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Weighted and Fractional Covering by Translates

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Abstract

We compare covering by finitely many weighted translates with covering defined by arbitrary finite Borel measures. For every compact set K and every convex body T in $\mathbb{R}^n$, we prove that the weighted and fractional covering numbers coincide. The proof applies the one-sided approximation theorem of Bukh and Nivasch, extends it to compactly supported probability measures, and discretizes a covering measure. Combined with the covering-separation duality of Artstein-Avidan and Slomka, this identifies the common value with the weighted separation number for the reflected covering body. We also give an independent proof of equality for compact K and arbitrary nonempty open T. Finally, we construct a compact, connected nonconvex set T, equal to the closure of its interior and with boundary of measure zero, for which weighted and fractional covering differ, even when K is a rectangle.

1 Introduction and main results

For sets $K, T \subseteq \mathbb{R}^n$, the classical covering number $N(K,T)$ is the least number of translates of T whose union contains K. A weighted covering allows nonnegative weights on the translates and requires total covering weight at least one at each point of K. The weights and translation centers can be encoded by a finitely supported measure. Allowing general measures gives a further relaxation. The question considered here is when these two relaxations have the same value.

Let $\mathcal{B}_+(\mathbb{R}^n)$ denote the finite nonnegative Borel measures on $\mathbb{R}^n$, and let $\mathcal{B}_{+,\mathrm{fin}}(\mathbb{R}^n)$ denote those with finite support. For a Borel function $u \geq 0$ and $\mu \in \mathcal{B}_+(\mathbb{R}^n)$, write

$$(\mu * u)(x) = \int_{\mathbb{R}^n} u(x-y)\, d\mu(y).$$

For Borel sets $K, T \subseteq \mathbb{R}^n$, define

$$N^*(K,T) = \inf\{\mu(\mathbb{R}^n) : \mu \in \mathcal{B}_+(\mathbb{R}^n),\ \mu * \mathbf{1}_T \geq \mathbf{1}_K\}, \tag{1}$$

$$N^w(K,T) = \inf\{\nu(\mathbb{R}^n) : \nu \in \mathcal{B}_{+,\mathrm{fin}}(\mathbb{R}^n),\ \nu * \mathbf{1}_T \geq \mathbf{1}_K\}. \tag{2}$$

All convolution inequalities in this paper are pointwise. The infimum of an empty feasible set is $+\infty$. The covering constraint is equivalently

$$\mu(x - T) \geq 1 \qquad (x \in K).$$

In particular, $N^w(K,T)$ is the infimum of $\sum_{i=1}^m w_i$ over finite families with $w_i \geq 0$ satisfying

$$\sum_{i=1}^m w_i \mathbf{1}_{a_i+T}(x) \geq 1 \qquad (x \in K).$$

The centers a_i are unrestricted. Clearly,

$$N^*(K,T) \leq N^w(K,T) \leq N(K,T). \tag{3}$$

When K is compact and T has nonempty interior, all three quantities are finite. We set the covering and separation numbers of the empty set equal to zero.

Weighted covering and separation were studied by Artstein-Avidan and Raz [1]. Artstein-Avidan and Slomka [2] developed their measure-theoretic counterparts and established a strong duality between fractional covering and weighted separation. Their Remark 1.4 distinguishes that duality from equality with the finite weighted covering number; Remark 2.5 discusses an error in an earlier approximation argument. Our approach to finite weighted covering uses one-sided approximation of measures on convex sets.

Throughout, a *convex body* is a compact convex subset of $\mathbb{R}^n$ with nonempty interior. Our main result is the following.

Theorem 1.1. *Let $K \subseteq \mathbb{R}^n$ be compact and let $T \subseteq \mathbb{R}^n$ be a convex body. Then*

$$\boxed{N^w(K,T) = N^*(K,T).}$$

No convexity assumption is imposed on K. The proof uses the one-sided weak approximation theorem of Bukh and Nivasch [3]. Its approximation points need not lie in the original point set. This freedom is compatible with the unrestricted translation centers in (2). Moreover, only a lower approximation to the mass of each set $x - T$ is required: an excess of covering mass is harmless.

For completeness, define the separation numbers by

$$M^*(K,T) = \sup\{\rho(K) : \rho \in \mathcal{B}_+(\mathbb{R}^n),\ \rho * \mathbf{1}_T \le 1\}, \tag{4}$$
$$M^w(K,T) = \sup\{\rho(K) : \rho \in \mathcal{B}_{+,\mathrm{fin}}(\mathbb{R}^n),\ \rho * \mathbf{1}_T \le 1\}. \tag{5}$$

Restricting a feasible measure to K preserves feasibility and its objective value, so the measures in these definitions may be taken to be concentrated on K. For compact K and a set T with nonempty interior, this restriction is finite even if one initially allows arbitrary nonnegative Borel measures: finitely many sets $x - T$ cover K, and each has mass at most one.

Theorem 1.3 of Artstein-Avidan and Slomka [2] asserts that, for compact K, T with $\operatorname{int} T \neq \varnothing$,

$$M^w(K,-T) = M^*(K,-T) = N^*(K,T). \tag{6}$$

Combining this with Theorem 1.1 gives the following corollary.

Corollary 1.2. *Let $K \subseteq \mathbb{R}^n$ be compact and let $T \subseteq \mathbb{R}^n$ be a convex body. Then*

$$\boxed{N^w(K,T) = N^*(K,T) = M^*(K,-T) = M^w(K,-T).}$$

There is also an independent discretization argument for open covering sets. It does not require convexity or the approximation theorem of Bukh and Nivasch.

Theorem 1.3. *Let $K \subseteq \mathbb{R}^n$ be compact and let $T \subseteq \mathbb{R}^n$ be a nonempty open set. Then*

$$\boxed{N^w(K,T) = N^*(K,T).}$$

In Theorem 1.3, the set T need not be bounded or have finite volume. We prove the convex case in Sections 2 and 3 and the open case in Section 4. Section 5 gives an example showing that a compact covering set may have nonempty interior, null boundary, and equal the closure of its interior, while still satisfying $N^w(K,T) > N^*(K,T)$.

2 One-sided approximation of measures

We first recall the finite-set approximation theorem used in the proof. If A is a finite multiset, all cardinalities involving A count multiplicity.

Theorem 2.1 (Bukh and Nivasch [3, Theorem 2])**.** *For every $n \geq 1$ and $0 < \eta < 1$, there exists an integer $G(n, \eta)$ such that every nonempty finite set $F \subseteq \mathbb{R}^n$ admits a nonempty finite multiset $A \subseteq \mathbb{R}^n$, with $\#A \leq G(n, \eta)$, satisfying*

$$\frac{\#(A \cap C)}{\#A} \geq \frac{\#(F \cap C)}{\#F} - \eta \qquad \text{for every convex set } C \subseteq \mathbb{R}^n.$$

The bound is independent of the cardinality of F. This allows a compactness argument in which the limit of the approximating measures remains finitely supported.

Lemma 2.2. *Let P be a compactly supported Borel probability measure on $\mathbb{R}^n$, and let $0 < \eta < 1$. There exists a probability measure*

$$Q = \frac{1}{m} \sum_{i=1}^{m} \delta_{a_i}, \qquad 1 \leq m \leq G(n, \eta),$$

such that

$$Q(C) \geq P(C) - \eta \qquad \text{for every closed convex set } C \subseteq \mathbb{R}^n. \tag{7}$$

Repeated points among the a_i are allowed.

Proof. Choose a closed ball D whose interior contains $\operatorname{supp} P$. There are nonempty finite sets $F_j \subseteq D$ such that their uniform probability measures

$$P_j = \frac{1}{\#F_j} \sum_{y \in F_j} \delta_y$$

converge weakly to P. One construction first approximates P by finitely supported probabilities, then approximates their weights by rational numbers with a common denominator. Replace the resulting repeated points by distinct points at distances tending to zero from the original locations. All these points can be kept in D. This construction applies whether or not P has atoms.

By Theorem 2.1, for each j there is a uniform probability measure Q_j on a multiset of at most $G(n, \eta)$ points such that

$$Q_j(C) \geq P_j(C) - \eta \qquad \text{for every convex set } C. \tag{8}$$

We may arrange that all atoms of Q_j belong to D. To see this, fix $b \in D$ and move every atom outside D to b, preserving its weight. Denote the resulting measure by $\widetilde{Q}_j$. Since $C \cap D$ is convex,

$$\widetilde{Q}_j(C) \geq Q_j(C \cap D) \geq P_j(C \cap D) - \eta = P_j(C) - \eta.$$

Thus this modification preserves (8); rename $\widetilde{Q}_j$ as Q_j.

Pass to a subsequence on which the multiset cardinality is a fixed integer $m \leq G(n, \eta)$, and write

$$Q_j = \frac{1}{m} \sum_{i=1}^{m} \delta_{a_{j,i}}, \qquad a_{j,i} \in D.$$

Compactness of D^m gives a further subsequence such that $a_{j,i} \to a_i$ for every i. Hence

$$Q_j \implies Q := \frac{1}{m} \sum_{i=1}^{m} \delta_{a_i}.$$

All subsequences have been chosen independently of the set C appearing below.

Fix a nonempty closed convex set C and a number $r > 0$. Put

$$C_r = \{z : \operatorname{dist}(z, C) \le r\}, \qquad U_r = \{z : \operatorname{dist}(z, C) < r\}.$$

The set C_r is closed and convex, while U_r is open and contains C. The Portmanteau inequalities and (8) imply

$$\begin{aligned} Q(C_r) &\ge \limsup_{j\to\infty} Q_j(C_r) \\ &\ge \liminf_{j\to\infty} P_j(U_r) - \eta \\ &\ge P(U_r) - \eta \\ &\ge P(C) - \eta. \end{aligned}$$

Since $C_r \downarrow C$ as $r \downarrow 0$, continuity from above gives (7). For $C = \varnothing$ the inequality is immediate. □

Remark 2.3. The lemma does not require $P(\partial C) = 0$. Enlarging C before passing to the weak limit, and then decreasing the enlargement, preserves the lower bound even when P charges the boundary. The measure Q is not asserted to be an empirical sample from P.

3 Convex covering bodies

Proof of Theorem 1.1. The case $K = \varnothing$ is immediate. Suppose $K \neq \varnothing$ and fix $0 < \varepsilon < 1$. Choose a covering measure μ such that

$$M := \mu(\mathbb{R}^n) < N^*(K, T) + \varepsilon.$$

Restrict μ to the compact set $K - T$. This preserves all covering constraints because $x - T \subseteq K - T$ for each $x \in K$, and it cannot increase the mass. Thus we may assume that μ is compactly supported and still denote its total mass by M. Since $\mu(x - T) \ge 1$ for $x \in K$, we have $M \ge 1$.

Apply Lemma 2.2 to the probability measure $P = \mu/M$ with $\eta = \varepsilon/M$. The resulting finitely supported probability measure Q satisfies

$$Q(x - T) \ge P(x - T) - \frac{\varepsilon}{M} \qquad (x \in K),$$

because every $x - T$ is closed and convex. It follows that

$$MQ(x - T) \ge \mu(x - T) - \varepsilon \ge 1 - \varepsilon \qquad (x \in K).$$

Therefore

$$\nu_\varepsilon = \frac{M}{1 - \varepsilon} Q$$

is a finitely supported covering measure of K by T. Its mass satisfies

$$N^w(K, T) \le \nu_\varepsilon(\mathbb{R}^n) = \frac{M}{1 - \varepsilon} < \frac{N^*(K, T) + \varepsilon}{1 - \varepsilon}.$$

Let $\varepsilon \downarrow 0$ and use (3). □

The reflection in Corollary 1.2 is forced by convolution duality. Indeed, if $\mu * \mathbf{1}_T \geq \mathbf{1}_K$ and $\rho * \mathbf{1}_{-T} \leq 1$, then Tonelli's theorem gives

$$\begin{aligned}\rho(K) &\leq \int_{\mathbb{R}^n} (\mu * \mathbf{1}_T)(x)\, d\rho(x) \\ &= \int_{\mathbb{R}^n} (\rho * \mathbf{1}_{-T})(y)\, d\mu(y) \\ &\leq \mu(\mathbb{R}^n).\end{aligned}$$

This proves the weak dual inequality $M^*(K, -T) \leq N^*(K, T)$. The reverse inequality, together with equality of weighted and fractional separation, is (6).

Proof of Corollary 1.2. Apply Theorem 1.1 and [2, Theorem 1.3], with the separation set reflected as in (6). □

Remark 3.1. The discretization proof retains arbitrary boundary mass. It does not replace a covering measure by an absolutely continuous one. Such a replacement can change the optimum even for intervals. For $K = T = [0, 1]$, the measure δ_0 covers K, so $N^*(K, T) = N^w(K, T) = 1$. Every absolutely continuous covering measure μ, however, satisfies

$$\mu([-1, 0]) \geq 1, \qquad \mu([0, 1]) \geq 1, \qquad \mu(\{0\}) = 0.$$

Consequently $\mu(\mathbb{R}) \geq 2$. Absolute continuity is therefore incompatible with arbitrarily near-optimal covering in this example.

4 Open covering sets

We give a separate proof of Theorem 1.3. The role of openness is to replace infinitely many covering constraints by finitely many measurable subsets, each contained in the relevant translate throughout a neighborhood of its parameter.

Proof of Theorem 1.3. Again assume $K \neq \varnothing$. Fix $0 < \varepsilon < 1$ and choose a finite covering measure μ with

$$M := \mu(\mathbb{R}^n) < N^*(K, T) + \varepsilon.$$

Choose $0 < \alpha, \beta < 1$ so that

$$\frac{1}{\alpha\beta} < 1 + \varepsilon. \tag{9}$$

For $r > 0$ define the closed inner portion

$$T_r = \{z \in \mathbb{R}^n : \operatorname{dist}(z, T^c) \geq r\}.$$

If $T = \mathbb{R}^n$, use the convention $\operatorname{dist}(z, \varnothing) = +\infty$. Because T is open, T_r increases to T as $r \downarrow 0$.

For each $x \in K$, continuity from below gives a number $r_x > 0$ such that the Borel set

$$S_x = x - T_{r_x}$$

satisfies

$$\mu(S_x) \geq \alpha,$$

since $\mu(x - T) \geq 1$. If $|y - x| < r_x$, then

$$S_x \subseteq y - T. \tag{10}$$

Indeed, if $u = x - z$ with $z \in T_{r_x}$, then $y - u = z + (y - x) \in T$, which is precisely the asserted inclusion.

The balls $B_{r_x}(x)$ cover K. By compactness, choose a finite subcover with centers $x_1, \dots, x_\ell$. Write $S_i = S_{x_i}$. For every $y \in K$, at least one S_i is contained in $y - T$, and each S_i has μ-mass at least α.

Let $\xi_1, \xi_2, \dots$ be independent random vectors with common law $P = \mu/M$, and let

$$P_m = \frac{1}{m} \sum_{j=1}^{m} \delta_{\xi_j}.$$

The strong law of large numbers, applied to the finite family of indicators $\mathbf{1}_{S_i}$, implies that some realization and some integer m satisfy

$$MP_m(S_i) \geq \beta\mu(S_i) \geq \alpha\beta \qquad (1 \leq i \leq \ell).$$

For that realization define

$$\nu = \frac{M}{\alpha\beta} P_m.$$

Given $y \in K$, choose i with $S_i \subseteq y - T$. Then

$$\nu(y - T) \geq \frac{M}{\alpha\beta} P_m(S_i) \geq 1.$$

Thus ν is a finitely supported covering measure. By (9),

$$N^w(K, T) \leq \nu(\mathbb{R}^n) = \frac{M}{\alpha\beta} < (1 + \varepsilon)(N^*(K, T) + \varepsilon).$$

Let $\varepsilon \downarrow 0$ and use (3). □

5 A nonconvex example

The next example shows that null boundary and approximation by the interior do not suffice to replace convexity in Theorem 1.1.

Example 5.1. There are compact sets $K, T \subseteq \mathbb{R}^2$ such that K is a rectangle, T is connected and has nonempty interior,

$$T = \overline{\operatorname{int} T}, \qquad \operatorname{vol}_2(\partial T) = 0,$$

but

$$N^*(K, T) \leq \frac{8}{5} < 2 = N^w(K, T).$$

Construction and verification. Choose a compact nowhere dense set $A \subseteq [0, 3]$ with one-dimensional Lebesgue measure $|A| = 5/2$. Such a set can be obtained by removing middle open intervals in a Cantor construction whose total removed length is $1/2$. Define

$$h(x) = -\min\{\operatorname{dist}(x, A), 1/2\}, \qquad 0 \leq x \leq 3,$$

and set

$$T = \{(x, y) : 0 \leq x \leq 3,\ -1 \leq y \leq h(x)\}, \qquad K = [0, 1] \times [-1, 0].$$

The function h is Lipschitz, takes values in $[-1/2, 0]$, and vanishes exactly on A. Therefore T is compact and contains $[0, 3] \times [-1, -1/2]$. Every point can be joined vertically to the

bottom edge, so T is path connected. Continuity of h and the strictly positive height of each vertical section imply $T = \overline{\operatorname{int} T}$. Its boundary consists of the graph of h, the bottom edge, and two vertical edges; each has planar Lebesgue measure zero.

We first construct a fractional covering of mass $8/5$. Let $\mathcal{L}^1$ denote one-dimensional Lebesgue measure and put

$$\mu = \frac{2}{5}(\mathcal{L}^1|_{[-3,1]}) \otimes \delta_0.$$

For $(x, y) \in K$ and $z \in A$, we have $(z, y) \in T$ and $x - z \in [-3, 1]$. Thus

$$\begin{aligned}(\mu * \mathbf{1}_T)(x, y) &= \frac{2}{5}\int_{-3}^{1} \mathbf{1}_T(x - u, y)\, du \\ &\geq \frac{2}{5}|A| = 1.\end{aligned}$$

Consequently

$$N^*(K, T) \leq \mu(\mathbb{R}^2) = \frac{8}{5}.$$

Next, consider an arbitrary finite weighted covering of K by translates $T + (u_i, v_i)$ with weights $w_i \geq 0$. A translate with $v_i < 0$ does not meet the top edge $[0, 1] \times \{0\}$. If $v_i = 0$, its intersection with that edge has horizontal coordinates in $u_i + A$. The finite union

$$\bigcup_{\{i:v_i=0\}} (u_i + A)$$

is nowhere dense. Choose $x_0 \in (0, 1)$ outside this union. Only translates with $v_i > 0$ can cover $(x_0, 0)$, hence

$$\sum_{\{i:v_i>0\}} w_i \geq 1. \tag{11}$$

On the other hand, a translate with $v_i > 0$ lies strictly above the line $y = -1$ at its lowest point. The covering constraint at any point of the bottom edge therefore gives

$$\sum_{\{i:v_i\leq 0\}} w_i \geq 1. \tag{12}$$

Adding (11) and (12) shows that every finite weighted cover has total weight at least 2.

Finally, T and $T + (0, 1/2)$ cover K, because T contains $[0, 3] \times [-1, -1/2]$. Giving both translates weight one proves $N^w(K, T) \leq 2$, and therefore $N^w(K, T) = 2$. □

By (6), the same example satisfies

$$M^w(K, -T) = M^*(K, -T) = N^*(K, T) \leq \frac{8}{5} < N^w(K, T).$$

Thus equality between weighted covering and reflected weighted separation also fails under these regularity assumptions.

The fractional cover in Example 5.1 is supported on a horizontal line. Sampling finitely many points from this measure produces finitely many translates whose top-edge traces are translates of A. Those traces leave some points of the top edge uncovered, regardless of how their weights are increased. This illustrates why pointwise covering cannot in general be recovered from an approximation that ignores boundary incidence. Convexity permits the different one-sided approximation used in Lemma 2.2; openness instead supplies the local inclusions (10).

References

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