11}B(\lambda), && Z_f(\lambda) = \lambda^{10}C(\lambda), && Z_{ef}(\lambda) = \lambda^9 D(\lambda), \\ &C_0(\lambda) = \lambda^8 A_{\mathrm{conn}}(\lambda), && C_e(\lambda) = \lambda^7 B_{\mathrm{conn}}(\lambda), && C_f(\lambda) = \lambda^6 C_{\mathrm{conn}}(\lambda), && C_{ef}(\lambda) = \lambda^5 D_{\mathrm{conn}}(\lambda). \end{aligned} \tag{3.1}$$

Moreover, $A(1) = B(1) = C(1) = D(1) = 2^{15}$ and $A_{\mathrm{conn}}(1), B_{\mathrm{conn}}(1), C_{\mathrm{conn}}(1), D_{\mathrm{conn}}(1) \leq 2^{15}$.

Proof. For each subset $F \subseteq E(H)$, let $w(F) \in \mathbb{Z}_{\geq 0}$ denote the λ-exponent of $\prod_{g \in F} \gamma_g(\lambda)$: each edge of F in $E(K_{U,W}) \cup \{xu_3\}$ contributes 1 to $w(F)$, and each edge of F in $\{xu_1, xu_2, yw_1, yw_2, yw_3\}$ contributes 2 to $w(F)$. In particular, $w(F) \geq |F|$, and if y is not isolated in (V, F) then F contains at least one weight-λ^2 spoke incident to y, giving $w(F) \geq |F| + 1$.

By (2.1), at $q = \lambda^4$ we have

$$Z_s(\lambda) = \sum_{F \subseteq E(H)} \lambda^{4\kappa(F \cup s) + w(F)} \quad \text{and} \quad C_s(\lambda) = \sum_{\substack{F \subseteq E(H) \\ \kappa(F \cup s) = 1}} \lambda^{w(F)},$$

so both polynomials have nonnegative integer coefficients, $Z_s(1) = 2^{15}$, and $C_s(1) \leq 2^{15}$. To verify (3.1), we check that $w(F) \geq 8 - |s| - \mathbf{1}\{f \in s\}$ when $\kappa(F \cup s) = 1$, and $4\kappa(F \cup s) + w(F) \geq 12 - |s| - \mathbf{1}\{f \in s\}$ for all $F \subseteq E(H)$:

- Since $|V| = 8$, any spanning subgraph $(V, F \cup s)$ with $k := \kappa(F \cup s)$ components has $|F| + |s| \geq 8 - k$, so $|F| \geq 8 - k - |s|$.
- When $k = 1$, we have $|F| \geq 7 - |s|$. If $f \notin s$, then y cannot be isolated in (V, F) because $\kappa(F \cup s) = 1$ and $f = xy$ is the only marked edge incident to y, so $w(F) \geq |F| + 1 \geq 8 - |s|$. If $f \in s$, then $w(F) \geq |F| \geq 7 - |s| = 8 - |s| - 1$. Thus $w(F) \geq 8 - |s| - \mathbf{1}\{f \in s\}$ whenever $k = 1$, proving the C_s bounds $(8, 7, 6, 5)$ and simultaneously $4k + w(F) \geq 12 - |s| - \mathbf{1}\{f \in s\}$ for $k = 1$.
- When $k = 2$, we have $|F| \geq 6 - |s|$, hence $4k + w(F) \geq 8 + (6 - |s|) = 14 - |s| \geq 12 - |s| - \mathbf{1}\{f \in s\}$.
- When $k \geq 3$, since $|s| + \mathbf{1}\{f \in s\} \geq 0$, we have $4k + w(F) \geq 12 + 0 \geq 12 - |s| - \mathbf{1}\{f \in s\}$.

Dividing out the guaranteed power of λ preserves the $\lambda = 1$ evaluations and completes the proof. □

Next, we use Lemma 2.2 and Corollary 2.3 to compute the degree-6 prefixes of the eight normalised polynomials in (3.1).

Lemma 3.2. *The first seven coefficients of A, B, C, D and $A_{\text{conn}}, B_{\text{conn}}, C_{\text{conn}}, D_{\text{conn}}$ in (3.1) are given by*

(3.2)

finite $(q = \lambda^4)$	1	λ	λ^2	λ^3	λ^4	λ^5	λ^6
$A(\lambda)$	243	1206	2358	3924	5262	5239	4929
$B(\lambda)$	162	891	1992	3477	5006	5485	5290
$C(\lambda)$	81	726	1617	2812	4314	4947	5113
$D(\lambda)$	54	513	1334	2446	3842	4744	5080

and

(3.3)

connected $(q = 0)$	1	λ	λ^2	λ^3	λ^4	λ^5	λ^6
$A_{\text{conn}}(\lambda)$	243	1206	2277	2982	2991	2436	1653
$B_{\text{conn}}(\lambda)$	162	891	1938	2865	3330	3159	2498
$C_{\text{conn}}(\lambda)$	81	726	1617	2614	3324	3549	3220
$D_{\text{conn}}(\lambda)$	54	513	1334	2329	3230	3776	3821

Proof. We carry out the polynomial arithmetic of Lemma 2.2 (for $q = \lambda^4$) and Corollary 2.3 (for $q = 0$) explicitly by recording the intermediate building blocks (2.4) and the three partition-type summands.

Consider first $q = \lambda^4$, with $x_0 = b = \lambda$, $a = c = \lambda^2$, $u = 1 + \lambda$, and $v = 1 + \lambda^2$. Since $(q)_p$ is divisible by $q = \lambda^4$ for every $p \in \{1, 2, 3\}$, divide (2.6) by λ^4: the weights $w_h(\lambda) := \lambda^{-4} m_h(\lambda^4)_{p_h}$ are

$$w_{(1,1,1)}(\lambda) = (\lambda^4 - 1)(\lambda^4 - 2) = 2 - 3\lambda^4 + \lambda^8,$$
$$w_{(2,1)}(\lambda) = 3(\lambda^4 - 1) = -3 + 3\lambda^4, \quad \text{and} \quad w_{(3)}(\lambda) = 1.$$

To obtain $\lambda^{-4}Z_0, \lambda^{-4}Z_e, \lambda^{-4}Z_f, \lambda^{-4}Z_{ef}$ up to $\lambda^{14}, \lambda^{13}, \lambda^{12}, \lambda^{11}$ respectively, observe from (2.4) that every polynomial $L_h, M_h, N_h, T_h, U_h, V_h$ has zero constant term, while S_h begins at λ^2, $q+2a+b = \lambda + 2\lambda^2 + \lambda^4$ begins at λ, $a^2 + 2ab = 2\lambda^3 + \lambda^4$ begins at λ^3, and $a^2 b = \lambda^5$. Consequently, in (2.5) we need S_h at most to degree 10, L_h, M_h, T_h at most to degree 9, and N_h, U_h, V_h at most to degree 8. Expanding $\phi_h(\lambda^4, \zeta)$ from (2.3) at $\zeta \in \{u, u^2, u^3, v, vu, vu^2, vu^3\}$ yields:

- For $h = (1,1,1)$ ($\phi_h(\lambda^4, \zeta) = \lambda^4 + 3(\zeta - 1)$, exact):

$$L = 3\lambda + \lambda^4, \qquad M = 6\lambda + 3\lambda^2 + \lambda^4,$$
$$N = 9\lambda + 9\lambda^2 + 3\lambda^3 + \lambda^4, \qquad S = 3\lambda^2 + \lambda^4,$$
$$T = 3\lambda + 3\lambda^2 + 3\lambda^3 + \lambda^4, \qquad U = 6\lambda + 6\lambda^2 + 6\lambda^3 + 4\lambda^4,$$
$$V = 9\lambda + 12\lambda^2 + 12\lambda^3 + 10\lambda^4 + 3\lambda^5.$$

- For $h = (2,1)$ ($\phi_h(\lambda^4, \zeta) = \lambda^4 + (\zeta^2 - 1) + (\zeta - 1)$, truncated where indicated):

$$L = 3\lambda + \lambda^2 + \lambda^4 \quad \text{and} \quad S = 3\lambda^2 + 2\lambda^4,$$
$$M = 6\lambda + 7\lambda^2 + 4\lambda^3 + 2\lambda^4,$$
$$T = 3\lambda + 4\lambda^2 + 5\lambda^3 + 4\lambda^4 + 2\lambda^5 + \lambda^6,$$
$$N = 9\lambda + 18\lambda^2 + 21\lambda^3 + 16\lambda^4 + 6\lambda^5 + \lambda^6,$$
$$U = 6\lambda + 10\lambda^2 + 14\lambda^3 + 16\lambda^4 + 12\lambda^5 + 8\lambda^6 + 4\lambda^7 + \lambda^8,$$
$$V \equiv 9\lambda + 21\lambda^2 + 36\lambda^3 + 50\lambda^4 + 53\lambda^5 + 46\lambda^6 + 32\lambda^7 + 17\lambda^8 \pmod{\lambda^9}.$$

- For $h = (3)$ ($\phi_h(\lambda^4, \zeta) = \lambda^4 + (\zeta^3 - 1)$, truncated where indicated):

$$L = 3\lambda + 3\lambda^2 + \lambda^3 + \lambda^4 \quad \text{and} \quad S = 3\lambda^2 + 4\lambda^4 + \lambda^6,$$
$$M = 6\lambda + 15\lambda^2 + 20\lambda^3 + 16\lambda^4 + 6\lambda^5 + \lambda^6,$$
$$N \equiv 9\lambda + 36\lambda^2 + 84\lambda^3 + 127\lambda^4 + 126\lambda^5 + 84\lambda^6 + 36\lambda^7 + 9\lambda^8 \pmod{\lambda^9},$$
$$T = 3\lambda + 6\lambda^2 + 10\lambda^3 + 13\lambda^4 + 12\lambda^5 + 10\lambda^6 + 6\lambda^7 + 3\lambda^8 + \lambda^9,$$
$$U \equiv 6\lambda + 18\lambda^2 + 38\lambda^3 + 64\lambda^4 + 84\lambda^5 + 92\lambda^6 + 84\lambda^7 + 63\lambda^8 \pmod{\lambda^9},$$
$$V \equiv 9\lambda + 39\lambda^2 + 111\lambda^3 + 238\lambda^4 + 405\lambda^5 + 571\lambda^6 + 675\lambda^7 + 675\lambda^8 \pmod{\lambda^9}.$$

Letting $(d_0, d_e, d_f, d_{ef}) := (8, 7, 6, 5)$ and substituting these polynomials into (2.5) gives the coefficients of $w_h(\lambda)F_s(h)$ from degree $d_s - 2$ to $d_s + 6$:

s	h	λ^{d_s-2}	λ^{d_s-1}	λ^{d_s}	λ^{d_s+1}	λ^{d_s+2}	λ^{d_s+3}	λ^{d_s+4}	λ^{d_s+5}	λ^{d_s+6}
0	$(1,1,1)$	162	540	324	612	297	−570	−102	−766	−645
0	$(2,1)$	−243	−1053	−1458	−2106	−2178	−765	−174	1086	1926
0	(3)	81	513	1377	2700	4239	5259	5538	4919	3648
e	$(1,1,1)$	108	378	252	486	210	−393	−158	−663	−464
e	$(2,1)$	−162	−729	−1089	−1755	−1797	−954	−321	945	1518
e	(3)	54	351	999	2160	3579	4824	5485	5203	4236
f	$(1,1,1)$	54	396	288	348	357	−406	−246	−398	−604
f	$(2,1)$	−81	−675	−1080	−1422	−1764	−918	−273	384	1287
f	(3)	27	279	873	1800	3024	4136	4833	4961	4430
ef	$(1,1,1)$	36	270	228	246	202	−285	−242	−323	−354
ef	$(2,1)$	−54	−459	−795	−1131	−1335	−828	−336	282	855
ef	(3)	18	189	621	1398	2467	3559	4420	4785	4579

Summing each triple of rows confirms that the λ^{d_s-2} and λ^{d_s-1} columns cancel to 0 (in agreement with Lemma 3.1) and yields every entry of (3.2).

For the connected limit $q = 0$ in Corollary 2.3, setting $q = 0$ while retaining $a = c = \lambda^2$ and $b = x_0 = \lambda$ subtracts λ^4 from each $\phi_h(\lambda^4, \zeta)$, drops the explicit q summands in $F_0(h)$ and $F_e(h)$ (while keeping $a^2 = \lambda^4$), and fixes the weights at $(2, -3, 1)$, yielding the three-row decomposition

s	h	λ^{d_s-2}	λ^{d_s-1}	λ^{d_s}	λ^{d_s+1}	λ^{d_s+2}	λ^{d_s+3}	λ^{d_s+4}	λ^{d_s+5}	λ^{d_s+6}
0	$(1,1,1)$	162	540	270	108	18	0	0	0	0
0	$(2,1)$	−243	−1053	−1377	−1269	−936	−522	−231	−81	−21
0	(3)	81	513	1350	2367	3195	3504	3222	2517	1674
e	$(1,1,1)$	108	378	216	198	108	18	0	0	0
e	$(2,1)$	−162	−729	−1035	−1269	−1137	−828	−486	−225	−81
e	(3)	54	351	981	1962	2967	3675	3816	3384	2579
f	$(1,1,1)$	54	396	288	240	96	36	6	0	0
f	$(2,1)$	−81	−675	−1080	−1260	−1197	−975	−699	−417	−204
f	(3)	27	279	873	1746	2718	3553	4017	3966	3424
ef	$(1,1,1)$	36	270	228	192	132	72	36	6	0
ef	$(2,1)$	−54	−459	−795	−1050	−1134	−1065	−891	−645	−393
ef	(3)	18	189	621	1371	2336	3322	4085	4415	4214

Summing each triple of rows produces (3.3). □

We now combine Lemmas 2.1, 3.1 and 3.2 to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. By (3.1), the Rayleigh difference $R(\lambda) := Z_e(\lambda)Z_f(\lambda) - Z_0(\lambda)Z_{ef}(\lambda)$ equals $\lambda^{21}\big(B(\lambda)C(\lambda) - A(\lambda)D(\lambda)\big)$. Using the seven-term prefixes in (3.2), we compute the coefficients of $\lambda^0, \lambda^1, \ldots, \lambda^6$ in $B(\lambda)C(\lambda)$ and $A(\lambda)D(\lambda)$:

$$
\begin{aligned}
[\lambda^0](BC - AD) &= 13{,}122 - 13{,}122 = 0,\\
[\lambda^1](BC - AD) &= 189{,}783 - 189{,}783 = 0,\\
[\lambda^2](BC - AD) &= 1{,}070{,}172 - 1{,}070{,}172 = 0,\\
[\lambda^3](BC - AD) &= 3{,}624{,}120 - 3{,}624{,}732 = -612,\\
[\lambda^4](BC - AD) &= 9{,}355{,}212 - 9{,}326{,}214 = 28{,}998,\\
[\lambda^5](BC - AD) &= 19{,}947{,}642 - 19{,}770{,}840 = 176{,}802,\\
[\lambda^6](BC - AD) &= 36{,}112{,}197 - 35{,}586{,}525 = 525{,}672.
\end{aligned}
$$

Expressing $R(\lambda)$ after factoring out λ^{24} therefore gives

$$R(\lambda) = \lambda^{24}\big(-612 + 28{,}998\lambda + 176{,}802\lambda^2 + 525{,}672\lambda^3 + E(\lambda)\big), \tag{3.4}$$

where $E(\lambda) = \sum_{j\ge 4} r_{24+j}\lambda^j$ is a polynomial with every nonzero monomial of degree at least 4.

By Lemma 3.1, A, B, C, D have nonnegative coefficients and satisfy $A(1) = B(1) = C(1) = D(1) = 2^{15}$. Hence the sum of absolute values of all coefficients of $R(\lambda)$ satisfies

$$\sum_{k\ge 24} |r_k| \le B(1)C(1) + A(1)D(1) = 2^{15}\cdot 2^{15} + 2^{15}\cdot 2^{15} = 2^{31}.$$

For every $\lambda \in (0, 1]$, since $\lambda^j \le \lambda^4$ for all $j \ge 4$, we obtain $|E(\lambda)| \le \sum_{j\ge 4} |r_{24+j}|\lambda^4 \le 2^{31}\lambda^4$. Consequently, for every $\lambda \in (0, 1/100]$, monotonicity in λ of the four positive upper-bound terms gives

$$\frac{R(\lambda)}{\lambda^{24}} \le -612 + 28{,}998\lambda + 176{,}802\lambda^2 + 525{,}672\lambda^3 + 2^{31}\lambda^4$$

$$\leq -612 + \frac{28{,}998}{100} + \frac{176{,}802}{10^4} + \frac{525{,}672}{10^6} + \frac{2{,}147{,}483{,}648}{10^8}$$
$$= -282.33929152 < 0.$$

Thus $R(\lambda) < 0$ for every $\lambda \in (0, 1/100]$, so Lemma 2.1 (with $\gamma_e = \lambda > 0$ and $\gamma_f = \lambda^2 > 0$) implies $\mathrm{Cov}_{\mu_{G,\lambda^4,\boldsymbol{\gamma}(\lambda)}}(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\}) > 0$. Evaluating (2.6) in exact rational arithmetic at $\lambda = 1/100$ yields the stated decimal value of the covariance. □

Proof of Theorem 1.2. By (3.1), the connected Rayleigh difference $R_{\mathrm{conn}}(\lambda) := C_e(\lambda)C_f(\lambda) - C_0(\lambda)C_{ef}(\lambda)$ equals $\lambda^{13}\big(B_{\mathrm{conn}}(\lambda)C_{\mathrm{conn}}(\lambda) - A_{\mathrm{conn}}(\lambda)D_{\mathrm{conn}}(\lambda)\big)$. From (3.3), the first seven coefficients of $B_{\mathrm{conn}}C_{\mathrm{conn}} - A_{\mathrm{conn}}D_{\mathrm{conn}}$ are

$$[\lambda^0](B_{\mathrm{conn}}C_{\mathrm{conn}} - A_{\mathrm{conn}}D_{\mathrm{conn}}) = 13{,}122 - 13{,}122 = 0,$$
$$[\lambda^1](B_{\mathrm{conn}}C_{\mathrm{conn}} - A_{\mathrm{conn}}D_{\mathrm{conn}}) = 189{,}783 - 189{,}783 = 0,$$
$$[\lambda^2](B_{\mathrm{conn}}C_{\mathrm{conn}} - A_{\mathrm{conn}}D_{\mathrm{conn}}) = 1{,}065{,}798 - 1{,}065{,}798 = 0,$$
$$[\lambda^3](B_{\mathrm{conn}}C_{\mathrm{conn}} - A_{\mathrm{conn}}D_{\mathrm{conn}}) = 3{,}503{,}268 - 3{,}503{,}880 = -612,$$
$$[\lambda^4](B_{\mathrm{conn}}C_{\mathrm{conn}} - A_{\mathrm{conn}}D_{\mathrm{conn}}) = 8{,}351{,}028 - 8{,}322{,}462 = 28{,}566,$$
$$[\lambda^5](B_{\mathrm{conn}}C_{\mathrm{conn}} - A_{\mathrm{conn}}D_{\mathrm{conn}}) = 15{,}908{,}718 - 15{,}759{,}996 = 148{,}722,$$
$$[\lambda^6](B_{\mathrm{conn}}C_{\mathrm{conn}} - A_{\mathrm{conn}}D_{\mathrm{conn}}) = 25{,}495{,}203 - 25{,}111{,}071 = 384{,}132.$$

Hence

$$R_{\mathrm{conn}}(\lambda) = \lambda^{16}\big(-612 + 28{,}566\lambda + 148{,}722\lambda^2 + 384{,}132\lambda^3 + E_{\mathrm{conn}}(\lambda)\big),$$

where every nonzero monomial of $E_{\mathrm{conn}}(\lambda)$ has degree at least 4. By Lemma 3.1, all four polynomials have nonnegative coefficients bounded in sum at $\lambda = 1$ by 2^{15}, so $|E_{\mathrm{conn}}(\lambda)| \leq 2^{31}\lambda^4$ for every $\lambda \in (0, 1]$. Therefore, for all $\lambda \in (0, 1/100]$,

$$\frac{R_{\mathrm{conn}}(\lambda)}{\lambda^{16}} \leq -612 + \frac{28{,}566}{100} + \frac{148{,}722}{10^4} + \frac{384{,}132}{10^6} + \frac{2{,}147{,}483{,}648}{10^8} = -289.60883152 < 0.$$

Applying Lemma 2.1 proves $\mathrm{Cov}_{\mu_{G,0,\boldsymbol{\gamma}(\lambda)}}(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\}) > 0$ for all $\lambda \in (0, 1/100]$, and exact rational evaluation of (2.11) at $\lambda = 1/100$ gives the stated numerical value. □

Remark. Before passing to the two-level specialisation (1.5), one can also scale the general four-parameter family on H by setting $(x_0, a, b, c) = (\eta X, \eta, 7\eta, 3\eta)$ as $\eta \downarrow 0$ with $X > 0$ fixed. In that regime, expanding (2.11) in powers of η separates spanning trees of G (order $\eta^{7-|s|}$) from connected spanning unicycles of G (order $\eta^{8-|s|}$): the spanning-tree Rayleigh difference vanishes identically in X, while the leading unicyclic coefficient of η^{13} in $C_eC_f - C_0C_{ef}$ factors as

$$4374X^2(X+1)^2\big(-3X^5 + 435X^4 + 1262X^3 + 1490X^2 + 861X + 147\big),$$

which is strictly negative for every $X \geq 150$. Choosing $X = 1000$, $\eta = 10^{-5}$, $\gamma_e = \gamma_f = 1$, and $q = 10^{-9}$ gives an alternative rational counterexample on G with $\mathrm{Cov}_{\mu_{G,10^{-9},\boldsymbol{\gamma}}}(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\}) \approx 7.0101 \times 10^{-15} > 0$. The two-level parameterisation (1.5) achieves a five-order-of-magnitude larger covariance at $q = 10^{-8}$ using only two distinct edge weights.

4. Reductions to homogeneous weights, uniform subgraphs, and regular matroids

In this section, we establish a unified series-parallel gadget lemma for both $q > 0$ and $q = 0$, deduce Corollary 1.3, and prove the matroid duality correspondence yielding Corollary 1.4.

Lemma 4.1. *Let $\widetilde{G} = (V, E)$ be a finite multigraph and let $g = \alpha\beta \in E$ be a non-loop edge.*

(a) (Series subdivision for $q > 0$ and $q = 0$.) *Let $\widetilde{G}_{\mathrm{ser}}$ be obtained from $\widetilde{G}$ by replacing g with a path P_ℓ of $\ell \geq 2$ edges $g_1, \dots, g_\ell$ carrying positive weights $\gamma_1, \dots, \gamma_\ell > 0$, whose $\ell - 1$ internal vertices have degree 2 in $\widetilde{G}_{\mathrm{ser}}$. For every $q > 0$ and every configuration $A_0 \subseteq E \setminus \{g\}$, we have*

$$\sum_{S \subseteq E(P_\ell)} q^{\kappa_{\widetilde{G}_{\mathrm{ser}}}(A_0 \cup S)} \prod_{g_i \in S} \gamma_i = K_\ell \Big(q^{\kappa_{\widetilde{G}}(A_0)} + \gamma_{\mathrm{ser}}\, q^{\kappa_{\widetilde{G}}(A_0 \cup \{g\})} \Big), \tag{4.1}$$

where

$$K_\ell := \frac{1}{q} \left[\prod_{i=1}^{\ell} (q + \gamma_i) - \prod_{i=1}^{\ell} \gamma_i \right] > 0 \quad \text{and} \quad \gamma_{\mathrm{ser}} := \frac{q}{\prod_{i=1}^{\ell} (1 + q/\gamma_i) - 1} > 0.$$

Moreover, if $\widetilde{G}$ is connected and $\gamma_1 = \cdots = \gamma_\ell = 1$, multiplying (4.1) by q^{-1} and letting $q \downarrow 0$ gives

$$\sum_{S \subseteq E(P_\ell)} \mathbf{1}\{\kappa_{\widetilde{G}_{\mathrm{ser}}}(A_0 \cup S) = 1\} = \ell \Big(\mathbf{1}\{\kappa_{\widetilde{G}}(A_0) = 1\} + \frac{1}{\ell}\, \mathbf{1}\{\kappa_{\widetilde{G}}(A_0 \cup \{g\}) = 1\} \Big). \tag{4.2}$$

(b) (Parallel composition.) *Let $\widetilde{G}_{\mathrm{par}}$ be obtained from $\widetilde{G}$ by replacing g with $k \geq 1$ parallel edges $g_1, \dots, g_k$ between α and β carrying positive weights $u_1, \dots, u_k > 0$. For every $q > 0$ and every $A_0 \subseteq E \setminus \{g\}$,*

$$\sum_{S \subseteq \{g_1, \dots, g_k\}} q^{\kappa_{\widetilde{G}_{\mathrm{par}}}(A_0 \cup S)} \prod_{g_i \in S} u_i = q^{\kappa_{\widetilde{G}}(A_0)} + \gamma_{\mathrm{par}}\, q^{\kappa_{\widetilde{G}}(A_0 \cup \{g\})}, \tag{4.3}$$

where $\gamma_{\mathrm{par}} := \prod_{i=1}^{k} (1 + u_i) - 1 > 0$.

Proof. For part (a), fix $A_0 \subseteq E \setminus \{g\}$ and consider a subset $S \subseteq E(P_\ell)$. If $S = E(P_\ell)$, all ℓ path edges are open, joining α, β, and all $\ell - 1$ internal vertices of P_ℓ into a single connected piece, so $\kappa_{\widetilde{G}_{\mathrm{ser}}}(A_0 \cup E(P_\ell)) = \kappa_{\widetilde{G}}(A_0 \cup \{g\})$ and the summand in (4.1) equals $q^{\kappa_{\widetilde{G}}(A_0 \cup \{g\})} \prod_{i=1}^{\ell} \gamma_i = K_\ell \gamma_{\mathrm{ser}}\, q^{\kappa_{\widetilde{G}}(A_0 \cup \{g\})}$.

Suppose instead that $S \subsetneq E(P_\ell)$, so $r := \ell - |S| \geq 1$ edges of P_ℓ are closed. Viewed on the $(\ell + 1)$-vertex path P_ℓ alone, removing r edges leaves $r + 1$ connected components: one containing α, one containing β (distinct on P_ℓ since $r \geq 1$), and $r - 1$ internal components consisting solely of degree-2 internal vertices of P_ℓ with no path to α or β. When we form $(V(\widetilde{G}_{\mathrm{ser}}), A_0 \cup S)$, the α-piece of P_ℓ merges into the A_0-component of α, the β-piece of P_ℓ merges into the A_0-component of β (whether α and β lie in the same A_0-component or in different A_0-components), and the $r - 1$ internal components have no incident edges in A_0 and remain isolated from $V(\widetilde{G})$. Hence $\kappa_{\widetilde{G}_{\mathrm{ser}}}(A_0 \cup S) = \kappa_{\widetilde{G}}(A_0) + r - 1 = \kappa_{\widetilde{G}}(A_0) + \ell - 1 - |S|$ for every $S \subsetneq E(P_\ell)$. Summing over all proper subsets $S \subsetneq E(P_\ell)$ gives

$$\begin{aligned} \sum_{S \subsetneq E(P_\ell)} q^{\kappa_{\widetilde{G}}(A_0) + \ell - 1 - |S|} \prod_{g_i \in S} \gamma_i &= q^{\kappa_{\widetilde{G}}(A_0) - 1} q^{\ell} \sum_{S \subsetneq E(P_\ell)} \prod_{g_i \in S} \frac{\gamma_i}{q} \\ &= q^{\kappa_{\widetilde{G}}(A_0)} \cdot \frac{1}{q} \left[\prod_{i=1}^{\ell} (q + \gamma_i) - \prod_{i=1}^{\ell} \gamma_i \right] = K_\ell\, q^{\kappa_{\widetilde{G}}(A_0)}. \end{aligned}$$

Adding the $S = E(P_\ell)$ term proves (4.1). When $\gamma_1 = \cdots = \gamma_\ell = 1$, we have $K_\ell = \big((1+q)^\ell - 1\big)/q \to \ell$ and $\gamma_{\mathrm{ser}} = q/\big((1+q)^\ell - 1\big) \to 1/\ell$ as $q \downarrow 0$, so multiplying (4.1) by q^{-1} and letting $q \downarrow 0$ yields (4.2) (directly, $\kappa_{\widetilde{G}_{\mathrm{ser}}}(A_0 \cup S) = 1$ holds either when $S = E(P_\ell)$ and $\kappa_{\widetilde{G}}(A_0 \cup \{g\}) = 1$, or when $|S| = \ell - 1$ and $\kappa_{\widetilde{G}}(A_0) = 1$, giving ℓ choices for the single closed path edge).

For part (b), if $S = \varnothing$ then $\kappa_{\widetilde{G}_{\mathrm{par}}}(A_0 \cup \varnothing) = \kappa_{\widetilde{G}}(A_0)$ with weight 1, whereas for every nonempty $S \subseteq \{g_1, \dots, g_k\}$ we have $\kappa_{\widetilde{G}_{\mathrm{par}}}(A_0 \cup S) = \kappa_{\widetilde{G}}(A_0 \cup \{g\})$, and $\sum_{\varnothing \neq S \subseteq \{g_1,\dots,g_k\}} \prod_{g_i \in S} u_i = \prod_{i=1}^{k}(1+u_i) - 1 = \gamma_{\mathrm{par}}$. □

Proof of Corollary 1.3. For part (a), fix $q \in (0, 10^{-8}]$ and $\gamma > 0$, and let $\lambda := q^{1/4} \in (0, 1/100]$. Let $\boldsymbol{\gamma}^* = (\gamma_g^*)_{g \in E}$ be the positive weight vector on G that sets $\gamma_g^* := \gamma_g(\lambda) \in \{\lambda, \lambda^2\}$ on the fifteen background edges $g \in E(H)$ and sets $\gamma_e^* = \gamma_f^* := \gamma$ on the two marked edges e and f. By Theorem 1.1 and Lemma 2.1, the Rayleigh difference $R_G(q, \boldsymbol{\gamma}^*) = R(\lambda)$ depends only on q and the background weights on H and satisfies $R_G(q, \boldsymbol{\gamma}^*) < 0$.

For each integer $\ell \geq 2$, let $u_\ell := q/\big((1 + q/\gamma)^\ell - 1\big) > 0$. Since $q/\gamma > 0$, we have $(1 + q/\gamma)^\ell \to \infty$ and hence $u_\ell \downarrow 0$ and $\log(1 + u_\ell) \downarrow 0$ as $\ell \to \infty$. For every ℓ large enough that $\log(1 + u_\ell) < \min_{g \in E(H)} \log(1 + \gamma_g^*)$, define for each $g \in E(H)$ the integer

$$k_g(\ell) := \left\lceil \frac{\log(1 + \gamma_g^*)}{\log(1 + u_\ell)} \right\rceil \geq 2 \quad \text{and} \quad \widetilde{\gamma}_g(\ell) := (1 + u_\ell)^{k_g(\ell)} - 1 > 0.$$

By definition of ceiling, $\log(1 + \gamma_g^*) \leq k_g(\ell) \log(1 + u_\ell) < \log(1 + \gamma_g^*) + \log(1 + u_\ell)$, so

$$\gamma_g^* \leq \widetilde{\gamma}_g(\ell) < (1 + \gamma_g^*)(1 + u_\ell) - 1 = \gamma_g^* + (1 + \gamma_g^*) u_\ell \xrightarrow{\ell \to \infty} \gamma_g^*.$$

Because $R_G(q, \cdot)$ is a polynomial in the fifteen background weights and $R_G(q, \boldsymbol{\gamma}^*) < 0$, continuity implies $R_G(q, \widetilde{\boldsymbol{\gamma}}(\ell)) < 0$ for a sufficiently large integer $\ell \geq 2$, where $\widetilde{\gamma}_e(\ell) = \widetilde{\gamma}_f(\ell) := \gamma$.

Construct G' from G by leaving e and f untouched as single edges of weight γ, and replacing each background edge $g \in E(H)$ by $k_g(\ell)$ internally vertex-disjoint paths of length $\ell \geq 2$, every edge of which receives weight γ. Because G is simple, $e, f \notin E(H)$, and every replacement path has length $\ell \geq 2$ with private internal vertices, G' is a finite simple graph in which every edge has weight γ. Applying (4.1) and (4.3) in Lemma 4.1 to each of the fifteen gadgets multiplies every background sum Z_s ($s \in \{0, e, f, ef\}$) by the same positive constant $K := \prod_{g \in E(H)} K_\ell^{k_g(\ell)} > 0$, where $K_\ell = \big((q + \gamma)^\ell - \gamma^\ell\big)/q$. Therefore $R_{G'}(q, \gamma) = K^2 R_G(q, \widetilde{\boldsymbol{\gamma}}(\ell)) < 0$, and Lemma 2.1 gives $\mathrm{Cov}_{\mu_{G', q, \gamma}}(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\}) > 0$.

For part (b), take $\lambda = 1/100$ in Theorem 1.2 and set the marked weights to $\gamma_e = \gamma_f = 1$, leaving $R_{G,\mathrm{conn}}(\boldsymbol{\gamma}(1/100)) = R_{\mathrm{conn}}(1/100) < 0$ unchanged. Construct G_{unif} from G by keeping e and f as single edges and subdividing each of the ten background edges of weight 10^{-2} into a path of length 100 (adding 99 internal vertices per edge) and each of the five background edges of weight 10^{-4} into a path of length 10,000 (adding 9999 internal vertices per edge). Then G_{unif} is simple and has

$$|V(G_{\mathrm{unif}})| = 8 + 10 \cdot 99 + 5 \cdot 9999 = 50{,}993 \quad \text{and} \quad |E(G_{\mathrm{unif}})| = 2 + 10 \cdot 100 + 5 \cdot 10{,}000 = 51{,}002.$$

By (4.2) in Lemma 4.1(a), summing over the internal states of all fifteen paths multiplies each connected sum C_s ($s \in \{0, e, f, ef\}$) by $K_0 := 100^{10} \cdot 10{,}000^5 = 10^{40} > 0$, so $R_{G_{\mathrm{unif}},\mathrm{conn}}(1) = K_0^2 R_{\mathrm{conn}}(1/100) < 0$, and Lemma 2.1 yields

$$\mathrm{Cov}_{\mu_{G_{\mathrm{unif}}, 0, 1}}(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\}) = 8.593016718113703\ldots \times 10^{-14} > 0.$$

This establishes part (b). □

We now turn to the matroid consequences. Let $M = (E, \mathcal{I}(M))$ be a finite matroid with rank function r_M, basis set $\mathcal{B}(M)$, and spanning-set family $\mathcal{S}(M) := \{A \subseteq E : r_M(A) = r_M(E)\}$, and let $M^* = (E, \mathcal{I}(M^*))$ denote the dual matroid [Oxl11]. Recall that M is *B-Rayleigh*, *I-Rayleigh*, or *S-Rayleigh* [CW06, SW08, Wag08] if for every positive weight vector $\boldsymbol{\gamma} \in \mathbb{R}^E_{>0}$ and every pair of distinct elements $e, f \in E$, the indicators $\mathbf{1}\{e \in A\}$ and $\mathbf{1}\{f \in A\}$ are nonpositively correlated under the probability measure proportional to $\gamma(A)$ supported on $\mathcal{B}(M)$, $\mathcal{I}(M)$, or $\mathcal{S}(M)$,

respectively. Similarly, M is *Z-Rayleigh* [Man11, ZZ26] if the probability measure proportional to $q^{r_M(E)-r_M(A)}\gamma(A)$ on 2^E satisfies pairwise negative correlation for all $\boldsymbol{\gamma} \in \mathbb{R}^E_{>0}$ and all $q \in (0,1]$.

Lemma 4.2. *Let M be a finite matroid on E, let $\boldsymbol{\gamma} \in \mathbb{R}^E_{>0}$, and define the reciprocal weight vector $\boldsymbol{\gamma}^{-1} := (\gamma_g^{-1})_{g\in E} \in \mathbb{R}^E_{>0}$. Let $\mu_{\mathcal{S}(M),\boldsymbol{\gamma}}$ and $\mu_{\mathcal{I}(M^*),\boldsymbol{\gamma}^{-1}}$ denote the probability measures on $\mathcal{S}(M)$ and $\mathcal{I}(M^*)$ proportional to $\gamma(A)$ and $\prod_{g\in B}\gamma_g^{-1}$, respectively. For every pair of elements $e, f \in E$,*

$$\mathrm{Cov}_{\mu_{\mathcal{I}(M^*),\boldsymbol{\gamma}^{-1}}}(\mathbf{1}\{e \in B\}, \mathbf{1}\{f \in B\}) = \mathrm{Cov}_{\mu_{\mathcal{S}(M),\boldsymbol{\gamma}}}(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\}).$$

Proof. By definition of the dual matroid [Oxl11, Proposition 2.1.9], $A \subseteq E$ contains a basis of M ($A \in \mathcal{S}(M)$) if and only if $B := E \setminus A$ is disjoint from a basis of M, which holds if and only if $B \in \mathcal{I}(M^*)$. Under the bijection $A \mapsto B = E \setminus A$ from $\mathcal{S}(M)$ onto $\mathcal{I}(M^*)$, we have $\gamma(A) = \big(\prod_{g\in E}\gamma_g\big)\prod_{g\in B}\gamma_g^{-1}$, so if $A \sim \mu_{\mathcal{S}(M),\boldsymbol{\gamma}}$ then $B = E \setminus A \sim \mu_{\mathcal{I}(M^*),\boldsymbol{\gamma}^{-1}}$. Since $\mathbf{1}\{e \in B\} = 1 - \mathbf{1}\{e \in A\}$ and $\mathbf{1}\{f \in B\} = 1 - \mathbf{1}\{f \in A\}$, shifting and negating both random variables leaves their covariance unchanged. □

Proof of Corollary 1.4. Because G is connected with $|V| = 8$ and $|E| = 17$, its cycle matroid $M(G)$ has rank $r_{M(G)}(E) = 8 - 1 = 7$ and corank $17 - 7 = 10$, and $\mathcal{S}(M(G)) = \{A \subseteq E : \kappa(A) = 1\}$. Thus $\mu_{\mathcal{S}(M(G)),\boldsymbol{\gamma}} = \mu_{G,0,\boldsymbol{\gamma}}$, while for $q > 0$ we have $q^{r_{M(G)}(E)-r_{M(G)}(A)}\gamma(A) = q^{\kappa(A)-1}\gamma(A)$, so the multivariate Tutte measure on $M(G)$ coincides with $\mu_{G,q,\boldsymbol{\gamma}}$. By Theorems 1.1 and 1.2, $M(G)$ is neither Z-Rayleigh nor S-Rayleigh.

By Lemma 4.2, the cographic dual $M(G)^*$ (of rank 10 and corank 7) at reciprocal weights $\boldsymbol{\gamma}(1/100)^{-1}$ (where every element weight equals 100 or 10,000) satisfies

$$\mathrm{Cov}_{\mu_{\mathcal{I}(M(G)^*),\boldsymbol{\gamma}(1/100)^{-1}}}(\mathbf{1}\{e \in B\}, \mathbf{1}\{f \in B\}) = \mathrm{Cov}_{\mu_{G,0,\boldsymbol{\gamma}(1/100)}}(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\}) > 0,$$

so $M(G)^*$ is not I-Rayleigh. Likewise, applying Lemma 4.2 to G_{unif} with $\boldsymbol{\gamma} = \mathbf{1}$ (so $\boldsymbol{\gamma}^{-1} = \mathbf{1}$) and invoking Corollary 1.3(b) shows that e and f have strictly positive covariance under the uniform distribution $\mu_{\mathcal{I}(M(G_{\mathrm{unif}})^*),\mathbf{1}}$ on independent sets of $M(G_{\mathrm{unif}})^*$. Finally, graphic matroids are regular and regular matroids are closed under duality [Oxl11, Proposition 2.2.21 and Theorem 2.2.26], so $M(G)^*$ and $M(G_{\mathrm{unif}})^*$ are regular matroids, and every regular matroid has a real stable basis generating polynomial [COSW04, Corollary 8.2] and is therefore B-Rayleigh [BBL09, Theorem 4.9]. □

5. Contrast with the unrooted spanning-forest limit

In this final section, we show that under the unrooted forest scaling (1.3), the marked edges $e = u_1u_2$ and $f = xy$ of G are *strictly negatively correlated* across the entire four-parameter symmetric family on H, and explain why neither limit arguments nor matroid duality convert our counterexamples into a graph-forest counterexample.

For positive forest weights $\boldsymbol{\beta} \in \mathbb{R}^E_{>0}$ on $G = (V, E)$, decompose the forest partition function (1.3) according to $A \cap \{e, f\}$ as

$$F_G(\boldsymbol{\beta}) = F_0 + \beta_e F_e + \beta_f F_f + \beta_e\beta_f F_{ef}, \quad \text{where} \quad F_s := \sum_{\substack{A\subseteq E(H)\\ \ell(A\cup s)=0}} \prod_{g\in A}\beta_g \quad (s \subseteq \{e, f\}).$$

Just as in Lemma 2.1, we have $\mathrm{Cov}_{\nu_{G,\boldsymbol{\beta}}}(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\}) = -\beta_e\beta_f R_{\mathcal{F}}/F_G(\boldsymbol{\beta})^2$, where $R_{\mathcal{F}} := F_eF_f - F_0F_{ef}$ is the forest Rayleigh difference, independent of $\beta_e, \beta_f > 0$.

Proposition 5.1. *Let $H = G\setminus\{e, f\}$ carry the two-level forest weights $\beta_g(\lambda) = \lambda$ for $g \in E(K_{U,W})\cup\{xu_3\}$ and $\beta_g(\lambda) = \lambda^2$ for $g \in \{xu_1, xu_2, yw_1, yw_2, yw_3\}$. Then the forest Rayleigh difference $R_{\mathcal{F}}(\lambda) := F_e(\lambda)F_f(\lambda) - F_0(\lambda)F_{ef}(\lambda)$ factors as*

$$R_{\mathcal{F}}(\lambda) = 6\lambda^5(1 + 3\lambda + \lambda^2)^2\big(1 + 13\lambda + 69\lambda^2 + 216\lambda^3 + 470\lambda^4 + 772\lambda^5$$

$$+\,963\lambda^6+979\lambda^7+553\lambda^8+152\lambda^9+16\lambda^{10}\big). \tag{5.1}$$

In particular, $R_{\mathcal{F}}(\lambda)>0$ *and* $\mathrm{Cov}_{\nu_{G,\boldsymbol{\beta}}}(\mathbf{1}\{e\in A\},\mathbf{1}\{f\in A\})<0$ *for every* $\lambda>0$ *and all* $\beta_e,\beta_f>0$.

Proof. We extract $F_0(\lambda),F_e(\lambda),F_f(\lambda),F_{ef}(\lambda)$ directly from Lemma 2.2 by scaling the background weights by q. Because G is simple and $s\subseteq\{e,f\}$ is disjoint from $E(H)$, every $A\subseteq E(H)$ satisfies $|A\cup s|=|A|+|s|$, so $\ell(A\cup s)=|A|+|s|-8+\kappa(A\cup s)$. Substituting $\gamma_g=q\beta_g(\lambda)$ for $g\in E(H)$ into (2.1) gives

$$q^{|s|-8}Z_s(q,q\boldsymbol{\beta}(\lambda))=\sum_{A\subseteq E(H)}q^{\ell(A\cup s)}\prod_{g\in A}\beta_g(\lambda)\xrightarrow{q\to0}F_s(\lambda),$$

so $F_0(\lambda),F_e(\lambda),F_f(\lambda),F_{ef}(\lambda)$ are the coefficients of q^8,q^7,q^7,q^6 in Z_0,Z_e,Z_f,Z_{ef} evaluated at $(x_0,a,b,c)=(q\lambda,q\lambda^2,q\lambda,q\lambda^2)$.

For any polynomial $\alpha=\alpha(q,\lambda)$, substituting $\zeta=1+q\alpha$ into (2.3) shows that $\phi_h(q,1+q\alpha)=q\psi_h(q,\alpha)$, where

$$\psi_{(1,1,1)}(q,\alpha)=1+3\alpha,\qquad \psi_{(2,1)}(q,\alpha)=1+3\alpha+q\alpha^2,$$
$$\psi_{(3)}(q,\alpha)=1+3\alpha+3q\alpha^2+q^2\alpha^3.$$

Thus $L_h,M_h,N_h,S_h,T_h,U_h,V_h$ in (2.4) are all divisible by q, and in (2.5) each $F_s(h)$ is divisible by $q^{5-|s|}$ (namely q^5,q^4,q^4,q^3 for $s=0,e,f,ef$). Define the scaled polynomials $X_s(h):=q^{|s|-5}F_s(h)\in\mathbb{Z}[q,\lambda]$. Since $m_{(1,1,1)}(q)_3=2q-3q^2+q^3$, $m_{(2,1)}(q)_2=-3q+3q^2$, and $m_{(3)}(q)_1=q$, we have for each $s\in\{0,e,f,ef\}$ the identity

$$\sum_h m_h(q)_{p_h}X_s(h)=q\big(X_s(3)-3X_s(2,1)+2X_s(1,1,1)\big)$$
$$+3q^2\big(X_s(2,1)-X_s(1,1,1)\big)+q^3X_s(1,1,1).$$

Extracting the coefficient of q^3 (which corresponds to $q^{3+(5-|s|)}=q^{8-|s|}$ in Z_s) decomposes $F_s(\lambda)$ into three explicit pieces:

$$F_s(\lambda)=[q^0]X_s(1,1,1)+3\,[q^1]\big(X_s(2,1)-X_s(1,1,1)\big)$$
$$+\,[q^2]\big(X_s(3)-3X_s(2,1)+2X_s(1,1,1)\big). \tag{5.2}$$

Let $D_s^{(0)}$, $D_s^{(1)}$, and $D_s^{(2)}$ denote the three respective summands on the right-hand side of (5.2). Expanding $X_s(h)$ to order q^2 via ψ_h gives their coefficients from λ^0 to λ^{12}:

s	piece	1	λ	λ^2	λ^3	λ^4	λ^5	λ^6	λ^7	λ^8	λ^9	λ^{10}	λ^{11}	λ^{12}
0	$D_0^{(0)}$	1	10	41	104	214	322	327	300	81	0	0	0	0
0	$D_0^{(1)}$	0	0	9	63	183	438	858	960	1155	543	81	0	0
0	$D_0^{(2)}$	0	0	0	3	39	132	307	784	938	1349	913	262	27
e	$D_e^{(0)}$	1	10	32	68	141	114	162	0	0	0	0	0	0
e	$D_e^{(1)}$	0	0	15	69	132	399	339	600	162	0	0	0	0
e	$D_e^{(2)}$	0	0	0	9	63	99	382	364	683	362	54	0	0
f	$D_f^{(0)}$	1	10	41	104	205	250	102	12	0	0	0	0	0
f	$D_f^{(1)}$	0	0	9	63	189	477	873	549	138	12	0	0	0
f	$D_f^{(2)}$	0	0	0	3	39	135	355	931	821	320	58	4	0
ef	$D_{ef}^{(0)}$	1	10	32	68	132	24	0	0	0	0	0	0	0
ef	$D_{ef}^{(1)}$	0	0	15	69	138	450	204	24	0	0	0	0	0
ef	$D_{ef}^{(2)}$	0	0	0	9	63	102	472	352	92	8	0	0	0

Summing each triple of rows gives the four forest polynomials:

$$\begin{aligned}
F_0(\lambda) &= 1 + 10\lambda + 50\lambda^2 + 170\lambda^3 + 436\lambda^4 + 892\lambda^5 + 1492\lambda^6 \\
&\quad + 2044\lambda^7 + 2174\lambda^8 + 1892\lambda^9 + 994\lambda^{10} + 262\lambda^{11} + 27\lambda^{12}, \\
F_e(\lambda) &= 1 + 10\lambda + 47\lambda^2 + 146\lambda^3 + 336\lambda^4 + 612\lambda^5 \\
&\quad + 883\lambda^6 + 964\lambda^7 + 845\lambda^8 + 362\lambda^9 + 54\lambda^{10}, \\
F_f(\lambda) &= 1 + 10\lambda + 50\lambda^2 + 170\lambda^3 + 433\lambda^4 + 862\lambda^5 \\
&\quad + 1330\lambda^6 + 1492\lambda^7 + 959\lambda^8 + 332\lambda^9 + 58\lambda^{10} + 4\lambda^{11}, \\
F_{ef}(\lambda) &= 1 + 10\lambda + 47\lambda^2 + 146\lambda^3 + 333\lambda^4 + 576\lambda^5 + 676\lambda^6 + 376\lambda^7 + 92\lambda^8 + 8\lambda^9.
\end{aligned}$$

Multiplying $F_e(\lambda)F_f(\lambda) - F_0(\lambda)F_{ef}(\lambda)$ and expanding the right-hand side of (5.1) gives the identical degree-19 polynomial

$$\begin{aligned}
R_{\mathcal{F}}(\lambda) &= 6\lambda^5 + 114\lambda^6 + 948\lambda^7 + 4674\lambda^8 + 15{,}624\lambda^9 + 38{,}370\lambda^{10} + 72{,}780\lambda^{11} + 109{,}710\lambda^{12} \\
&\quad + 132{,}732\lambda^{13} + 124{,}734\lambda^{14} + 83{,}088\lambda^{15} + 36{,}390\lambda^{16} + 9846\lambda^{17} + 1488\lambda^{18} + 96\lambda^{19}. \quad \square
\end{aligned}$$

More generally, expanding $q^{|s|-8}Z_s(q, q\boldsymbol{\beta})|_{q=0}$ across all four independent variables (x_0, a, b, c) of Lemma 2.2 yields a four-variable forest Rayleigh polynomial $R_{\mathcal{F}}(x_0, a, b, c)$ with 202 nonzero monomials, *every one of which has a strictly positive integer coefficient.* Thus no choice of positive weights $(x_0, a, b, c) \in \mathbb{R}^4_{>0}$ on (G, e, f) can violate pairwise negative correlation in the unrooted forest model.

We conclude by recording three structural reasons why known reductions do not transfer our counterexamples to the unrooted forest measure $\nu_{\widetilde{G},\boldsymbol{\beta}}$:

(1) *Inequivalence of the two $q \downarrow 0$ limits.* Holding $0 < q < 1$ Rayleigh positivity for all graphs and all positive weights would imply both the connected-subgraph (S-Rayleigh) and forest (I-Rayleigh) conjectures via the two limits (1.2) and (1.3) [SW08, Section 3], but refuting the premise and its fixed-weight limit (1.2) says nothing about the scaled limit (1.3), which suppresses all cyclic configurations responsible for the negative term in (3.4).

(2) *Non-planarity and Whitney's theorem.* On a connected planar graph $\widetilde{G}$, the cographic matroid $M(\widetilde{G})^*$ is isomorphic to the graphic matroid $M(\widetilde{G}^*)$ of the planar dual $\widetilde{G}^*$, so Lemma 4.2 identifies connected spanning subgraphs of $\widetilde{G}$ with spanning forests of $\widetilde{G}^*$. Our graph G contains $K_{3,3}$ on $U \cup W$, and G_{unif} contains a subdivision of $K_{3,3}$. By Kuratowski's and Whitney's theorems [Oxl11, Theorem 5.2.2], both $M(G)^*$ and $M(G_{\text{unif}})^*$ have the non-graphic minor $M(K_{3,3})^*$ and cannot be represented as $M(\widetilde{G})$ for any graph $\widetilde{G}$. Similarly, in terms of Seymour's regular-matroid decomposition into graphic matroids, cographic matroids, and R_{10}, Corollary 1.4 refutes S-Rayleigh on the graphic summands (via $M(G)$) and I-Rayleigh on the cographic summands (via $M(G)^*$), while leaving I-Rayleigh on the graphic summands (equivalently, S-Rayleigh on the cographic summands) open.

(3) *Distinction from fixed-cardinality forests.* Conditioning $\nu_{\widetilde{G},1}$ on a feasible edge count $|A| = k$ ($0 \le k \le r_{M(\widetilde{G})}(E)$) yields the uniform *basis* distribution of the rank-k truncation of $M(\widetilde{G})$, whose independent sets are all forests of size at most k. In the example of Seymour, Winkler, and Sudan [FM92, HSW22] ($K_{2,5}$ with hub edge $e = uv$ and pendant edge $f = uw$, conditioned on $|A| = 6$), the six-edge forest counts in states $(00, 10, 01, 11)$ are $(80, 32, 192, 80)$, giving positive covariance $(80 \cdot 80 - 32 \cdot 192)/384^2 = 1/576 > 0$. In contrast, under the full forest measure $\nu_{\widetilde{G},\boldsymbol{\beta}}$, the pendant edge f is included or excluded independently of every forest on $\widetilde{G} \setminus \{f\}$, so $F_{\widetilde{G}}(\boldsymbol{\beta}) = (1 + \beta_f)F_{\widetilde{G}\setminus\{f\}}(\boldsymbol{\beta})$ and $\operatorname{Cov}_{\nu_{\widetilde{G},\boldsymbol{\beta}}}(\mathbf{1}\{e \in A\}, \mathbf{1}\{f \in A\}) = 0$.

Whether pairwise edge negative correlation holds for the unrooted forest measure $\nu_{\widetilde{G},\boldsymbol{\beta}}$ on every finite graph $\widetilde{G}$ with positive edge weights remains an intriguing open problem.

References

[ALO+21] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, Cynthia Vinzant, and Thuy-Duong Vuong. Log-concave polynomials IV: approximate exchange, tight mixing times, and near-optimal sampling of forests. In *STOC*, pages 408–420. ACM, 2021.

[BBL09] Julius Borcea, Petter Brändén, and Thomas M. Liggett. Negative dependence and the geometry of polynomials. *J. Amer. Math. Soc.*, 22(2):521–567, 2009.

[BH20] Petter Brändén and June Huh. Lorentzian polynomials. *Ann. of Math. (2)*, 192(3):821–891, 2020.

[BP93] Robert Burton and Robin Pemantle. Local characteristics, entropy and limit theorems for spanning trees and domino tilings via transfer-impedances. *Ann. Probab.*, 21(3):1329–1371, 1993.

[BSST40] R. L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte. The dissection of rectangles into squares. *Duke Math. J.*, 7:312–340, 1940.

[Coc08] Clifford C. Cocks. Correlated matroids. *Comb. Probab. Comput.*, 17(4):511–518, 2008.

[COSW04] Young-Bin Choe, James G. Oxley, Alan D. Sokal, and David G. Wagner. Homogeneous multivariate polynomials with the half-plane property. *Adv. in Appl. Math.*, 32(1–2):88–187, 2004.

[ÇR26] Recep Altar Çiçeksiz and Mohan Ravichandran. Negative correlations for forests and the $q < 1$ random-cluster model. *arXiv*, abs/2610.00549, 2026. Numbering as in arXiv:2610.00549v1.

[CW06] Young-Bin Choe and David G. Wagner. Rayleigh matroids. *Comb. Probab. Comput.*, 15(5):765–781, 2006.

[Eri12] Alejandro Erickson. Sums of squares and negative correlation for spanning forests of series-parallel graphs. *Australas. J. Combin.*, 52:75–87, 2012.

[FK72] Cornelius M. Fortuin and Pieter W. Kasteleyn. On the random-cluster model. I. Introduction and relation to other models. *Physica*, 57:536–564, 1972.

[FKG71] Cornelius M. Fortuin, Pieter W. Kasteleyn, and Jean Ginibre. Correlation inequalities on some partially ordered sets. *Comm. Math. Phys.*, 22(2):89–103, 1971.

[FM92] Tomás Feder and Milena Mihail. Balanced matroids. In *STOC*, pages 26–38. ACM, 1992.

[FM26] Hao Fang and Biao Ma. Gårding polynomials. *arXiv*, abs/2604.27755, 2026. Numbering as in arXiv:2604.27755v3.

[Gri06] Geoffrey Grimmett. *The random-cluster model*, volume 333 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 2006.

[Gup26] Anish Gupta. Edges of the uniform random forest of K_n are pairwise negatively correlated for every n. *arXiv*, abs/2609.26134, 2026.

[GW04] Geoffrey R. Grimmett and Stephanie N. Winkler. Negative association in uniform forests and connected graphs. *Random Structures Algorithms*, 24(4):444–460, 2004.

[HSW22] June Huh, Benjamin Schröter, and Botong Wang. Correlation bounds for fields and matroids. *J. Eur. Math. Soc. (JEMS)*, 24(4):1335–1351, 2022.

[Kah00] Jeff Kahn. A normal law for matchings. *Combinatorica*, 20(3):339–391, 2000.

[KN10] Jeff Kahn and Michael Neiman. Negative correlation and log-concavity. *Random Structures Algorithms*, 37(3):367–388, 2010.

[Man11] Arun P. Mani. An extension of matroid rank submodularity and the Z-Rayleigh property. *Electron. J. Combin.*, 18(1):Paper 113, 2011.

[NP25] Son Nguyen and Pavlo Pylyavskyy. Correlations in random cluster model at $q = 1$. *arXiv*, abs/2507.09520, 2025. Numbering as in arXiv:2507.09520v1.

[Oxl11] James Oxley. *Matroid theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, second edition, 2011.

[Par26] Heehyun Park. The $q < 1$ random-cluster model on wired trees: uniqueness and negative dependence. *arXiv*, abs/2608.08565, 2026. Numbering as in arXiv:2608.08565v1.

[Pem00] Robin Pemantle. Towards a theory of negative dependence. *J. Math. Phys.*, 41(3):1371–1390, 2000.

[RS04] Gordon Royle and Alan D. Sokal. The Brown–Colbourn conjecture on zeros of reliability polynomials is false. *J. Combin. Theory Ser. B*, 91(2):345–360, 2004.

[Sok05] Alan D. Sokal. The multivariate Tutte polynomial (alias Potts model) for graphs and matroids. In *Surveys in combinatorics 2005*, volume 327 of *London Math. Soc. Lecture Note Ser.*, pages 173–226. Cambridge Univ. Press, Cambridge, 2005.

[Sta11] Dudley Stark. The edge correlation of random forests. *Ann. Comb.*, 15(3):529–539, 2011.

[SW75] P. D. Seymour and D. J. A. Welsh. Combinatorial applications of an inequality from statistical mechanics. *Math. Proc. Cambridge Philos. Soc.*, 77(3):485–495, 1975.

[SW08] Charles Semple and Dominic Welsh. Negative correlation in graphs and matroids. *Comb. Probab. Comput.*, 17(3):423–435, 2008.

[TZ26a] Pengfei Tang and Zibo Zhang. Negative correlation for the random-cluster model below one on high-degree regular graphs. *arXiv*, abs/2609.11578, 2026. Numbering as in arXiv:2609.11578v2.

[TZ26b] Pengfei Tang and Zibo Zhang. Pairwise correlation bounds for the random-cluster model with cluster weight at most one. *arXiv*, abs/2611.11993, 2026. Numbering as in arXiv:2611.11993v1.

[TZ26c] Pengfei Tang and Zibo Zhang. Pairwise negative correlation for uniform spanning subgraphs of the complete graph. *arXiv*, abs/2603.10738, 2026. Numbering as in arXiv:2603.10738v2.

[Wag08] David G. Wagner. Negatively correlated random variables and Mason's conjecture for independent sets in matroids. *Ann. Comb.*, 12(2):211–239, 2008.

[ZZ26] Qinghong Zhao and Xiangqian Zhou. Every rank-three matroid is Z-Rayleigh. *arXiv*, abs/2611.15512, 2026. Numbering as in arXiv:2611.15512v1.