PULL-BACK GEOMETRY AND INVARIANT DENSITIES IN ONE-DIMENSIONAL DYNAMICS

Abstract. We construct two one-dimensional maps that distinguish shrinking of individual pull-backs, summability of bad pull-backs, and endpoint integrability of invariant densities. The first is a topologically exact C^3 interval map with quadratic critical points. Its pull-backs satisfy a stretched-exponential shrinking estimate, but its badness exponent is positive, even when the degree weights are omitted. The second is a semi-hyperbolic real cubic polynomial $F(z) = tz(z-1)^2$ with geometric pull-back exponent $q(F) = 4$, whose invariant density relative to the geometric conformal measure belongs to $L^{4/3}$. These examples answer negatively Questions 2.13 and 2.18 of Rivera–Letelier and Shen. The interval construction preserves critical-orbit records while inserting two-branch returns at arbitrarily small scales. The polynomial construction uses a quadratic branched quotient to separate quartic pull-back geometry from a summable family of postcritical density spikes.

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AI Contribution Statement

AI systems developed all of the substantive mathematical contributions in this paper and carried out the entire Lean formalization. The work took many rounds of development, revision, and verification. The author made only minor edits to the exposition.

1. Introduction and main results

Pull-back estimates enter the study of one-dimensional dynamics in several ways. They control the size of individual inverse images, the summability of families of inverse images, and the regularity of invariant measures. In the presence of critical points these uses require different information. This paper gives two examples that distinguish them within the framework of Rivera–Letelier and Shen [25].

For a real or complex one-dimensional map f, let $\beta_{\max}(f)$ be the supremum of its polynomial shrinking exponents, and let $\delta_{\mathrm{bad}}(f)$ be its badness exponent. The first quantity concerns the diameter of each pull-back of a fixed-radius ball. The second concerns a weighted sum over pull-backs that are not contained in an earlier diffeomorphic pull-back of a nice neighborhood of the critical set. Rivera–Letelier and Shen proved that backward contraction, together with expansion away from critical points, implies both $\beta_{\max}(f) = \infty$ and $\delta_{\mathrm{bad}}(f) = 0$. Their Question 2.13 asks whether the former conclusion alone implies the latter. Our first result shows that it does not.

Theorem A. *There exists a boundary-anchored, topologically exact C^3 map $f : [0,1] \to [0,1]$ with a nonempty finite set of quadratic critical points and only hyperbolic repelling periodic orbits, with the following properties. There are $\rho, C, \kappa > 0$ such that every component W of $f^{-m}(B(y,\rho))$ satisfies*

$$|W| \le C \exp(-\kappa m^{1/3}) \qquad (y \in [0,1],\ m \ge 1). \tag{1.1}$$

There is also $s_* > 0$ such that, for every nice set V,

$$\sum_{m \ge 1} \sum_{W \in \mathfrak{B}_m(V)} |W|^{s_*} = \infty, \tag{1.2}$$

where $\mathfrak{B}_m(V)$ denotes the family of bad pull-backs of V by f^m. In particular, $\mathcal{J}(f) = [0,1]$, $\beta_{\max}(f) = \infty$, and $\delta_{\mathrm{bad}}(f) \ge s_* > 0$.

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The exponent s_* is independent of V. The divergence in (1.2) does not use the degree weights in the definition of badness. Positive badness was already known: [25, Remark 2.11] derives it for certain Fibonacci maps from Keller and Nowicki [9]. The additional assertion here is the uniform shrinking estimate (1.1) for a single map with quadratic critical points. Results on stretched-exponential decay of correlations for generalized Fibonacci maps, such as Gao and Shen [7], concern statistical decay, rather than the uniform pull-back estimate asserted here.

The second example concerns the relation between pull-back geometry and invariant densities. In the notation of [25], $\gamma(f) = \beta_{\max}(f)(\mathrm{HD}\,\mathcal{J}(f) - \delta_{\mathrm{bad}}(f))$. For $f \in \mathscr{A}^*$ with $\gamma(f) > 2$, their Corollary D gives a geometric conformal probability μ and an invariant probability $\nu \ll \mu$. Their geometric pull-back exponent $q(f)$ controls the size of diffeomorphic pull-backs with distortion at most two; its associated endpoint is $p(f) = q(f)/(q(f) - 1)$. Question 2.18 asks whether $\gamma(f) = \infty$ always forces $d\nu/d\mu \notin L^{p(f)}(\mu)$.

Theorem B. *There exists $t > 1$ such that the polynomial $F(z) = tz(z-1)^2$ is semi-hyperbolic and satisfies*

$$F \in \mathscr{A}^*_{\mathbb{C}}, \qquad \gamma(F) = \infty, \qquad q(F) = 4.$$

Let μ_F be its geometric conformal probability and let ν_F be the invariant probability absolutely continuous with respect to μ_F. Then

$$\frac{d\nu_F}{d\mu_F} \in L^{4/3}(\mu_F), \qquad \frac{4}{3} = \frac{q(F)}{q(F)-1}.$$

The polynomial in Theorem B has exponential shrinking and zero badness. Thus this example does not rely on the obstruction in Theorem A. Even in this stronger setting, the worst pull-back geometry does not by itself imply nonintegrability at the associated endpoint. The assertion is in complex rational dynamics; it is not a statement about Lebesgue densities for an interval restriction. We do not determine integrability for $p > 4/3$.

Relation to earlier work. Derivative growth along critical orbits provides an important route to invariant densities and statistical estimates. For interval maps, see Bruin–Shen–van Strien [4], Bruin–Luzzatto–van Strien [2], and Bruin–Rivera–Letelier–Shen–van Strien [3]. The last paper relates large derivatives to backward contraction. Li and Shen [12] establish related implications for polynomials. The interval construction below does not impose pointwise growth of every cumulative critical derivative. It instead controls the running maximum of a derivative–distance quantity and the distortion accumulated before that maximum is reached.

In complex dynamics, the distinction between Collet–Eckmann expansion and its topological versions is developed in Przytycki–Rivera–Letelier–Smirnov [23]. Semi-hyperbolicity and bounded-degree pull-back geometry are treated by Carleson–Jones–Yoccoz [5] and Mihalache [15]. The statistical theory for topological Collet–Eckmann maps and the use of nice inducing schemes are developed in [20, 21]. Our parameter selection adapts the realization and extremal-bracketing arguments of Mihalache [14]; the prescribed passage depths and their use in the endpoint problem are established here.

For conformal and invariant measures under nonuniform hyperbolicity, we use the transfer-operator estimate of Przytycki [18]; see also [19, 8]. Conformal mass estimates under backward contraction appear in Li–Shen [11] and in [25]. The endpoint argument below needs an exact upper bound Cr^h at the postcritical centers of the lifted map, not merely $Cr^{h-\varepsilon}$. We prove this localized bound from expansion of the postcritical set and then combine it with the weight introduced by the branched quotient.

The first example need not be statistically irregular for every potential. By Li–Rivera–Letelier [10, Corollary 2.5], Theorem A implies that every Hölder continuous potential has a unique equilibrium state, with exponential decay of correlations. Indeed, for a Hölder exponent $\alpha \in (0, 1]$ one may choose a polynomial shrinking exponent $\beta > 1/\alpha$. This observation does not apply to the singular geometric

potentials $-t \log |Df|$. The relation between periodic Lyapunov exponents and asymptotic expansion is studied in Rivera–Letelier [24], and the thermodynamics of geometric potentials for multimodal maps in Przytycki–Rivera–Letelier [22]. For our map every periodic exponent is positive, but their infimum is zero.

The constructions. For the interval example, we insert returns f^{r_n} at arbitrarily small neighborhoods of a distinguished critical point. Each return has two branches with derivatives bounded above and below independently of n. For each nice set V, choose a sufficiently small return from this sequence. Its two branches give a Cantor set K_V with a positive dimension lower bound independent of V. For each positive integer k, suitable bad pull-backs of V at the k-fold return time cover K_V. The return times tend to infinity. Critical-orbit records nevertheless give a logarithmic bound for covering times and hence (1.1). Thus the spatial obstruction to summability persists while the cost in the original time variable is controlled.

For the polynomial example, the selected critical-value orbit approaches the other critical point at times n_k with logarithmic depth comparable to n_k^2. Introduce the lift

$$S(\zeta) = a\zeta(\zeta^2 - 1), \qquad \pi(\zeta) = \zeta^2, \qquad a^2 = t, \qquad \pi \circ S = F \circ \pi.$$

The postcritical closure of S is separated from its critical set and is uniformly expanding. Downstairs, the selected passages yield quartic normal models and force $q(F) = 4$. Upstairs, the invariant density is bounded by an exponentially weighted sum of postcritical $h/2$-spikes, where $h = \mathrm{HD}\,\mathcal{J}(S) = \mathrm{HD}\,\mathcal{J}(F)$. The quotient formula reduces the endpoint to

$$\int \rho_S^{4/3} (\pi^{\#})^{-h/3}\, d\mu_S < \infty.$$

The two singular orders add to h, so the interaction of a spike with the quotient weight has only logarithmic growth as their centers approach. The complete itinerary controls this growth at every time, and the exponential coefficients make the resulting norms summable.

Section 2 fixes the definitions. Sections 3–5 prove Theorem A. Section 6 constructs the polynomial and its lift, Section 7 computes the geometric exponent, and Section 8 proves the endpoint estimate.

2. Pull-back exponents

We use the conventions of [25, Sections 2.1–2.2]. For background on real and complex dynamics, see [6, 16]. All balls and topological notions for an interval are relative to that interval. Complex maps act on the Riemann sphere $\widehat{\mathbb{C}}$.

2.1. **Maps and shrinking.** The class $\mathscr{A}_{\mathbb{R}}$ consists of boundary-anchored C^1 interval maps, C^3 away from their critical points, with non-flat critical points and no neutral periodic orbit. Non-flatness means that for each critical point c there are local C^3 diffeomorphisms ϕ, ψ, vanishing at c, $f(c)$ respectively, such that $|\psi(f(x))| = |\phi(x)|^{\ell_c}$ for some $\ell_c > 1$. Boundary-anchored means $f(\partial I) \subset \partial I$ and $Df \neq 0$ at the endpoints. The map constructed below is globally C^3 and has $\ell_c = 2$ at every critical point. The class $\mathscr{A}_{\mathbb{C}}$ consists of rational maps of degree at least two without neutral periodic orbits. Put $\mathscr{A} = \mathscr{A}_{\mathbb{R}} \cup \mathscr{A}_{\mathbb{C}}$ and $\mathrm{Crit}_0(f) = \mathrm{Crit}(f) \cap \mathcal{J}(f)$, where $\mathcal{J}(f)$ is the set on which the iterates are not locally equicontinuous. The critical set in the Julia set is assumed nonempty whenever an exponent from [25] is used.

An interval map is *topologically exact* if every nonempty relatively open set eventually maps onto the whole interval. For $x \in \mathcal{J}(f)$, let $W(x, m, \rho)$ be the component of $f^{-m}(B(f^m(x), \rho))$ containing x. Polynomial shrinking with exponent $\beta \geq 0$ means that there are $C, \rho > 0$ such that $\operatorname{diam} W(x, m, \rho) \leq C m^{-\beta}$ for every $x \in \mathcal{J}(f)$ and $m \geq 1$. The supremum of these exponents is $\beta_{\max}(f)$. In the complex case all diameters are spherical.

2.2. **Nice sets and bad pull-backs.** A *nice set* V contains $\mathrm{Crit}_0(f)$, each component is an open interval or topological disk containing exactly one point of $\mathrm{Crit}_0(f)$, and $f^n(\partial V) \cap V = \varnothing$ for every $n \geq 1$. In addition, V is disjoint from the forward orbits of critical points outside the Julia set and from all periodic orbits outside the Julia set. These exclusions are vacuous for our interval map, whose Julia set is the whole interval.

A pull-back of V by f^m is a component of $f^{-m}(V)$. It is *diffeomorphic* if f^m maps it diffeomorphically onto a component of V; in the complex case this means biholomorphically. A pull-back W by f^m is *bad* if, for every $1 \leq m' \leq m$ with $f^{m'}(W) \subset V$, the component of $f^{-m'}(V)$ containing W is not diffeomorphic. Denote this family by $\mathfrak{B}_m(V)$.

For an interval map put $d_V(W) = 2^N$, where N counts the times $0 \leq j < m$ at which the component of $f^{-(m-j)}(V)$ containing $f^j(W)$ meets $\mathrm{Crit}(f)$. For a rational map, $d_V(W)$ is the degree of $f^m : W \to f^m(W)$. Define

$$S_V(s) = \sum_{m \geq 1} \sum_{W \in \mathfrak{B}_m(V)} d_V(W)\, \mathrm{diam}(W)^s,$$

$$\delta_{\mathrm{bad}}(V) = \inf\{s > 0 : S_V(s) < \infty\}, \qquad \delta_{\mathrm{bad}}(f) = \inf_{V \text{ nice}} \delta_{\mathrm{bad}}(V).$$

An empty infimum is $+\infty$. Arbitrarily small nice sets for our interval map are constructed in Section 5.

A nice set is *symmetric* if $f(\partial V^c) \subset \partial f(V^c)$ for each critical component V^c. A nice couple $(V, \widehat{V})$ is a nested pair of nice sets with $\overline{V} \subset \widehat{V}$ and $f^n(\partial V) \cap \widehat{V} = \varnothing$ for $n \geq 1$. The class $\mathscr{A}^*$ in [25] requires expansion away from $\mathrm{Crit}_0(f)$, arbitrarily small symmetric nice couples, and, in the real case, essential topological exactness on the Julia set. Only the complex examples below are asserted to belong to this class. For them, membership follows from the TCE condition by [25, Corollary 2.19].

2.3. **The geometric exponent and conformal measures.** For a rational map R, write $R^\#$ for its spherical derivative and $\mathrm{Dist}_\#(G|_U) = \sup_{z,w \in U} G^\#(z)/G^\#(w)$ for the spherical distortion of a univalent map. In an affine coordinate, $R^\#(z) = |R'(z)|(1 + |z|^2)/(1 + |R(z)|^2)$.

Definition 2.1. The geometric pull-back exponent $q(R)$ is the infimum of the numbers $q > 0$ for which there is $C_q > 0$ such that

$$\mathrm{diam}_\# W \leq C_q \delta^{1/q}$$

for every $x \in \mathcal{J}(R)$, $\delta > 0$, $m \geq 1$, and component W of $R^{-m}(B_\#(x, \delta))$ for which $R^m : W \to B_\#(x, \delta)$ is biholomorphic with spherical distortion at most two. If there is no such q, set $q(R) = \infty$.

A Borel probability μ is h-conformal for R if $\mu(R(E)) = \int_E (R^\#)^h \, d\mu$ whenever E is Borel and $R|_E$ is injective. The geometric exponent is $h = \mathrm{HD}\, \mathcal{J}(R)$. Following [25], put $\gamma(R) = \beta_{\max}(R)(h - \delta_{\mathrm{bad}}(R))$, with the extended-product convention that $+\infty$ times a positive number is $+\infty$ and $+\infty$ times zero is zero. We use this quantity only where the factors are defined and $h - \delta_{\mathrm{bad}}(R) > 0$.

A rational map is *semi-hyperbolic* if it has no parabolic periodic orbit and every critical point in its Julia set is non-recurrent. Such maps have exponential shrinking and satisfy the topological Collet–Eckmann condition; see [5, 15, 23]. Consequently, for a semi-hyperbolic map with a critical point in the Julia set, [25, Remark 2.12 and Corollary 2.19] give $\delta_{\mathrm{bad}} = 0$, $\gamma = \infty$, and the conformal and invariant probabilities used in Theorem B. Their uniqueness will be recalled when the quotient formula is proved.

3. Small critical returns and bad pull-backs

The following criterion isolates the obstruction used in the interval construction. The return times may vary, but the derivative bounds must be uniform. Throughout Sections 3–5, $I = [0, 1]$ and $|J|$ denotes interval length.

Proposition 3.1. *Let $f : I \to I$ be C^1, with finitely many critical points and $\mathcal{J}(f) = I$. Suppose that $c \in \mathrm{Crit}(f)$, $1 < \lambda \le L < \infty$, and that closed intervals H_n and positive integers r_n satisfy*

$$c \in \operatorname{int} H_n, \qquad \sup_{x \in H_n} |x - c| \longrightarrow 0, \qquad \sup_{x \in f^{r_n}(H_n)} |x - c| \longrightarrow 0.$$

Assume that f^{r_n} maps two disjoint closed subintervals of $\operatorname{int} H_n$ *diffeomorphically onto H_n, with*

$$\lambda \le |Df^{r_n}| \le L$$

on those intervals. Put $s = \log 2/\log L$. For every nice set V, there exist $r \ge 1$ and $c_V > 0$ such that

$$\sum_{W \in \mathfrak{B}_{kr}(V)} |W|^s \ge c_V \qquad (k \ge 1). \tag{3.1}$$

Consequently, $\delta_{\mathrm{bad}}(f) \ge s$.

Proof. Fix a nice set V and choose a return $\mathcal{R} = f^r$ on an interval H for which $H \cup \mathcal{R}(H) \subset V$. For $0 \le j < r$, the interval $f^j(H)$ avoids ∂V: otherwise a point of ∂V would enter V after $r - j$ further iterates. Hence

$$f^j(H) \cap V \ne \varnothing \quad \Longrightarrow \quad f^j(H) \subset V. \tag{3.2}$$

Let $h_0, h_1 : H \to H$ be the two inverse branches. Their images are disjoint and

$$L^{-1}|x - y| \le |h_i(x) - h_i(y)| \le \lambda^{-1}|x - y|.$$

For a word $w = (i_1, \dots, i_k)$, write $H_w = h_{i_1} \circ \cdots \circ h_{i_k}(H)$ and define

$$K = \bigcap_{k \ge 1} \bigcup_{|w| = k} H_w.$$

Then $\mathcal{R}(K) = K$. The coding of K sends the $(1/2, 1/2)$-Bernoulli measure to a probability measure μ with $\mu(H_w) = 2^{-|w|}$. Put $\Delta = \operatorname{dist}(h_0(H), h_1(H)) > 0$. Distinct level-$k$ intervals are separated by at least $\Delta L^{-(k-1)}$. An interval J with $\Delta L^{-k} \le |J| < \Delta L^{-(k-1)}$ therefore meets at most one, giving

$$\mu(J) \le \Delta^{-s}|J|^s. \tag{3.3}$$

For $|J| \ge \Delta$ this follows from $\mu(J) \le 1$; singleton sets have mass zero.

Fix $k \ge 1$ and a pull-back W of V by f^{kr} that meets K. Choose $x \in W \cap K$ and any time $1 \le m' \le kr$ with $f^{m'}(W) \subset V$. Write $m' = qr + j$, where $0 \le j < r$.

If $j > 0$, then $\mathcal{R}^q(x) \in H$ and (3.2) gives $f^j(H) \subset V$. The level-q interval H_w containing x satisfies

$$\mathcal{R}^q(H_w) = H, \qquad f^{m'}(H_w) \subset V.$$

For $q = 0$, use $H_w = H$. Since H_w meets W, it lies in the component of $f^{-m'}(V)$ containing W. Some $y \in H_w$ satisfies $\mathcal{R}^q(y) = c$, and $Df^{m'}(y) = 0$.

If $j = 0$, then $q \ge 1$. Use the level-$(q-1)$ interval containing x. Its image under $f^{m'}$ is $\mathcal{R}(H) \subset V$, and it contains a point whose image under $\mathcal{R}^{q-1}$ is c. Again $Df^{m'}$ vanishes in the relevant component. The pull-back W is therefore bad.

Since $f^{kr}(K) = K \subset V$, these bad pull-backs cover K. By (3.3),

$$1 \le \Delta^{-s} \sum_{W \in \mathfrak{B}_{kr}(V)} |W|^s.$$

This proves (3.1). For $0 < t \le s$, the inequalities $d_V(W) \ge 1$ and $|W| \le 1$ give $S_V(t) = \infty$. □

The exponent in Proposition 3.1 is independent of V; the return time and the positive lower bound in (3.1) need not be. No first-return or restrictive-interval assumption is used.

4. Critical-orbit records and shrinking

We work temporarily with a continuous piecewise monotone map $g : I \to I$. Branches are restrictions to closed partition intervals. Let Γ be its finite partition set, containing $0, 1$ and a distinguished point c. Assume that g is C^1 with nonzero derivative on each component of $I \setminus \Gamma$, and that every branch except the two adjacent to c maps onto I. Suppose

$$g(c + x) = v - \alpha x^2 \qquad (|x| \le u_0), \tag{4.1}$$

where $[c - u_0, c + u_0] \subset (0, 1)$ contains no other point of Γ. On the open branches assume

$$|g'| \le L_0, \qquad |g'| \ge 256 \quad \text{if } |x - c| \ge u_0, \tag{4.2}$$

and the distortion bound

$$\bigl|\log |g'(x)| - \log |g'(y)|\bigr| \le C_d \frac{|x - y|}{\operatorname{dist}(x, \Gamma)} \tag{4.3}$$

whenever x, y lie in the interior of one branch and $|x - y| \le \operatorname{dist}(x, \Gamma)/2$.

Suppose $b_j = g^j(c) \notin \Gamma$ for $j \ge 1$, and define

$$D_0 = 1, \quad D_j = |Dg^j(v)|, \quad d_j = \operatorname{dist}(b_j, \Gamma), \quad a_j = \frac{D_{j-1}}{d_j}, \quad Q_m = \max_{1 \le j \le m} a_j.$$

The sum $\sum_{j=1}^m a_j$ is the derivative–distance quantity used in telescope distortion estimates; see Shen [26, §2.4, (2.8), and Lemma 2.1]. Here distances are measured to the full partition set Γ. We will maintain the bounds

$$\sum_{j=1}^{m} a_j \le C_{\mathrm{rec}} Q_m \qquad (m \ge 1), \tag{4.4}$$

$$\alpha D_{j-1} d_j \ge K \qquad (j \ge 1), \tag{4.5}$$

$$Q_m \ge e^{\sqrt{m}} \qquad (m \ge 1). \tag{4.6}$$

The first controls accumulated distortion, the second converts a stopping threshold into expansion, and the third bounds the stopping time. These conditions concern the running maximum Q_m; they do not require pointwise growth of D_m.

4.1. **A binding estimate.**

We use the standard binding argument; compare Bruin–Luzzatto–van Strien [2, §2.2, Lemmas 2.5–2.6]. We include the proof for the present piecewise-smooth setting and the stopping rule defined by Q_m.

Lemma 4.1. *Assume* (4.1)–(4.6). *Choose* $\eta > 0$ *so that*

$$8C_d C_{\mathrm{rec}} \eta < \log 2, \qquad 8\eta < \tfrac{1}{2}, \qquad \alpha u_0^2 a_1 < \eta, \qquad \frac{\sqrt{\eta K}}{L_0} > 128. \tag{4.7}$$

For $0 < u \le u_0$, *set*

$$b(u) = \min\{m \ge 1 : \alpha u^2 Q_m > \eta\}.$$

Then $2 \le b(u) \le C(1 + \log u^{-1})^2$. *If* E *is a nondegenerate closed interval on one side of* c *contained in*

$$\{x : u \le |x - c| \le 2u\} \cap [c - u_0, c + u_0],$$

then $g^{b(u)-1}$ *is a diffeomorphism on* E *and*

$$|Dg^{b(u)-1}(x)| > 128 \qquad (x \in E).$$

Proof. The bound on $b(u)$ follows from (4.6); the lower bound follows from $\alpha u_0^2 a_1 < \eta$. Write $b = b(u)$. Let J_1 be the interval spanned by v and $g(E)$, so $|J_1| \le 4\alpha u^2$.

Inductively, for $1 \le j < b$, the interval $g^{j-1}(J_1)$ contains b_j, stays in its branch, and has length at most

$$8\alpha u^2 D_{j-1}. \tag{4.8}$$

Indeed, the accumulated logarithmic distortion through the preceding images is at most

$$8C_d\alpha u^2 \sum_{i<b} a_i \le 8C_d C_{\mathrm{rec}}\eta < \log 2.$$

The derivative ratio is therefore between $1/2$ and 2, yielding (4.8). Its ratio to d_j is at most $8\alpha u^2 a_j \le 8\eta < 1/2$, which also proves that the new image stays in the branch. The same inequalities start the induction at $j = 1$.

It follows that g^{b-1} is monotone on E and

$$|Dg^{b-1}(x)| \ge \alpha u D_{b-2}.$$

By minimality of b, the new maximum is $Q_b = a_b$, and

$$\alpha u^2 \frac{D_{b-1}}{d_b} > \eta.$$

Multiplication by (4.5) at time b gives $(\alpha u D_{b-1})^2 > \eta K$. Hence

$$\alpha u D_{b-2} \ge \frac{\alpha u D_{b-1}}{L_0} > \frac{\sqrt{\eta K}}{L_0} > 128.$$

□

4.2. **Covering time and pull-backs.** For a surjection $g : I \to I$ and a nondegenerate closed interval J, let

$$\tau_g(J) = \min\{n \ge 0 : g^n(J) = I\},$$

with value ∞ if the set is empty.

Proposition 4.2. *Under the hypotheses of Lemma 4.1, there is $C > 0$ such that*

$$\tau_g(J) \le C(1 + \log|J|^{-1})^3 \tag{4.9}$$

for every nondegenerate closed interval $J \subset I$.

Proof. If J contains a full branch, then $g(J) = I$. Otherwise it meets at most four branches in positive length: the two end pieces and at most two complete non-full branches. Select a branch piece of length at least $|J|/4$, and split it at the boundary of $(c - u_0, c + u_0)$ if necessary.

One obtains a closed subinterval E of length at least $|J|/32$ either outside this neighborhood, in the interior of a branch, or in a one-sided annulus as in Lemma 4.1, with $u \ge |J|/32$. In the second case, take the half farther from c of the selected inside piece, and then a closed subinterval of half its length. In the first case, take an interior closed subinterval of the outside piece.

By (4.2) or Lemma 4.1, in at most $C'(1 + \log|J|^{-1})^2$ iterates the image of E has length at least $4|J|$. Repeat with this image interval. After at most $O(1 + \log|J|^{-1})$ repetitions a full branch must occur, since otherwise lengths would exceed 1. The successive intervals are no shorter than the first, so summing the time bounds proves (4.9). □

Lemma 4.3. *Suppose a continuous surjection $f : I \to I$ satisfies* (4.9)*. Then f is topologically exact, $\mathcal{J}(f) = I$, and every component W of $f^{-m}(B(y, 1/10))$ satisfies* (1.1) *for suitable $C, \kappa > 0$.*

Proof. Every nonempty open set contains a nondegenerate closed interval, so f is exact. Given x and an arbitrarily small interval containing it, some iterate maps that interval onto I; one point of it has image at distance at least $1/2$ from the image of x. Thus the iterates are not equicontinuous at x.

Choose a closed $J \subset W$ with $|J| \ge |W|/2$. Since $f^m(J)$ lies in a proper ball and f is onto, $\tau_f(J) > m$. Consequently

$$m < C\big(1 + \log(2/|W|)\big)^3,$$

and solving for $|W|$ proves the assertion. □

5. Construction of the interval map

5.1. **A fixed auxiliary model.** Fix $M = 1024$, $L_0 = 3M$, $C_d = 100$, and $C_{\mathrm{rec}} = 20$. Choose the remaining constants in the following order:

$$0 < \eta < \min\left\{\frac{1}{16}, \frac{\log 2}{16 C_d C_{\mathrm{rec}}}\right\}, \qquad K > \max\left\{1, \frac{(128 L_0)^2}{\eta}\right\},$$
$$a_- = 100K, \qquad a_+ = a_-^8, \tag{5.1}$$
$$\alpha > \max\{200M^2, M^3/\eta, 40MK\}, \qquad u_0 = \frac{M}{2\alpha}.$$

These constants remain fixed throughout the construction.

Lemma 5.1. *There are points $c, p_0, v \in (0,1)$ and a continuous map $g_0 : I \to I$ with partition $\Gamma = \{j/M : 0 \le j \le M\}$ having the following properties.*

(i) *The point c belongs to Γ, and*
$$\frac{1}{4} < p_0 < \frac{1}{3}, \qquad c - p_0 > \frac{1}{2}, \qquad c < \frac{4}{5}, \qquad \frac{89}{100} < v < \frac{9}{10}.$$
Moreover,
$$d_0 := \operatorname{dist}(p_0, \Gamma) = \operatorname{dist}(v, \Gamma), \qquad \frac{1}{4M} < d_0 < \frac{1}{3M}.$$

(ii) *The map g_0 is C^3 except at the interior points of $\Gamma \setminus \{c\}$, where it has fixed linear corners. Near the endpoints it is affine. It satisfies* (4.1)*, and $c \mapsto v \mapsto p_0 \mapsto p_0$. All branches except the two adjacent to c are full.*

(iii) *Put $E_* = [v - d_0/4, v + d_0/4]$. Every map in*
$$\mathcal{U} = \{g_0 + h : h \in C^3(I),\ \operatorname{supp} h \subset E_*,\ \|h\|_{C^3} < 1\} \tag{5.2}$$
has the same branch structure, satisfies (4.1)*,* (4.2)*, and* (4.3)*, and agrees with g_0 near Γ and the endpoints.*

(iv) *Two full affine branches ℓ_+, ℓ_-, of slopes $M, -M$, are fixed throughout $\mathcal{U}$, with $\ell_+(p_0) = p_0$. Their inverse branches h_+, h_- map $[1/10, 9/10]$ into itself. The domains of ℓ_+ and ℓ_-, as well as the quadratic neighborhood of c, are disjoint from E_*.*

For g_0, the three inequalities (4.4)–(4.6) *are strict.*

Proof. Let $T : I \to I$ be the full-branch sawtooth map of slope magnitude M: on $[j/M, (j+1)/M]$ it is $Mx - j$ for even j and $j + 1 - Mx$ for odd j. Choose
$$c = \frac{769}{M}, \qquad p_0 = \frac{256}{M - 1}, \qquad v = \frac{920 + p_0}{M}.$$
Then $T(v) = T(p_0) = p_0$, and the geometric inequalities of the lemma hold, with $d_0 = p_0/M$. Reserve the two affine branches
$$\ell_+(x) = Mx - 256, \qquad \ell_-(x) = 128 - Mx,$$
whose inverses map $[1/10, 9/10]$ into itself. The two affine branch domains are disjoint from $E_* = [v - d_0/4, v + d_0/4]$ and from the interval to be modified at c.

Put $b = M^{-1}$, $\varepsilon_0 = b/100$, and
$$\vartheta(z) = 140 \int_0^z t^3 (1 - t)^3 \, \mathrm{d}t.$$
Direct differentiation and integration give
$$\vartheta'(z) = 140 z^3 (1 - z)^3, \quad \max_{[0,1]} \vartheta' = \frac{35}{16}, \quad \int_0^1 \vartheta(z) \, \mathrm{d}z = \frac{1}{2}, \quad \int_0^1 z\vartheta(z) \, \mathrm{d}z = \frac{13}{36}. \tag{5.3}$$

The first three derivatives of ϑ vanish at both endpoints. Since $\alpha > 200M^2$, we have $2u_0 < \varepsilon_0$. Define

$$s_0 = \frac{v - 3M\varepsilon_0 + (13/36)Mu_0}{97\varepsilon_0}.$$

Then $M/2 < s_0 < M$: in the integral below the choices $s_0 = M/2$ and $s_0 = M$ give values less than 0.515 and greater than 0.999, respectively, whereas $0.89 < v < 0.9$.

For $0 \le u \le b$, let

$$q(u) = \begin{cases} 2\alpha u, & 0 \le u \le u_0, \\ M[1 + z(1 - \vartheta(z))], & u_0 \le u \le 2u_0, \\ M, & 2u_0 \le u \le \varepsilon_0, \\ M + (s_0 - M)\vartheta((u - \varepsilon_0)/\varepsilon_0), & \varepsilon_0 \le u \le 2\varepsilon_0, \\ s_0, & 2\varepsilon_0 \le u \le 98\varepsilon_0, \\ s_0 + (M - s_0)\vartheta((u - 98\varepsilon_0)/\varepsilon_0), & 98\varepsilon_0 \le u \le 99\varepsilon_0, \\ M, & 99\varepsilon_0 \le u \le b, \end{cases}$$

where $z = (u - u_0)/u_0$ in the second line. At u_0 the second expression agrees through order three with $2\alpha u$; at $2u_0$ it agrees through order three with M. The remaining joins follow from the endpoint derivatives of ϑ. Thus q is C^3. Equation (5.3) yields

$$\int_0^b q(u)\,\mathrm{d}u = 3M\varepsilon_0 + 97s_0\varepsilon_0 - \frac{13}{36}Mu_0 = v. \tag{5.4}$$

Define

$$g_0(c \pm u) = v - \int_0^u q(t)\,\mathrm{d}t \quad (0 \le u \le b), \qquad g_0 = T \quad \text{outside } [c - b, c + b].$$

The central expression is exactly $v - \alpha u^2$. Near b, $q = M$ and (5.4) makes the modified peak agree with the original affine branches. All other corners and both endpoint germs are unchanged.

We verify the distortion bound. On the modified branches,

$$\left|\frac{g_0''(c \pm u)}{g_0'(c \pm u)}\right| = \frac{|q'(u)|}{q(u)}.$$

On the quadratic part, multiplication by the distance to Γ gives at most one. On the first transition,

$$\frac{u|q'(u)|}{q(u)} = \frac{(1 + z)|1 - \vartheta(z) - z\vartheta'(z)|}{1 + z(1 - \vartheta(z))} < 5.$$

If the expression inside the absolute value is nonnegative, the quotient is at most two; otherwise it is at most $2 \max \vartheta' = 35/8$. On each of the other transitions, $\operatorname{dist}(c \pm u, \Gamma) \le 2\varepsilon_0$, $q \ge M/2$, and $|q'| \le (M/(2\varepsilon_0)) \max \vartheta'$, so the same bound $35/8$ applies. Consequently

$$\operatorname{dist}(x, \Gamma)\left|\frac{g_0''(x)}{g_0'(x)}\right| < 5 \tag{5.5}$$

on every open branch.

The support E_* lies inside the original increasing branch through v. A perturbation as in (5.2) keeps its endpoint values and has $M - 1 \le g' \le M + 1$ and $|g''| \le 1$ there. Hence it creates no new turn and still maps into I. Elsewhere the map is unchanged. In particular, all full branches remain full and the distinguished quadratic germ is fixed.

The bound (5.5) remains valid throughout $\mathcal{U}$. If $|x - y| \le \operatorname{dist}(x, \Gamma)/2$, then the intervening segment has distance at least $\operatorname{dist}(x, \Gamma)/2$ from Γ. Integrating (5.5) along it proves (4.3), with $C_d = 100$. Finally $q \le 2M$, and $q \ge M/2$ outside the quadratic core. Together with the perturbation bounds this proves (4.2) with $L_0 = 3M$.

Along the distinguished critical orbit, $D_j = M^j$ and $d_j = d_0$. Thus $\sum_{j\le m} a_j < 2Q_m$; the product and growth inequalities follow from (5.1). □

The distance on $\mathcal{U}$ is the C^3 norm of the difference of the perturbations. The maps themselves have the same fixed corners. Let $\mathcal{G} \subset \mathcal{U}$ consist of maps whose distinguished critical orbit avoids Γ at all positive times, eventually lands at p_0, and satisfies

$$\sum_{j=1}^{m} a_j < 20Q_m, \qquad \alpha D_{j-1} d_j > K, \qquad Q_m > e^{\sqrt{m}} \quad (m, j \ge 1). \tag{5.6}$$

In particular, $g_0 \in \mathcal{G}$.

5.2. **The insertion proposition.**

Proposition 5.2. *Let $g \in \mathcal{G}$, let O be a neighborhood of g in $\mathcal{U}$, and let $\varepsilon, \delta > 0$ and $R \ge 1$. There exist an integer $r > R$, a map $\widetilde{g} \in \mathcal{G} \cap O$ with $\|\widetilde{g} - g\|_{C^3} < \varepsilon$, and a closed interval H with $c \in \operatorname{int} H$, such that*

$$H \cup \widetilde{g}^r(H) \subset B(c, \delta). \tag{5.7}$$

There also exist disjoint closed intervals $K_0, K_1 \subset \operatorname{int} H$ and a neighborhood $\widetilde{O}$ of $\widetilde{g}$ in $\mathcal{U}$ such that every $\widehat{g} \in \widetilde{O}$ satisfies

$$H \cup \widehat{g}^r(H) \subset B(c, \delta)$$

and

$$H \subset \operatorname{int} \widehat{g}^r(K_i), \qquad 4 < |D\widehat{g}^r| < 8\sqrt{a_+} \quad \text{on } K_i, \qquad i = 0, 1. \tag{5.8}$$

In particular, $\widehat{g}^r$ has two branches mapping diffeomorphically onto H.

The proof has three parts: realization and landing, uniform block estimates, and persistence. At each stage, the old map, the finite prefix, the cutoff, and any protected finite-time data are fixed before N is chosen. Constants $C(g)$ may depend on these data, but not on N, $a \in [a_-, a_+]$, or the return index; all limits are uniform in a.

5.3. **Realization and landing.** Fix $g \in \mathcal{G}$. By extending its fixed-point tail, choose $P \ge 3$ so that

$$g^P(c) = p_0, \qquad \sum_{j=1}^{P} a_j < 2a_P, \qquad D_{P-1} > e^{2\sqrt{P}+10}. \tag{5.9}$$

Indeed, along that tail a_j grows by a factor M at each step, and the ratio of the partial sum to its last term tends to $M/(M-1) < 2$.

Lemma 5.3. *There are a fixed C^3 cutoff χ supported in $\operatorname{int} E_*$, an integer $e \in \{0, 1\}$, and $N_0(g)$ with the following properties. For every $N \ge N_0(g)$ and $a \in [a_-, a_+]$, there are continuous choices $t_N(a)$, $\sigma_{N,a} > 0$, and $u_{N,a} > 0$. Put*

$$g_{N,a} = g + t_N(a)\chi, \qquad r = P + N + e, \qquad \mathcal{R}_{N,a} = g_{N,a}^r.$$

Then $g_{N,a} \in \mathcal{U}$, $t_N(a) = O_g(M^{-N})$, $\sigma_{N,a} \asymp_g M^{-N}$, and, writing $\sigma = \sigma_{N,a}$ and $u = u_{N,a}$,

$$\mathcal{R}_{N,a}(c + x) = c + a\sigma - \frac{x^2}{\sigma}\big(1 + \epsilon_{N,a}(x)\big), \tag{5.10}$$

$$D\mathcal{R}_{N,a}(c + x) = -\frac{2x}{\sigma}\big(1 + \widetilde{\epsilon}_{N,a}(x)\big) \qquad (|x| \le u), \tag{5.11}$$

where

$$|\epsilon_{N,a}(x)| + |\widetilde{\epsilon}_{N,a}(x)| \le C(g)x^2, \qquad |\epsilon_{N,a}(x)|, |\widetilde{\epsilon}_{N,a}(x)| \le \frac{1}{100}. \tag{5.12}$$

Moreover, $u < u_0$, $\mathcal{R}_{N,a}(c \pm u) = p_0$, and

$$\frac{u^2}{\sigma}\big(1 + \epsilon_{N,a}(u)\big) = c - p_0 + a\sigma, \qquad u^2 = (c - p_0 + o(1))\sigma. \tag{5.13}$$

For $1 \le s \le P$ and $|y - c| \le u$, the finite prefix satisfies

$$\begin{aligned} \tfrac{1}{2} d_s(g) &\le \operatorname{dist}(g^s_{N,a}(y), \Gamma) \le 2 d_s(g), \\ \tfrac{1}{2} D_{s-1}(g) &\le |D g^{s-1}_{N,a}(g_{N,a}(y))| \le 2 D_{s-1}(g). \end{aligned} \tag{5.14}$$

There is also a choice $a_N \in (a_-, a_+)$ and an integer $k = k(N) = O_g(\log N)$ for which

$$\mathcal{R}^k_{N,a_N}(c) = c - u_{N,a_N}, \qquad \mathcal{R}^{k+1}_{N,a_N}(c) = p_0. \tag{5.15}$$

The orbit of c under g_{N,a_N} avoids Γ at every positive time.

Proof. The old orbit does not revisit v after time one: a revisit would make v periodic, contrary to eventual landing at $p_0 \ne v$. Choose a small neighborhood $E \ni v$ with $\overline{E} \subset \operatorname{int} E_*$ that avoids $g^j(c)$ for $2 \le j \le P$. Let χ be supported in E and equal to one near v. Write $g_t = g + t\chi$. On a fixed neighborhood of v, for small t,

$$G_t(z) := g_t^{P-1}(z) = g^{P-2}(g(z) + t), \qquad G_0(v) = p_0, \qquad \partial_t G_t(v)|_{t=0} \ne 0. \tag{5.16}$$

This is a family of local C^3 diffeomorphisms.

Choose $e \in \{0, 1\}$ so that the affine composition

$$\mathcal{L}_N = \begin{cases} \ell_+^N, & e = 0, \\ \ell_- \circ \ell_+^N, & e = 1 \end{cases} \quad \text{satisfies} \quad \mathcal{L}_N' G_0'(v) > 0.$$

For small t, set $\sigma(t) = (\alpha \mathcal{L}_N' G_t'(v))^{-1} > 0$ and solve

$$G_t(v) = \mathcal{L}_N^{-1}(c + a\sigma(t)). \tag{5.17}$$

Writing $h_e = \mathrm{id}$ for $e = 0$ and $h_e = h_-$ for $e = 1$, the inverse is

$$\mathcal{L}_N^{-1}(z) = p_0 + M^{-N}(h_e(z) - p_0). \tag{5.18}$$

The difference of the two sides of (5.17) is $O_g(M^{-N})$ at $t = 0$. Its derivative in t tends uniformly to the nonzero derivative in (5.16), since the derivative of the right side is $O_g(M^{-2N})$. On a sufficiently large fixed multiple of $[-M^{-N}, M^{-N}]$ the difference is strictly monotone and has opposite signs at the endpoints. Thus it has a unique small zero $t_N(a)$, continuous in a, and

$$|t_N(a)| \le C(g) M^{-N}, \qquad 0 < c(g) M^{-N} \le \sigma(t_N(a)) \le C(g) M^{-N}. \tag{5.19}$$

Initially regard $\mathcal{L}_N(G_{t_N(a)}(v - \alpha x^2))$ as a local expression. The fundamental theorem of calculus yields (5.10)–(5.11), with

$$\begin{aligned} 1 + \epsilon_{N,a}(x) &= \int_0^1 \frac{G'_{t_N(a)}(v - s\alpha x^2)}{G'_{t_N(a)}(v)}\, \mathrm{d}s, \\ 1 + \widetilde{\epsilon}_{N,a}(x) &= \frac{G'_{t_N(a)}(v - \alpha x^2)}{G'_{t_N(a)}(v)}. \end{aligned}$$

The first bound in (5.12) follows from a uniform upper bound for $|G_t''|$ and a positive lower bound for $|G_t'|$ on the fixed neighborhood. For small $x > 0$ the expression is strictly decreasing. Evaluating at $x^2 = (c - p_0)\sigma/2$ and $x^2 = 2(c - p_0)\sigma$ gives a unique root $u > 0$ at level p_0, with (5.13). The expression is even in x.

For $|x| \le u$, its image lies in $[p_0, c + a\sigma] \subset (0, 1)$. Applying the inverse branches in (5.18) verifies every prescribed affine step. The preceding finite prefix stays in the old monotonicity intervals by

continuity. Thus the local expression equals $g^r_{N,a}(c+x)$ throughout $[-u,u]$. Since $u \to 0$, increasing $N_0(g)$ gives $u < u_0$, the one-percent bounds in (5.12), and the uniform finite-prefix comparisons (5.14).

It remains to select the landing parameter. As long as successive returns stay in $[c-u, c+u]$, write

$$\mathcal{R}^j_{N,a}(c) = c + \sigma z_j.$$

Here $z_1 = a$, and $a_- > 100$ together with (5.12) gives

$$\tfrac{1}{2}|z_j|^2 \le |z_{j+1}| \le 2|z_j|^2, \qquad z_j < 0 \quad (j \ge 2). \tag{5.20}$$

The estimate also applies to the first exiting image. It follows that

$$2^{j-1}(\log a - \log 2) \le \log|z_j| \le 2^{j-1}(\log a + \log 2). \tag{5.21}$$

The exit threshold is

$$T_a := \log(u_{N,a}/\sigma_{N,a}) = \tfrac{1}{2}N\log M + O_g(1).$$

In particular $a\sigma < u$ for all parameters when N is large.

Let k be the first strict exit time for $a = a_+$. It is finite by (5.20), and

$$2^{k-2}(\log a_+ - \log 2) \le T_{a_+}, \qquad k = O_g(\log N).$$

Since $a_+ = a_-^8$,

$$q_* := \frac{2(\log a_- + \log 2)}{\log a_+ - \log 2} < \frac{1}{2}, \qquad T_{a_-}/T_{a_+} \longrightarrow 1.$$

If the orbit for a_- were to exit by time k, the upper bound in (5.21) at its first exit would give $\log|z_j| \le q_* T_{a_+} < T_{a_-}$, a contradiction. The same bound shows that its first k returns lie strictly inside the interval.

For this fixed N and k, let

$$\mathcal{E}_N = \{a \in [a_-, a_+] : \exists\, 1 \le j \le k,\ |\mathcal{R}^j_{N,a}(c) - c| > u_{N,a}\}.$$

This set is relatively open, contains a left neighborhood of a_+, and misses a right neighborhood of a_-. Its leftmost boundary point $a_N = \inf \mathcal{E}_N$ lies in (a_-, a_+) and is not in $\mathcal{E}_N$. At least one of the first k returns then meets the boundary. While the input remains in the interval,

$$\mathcal{R}_{N,a}(x) \le \mathcal{R}_{N,a}(c) = c + a\sigma < c + u,$$

so only the left endpoint can be reached. A boundary hit before time k would be followed by the value p_0, which is outside $[c-u, c+u]$, contradicting $a_N \notin \mathcal{E}_N$. This proves (5.15).

The positive-time orbit avoids Γ. During each finite prefix this follows from (5.14); during each affine segment it follows from the realized itinerary; and the central returns are distinct from c by (5.20). After landing the orbit stays at p_0. □

5.4. **Uniform block estimates.**

Lemma 5.4. *For all sufficiently large N, the map $\widetilde{g}_N = g_{N,a_N}$ of Lemma 5.3 belongs to $\mathcal{G}$. With*

$$T_{\text{land}} = (k+1)r,$$

it also satisfies

$$\sum_{i=1}^{T_{\text{land}}} a_i < 2a_{T_{\text{land}}}, \qquad D_{T_{\text{land}}-1} > e^{2\sqrt{T_{\text{land}}}+10}. \tag{5.22}$$

Proof. All quantities without a superscript refer to $\widetilde{g}_N$; old quantities refer to g. Abbreviate $T = T_{\text{land}}$, $\sigma = \sigma_{N,a_N}$, and $u = u_{N,a_N}$. For $1 \le j \le k$ put

$$y_j = \widetilde{g}_N^{jr}(c), \quad \delta_j = |y_j - c|, \quad Z_j = \frac{D_{jr-1}}{\delta_j}, \quad \Pi_j = \alpha D_{jr-1}\delta_j.$$

Since $u < 1/(2M)$ for large N, $d_{jr} = \delta_j$. At the first return,

$$D_{r-1} = \frac{1}{\alpha\sigma}, \qquad Z_1 = \frac{1}{\alpha a_N \sigma^2}, \qquad \Pi_1 = a_N. \tag{5.23}$$

The times in each later block are partitioned into the prefix $jr+1, \dots, jr+P$, the affine segment $jr+P+1, \dots, (j+1)r-1$, and the endpoint $(j+1)r$. The initial block and the final fixed-point tail will be treated separately.

Central returns and finite prefixes. For $j < k$, formulas (5.10)–(5.11) give

$$\frac{Z_{j+1}}{Z_j} = \frac{2(1+\widetilde{\epsilon}_{N,a_N}(y_j - c))}{1+\epsilon_{N,a_N}(y_j-c) - a_N\sigma^2/\delta_j^2} \geq \frac{198}{101} > \frac{3}{2}.$$

The denominator is positive by (5.20). Hence

$$\sum_{i=1}^{j} Z_i \leq 3Z_j. \tag{5.24}$$

The derivative magnitude at y_j equals $2\alpha\delta_j$. Using (5.14),

$$\sum_{s=1}^{P} a_{jr+s} \leq 8\alpha\delta_j^2 \left(\sum_{s=1}^{P} a_s^{\mathrm{old}}\right) Z_j \leq \theta Z_j, \qquad \theta = \frac{1}{100}. \tag{5.25}$$

The last inequality holds simultaneously for every j, since $\delta_j \leq u$ and $u^2 = O_g(\sigma)$. The old finite sum is fixed before N is chosen.

Affine segments. Every point of a prescribed affine segment has distance from Γ between

$$d_* := \frac{1}{20M} \quad \text{and} \quad d^* := \frac{1}{2M}.$$

Indeed, its terminal image lies in $[p_0, c + a_N\sigma] \subset [1/5, 4/5]$. All intervening inverse images under $h_\pm$ lie in $[1/10, 9/10]$. The distance of an affine inverse image to the ends of its branch is the distance of its image to $\{0,1\}$ divided by M, giving even the lower bound $1/(10M)$.

Cumulative derivatives multiply by M at each affine step. If the terminal time of a complete segment is t, its terms excluding the terminal point satisfy

$$\sum_{i \text{ in the segment}} a_i \leq \frac{D_{t-1}}{(M-1)d_*} < 21 D_{t-1}. \tag{5.26}$$

If $t = (j+1)r$ is a central return, then $D_{t-1} = \delta_{j+1} Z_{j+1}$, so this sum is at most θZ_{j+1} for large N. For a segment stopped at time l before its terminal return,

$$\sum_{i \text{ in the partial segment}} a_i \leq \frac{M}{M-1}\frac{d^*}{d_*} a_l < 11 a_l. \tag{5.27}$$

The constants in (5.26)–(5.27) depend only on the fixed affine geometry.

The initial finite prefix has bounded sum as $N \to \infty$. Thus, by (5.23) and (5.26),

$$\frac{\sum_{i<r} a_i}{Z_1} \leq C(g)\alpha a_+ \sigma^2 + 21 a_+ \sigma \longrightarrow 0.$$

Choose N so that this ratio is at most θ. Combining (5.24), (5.25), and the complete-segment bound yields

$$\sum_{i=1}^{jr} a_i < 4 Z_j \qquad (1 \leq j \leq k). \tag{5.28}$$

Inside a subsequent prefix the sum increases by at most θZ_j. Inside a partial affine segment, (5.27) then gives

$$\sum_{i=1}^{m} a_i < (4+\theta)Z_j + 11a_m < 16Q_m.$$

For the initial prefix, all old strict inequalities up to time P, as well as $\sum_{i\le P} a_i < 2a_P$ and $D_{P-1} > e^{2\sqrt{P}+10}$, hold for large N by finite-time continuity. Consequently the initial partial affine segment has sum less than $13Q_m$.

Landing and its tail. The last central point is $c-u$. Equations (5.11) and (5.13) give

$$\frac{D_{T-1}}{Z_k} = 2(c-p_0+a_N\sigma)\frac{1+\widetilde{\epsilon}_{N,a_N}(u)}{1+\epsilon_{N,a_N}(u)} \ge c-p_0. \tag{5.29}$$

The errors are even, so the formula applies at $-u$. Since $a_T = D_{T-1}/d_0$, (5.26) and (5.28) imply

$$\frac{\sum_{i=1}^{T} a_i}{a_T} \le 1 + \frac{(4+\theta)d_0}{c-p_0} + 21d_0 < 2. \tag{5.30}$$

Here $c-p_0 > 1/2$, $d_0 < 1/(3M)$, and $M = 1024$. For $s \ge 0$ the orbit is fixed at p_0 and $a_{T+s} = M^s a_T$. The ratio of the sum through time $T+s$ to its last term is less than

$$2M^{-s} + \frac{1-M^{-s}}{1-M^{-1}} \le 2.$$

Together with the estimates above, this proves the first inequality of (5.6) at every time.

The product bound. From (5.23), $\Pi_1 = a_N \ge a_-$. Equations (5.20) and (5.11) give

$$\frac{\Pi_{j+1}}{\Pi_j} = |D\mathcal{R}_{N,a_N}(y_j)|\frac{\delta_{j+1}}{\delta_j} \ge \frac{99}{100}(\delta_j/\sigma)^2 > 1.$$

Thus $\Pi_j \ge a_-$. On the following finite prefix, (5.14) gives

$$\alpha D_{jr+s-1} d_{jr+s} \ge \tfrac{1}{2}\Pi_j\big(\alpha D^{\mathrm{old}}_{s-1} d^{\mathrm{old}}_s\big) > K \qquad (1 \le s \le P).$$

The cumulative derivative at the entrance to such an affine segment is at least $\Pi_j D^{\mathrm{old}}_{P-1} > 1$. At the entrance to the initial affine segment it is also greater than one, by the preserved inequality (5.9). Subsequent affine derivatives multiply by M. Since $\alpha d_* > K$, all affine times satisfy the product bound. It also holds at the landing point and on its tail, where the distance is $d_0 > d_*$. The initial finite prefix was already protected by continuity. This proves the second inequality of (5.6).

Growth of the running record. By (5.19) and (5.23),

$$\log Z_1 = 2N\log M + O_g(1).$$

As $r = P+N+e$ and $2\log M > 4$, we may require

$$Z_1 > e^{4r}, \qquad T < r^2. \tag{5.31}$$

The second inequality follows from $k = O_g(\log N)$. Hence $Q_m \ge Z_1 > e^{\sqrt{m}}$ for $r \le m \le T$.

Before the first return, the first P strict inequalities persist. On the following affine segment,

$$D_{m-1} = D_{P-1}M^{m-P}, \qquad d_m \le \frac{1}{2M}.$$

The preserved bound $D_{P-1} > e^{2\sqrt{P}+10}$ implies $a_m > e^{\sqrt{m}}$; use $\sqrt{P+s} \le \sqrt{P} + s/(2\sqrt{P})$. Finally, (5.29) and (5.31) give

$$D_{T-1} \ge (c-p_0)Z_1 > e^{2\sqrt{T}+10}$$

for large N. On the fixed-point tail the same affine estimate applies. This proves the third inequality of (5.6) and, together with (5.30), proves (5.22). □

At the first new central return, $D_r = 2a_N$, whereas Z_1 has order σ^{-2}. The construction controls the running maximum Q_m, rather than imposing pointwise growth on all cumulative derivatives D_m.

5.5. **Persistent branches and proof of the insertion proposition.**

Lemma 5.5. *For all sufficiently large N, the map $\widetilde{g}_N$ has an interval $H_N \ni c$ satisfying*

$$H_N \subset B(c, 2\sqrt{a_+}\,\sigma), \qquad \widetilde{g}_N^r(H_N) \subset B(c, 2a_+\sigma), \tag{5.32}$$

where $\sigma = \sigma_{N,a_N}$. It has two branches over H_N with the persistent bounds (5.8)*. The containments in* (5.32) *persist for sufficiently small perturbations in $\mathcal{U}$.*

Proof. Equations (5.10)–(5.12) imply

$$\frac{\mathcal{R}_{N,a}(c + \sigma_{N,a} z) - c}{\sigma_{N,a}} \longrightarrow Q_a(z) := a - z^2$$

in C^1 on every fixed bounded interval, uniformly for $a \in [a_-, a_+]$. Put $h_a = \sqrt{a}+1$ and $J_a = [-h_a, h_a]$. The intervals

$$I_a^+ = \left[\sqrt{a - h_a - \tfrac{1}{4}}, \sqrt{a + h_a + \tfrac{1}{4}}\right], \qquad I_a^- = -I_a^+$$

are disjoint compact subintervals of $\operatorname{int} J_a$. Each maps onto $[-h_a - 1/4, h_a + 1/4]$ under Q_a, and

$$8 < |Q_a'| < 4\sqrt{a_+} \quad \text{on } I_a^+ \cup I_a^-.$$

Set $H_N = c + \sigma J_{a_N}$ and $K_i = c + \sigma I_{a_N}^{\pm}$. For large N, the images of these fixed intervals under $\widetilde{g}_N^r$ contain H_N in their interiors, and

$$6 < |D\widetilde{g}_N^r| < 6\sqrt{a_+} \quad \text{on } K_0 \cup K_1.$$

Also $Q_a(J_a) = [-2\sqrt{a} - 1, a]$, so (5.32) holds with strict margins.

The interval H_N is contained in $[c - u_{N,a_N}, c + u_{N,a_N}]$. Its positive-time images before time r avoid Γ by the realized itinerary and (5.14). On H_N itself the map is the fixed smooth quadratic map. Thus finite composition on these compact intervals depends continuously in C^1 on the perturbation in $\mathcal{U}$. The strict covering margins and the wider derivative bounds (5.8) persist, as do (5.32). Taking the inverse image of H_N inside each K_i supplies the exact onto branches. □

Proof of Proposition 5.2. For all sufficiently large N, Lemma 5.4 gives $\widetilde{g}_N \in \mathcal{G}$, and Lemma 5.5 gives the persistent return. Moreover,

$$\|\widetilde{g}_N - g\|_{C^3} \le |t_N(a_N)| \|\chi\|_{C^3} \longrightarrow 0, \qquad \sigma_{N,a_N} \longrightarrow 0, \qquad r \longrightarrow \infty.$$

Choose such an N so that $\widetilde{g}_N \in \mathcal{O}$, the perturbation is less than ε, $r > R$, and $2a_+\sigma_{N,a_N} < \delta$. The persistence neighborhood can then be restricted to retain (5.7). □

5.6. **The diagonal construction.**

Proposition 5.6. *There exists $g \in \mathcal{U}$ satisfying the hypotheses of Proposition 4.2, together with $r_n \to \infty$ and closed intervals $H_n \ni c$, such that*

$$c \in \operatorname{int} H_n, \qquad H_n \cup g^{r_n}(H_n) \subset B(c, 2^{-n}). \tag{5.33}$$

Each g^{r_n} has two branches onto H_n, with derivatives between 4 and $8\sqrt{a_+}$ in absolute value.

Proof. Start with $g_0 \in \mathcal{G}$, $r_0 = 0$, and $\omega_0 = 1/4$. At stage n, apply Proposition 5.2 to g_{n-1} with perturbation size less than $\omega_{n-1}/4$, spatial radius 2^{-n}, and return time $r_n > r_{n-1} + 1$. All earlier strict covering and derivative conditions are retained by choosing the prescribed neighborhood within their protection neighborhoods.

After constructing g_n, choose $0 < \omega_n < \omega_{n-1}/4$ so that the ball of radius ω_n about g_n in $\mathcal{U}$ retains every protected return, including its strict coverings, derivative bounds, and spatial containment. Such

a radius exists by the finite-time persistence in Proposition 5.2. The preceding perturbation budget keeps g_n inside all earlier protected neighborhoods.

The perturbations satisfy

$$\sum_{n\geq 1}\|g_n - g_{n-1}\|_{C^3} < \omega_0/3 < 1.$$

Thus their sum defines $g \in \mathcal{U}$. For each fixed n, the remaining tail has norm less than $\omega_n/3$, so the nth return persists in the limit and (5.33) follows.

To pass the record estimates to g, fix $j \geq 1$. Every g_n has Lipschitz constant L_0, and its product bound gives

$$d_j(g_n) > \frac{K}{\alpha L_0^{j-1}}.$$

Finite-time iterates converge uniformly; explicitly,

$$\|g_n^j - g^j\|_\infty \leq \|g_n - g\|_\infty \sum_{i=0}^{j-1} L_0^i.$$

The limiting orbit therefore avoids Γ at every positive time. On each of these finitely many branches the maps converge in C^3, so the corresponding derivative products converge. Taking the limit in the finite sums, products, and maxima in (5.6) gives (4.4)–(4.6) for g.

The remaining hypotheses are uniform over $\mathcal{U}$. In particular,

$$\alpha u_0^2 a_1 = \frac{M^2}{4\alpha d_0} < \frac{M^3}{\alpha} < \eta,$$

and the other conditions in (4.7) follow from (5.1). □

5.7. **Smoothing and shrinking.** Take g from Proposition 5.6, and define

$$\Theta(t) = \frac{1-\cos(\pi t)}{2}, \qquad f = \Theta \circ g \circ \Theta^{-1}.$$

On $(0,1)$, Θ is an analytic diffeomorphism. At each unmodified corner $q \in \Gamma \setminus \{0,1,c\}$, the local expression for g is $M|t-q|$ or $1-M|t-q|$. Hence

$$f(\Theta(t)) - f(\Theta(q)) = \pm \sin^2\left(\frac{\pi M(t-q)}{2}\right).$$

The sine factor is a local coordinate with nonzero derivative, so these points become quadratic critical points. At $c_* := \Theta(c)$, use

$$\phi(x) = \sqrt{\alpha}\,(\Theta^{-1}(x) - c), \qquad \psi(y) = \Theta^{-1}(y) - v.$$

Equation (4.1) gives $|\psi(f(x))| = |\phi(x)|^2$.

Near the endpoints, g agrees with the original affine end branches, so

$$f(x) = \frac{1 - T_M(1-2x)}{2},$$

where T_M is the Chebyshev polynomial. Thus f is smooth there and

$$f(0) = f(1) = 0, \qquad Df(0) = M^2, \qquad Df(1) = -M^2.$$

Hence f is globally C^3, is boundary-anchored, and has precisely the $M-1$ quadratic critical points $\Theta(\Gamma \setminus \{0,1\})$.

Proposition 4.2 applies to g. Since Θ has Lipschitz constant $\pi/2$,

$$|\Theta^{-1}(J)| \geq \frac{2}{\pi}|J|$$

for every interval J. Applying (4.9) to $\Theta^{-1}(J)$ therefore gives

$$\tau_f(J) \le C(1 + \log|J|^{-1})^3.$$

Lemma 4.3 now proves exactness, $\mathcal{J}(f) = I$, and (1.1), with $\rho = 1/10$. Since

$$\sup_{m\ge 1} m^{\beta} e^{-\kappa m^{1/3}} < \infty \qquad (\beta > 0),$$

we have $\beta_{\max}(f) = \infty$.

5.8. **Periodic points and nice sets.** Exactness excludes attracting periodic orbits. Indeed, a small closed neighborhood of an attracting point is mapped into itself by its period iterate. If any iterate of that neighborhood were I, surjectivity would give the same at a later multiple of the period, a contradiction.

Suppose that z is a neutral periodic point of period q and put $\mathcal{R} = f^{2q}$. Then $\mathcal{R}(z) = z$ and $D\mathcal{R}(z) = (Df^q(z))^2 = 1$, so Taylor's theorem gives, near z,

$$|\mathcal{R}(x) - z| \le |x - z| + C|x - z|^2.$$

Choose $a > 0$ with $e^{2Ca} < 2$. For large m, take a one-sided interval $J_m \subset I$ of length a/m with endpoint z. Induction gives

$$\sup_{x\in J_m} |\mathcal{R}^j(x) - z| \le \frac{a}{m}\left(1 + \frac{2Ca}{m}\right)^j < \frac{2a}{m} \qquad (0 \le j \le m).$$

These inequalities keep the orbit in the neighborhood where the Taylor estimate is valid. Thus J_m lies in the component of $f^{-2qm}(B(z, 1/10))$ containing z. Its length a/m contradicts (1.1). Every periodic orbit is therefore hyperbolic repelling, and f belongs to the real class of [25, Section 2.1].

Lemma 5.7. *The map f has arbitrarily small nice sets containing its critical set.*

Proof. Periodic points are dense. Given a closed nondegenerate interval $J \subset (0, 1)$, exactness gives $f^n(J) = I$. At points of J mapped to 0 and 1, the function $f^n(x) - x$ has opposite signs. It therefore vanishes somewhere in J.

Choose periodic points on both sides of each critical point, as close to it as desired. Let P be the union of their full finite periodic orbits. No critical point belongs to P, because all periodic orbits are repelling. The components of $I \setminus P$ containing critical points form an open set V. Choosing the bounding points sufficiently close makes these components arbitrarily small and ensures that each contains one critical point. Since

$$\partial V \subset P, \qquad f(P) = P, \qquad P \cap V = \varnothing,$$

the niceness condition holds. □

5.9. **Completion of the theorem.** Set $\widehat{H}_n = \Theta(H_n)$. Equation (5.33) gives

$$\widehat{H}_n \longrightarrow \{c_*\}, \qquad f^{r_n}(\widehat{H}_n) \longrightarrow \{c_*\}.$$

On each of the two full branches,

$$Df^{r_n}(\Theta(x)) = \frac{\Theta'(g^{r_n}(x))}{\Theta'(x)} Dg^{r_n}(x).$$

Both x and $g^{r_n}(x)$ belong to H_n, and $\Theta'(c) > 0$. The quotient tends uniformly to one. After discarding finitely many terms, the branch derivatives lie between 2 and $L := 16\sqrt{a_+}$. Proposition 3.1 applies with

$$s_* = \frac{\log 2}{\log(16\sqrt{a_+})} > 0.$$

It yields (1.2) and $\delta_{\mathrm{bad}}(f) \ge s_*$. This completes the proof of Theorem A.

Every periodic orbit is repelling, but the periodic Lyapunov exponents have infimum zero. Indeed, an inverse branch of f^{r_n} on $\widehat{H}_n$ has a fixed point p_n. Its least period divides r_n, and

$$0 < \chi(p_n) = \frac{1}{r_n} \log |Df^{r_n}(p_n)| \le \frac{\log L}{r_n} \longrightarrow 0.$$

The argument gives no upper bound for $\delta_{\mathrm{bad}}(f)$ and does not claim that it is finite. The rate $1/3$ and the number of critical points have not been optimized. Unimodal, real-analytic, and rational realizations are not addressed here.

6. A cubic polynomial with prescribed critical passages

We now construct the polynomial used in Theorem B. The parameter argument prescribes two kinds of long blocks: one keeps the critical orbit away from its own critical point, and the other fixes the depth of its passages near the other critical point. We first carry out this argument in a normalized interval family and then pass to the quadratic lift.

6.1. **The parameter family.** Let $T(x) = x^3 - 3x$ and, for $0 \le \tau \le \tau_0$, put

$$A_\tau(x) = (4+\tau)x - 2 - \tau, \qquad B_\tau(y) = \frac{y - T(-2-\tau)}{2 - T(-2-\tau)}, \qquad P_\tau = B_\tau \circ T \circ A_\tau.$$

Here $\tau_0 > 0$ is small. The map P_τ sends $[0,1]$ into itself, fixes both endpoints, and has critical points $c_1(\tau) = (1+\tau)/(4+\tau)$ and $c_2(\tau) = (3+\tau)/(4+\tau)$. We have $P_\tau(c_1(\tau)) = 1$; write $v_\tau = P_\tau(c_2(\tau))$. The three monotonicity intervals are

$$I_1 = [0, c_1), \qquad I_2 = (c_1, c_2), \qquad I_3 = (c_2, 1],$$

with parameter dependence understood. Their orientation signs are $+1, -1, +1$, respectively. At $\tau = 0$ all three branches are full, $v_0 = 0$, and the fixed point $r_0 = 1/2$ has derivative -3. After decreasing τ_0, the fixed points $r_\tau \in I_2$ and 1 are repelling throughout the family.

We use the natural family studied in [14, Section 4.1]. We decrease τ_0 so that v_τ and $P_\tau(v_\tau)$ lie in I_1, the endpoints and the unique interior fixed point are repelling, and there is no two-cycle entirely in I_2. Finite admissible itinerary cylinders then have endpoints continuous in the parameter. These are relation (6) and Lemmas 3.1–3.3 of that paper; the realization and bracketing statements needed below are stated explicitly. The itinerary $\iota_\tau(x)$ records the successive symbols in $\{I_1, c_1, I_2, c_2, I_3\}$ and terminates at the first critical symbol, if there is one. Put $\underline{k}(\tau) = \iota_\tau(v_\tau)$. The order on itineraries is the signed lexicographic order for $I_1 < c_1 < I_2 < c_2 < I_3$: after a common interval word W, the next symbols are compared in the usual order when its orientation $\epsilon(W)$ is positive, and in the reverse order otherwise.

Let Σ consist of infinite interval words and finite interval words followed by a single critical symbol. We use the itinerary space

$$\Sigma_0 = \big(\Sigma \setminus (\sigma^-(I_1^\infty) \cup \sigma^-(I_3^\infty))\big) \cup \{I_1^\infty, I_3^\infty\},$$

where $\sigma^-(\underline{i})$ denotes the words eventually equal to $\underline{i}$; this is [14, equation (7)]. A word is minimal if it is no larger than any of its shifts. Every kneading itinerary is minimal. In particular, finite words ending at a critical symbol and words eventually equal to I_2^∞ belong to Σ_0.

Lemma 6.1. *The I_2^ℓ-cylinders shrink uniformly in τ to r_τ. For every finite interval word W, the parameter fiber of WI_2^∞ is finite.*

Proof. The endpoints of the cylinders are continuous in τ by [14, Lemma 3.3]. Their alternating subsequences are monotone, and their limiting endpoints are interchanged by P_τ. If the endpoints were distinct, they would form an I_2^∞ two-cycle, excluded by [14, Lemma 3.2]. Thus both limits are r_τ. Dini's theorem gives uniform convergence on $[0, \tau_0]$; this is also [14, equation (10)].

If $m = |W|$, a parameter in the fiber satisfies $P_\tau^m(v_\tau) = r_\tau$. The latter equation is real analytic on a neighborhood of $[0, \tau_0]$. Its left side minus its right side is not identically zero: at $\tau = 0$ it equals $-1/2$. It therefore has only finitely many zeros on this compact interval. □

We recall the three parameter inputs in the form needed here. The realization statement [14, Proposition 3.5] says that every minimal word in Σ_0 strictly between the kneadings at the endpoints of a parameter interval is realized in its interior. Next, suppose that $J = [\alpha, \beta]$ and W satisfy

$$\underline{k}(\alpha) \prec WI_2^\infty \prec \underline{k}(\beta), \qquad WI_2^\infty \text{ is minimal.} \tag{6.1}$$

For every sufficiently large u with $\epsilon(WI_2^u) = 1$, define the extremal parameters

$$\eta_u^- = \max\big(\underline{k}^{-1}(WI_2^u c_1) \cap (\alpha, \beta)\big),$$
$$\eta_u^+ = \min\big(\underline{k}^{-1}(WI_2^u c_2) \cap (\eta_u^-, \beta)\big).$$

By [14, Proposition 3.7 and Corollary 3.8] and Lemma 6.1, they exist, satisfy $\eta_u^- < \eta_u^+$, and converge to the same parameter $\eta^\infty \in (\alpha, \beta)$ with kneading WI_2^∞. This limit depends on J, W, not on u.

The quantitative input is [14, proof of Proposition 4.1, equation (22)]. Put $\lambda_3 = |P'_{\eta^\infty}(1)| > 1$. For each $\vartheta \in (0, 1)$ there are thresholds U, L_0 such that, simultaneously for every allowed $u \geq U$, every $L \geq L_0$, every $\tau \in [\eta_u^-, \eta_u^+]$, and every point with itinerary $I_2 I_3^L I_2 \cdots$,

$$\lambda_3^{-L(1+\vartheta)} < 1 - P_\tau(y) < \lambda_3^{-(L-1)(1-\vartheta)}. \tag{6.2}$$

The source parameters k_1, k_2 are u, L here. In particular, L_0 is independent of the choice of $u \geq U$. This order of quantifiers allows us to set $L = (|W| + u)^2$ below.

6.2. **Extending a kneading word.** We use two elementary consequences of the signed order. The itineraries with a given interval prefix form an order-convex set. Also, if W begins with I_1, WI_2^∞ is minimal, and $k \geq |W|$, then $WI_2^k\xi$ is minimal whenever it belongs to Σ_0 and the tail ξ contains no I_1. For the latter assertion, write $n = |W|$ and $\underline{i} = WI_2^k\xi$. A shift $\sigma^j\underline{i}$ smaller than $\underline{i}$ must start with I_1, so $0 < j < n$. If its first disagreement with $\underline{i}$ occurred at an index $m \geq n$, then $\underline{i}(p) = \underline{i}(p + j)$ for $0 \leq p < n$. Since $\underline{i}$ agrees with WI_2^∞ on its first $2n$ symbols, this equality would make WI_2^∞ j-periodic: for $p \geq n$ both compared symbols are I_2. This is impossible because it begins with I_1 and is eventually I_2. Thus $m < n$; the two symbols at the first disagreement, and the orientation of their common prefix, are then the same as in the comparison of WI_2^∞ with its jth shift. Minimality of that word gives the contradiction.

There are constants $M > 1$ and $\delta > 0$, uniform over the family, such that $[c_1, c_1 + 2\delta] \setminus \{c_1\} \subset I_2$ and

$$M^{-1}|y - c_1(\tau)|^2 \leq 1 - P_\tau(y) \leq M|y - c_1(\tau)|^2 \qquad (|y - c_1(\tau)| \leq 2\delta). \tag{6.3}$$

This follows from the nonzero second derivative at c_1 and compactness.

Lemma 6.2. *Suppose that W begins with I_1 and satisfies* (6.1)*. For every sufficiently large allowed u and every sufficiently large L, with the threshold for L independent of u, put $W' = WI_2^{u+1}I_3^L$. Then $W'I_2^\infty$ is minimal and is realized inside (η_u^-, η_u^+). For arbitrarily large b with $\epsilon(W'I_2^{b+1}) = 1$, there is a compact subinterval $J' = [\alpha', \beta'] \Subset (\eta_u^-, \eta_u^+)$ such that*

$$\underline{k}(\alpha') = W'I_2^{b+1}c_1, \qquad \underline{k}(\beta') = W'I_2^{b+1}c_2.$$

Every parameter in J' has prefix $W'I_2^{b+1}$, and $|J'| \to 0$ as $b \to \infty$ through the allowed parity class. For every parameter in this subinterval, the point at time $|W| + u$ belongs to $(c_1, c_1 + \delta)$ and satisfies (6.2) *and* (6.3)*.*

Proof. First, the entire first bracket has prefix WI_2^u. Indeed, a kneading below $WI_2^u c_1$ inside the bracket, together with realization on the interval to η_u^+, would produce another $WI_2^u c_1$-parameter to the right of η_u^-. This contradicts extremality. The upper inequality is proved in the same way, using minimality of η_u^+. Order convexity gives the common prefix.

Take $u \geq |W|$. Tail insertion gives minimality of $W'I_2^\infty$, and this word belongs to Σ_0. After the common prefix WI_2^u, the next symbols of the two endpoint words and of the central word are c_1, I_2, c_2. Their order is the displayed one because $\epsilon(WI_2^u) = 1$. Realization gives an interior parameter, and Lemma 6.1 gives finite fiber. A second application of the extremal-bracketing result, now to W', gives J' and its shrinking width. Taking $b + 1 \geq |W'|$, the same extremality argument gives the common prefix throughout J'.

It remains to ensure that the passage uses the quadratic branch at c_1. The positive number $\kappa_\delta = \min_{\tau \in J}(1 - P_\tau(c_1(\tau) + \delta))$ is independent of u. Increase L_0 until $\lambda_3^{-(L_0-1)(1-\vartheta)} < \kappa_\delta$. The point in question lies in I_2, where P_τ is decreasing. Thus (6.2) forces it into $(c_1, c_1 + \delta)$, and (6.3) applies. □

6.3. **The limiting parameter.** The seed construction in [14, Section 4.1, equation (15) and the paragraph preceding equation (16)] gives a compact interval J_0 and a word $W_0 = I_1^{t_0}$ satisfying (6.1). Fix $\vartheta \in (0, 1)$. One extremal bracket, chosen sufficiently far along, gives $J_1 \Subset \operatorname{int} J_0$ of length less than $1/2$ and a word W_1 beginning with I_1 such that every parameter of J_1 has prefix $W_1 I_2$, while its endpoint kneadings bracket $W_1 I_2^\infty$. For example, from the bracket for $W_0 I_2^u$ take $W_1 = W_0 I_2^{u-1}$.

Inductively suppose that J_k, W_k have these properties, and put $t_k = |W_k|$. Choose an allowed $u_k \geq \max\{t_k + 1, k\}$ beyond the thresholds in Lemma 6.2, and set

$$n_k = t_k + u_k, \qquad L_k = n_k^2, \qquad W_{k+1} = W_k I_2^{u_k+1} I_3^{L_k} I_2^{b_k}. \tag{6.4}$$

Increase u_k so that L_k exceeds the required depth threshold. This is possible because that threshold is independent of u_k. Choose $b_k \geq \max\{k, |W_k I_2^{u_k+1} I_3^{L_k}|\}$ in the allowed parity class, and choose the second bracket J_{k+1} so that

$$J_{k+1} \Subset \operatorname{int} J_k, \qquad |J_{k+1}| < 2^{-k-1}.$$

The first bracket may also be taken compactly inside J_k. Every parameter in J_{k+1} has prefix $W_{k+1} I_2$, and its endpoint kneadings bracket the minimal word $W_{k+1} I_2^\infty$. Thus the induction continues.

Proposition 6.3. *Let $\{\tau_*\} = \bigcap_{k \geq 1} J_k$ and write $P = P_{\tau_*}$, $c_i = c_i(\tau_*)$, $v = v_{\tau_*}$, and $r = r_{\tau_*}$. The orbit of v avoids both critical points and satisfies*

$$\inf_{m \geq 0} |P^m(v) - c_2| > 0, \qquad r, 1 \in \omega(v). \tag{6.5}$$

Its itinerary is

$$\omega = W_1 I_2^{B_1} I_3^{L_1} I_2^{B_2} I_3^{L_2} \cdots, \qquad B_k, L_k \longrightarrow \infty, \tag{6.6}$$

where the kth displayed I_3-block occupies times $n_k + 1, \ldots, n_k + L_k$. The integers n_k increase strictly, and there are $A_1, A_2 > 0$ and $C \geq 0$ such that

$$A_1 n_k^2 - C \leq \log |P^{n_k}(v) - c_1|^{-1} \leq A_2 n_k^2 + C. \tag{6.7}$$

Proof. Nestedness gives a unique parameter. Since every J_k has the entire prefix $W_k I_2$, the limiting itinerary is infinite and contains no critical symbol. Expanding (6.4) gives $B_1 = u_1 + 1$ and $B_k = b_{k-1} + u_k + 1$ for $k \geq 2$. It also gives the stated time indices and $n_{k+1} > t_{k+1} > n_k$.

Let $\lambda_- \leq \lambda_+$ be the minimum and maximum of $|P'_\tau(1)|$ on J_0; thus $1 < \lambda_- \leq \lambda_+ < \infty$. The common limiting parameter of each first bracket belongs to J_0. Combining (6.2) and (6.3) at $y_k = P^{n_k}(v)$ yields

$$M^{-1/2} \lambda_+^{-L_k(1+\vartheta)/2} < |y_k - c_1| < M^{1/2} \lambda_-^{-(L_k-1)(1-\vartheta)/2}.$$

This proves (6.7) with constants independent of k.

To prove nonrecurrence, use the absence of I_1 from the itinerary after the fixed prefix W_1. Since $P(c_2) = v < c_1$, there is a neighborhood U of c_2 with $P(U) \subset I_1$. No point $P^m(v)$ with $m \geq |W_1|$ can belong to U, since its next symbol would then be I_1. The finitely many earlier orbit points are also different from c_2. This proves the positive distance in (6.5).

At time t_k, the itinerary begins with $I_2^{u_k+1}$, so Lemma 6.1 gives $P^{t_k}(v) \to r$. The upper bound in (6.2) gives $P^{n_k+1}(v) \to 1$ directly. Both time sequences tend to infinity, which proves (6.5). □

6.4. **Normalization and the lift.** Let $A(z) = (z-1)/(c_1-1)$ and $F = A \circ P \circ A^{-1}$. The point 0 is a simple zero of F, the point 1 is a double zero, and $F'(0) = P'(1) = t > 1$. Therefore

$$F(z) = tz(z-1)^2, \qquad S(\zeta) = a\zeta(\zeta^2-1), \qquad \pi(\zeta) = \zeta^2, \qquad a = \sqrt{t}, \qquad \pi \circ S = F \circ \pi. \tag{6.8}$$

The finite critical points are $c_F = 1/3, d_F = 1$ for F and $c_\pm = \pm 1/\sqrt{3}$ for S. Put $V = F(c_F)$ and $w_\pm = S(c_\pm)$. We record all the dynamical properties needed in the remaining sections.

Proposition 6.4. *The maps F and S are semi-hyperbolic. Their finite critical points belong to their Julia sets, and $\mathcal{J}(S) = \pi^{-1}(\mathcal{J}(F))$. There is a compact invariant real interval $I_F \subset \mathcal{J}(F) \cap [0,\infty)$ with c_F in its interior. Moreover,*

$$F^m(V) \notin \{c_F, d_F\}, \qquad \inf_{m\ge 0} |F^m(V) - c_F| > 0, \qquad d_F \mapsto 0 \mapsto 0, \tag{6.9}$$

and 0 and a second repelling fixed point belong to $\omega(V)$. There are nonzero real numbers ε_k such that

$$F^{n_k}(V) = 1 + \varepsilon_k, \qquad A_1' n_k^2 - C' \le \log|\varepsilon_k|^{-1} \le A_2' n_k^2 + C'. \tag{6.10}$$

The compact postcritical set

$$X = \overline{\{S^j(w_+), S^j(w_-) : j \ge 0\}}$$

contains 0, is contained in $\mathcal{J}(S) \cap \mathbb{C}$, and is separated from $\mathrm{Crit}(S)$. *There are $C_{\exp} > 0$ and $\Lambda > 1$ such that*

$$(S^n)^\#(x) \ge C_{\exp}\Lambda^n \qquad (x \in X,\ n \ge 0). \tag{6.11}$$

Both maps belong to $\mathscr{A}^*_{\mathbb{C}}$, have zero badness, and satisfy $\gamma = \infty$.

Proof. The affine normalization transfers (6.5)–(6.7) to (6.9)–(6.10); the second fixed point is $A(r)$. Also $I_F = A([0,1])$ is compact and invariant. Since $\omega(V)$ contains a repelling fixed point, V belongs to the Julia set. Here we use the classification of periodic Fatou components and Sullivan's no-wandering-domain theorem: the orbit of a Fatou point cannot accumulate on a repelling periodic point [17, Theorem 13.1], [27]. Complete invariance gives $c_F, d_F \in \mathcal{J}(F)$, and (6.9) shows that neither is recurrent. Every cycle of immediate attracting or parabolic basins contains a critical point of an iterate and hence a forward image that is a critical point of the original map in the Fatou set [17, Lemmas 10.2–10.3], [1]. As all finite critical points are in $\mathcal{J}(F)$, such a finite cycle is impossible. Thus F has no parabolic cycle and is semi-hyperbolic.

Semi-hyperbolicity gives exponential shrinking and excludes rotation domains and Cremer points; see [5, 15]. There are therefore no bounded Fatou components. It follows that $K(F) = \mathcal{J}(F)$ and $I_F \subset \mathcal{J}(F)$. The semiconjugacy in (6.8) gives $K(S) = \pi^{-1}(K(F))$. Taking boundaries is legitimate because π is open, so $\mathcal{J}(S) = \pi^{-1}(\mathcal{J}(F))$. In particular, $c_\pm \in \mathcal{J}(S)$ and the two critical-value orbits are bounded. Their closure X satisfies

$$\pi(X) = \overline{\{F^j(V) : j \ge 0\}}.$$

The right side avoids c_F, while $\pi(c_\pm) = c_F$. Thus X is disjoint from $c_\pm$ and from infinity. It contains 0 since its image contains 0 and $\pi^{-1}(0) = \{0\}$. Both finite critical points of S are nonrecurrent. The same critical-basin argument excludes parabolic cycles of S, so S is semi-hyperbolic as well.

To apply Mañé's theorem [13], note that X is a compact forward-invariant subset of $\mathcal{J}(S)$, avoids the critical set, and contains no parabolic point. The finite critical points are nonrecurrent, and the only other recurrent critical point is infinity, whose ω-limit is disjoint from X. Hence some iterate S^N has spherical derivative at least $\lambda_0 > 1$ on X. The remaining N possible iterate lengths have a uniform positive lower derivative bound on X, which gives (6.11).

Finally, semi-hyperbolicity implies exponential shrinking and the TCE condition [5, 15, 23]. Neither map is uniformly hyperbolic, since it has critical points in its Julia set. Therefore [25, Remark 2.12 and Corollary 2.19] gives the last assertions. □

6.5. **Passage scales and inverse branches.** For all sufficiently large k, choose $d_k \in \{-1, 1\}$ so that $S^{n_k}(w_+) = d_k + e_k$, with $e_k \to 0$. Then $\varepsilon_k = e_k(2d_k + e_k)$, so $e_k \neq 0$ and

$$A_1'' n_k^2 - C'' \le \log |e_k|^{-1} \le A_2'' n_k^2 + C''. \tag{6.12}$$

We discard the finitely many earlier indices. The selected passages control geometry; the following all-time bound will control the density.

Lemma 6.5. *For $b_{\sigma,j} = S^j(w_\sigma)$, $\sigma \in \{+,-\}$ and $j \ge 0$, one has $b_{\sigma,j} \neq 0$ and*

$$\log^+ \frac{1}{\operatorname{dist}_\#(b_{\sigma,j}, 0)} \le C(1+j)^2. \tag{6.13}$$

Proof. The centers of the two orbits differ by sign. It is therefore enough to consider $b_j = S^j(w_+)$. None is zero, since an orbit landing at zero could not have the later passages in (6.10). Transport the symbols in (6.6) by A. The intervals carrying I_3, I_2 become $(0, c_F)$ and (c_F, d_F), respectively. Since $b_j^2 = F^j(V)$, outside the displayed I_3-blocks and the finite initial prefix we have $|b_j| \ge \sqrt{c_F}$. At the first time $s_k = n_k + 1$ of an I_3-block,

$$|b_{s_k}|^2 = t(1+\varepsilon_k)\varepsilon_k^2, \qquad \log |b_{s_k}|^{-1} \le C(1+n_k)^2.$$

Inside such a block, $b_j \in (-c_+, c_+)$, the central interval on which S is one-to-one. Choose $r_0 > 0$ so small that $a(1-r_0^2) > 1$ and $r_0 < c_+$. The inverse branch of S on this central interval maps $(-r_0, r_0)$ into itself. If a point of the block lies in this smaller interval, all its predecessors in the same block lie there too. At each such step $|S(z)| \ge a(1-r_0^2)|z|$, and hence $|b_j| \ge |b_{s_k}|$. If it does not lie there, $|b_j| \ge r_0$. This proves the Euclidean version of (6.13); the spherical version follows by compactness. □

Lemma 6.6 (Uniform inverse branches)**.** *There are $\delta > 0$ and $0 < c_K \le C_K < \infty$ such that for every $x \in X$ and $n \ge 0$, writing $y = S^n(x)$ and $D_n = (S^n)^\#(x)$, there is a univalent inverse branch $g_{x,n}$ on $B_\#(y, \delta)$ taking y to x. Its image $W_{x,n} = g_{x,n}(B_\#(y, \delta/2))$ satisfies*

$$B_\#(x, c_K D_n^{-1}) \subset W_{x,n} \subset B_\#(x, C_K D_n^{-1}), \qquad C_K^{-1} D_n \le (S^n)^\# \le C_K D_n \quad \text{on } W_{x,n}. \tag{6.14}$$

Moreover, in Euclidean coordinates, with $E_n = (S^n)'(x)$,

$$S^n(x+\tau) = S^n(x) + E_n\tau + O(|E_n|^2|\tau|^2) \qquad (|\tau| \le c_0|E_n|^{-1}), \tag{6.15}$$

where $c_0 > 0$ and the implied constant are independent of x, n.

Proof. Choose N so that $T = S^N$ has derivative at least $\lambda_0 > 1$ on X, and choose $\lambda_0^{-1} < \kappa < 1$. Compactness and the inverse-function theorem give inverse branches of T on balls of a common radius R centered at Tx, taking Tx to x, with spherical derivative at most κ. Decrease R below the spherical convexity radius and below $\operatorname{dist}_\#(X, \infty)/4$. Each such branch maps its ball into $B_\#(x, \kappa R)$. For the finitely many remainders $0 \le r < N$, choose a common $\delta < R$ on which the inverse branches of S^r along X are univalent and have image in the corresponding R-balls. Decrease δ further so that all target balls lie in a fixed bounded affine neighborhood of X. Writing $n = qN + r$, compose the remainder branch with the q contracting branches. This gives $g_{x,n}$ on the fixed δ-ball. Its center derivative is D_n^{-1}. Koebe's growth and derivative theorems on the half-radius ball give (6.14).

All centers lie in a compact subset of the finite plane. Thus, in Euclidean coordinates, S^n is univalent on a disk of radius $c|E_n|^{-1}$ about x and its image there lies in a fixed bounded neighborhood of X. Cauchy's estimate on a smaller disk bounds its second derivative by $C|E_n|^2$. Taylor's formula gives (6.15). □

7. The geometric pull-back exponent

Fix the maps and passages from Section 6. The upper bound for $q(F)$ comes from the order in which small pull-back chains can encounter the critical points. The lower bound uses the selected passages and requires complete univalent pull-backs, not just local inverse images.

7.1. **A degree-four upper bound.**

Proposition 7.1. *There is $R_0 > 0$ such that, for every $x \in \mathcal{J}(F)$, every $m \geq 1$, and every component U of $F^{-m}(B_\#(x, R_0))$, the map $F^m : U \to B_\#(x, R_0)$ is proper of degree at most four and* $\operatorname{diam}_\# U < 1$. *Consequently, $q(F) \leq 4$.*

Proof. By (6.9), $\delta_F = \inf_{n\geq 1} \operatorname{dist}_\#(F^n(c_F), c_F) > 0$. Choose $\eta \in (0, 1)$ smaller than the distances separating $c_F, d_F, 0$ from one another and smaller than δ_F. Take it small enough that a domain of diameter less than η containing either critical point, together with its image of diameter less than η, lies in fixed quadratic coordinate neighborhoods.

Exponential shrinking, together with finite-depth compactness, allows us to choose R_0 so that every pull-back of $B_\#(x, R_0)$, including the ball itself, has diameter less than η. For the compactness assertion, if pull-backs of bounded depth and radii tending to zero had diameter bounded away from zero, a Hausdorff limit of their closures would be a nondegenerate continuum in a finite fiber of an iterate. This is impossible. For large depths, exponential shrinking gives the required bound. We also choose the target balls disjoint from infinity.

Let $U_j = F^j(U)$, $0 \leq j \leq m$. Properness of the rational map shows that each U_j is a full intermediate pull-back component and that $F : U_j \to U_{j+1}$ is proper and onto. Every level is thin and avoids infinity. Starting from the terminal disk and proceeding backwards, an unbranched step is a conformal isomorphism. A step containing a critical point is contained in its quadratic coordinate neighborhood; the local model $z \mapsto z^2$ gives degree two and a simply connected preimage. Thus all levels are simply connected, and a step has degree two exactly when it contains a finite critical point.

The point c_F cannot occur at two levels $j < k$: the later level would contain both c_F and $F^{k-j}(c_F)$, contradicting its diameter bound. If a level contains d_F, every later level contains 0 and therefore neither critical point. Hence each critical point occurs at most once, and the only possible order of two critical encounters is c_F followed by d_F. Multiplying the one-step degrees gives the bound four.

We use the bounded-degree disk estimate [15, Lemma 7 in the arXiv version]: if U' is a component of the pull-back of $B_\#(x, r)$ inside a component U over $B_\#(x, R)$, $0 < r < R < 1$, $\operatorname{diam}_\# U < 1$, and the proper degree over the larger ball is at most D, then

$$\operatorname{diam}_\# U' \leq 64 \operatorname{diam}_\# U \, (r/R)^{1/D}.$$

Apply this with $R = R_0$, $r = \delta$ and $D = 4$. It gives $\operatorname{diam}_\# W \leq C\delta^{1/4}$ for every pull-back of $B_\#(x, \delta)$ when $\delta < R_0$. For $\delta \geq R_0$, enlarge C and use the diameter of the sphere. This is stronger than the estimate required in Definition 2.1. □

7.2. **The quartic normal models.** Put $D_k = (S^{n_k})'(w_+)$ and $\alpha_c = S''(c_+)/2 = 3ac_+$, $\beta_c = S'(1) = S'(-1) = 2a$. All these numbers are real, and $D_k \neq 0$ by (6.11). Define

$$\sigma_k = \operatorname{sgn}(\alpha_c D_k / e_k), \qquad \varrho_k = \left|\frac{e_k}{\alpha_c D_k}\right|^{1/2}, \qquad \Phi_k(s) = (c_+ + \varrho_k s)^2.$$

In particular, $\varrho_k \to 0$.

Lemma 7.2. *The functions*

$$H_k(s) = \frac{F^{n_k+2}(\Phi_k(s))}{\beta_c^2 e_k^2}$$

satisfy

$$H_k(s) - (1 + \sigma_k s^2)^2 \longrightarrow 0 \quad \text{in } C^1 \text{ on each compact subset of } \mathbb{C}. \tag{7.1}$$

Proof. The exact Taylor expansion at c_+ gives

$$S(c_+ + \varrho_k s) = w_+ + \alpha_c \varrho_k^2 s^2 + a\varrho_k^3 s^3 = w_+ + \frac{e_k}{D_k}\big(\sigma_k s^2 + O(\varrho_k)\big),$$

where the last error is uniform in C^1 on each fixed compact set. The perturbation has size $O(|e_k/D_k|) = o(|D_k|^{-1})$. Therefore (6.15), applied on a slightly larger compact set, yields

$$S^{n_k+1}(c_+ + \varrho_k s) = d_k + e_k\big(1 + \sigma_k s^2 + o(1)\big).$$

The error is also $o(1)$ in C^1: the error in (6.15), divided by e_k, is $O(|e_k|)$, and holomorphic convergence on a larger compact set controls the derivative. Since $S(d_k) = 0$ and $S'(d_k) = \beta_c$,

$$S^{n_k+2}(c_+ + \varrho_k s) = \beta_c e_k\big(1 + \sigma_k s^2 + o(1)\big).$$

The semiconjugacy says that $F^{n_k+2}(\Phi_k(s))$ is the square of this last expression. This proves (7.1). The first quadratic operation comes from c_+, and the second from the branch point of π at zero. □

7.3. **Complete pull-backs.**

Proposition 7.3. *For all sufficiently large k, there are $x_k \in \mathcal{J}(F)$, $\delta_k > 0$, and a component W_k of $F^{-(n_k+2)}(B_\#(x_k, \delta_k))$ such that $F^{n_k+2} : W_k \to B_\#(x_k, \delta_k)$ is biholomorphic,* $\operatorname{Dist}_\#(F^{n_k+2}|_{W_k}) < 2$, *and*

$$\delta_k \asymp |e_k|^2, \qquad \operatorname{diam}_\# W_k \asymp |D_k|^{-1/2}|e_k|^{1/2}. \tag{7.2}$$

All comparison constants are independent of k.

Proof. For $\sigma = \pm 1$, write $H_\sigma(s) = (1 + \sigma s^2)^2$. At $s_0 = 1/2$ both derivatives are nonzero. Choose concentric disks $\Delta = B(s_0, r)$ and $\Delta^\sharp = B(s_0, 2r)$ such that both models are univalent on a neighborhood of $\overline{\Delta^\sharp}$ and their derivative distortions there are less than $5/4$. Their difference quotients on the product of this compact disk with itself are bounded above and away from zero. By (7.1) and convexity of the disk, the same is true uniformly for H_k, for large k. In particular, H_k is univalent, its derivative distortion on Δ is less than $4/3$, and

$$m_H|s - t| \le |H_k(s) - H_k(t)| \le M_H|s - t| \qquad (s, t \in \overline{\Delta}) \tag{7.3}$$

for constants $m_H, M_H > 0$. The model boundary gaps are positive, so there is $\eta > 0$ with

$$\operatorname{dist}(H_k(s_0), H_k(\partial\Delta)) > 4\eta \tag{7.4}$$

for every sufficiently large k.

The map Φ_k is univalent near $\overline{\Delta^\sharp}$, since

$$\Phi_k(s) - \Phi_k(t) = \varrho_k(s - t)\big(2c_+ + \varrho_k(s + t)\big). \tag{7.5}$$

Set $z_k^* = \Phi_k(s_0)$ and $x_k = F^{n_k+2}(z_k^*) = \beta_c^2 e_k^2 H_k(s_0)$. The point z_k^* is real and tends to c_F, which is in the interior of I_F. Thus $z_k^* \in I_F \subset \mathcal{J}(F)$ for large k, and $x_k \in \mathcal{J}(F)$ by forward invariance. Also $x_k \to 0$.

Euclidean and spherical balls near zero satisfy $B(x, c_\# r) \subset B_\#(x, r) \subset B(x, C_\# r)$ with fixed positive constants. Choose $\kappa > 0$ so small that $C_\#\kappa < \eta$, and put $\delta_k = \kappa\beta_c^2|e_k|^2$. The normalized target $\widehat{B}_k = (\beta_c^2 e_k^2)^{-1} B_\#(x_k, \delta_k)$ satisfies

$$B(H_k(s_0), c_\#\kappa) \subset \widehat{B}_k \subset B(H_k(s_0), C_\#\kappa) \Subset H_k(\Delta). \tag{7.6}$$

The last inclusion follows from (7.4). Define $\Omega_k = (H_k|_\Delta)^{-1}(\widehat{B}_k)$ and $W_k = \Phi_k(\Omega_k)$. The identity

$$F^{n_k+2}(\Phi_k(s)) = \beta_c^2 e_k^2 H_k(s) \tag{7.7}$$

shows that $F^{n_k+2} : W_k \to B_\#(x_k, \delta_k)$ is biholomorphic.

We verify completeness of this pull-back. By (7.4)–(7.6), the image of $\Phi_k(\partial\Delta)$ under F^{n_k+2} is disjoint from the closure of the target ball. The full preimage component containing z_k^* cannot leave $\Phi_k(\Delta)$: otherwise a path in that component would cross this boundary. Inside $\Phi_k(\Delta)$ the map is univalent, so the component is exactly W_k.

Equations (7.3) and (7.6) bound $\operatorname{diam}\Omega_k$ above and away from zero independently of k. Equation (7.5) therefore gives $\operatorname{diam}_\# W_k \asymp \varrho_k$, proving (7.2). Finally, differentiating (7.7) gives

$$(F^{n_k+2})'(\Phi_k(s)) = \frac{\beta_c^2 e_k^2 H_k'(s)}{2\varrho_k(c_+ + \varrho_k s)}.$$

The first factor contributing to the derivative ratio has distortion less than $4/3$, and the ratio of the denominators tends uniformly to one. Thus the Euclidean distortion on W_k is less than $3/2$ for large k. The source domains tend to c_F and the targets to zero, so the ratio of the spherical metric factors also tends uniformly to one and is less than $4/3$. The spherical distortion is consequently less than $(3/2)(4/3) = 2$. □

Proposition 7.4. *The polynomial in Proposition 6.4 satisfies $q(F) = 4$.*

Proof. The upper bound is Proposition 7.1. Choose $M > \max\{1, \sup_X |S'|\}$, so that $|D_k| \le M^{n_k}$. For any $0 < q < 4$, Proposition 7.3 gives

$$\log \frac{\operatorname{diam}_\# W_k}{\delta_k^{1/q}} \ge \left(\frac{2}{q} - \frac{1}{2}\right) \log |e_k|^{-1} - \frac{n_k}{2} \log M - O_q(1).$$

By (6.12), the right side tends to infinity. The balls are centered in $\mathcal{J}(F)$, and their complete pull-backs have spherical distortion less than two. Thus every $q < 4$ fails the uniform estimate in Definition 2.1, as required. □

8. Conformal measures and endpoint integrability

The quadratic quotient introduces a weight at zero. We first identify that weight, then estimate its interaction with the postcritical spikes. All conformal measures are spherical unless another metric is specified.

8.1. **Measures under the quotient.** The two Julia sets have the same Hausdorff dimension $h = \operatorname{HD}\mathcal{J}(F) = \operatorname{HD}\mathcal{J}(S) \ge 1$. Indeed, π is Lipschitz on the sphere, and away from its two critical points the source is covered by countably many sets on which π is bi-Lipschitz. The Julia-set identity in Proposition 6.4 gives equality of the dimensions; the interval $I_F \subset \mathcal{J}(F)$ gives the lower bound.

For $R = F, S$, let μ_R be the geometric conformal probability and let $\nu_R \ll \mu_R$ be its associated invariant probability. These are the measures in [25, Corollary D]. We recall why they agree with the measures in [20]. Theorem A of that paper gives a non-atomic conformal probability of minimal exponent, supported on the conical Julia set. Its exponent is at most h, since an h-conformal probability exists. The exclusivity assertion of RLS Corollary D forces it to coincide with μ_R; otherwise its exponent would be strictly larger than h. The same assertion gives uniqueness at exponent h, and [20, Theorem B] gives uniqueness of ν_R among invariant probabilities absolutely continuous with respect to μ_R. Write $d\nu_R = \rho_R\, d\mu_R$.

Since S commutes with $\zeta \mapsto -\zeta$, a spherical isometry, uniqueness implies that μ_S and ν_S are even. Averaging a Borel representative with its reflection, we take ρ_S even. The spherical derivative

$$\pi^\#(\zeta) = \frac{2|\zeta|(1 + |\zeta|^2)}{1 + |\zeta|^4}$$

is even and vanishes only at 0 and infinity.

Proposition 8.1. *Put $Z = \int (\pi^\#)^h \, d\mu_S$. Then $0 < Z < \infty$ and*

$$\mu_F = Z^{-1}\pi_*\big((\pi^\#)^h \mu_S\big), \qquad \nu_F = \pi_* \nu_S. \tag{8.1}$$

There is a Borel representative of ρ_F for which

$$\rho_F(\pi\zeta) = Z\rho_S(\zeta)(\pi^\#(\zeta))^{-h} \quad \text{for } \mu_S\text{-almost every } \zeta. \tag{8.2}$$

For every $p > 0$ the following is an equality of extended nonnegative integrals:

$$\int \rho_F^p \, d\mu_F = Z^{p-1} \int \rho_S^p (\pi^\#)^{-h(p-1)} \, d\mu_S. \tag{8.3}$$

At the two branch points the integrand on the right can be defined to be zero.

Proof. Write $w = (\pi^\#)^h$ and $\widetilde{\mu} = Z^{-1}\pi_*(w\mu_S)$. Boundedness of w gives $Z < \infty$, and non-atomicity and positivity of w off the branch points give $Z > 0$. The measure $\widetilde{\mu}$ is non-atomic because π has finite fibers.

To check conformality, let $A \subset \mathcal{J}(F)$ be a Borel injectivity set for F. Remove the finite set $\{0, c_F, d_F, \infty\}$, which changes neither side of the desired identity. Split $\pi^{-1}(A)$ into its two Borel sheets $A_+, A_- = -A_+$. The map S is injective on each sheet, and

$$\pi^{-1}(F(A)) = S(A_+) \,\dot\cup\, S(A_-).$$

Indeed, the images of the two nonzero lifts are opposite and nonzero; the removed points include the preimages of the branch values. The weighted conformal change of variables and the chain rule give

$$\begin{aligned} Z\widetilde{\mu}(F(A)) &= \sum_{\sigma=\pm} \int_{A_\sigma} w(S\zeta)(S^\#(\zeta))^h \, d\mu_S(\zeta) \\ &= \int_{\pi^{-1}(A)} (F^\#(\pi\zeta))^h w(\zeta) \, d\mu_S(\zeta) = Z \int_A (F^\#)^h \, d\widetilde{\mu}. \end{aligned}$$

Thus $\widetilde{\mu}$ is h-conformal, and uniqueness gives the first identity in (8.1). The semiconjugacy makes $\pi_*\nu_S$ invariant. Since $w > 0$ μ_S-almost everywhere, it is absolutely continuous with respect to $\widetilde{\mu}$, so uniqueness gives the second identity.

The function $Z\rho_S/w$ is even off the branch points and therefore descends to a Borel function on their complement. For every Borel set A, its integral against μ_F is $\int_{\pi^{-1}(A)} \rho_S \, d\mu_S = \nu_F(A)$. This proves (8.2). Changing variables in its pth power using (8.1) proves (8.3), including when an integral is infinite. □

8.2. **Mass at the postcritical centers.**

Lemma 8.2. *There is $C_A > 0$ such that*

$$\mu_S(B_\#(x, r)) \le C_A r^h \qquad (x \in X,\ r > 0). \tag{8.4}$$

For each $0 < \alpha < h$ there is $C_\alpha > 0$ such that

$$\int_{B_\#(x,R)} \operatorname{dist}_\#(z, x)^{-\alpha} \, d\mu_S(z) \le C_\alpha R^{h-\alpha} \qquad (x \in X,\ R > 0). \tag{8.5}$$

In particular, the corresponding potentials over $\mathcal{J}(S)$ are uniformly bounded for centers in X.

Proof. Let c_K, C_K be the constants in (6.14), and put $M_1 = \max\{1, \max_X S^\#\}$ and $A = c_K/(2M_1)$. For $0 < r < A$, take the least $n \ge 1$ with $D_n = (S^n)^\#(x) \ge A/r$. It exists by (6.11). Minimality gives $D_n < M_1 A/r = c_K/(2r)$, so (6.14) implies $B_\#(x, r) \subset W_{x,n}$. Conformality along the univalent branch gives

$$1 \ge \mu_S(S^n(W_{x,n})) = \int_{W_{x,n}} ((S^n)^\#)^h \, d\mu_S \ge C_K^{-h} D_n^h \mu_S(W_{x,n}).$$

It follows that $\mu_S(B_\#(x, r)) \le C_K^h A^{-h} r^h$. For $r \ge A$ use that μ_S is a probability. Decomposing $B_\#(x, R)$ into the annuli of radii $2^{-j}R$ now gives (8.5), since $h - \alpha > 0$. □

The exact exponent h in (8.4) is used in the next estimate. An upper bound with exponent $h - \varepsilon$ would not give the logarithmic loss at this endpoint. For the next two lemmas we work in the fixed affine chart containing $\mathcal{J}(S)$; Euclidean and spherical distances are uniformly comparable there.

Lemma 8.3 (A two-center estimate)**.** *For every $b \in X \setminus \{0\}$,*

$$\int |z-b|^{-2h/3}|z|^{-h/3}\, d\mu_S(z) \le C\left(1+\log^+\frac{1}{|b|}\right). \tag{8.6}$$

Proof. Fix a small $r_0 > 0$. If $d = |b| \ge r_0/4$, split at $B(b, d/2)$. On this ball the zero-centered factor is bounded; on its complement the b-centered factor is bounded. Lemma 8.2 gives a uniform bound.

Suppose that $0 < d < r_0/4$. On $B(b, d/2)$, the integral is at most $Cd^{-h/3}d^{h-2h/3} = C$. On $B(0, 2d) \setminus B(b, d/2)$, it is at most $Cd^{-2h/3}d^{h-h/3} = C$, again by (8.5). On $2d < |z| < r_0$ the kernel is at most $C|z|^{-h}$. Each dyadic annulus has integral at most a uniform constant by (8.4) centered at $0 \in X$. There are $O(1 + \log(r_0/d))$ such annuli. On $|z| \ge r_0$ both factors are bounded. This proves (8.6). □

Lemma 8.4 (Weighted spike summation)**.** *Let $b_j \in X \setminus \{0\}$ and $u_j \ge 0$. If*

$$\sum_j u_j\left(1+\log^+\frac{1}{|b_j|}\right)^{3/4} < \infty, \tag{8.7}$$

then

$$1 + \sum_j u_j|z-b_j|^{-h/2} \in L^{4/3}\big((1+|z|^{-h/3})\, d\mu_S(z)\big).$$

The weight at $z = 0$ may be assigned any value.

Proof. The weighted measure is finite by Lemma 8.2 at the center zero. The unweighted $2h/3$-potential and (8.6) give

$$\big\||\cdot - b|^{-h/2}\big\|_{L^{4/3}((1+|z|^{-h/3})\mu_S)} \le C\left(1+\log^+\frac{1}{|b|}\right)^{3/4}.$$

Apply Minkowski's inequality to finite partial sums and then use monotone convergence and (8.7). □

8.3. **The invariant density and the endpoint.** Let $\mathcal{L}$ be the transfer operator

$$(\mathcal{L}\varphi)(x) = \sum_{S(y)=x} (S^{\#}(y))^{-h}\varphi(y),$$

defined off the critical-value set; values on that null set are irrelevant. Conformality gives the duality identity $\int \psi\,\mathcal{L}\varphi\, d\mu_S = \int (\psi \circ S)\varphi\, d\mu_S$. Thus $\mathcal{L}$ is a positive contraction on $L^1(\mu_S)$ and preserves the integral of nonnegative functions.

Proposition 8.5. *There are $C_P > 0$ and $\theta \in (0, 1)$ such that, almost everywhere,*

$$\mathcal{L}^N\mathbf{1}(z) \le G(z) := C_P + C_P \sum_{\sigma=\pm}\sum_{j\ge 0} \theta^j\, \mathrm{dist}_{\#}(z, b_{\sigma,j})^{-h/2} \qquad (N \ge 0). \tag{8.8}$$

Moreover, $\rho_S \le G$ almost everywhere.

Proof. We verify the hypotheses of the estimate in [18, Theorem 0.1 and equation (1.3)]. The only critical points in $\mathcal{J}(S)$ are c_+, c_-; their positive orbits avoid the critical set. Thus the critical multiplicity parameter in that estimate is two, not the geometric exponent $q(F)$. The map has no parabolic cycle, $\mathcal{J}(S) \ne \widehat{\mathbb{C}}$, and μ_S is non-atomic. Equation (6.11) gives the Collet–Eckmann lower bound along both critical-value orbits.

For clarity, the affine and spherical normalizations are related as follows. Set $\chi(z) = (1 + |z|^2)^h$, $Q = \int \chi\, d\mu_S$, and $d\mu_e = Q^{-1}\chi\, d\mu_S$. The chain rule shows that μ_e is Euclidean h-conformal and that

$$\mathcal{L} = M_\chi \mathcal{L}_e M_{\chi^{-1}}, \qquad \mathcal{L}^N\mathbf{1} \le \frac{\max_{\mathcal{J}(S)}\chi}{\min_{\mathcal{J}(S)}\chi}\,\mathcal{L}_e^N\mathbf{1}. \tag{8.9}$$

Both the measures and the two metrics are uniformly comparable on $\mathcal{J}(S)$. Lemma 8.2 gives

$$M_e := \sup_{\sigma,j}\int |z-b_{\sigma,j}|^{-h/2}\, d\mu_e(z) < \infty, \qquad |(S^n)'(w_\sigma)| \ge C_e\Lambda^n.$$

Choose $1 < \Lambda' < \Lambda$ and $0 < C_* \le \min\{C_e, (\Lambda')^{h/2}/(2M_e)\}$. Then, for every $n \ge 1$,

$$\int |z - S^n(c_\sigma)|^{-h/2}\, d\mu_e(z) \le M_e < C_*^{-1}(\Lambda')^{hn/2}, \qquad |(S^n)'(w_\sigma)| \ge C_*\Lambda^n.$$

These are the required derivative and potential conditions. In the author version of [18], non-atomicity and the potential condition are listed separately as (0.3) and (0.4).

Equation (1.3) of that paper consequently gives, for every $\Gamma > 1$, a constant C_Γ independent of N such that

$$\mathcal{L}_e^N \mathbf{1}(z) \le C_\Gamma + C_\Gamma \sum_{\sigma=\pm} \sum_{j\ge 0} \Gamma^j \Lambda^{-jh/2} |z - b_{\sigma,j}|^{-h/2}.$$

Take $1 < \Gamma < \Lambda^{h/2}$ and $\theta = \Gamma\Lambda^{-h/2} < 1$. Equation (8.9) gives (8.8); enlarge C_P to include $N = 0$ and take one null set for all N. The center symmetry makes G even, and Lemma 8.2 gives $G \in L^1(\mu_S)$.

The Cesàro densities $\widehat{\rho}_N = N^{-1}\sum_{k=0}^{N-1} \mathcal{L}^k \mathbf{1}$ are bounded by G, have integral one, and are uniformly integrable. A subsequence converges weakly in $L^1(\mu_S)$ to $\widehat{\rho}$, with $0 \le \widehat{\rho} \le G$ and integral one. The identity

$$\mathcal{L}\widehat{\rho}_N - \widehat{\rho}_N = N^{-1}(\mathcal{L}^N \mathbf{1} - \mathbf{1}), \qquad \|\mathcal{L}\widehat{\rho}_N - \widehat{\rho}_N\|_1 \le 2/N$$

shows that $\mathcal{L}\widehat{\rho} = \widehat{\rho}$. Thus $\widehat{\rho}\mu_S$ is invariant, and uniqueness identifies it with ν_S. Hence $\rho_S \le G$ almost everywhere. □

Completion of the proof of Theorem B. Since $\mathcal{J}(S)$ is bounded, the formula for $\pi^\#$ gives

$$(\pi^\#(z))^{-h/3} \le C(1 + |z|^{-h/3}) \qquad (z \ne 0).$$

By Lemma 6.5, the centers in (8.8) satisfy $1 + \log^+(|b_{\sigma,j}|^{-1}) \le C(1 + j)^2$. Thus the weighted summability condition in Lemma 8.4 follows from

$$\sum_{j\ge 0} \theta^j (1 + j)^{3/2} < \infty.$$

Using metric comparison and $\rho_S \le G$, we conclude that

$$\int \rho_S^{4/3} (\pi^\#)^{-h/3}\, d\mu_S < \infty.$$

The moment identity (8.3) with $p = 4/3$ proves $\rho_F \in L^{4/3}(\mu_F)$. Propositions 6.4 and 7.4 give the remaining assertions of the theorem. □

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