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SOME REMARKS ON POTENTIAL AUTOMORPHY: S_N EXTENSIONS, COMPATIBLE SYSTEMS AND DEFORMATION RINGS

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Abstract. We observe that in the potential modularity (automorphy) theorems of Taylor and of Barnet-Lamb–Gee–Geraghty–Taylor, the totally real field K over which modularity is obtained can be chosen to be Galois over the base field F with $\mathrm{Gal}(K/F) \cong S_N$, with N arbitrarily large, linearly disjoint from any prescribed finite extension of F, and such that every place in a prescribed finite set splits completely in K. The only possible loss of residual image after restriction to K comes from the quadratic field K^{A_N}. In dimension two this yields: (a) for finitely ramified, totally odd, regular de Rham representations over a totally real field F in which p splits completely, under the residual hypotheses of Section 6 and without assuming potential diagonalisability, potential modularity over an S_N extension in which p still splits completely, via residual modularity over K followed by Kisin's method; and (b) for applications to safe chains, a choice of K, made a priori in terms of a bound C, which preserves the residual images of all members of the compatible system in characteristic $p < C$ and their local behaviour at the specified places. We also show that, when $\mathrm{Gal}(K/F) \cong S_N$, the compatible system attached to a potentially modular two-dimensional representation can be constructed without Brauer's induction theorem, by gluing the automorphic representations over the fixed fields of the soluble subgroups of S_N. In a final section we record a direct way of proving finiteness of deformation rings from potential modularity: for $p \nmid n$ and fixed determinant, assuming absolute residual irreducibility over the Galois closure of K/F, the deformation ring over F is a quotient of the corresponding ring over K.

1. Introduction and statement

Throughout, F is a totally real number field, p is a prime, $n \geq 2$, and

$$\rho\colon G_F \longrightarrow \mathrm{GL}_n(\bar{\mathbb{Q}}_p)$$

is a continuous representation with semisimplified residual representation $\bar{\rho}\colon G_F \to \mathrm{GL}_n(\bar{\mathbb{F}}_p)$. We write ζ_p for a primitive p-th root of unity.

We say that ρ satisfies **(H)** if it admits a polarisation μ such that (ρ, μ) is a totally odd, regular algebraic, polarised representation in the sense of [3, §2.1, Cor. 4.5.2], ρ is unramified outside finitely many places and potentially diagonalisable at every place above p, $\bar{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible, and $p \geq 2(n+1)$. The proof uses this numerical bound both for the original representation and, through the constituent bound of [3, Prop. 2.1.2, Prop. 3.3.1], for an auxiliary induced representation of dimension $2n$. In dimension two, Remark 3.2 treats $p = 3, 5$ under the adequacy and projective image assumptions used in the proof of [8, Thm. 3.6(2)].

Theorem 1.1. *Assume that ρ satisfies* (H). *Let $N_0 \geq 1$ and let F^{avoid}/F be a finite extension. Then there exist an integer $N \geq N_0$ and a finite Galois extension K/F such that:*

(i) *K is totally real and $\mathrm{Gal}(K/F) \cong S_N$;*
(ii) *K is linearly disjoint from F^{avoid} over F;*
(iii) *$\rho|_{G_K}$ is automorphic.*

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Moreover, given any finite set S of finite places of F (for instance the places above p), K can be chosen so that in addition

(iv) *every place in S splits completely in K.*

The construction uses Moret-Bailly to find closed points of an auxiliary curve with prescribed local behaviour. Their sum is the polar divisor of a function with monodromy S_N (Lemma 2.2). Hilbert irreducibility then gives a fibre whose splitting field has group S_N and whose geometric points satisfy the local conditions (Proposition 2.3). Kadets also constructs closed points with symmetric Galois closure in sufficiently divisible degrees [25, Thm. 1.1]. Here the local condition is imposed on every conjugate of the point, as needed for automorphy.

Sections 3 and 4 prove (i)–(iv) together. Following the restriction of scalars argument of Snowden [38, §8], we absorb local conditions in a soluble auxiliary extension, construct the S_N field over the original base and apply soluble descent. Section 5 discusses the behaviour of residual images under restriction to S_N extensions, including a uniform version for families of residual representations with bounded ramification. Section 6 specialises to dimension two: uniformity in p via Dickson's classification, and, when p splits completely in $F/\mathbb{Q}$, potential modularity without potential diagonalisability over S_N extensions K/F in which every place above p splits completely (so $K_w = \mathbb{Q}_p$), via Kisin's method. Section 7 explains how to choose K a priori for applications to safe chains. Section 8 shows that, when $\mathrm{Gal}(K/F) \cong S_N$, the compatible system attached to a potentially automorphic two-dimensional ρ can be constructed without Brauer's induction theorem, by gluing the automorphic representations over the fields K^H with H soluble. Section 9 contains further remarks, in particular the reduction of modularity over F to descent along A_N extensions. Finally, Section 10 records a direct way of deducing finiteness of deformation rings over F from an $R = \mathbb{T}$ theorem over K: for $p \nmid n$ and fixed determinant, the deformation ring over F is a *quotient* of the one over K, for any finite extension K/F over whose Galois closure $\bar{\rho}$ stays absolutely irreducible.

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2. Points with S_N residue fields

2.1. **Moret-Bailly's theorem.** We use Moret-Bailly's theorem [32] in the prescribed completion form of [3, Prop. 3.1.1], with its two base fields equal and the avoidance field replaced by its Galois closure. The refinement asserting that the field can be taken linearly disjoint from a prescribed finite extension F^{avoid} (the "avoid" condition), which is what guarantees that residual images do not change, is stated in [23, Prop. 2.1]. It was also obtained by the author, who presented it at a conference in Strasbourg in July 2005, as is recorded in [23] (see also [38, Prop. 5.2.2]).

Theorem 2.1 (Moret-Bailly)**.** *Let T/F be a smooth, geometrically irreducible variety, and S a finite set of places of F. For each $v \in S$ let L_v/F_v be a finite Galois extension (with $L_v = \mathbb{R}$ if v is real) and $\Omega_v \subset T(L_v)$ a non-empty open $\mathrm{Gal}(L_v/F_v)$-invariant subset. Let F^{avoid}/F be finite. Then there exist a finite Galois extension F'/F, linearly disjoint from F^{avoid}, and $P \in T(F')$ such that for every $v \in S$ and every place $w \mid v$ of F' one has $F'_w \cong L_v$ over F_v and $P \in \Omega_v \subset T(F'_w)$.*

2.2. **The monodromy lemma.**

Lemma 2.2. *Let k be a field of characteristic 0, $\bar{T}/k$ a smooth projective geometrically irreducible curve of genus g, and D a reduced effective k-rational divisor on $\bar{T}$ of degree $N \geq 2g+3$. Let $V = L(D)$. There is a non-empty Zariski open subset $U \subset V$, defined over k, such that for every $f \in U$:*

(1) *f has simple poles exactly at the points of D;*
(2) *$f\colon \bar{T} \to \mathbb{P}^1$ has simple branching: above each branch value there is exactly one ramification point, with ramification index 2;*
(3) *the geometric and the arithmetic monodromy groups of f are both equal to S_N.*

Proof. We work over $\bar{k}$; the unions of the bad loci below and their Zariski closures are defined over k. If E is an effective divisor with support disjoint from D and $\deg(D-E) \geq 2g-1$, then $D-E$ is non-special, so $h^0(D) - h^0(D-E) = \deg E$ by Riemann–Roch, and the evaluation map

$$V \longrightarrow \bigoplus_{P \in \operatorname{supp} E} \mathcal{O}_P/\mathfrak{m}_P^{e_P}$$

is surjective (we use that $\mathcal{O}(D)_P = \mathcal{O}_P$ for $P \notin D$).

(1) The functions with pole at most $D - P_i$ for some $P_i \in D$ form $\bigcup_i L(D-P_i)$; since $\deg(D-P_i) \geq 2g-1$, each $L(D-P_i)$ is a hyperplane of V.

(2a) Let $\Sigma_1 = \{(f,P) : P \in \bar{T} \setminus D,\ f'(P) = f''(P) = 0\}$, derivatives taken in a local parameter. Taking $E = 3P$ (note $N - 3 \geq 2g-1$), the evaluation $V \to \mathcal{O}_P/\mathfrak{m}_P^3$ is surjective, so the two linear conditions are independent. Hence the fibres of $\Sigma_1 \to \bar{T}$ have dimension $\dim V - 2$, $\dim \Sigma_1 \leq \dim V - 1$, and the projection $\Sigma_1 \to V$ is not dominant.

(2b) Let $\Sigma_2 = \{(f,P,Q) : P \neq Q \in \bar{T} \setminus D,\ f'(P) = f'(Q) = 0,\ f(P) = f(Q)\}$. Taking $E = 2P + 2Q$ (here $N - 4 \geq 2g-1$ is used), the evaluation $V \to \mathcal{O}_P/\mathfrak{m}_P^2 \oplus \mathcal{O}_Q/\mathfrak{m}_Q^2 \cong \bar{k}^4$ is surjective, and the three linear forms $f'(P)$, $f'(Q)$, $f(P) - f(Q)$ are independent on $\bar{k}^4$. Hence $\dim \Sigma_2 \leq 2 + \dim V - 3 = \dim V - 1$ and $\Sigma_2 \to V$ is not dominant.

Points of D are simple poles, hence unramified. Let U be the complement in V of the closures of the bad loci in (1), (2a), (2b).

(3) With simple branching, the local monodromy around each branch value is a transposition, and their conjugates generate the geometric monodromy group G. Indeed, the quotient of the geometric Galois closure by the normal subgroup generated by the inertia groups is an everywhere unramified connected cover of $\mathbb{P}^1_{\bar{k}}$, hence trivial by Riemann–Hurwitz. It is transitive because $\bar{T}$ is geometrically irreducible. A transitive subgroup of S_N generated by transpositions is S_N (the graph of the transpositions is connected). The arithmetic monodromy group A satisfies $G \trianglelefteq A \leq S_N$, so $A = S_N$; in particular the Galois closure of $k(\bar{T})/k(f)$ is regular over k. □

2.3. **The S_N refinement of Moret-Bailly.**

Proposition 2.3. *Let T/F be a smooth geometrically irreducible curve, and let S, $(L_v, \Omega_v)_{v \in S}$ and F^{avoid} be as in Theorem 2.1; we assume S contains all archimedean places. Let $N_0 \geq 1$. Then there exist $N \geq N_0$ and a closed point $x \in T$ of degree N, with residue field F_x and Galois closure K of F_x/F, such that:*

(a) *$\mathrm{Gal}(K/F) \cong S_N$;*
(b) *K is linearly disjoint from F^{avoid} over F;*
(c) *for every $v \in S$, every F-embedding $F_x \hookrightarrow \bar{F}_v$ has image in L_v and sends x into Ω_v. Consequently, for every place $w \mid v$ of K, K_w is F_v-isomorphic to a subfield of L_v, and the image of $x \in T(K)$ in $T(L_v)$ lies in Ω_v.*

In particular K is totally real, and if $L_v = F_v$ (resp. L_v/F_v is unramified) then v splits completely (resp. is unramified) in K. One may further require that $F_v(\iota(F_x)) = L_v$ for each embedding ι in (c)*, and hence $K_w \cong L_v$ for every $w \mid v$.*

Proof. Let $\bar{T}$ be the smooth compactification of T, of genus g. We may and do replace F^{avoid} by its Galois closure over F; put $H = \mathrm{Gal}(F^{\mathrm{avoid}}/F)$.

For the additional assertion, replace Ω_v by the complement in it of all $T(L')$ for proper intermediate fields $F_v \subseteq L' \subsetneq L_v$. These are finitely many closed sets with empty interior. Indeed, near any L'-point an étale parameter defined over L' takes the L'-points into L', which has empty interior in L_v. The remaining open set is nonempty and Galois stable. A point in it generates L_v over F_v.

Step 1: seeds. Apply Theorem 2.1 to obtain $P_1 \in T(F_1')$ as there, and let x_1 be the closed point of T under P_1. Every F-embedding of the residue field of x_1 into $\bar{F}_v$ factors through some $F'_{1,w} \cong L_v$, and since Ω_v is $\mathrm{Gal}(L_v/F_v)$-stable, every geometric point of x_1 (viewed over $\bar{F}_v$) lies in $\Omega_v \subset T(L_v)$. Repeat with Ω_v replaced by Ω_v minus the finitely many geometric points already obtained (still non-empty, open and Galois-stable), obtaining pairwise distinct closed points $x_1, \dots, x_k$. Put $D = x_1 + \cdots + x_k$, a reduced effective F-rational divisor; choose k so that $N := \deg D \geq \max(N_0, 2g+3)$. For every $v \in S$, all N geometric points of D lie in $\Omega_v \subset T(L_v)$.

Step 2: the function. By Lemma 2.2 (with $k = F$, which is infinite) choose $f \in U(F)$. Let M be the Galois closure of $F(\bar{T})/F(f)$; then $\mathrm{Gal}(M/F(f)) \cong S_N$ and M is regular over F.

Step 3: local analysis near the poles. Fix $v \in S$ and an embedding $\bar{F} \hookrightarrow \bar{F}_v$. For each geometric point P of D, the function $u = 1/f$ is regular at P with a simple zero, so by the v-adic (or real) inverse function theorem there is a neighbourhood U_P of P in $T(L_v)$ mapped by u bijectively onto a disc $\{z \in L_v : |z|_v < \varepsilon_v\}$. Shrinking, we may assume the U_P are pairwise disjoint and contained in Ω_v. If $t \in F$ satisfies $|t|_v > \varepsilon_v^{-1}$, each U_P contains exactly one point Q with $f(Q) = t$, and $Q \in T(L_v)$. This produces N distinct points of $f^{-1}(t)$; since $\deg f = N$, these are all the geometric points of the fibre (and t is not a branch value).

Step 4: Hilbert irreducibility. Since M is regular over F and $F^{\mathrm{avoid}}(f)/F(f)$ is a constant extension, $M \cap F^{\mathrm{avoid}}(f) = F(f)$ and $\mathrm{Gal}(MF^{\mathrm{avoid}}/F(f)) \cong S_N \times H$. The set of $t \in F$ at which this Galois extension specialises with full Galois group $S_N \times H$ contains a Hilbert subset of F [14, Ch. 13, 16]. Hilbert subsets of a number field are dense in $\prod_{v \in S} F_v$ (Geyer's lemma; see [15] and [14, Ch. 13]). Choose such t with $|t|_v > \varepsilon_v^{-1}$ for all $v \in S$ and $t \notin f(\bar{T} \setminus T)$.

Step 5: conclusion. Since S_N acts transitively on the fibre, $f^{-1}(t)$ is a single closed point $x \in T$ of degree N; its residue field is F_x and the specialisation of M is its Galois closure K, with $\mathrm{Gal}(K/F) \cong S_N$, proving (a). The specialisation of MF^{avoid} is KF^{avoid}, of degree $N!\,|H|$ over F, proving (b). Step 3 gives (c): the N geometric points of the fibre are the F-embeddings of F_x, all landing in L_v and in Ω_v, and K is generated by their images. The last assertions follow: if all images of F_x lie in F_v (resp. in an unramified L_v), the same holds for the compositum K of its conjugates. For real v we have $L_v = \mathbb{R}$, so K is totally real. □

Remark 2.4. The integers N produced are degrees of admissible divisors D as in Step 1. Once one such divisor D_0 of degree $N_1 \geq 2g+3$ is available, the fibres obtained in Step 5 are pairwise disjoint admissible points of degree N_1, and sums of j of them are admissible divisors of degree jN_1. Hence every multiple jN_1, $j \geq 1$, occurs as an N.

3. Proof of Theorem 1.1

We go through the proof of [3, Cor. 4.5.2], which is the totally real case of [3, Thm. 4.5.1] and rests on [3, Prop. 3.3.1, Thm. 3.1.2], keeping track of the fields involved. (The method goes back to [40, 23].)

Step A: a CM field [3, proof of Cor. 4.5.2]. Let D be the Galois closure over F of $F^{\mathrm{avoid}\,\bar{F}^{\ker\bar\rho}}(\zeta_p)$. Choose a totally imaginary quadratic extension E/F, split at every place above p and linearly disjoint from D. For instance, weak approximation gives a defining square class which is negative at every real place, a square at the places above p, and of odd valuation at a further odd prime unramified in D. Hence $\zeta_p \notin E$ and the residual image over $E(\zeta_p)$ is the original image over $F(\zeta_p)$. Use the totally odd CM polarisation of $\rho|_{G_E}$ as in the proof of the cited corollary (in the orthogonal case one multiplies the even multiplier by the quadratic character of E/F). It suffices to prove that $\rho|_{G_{EF'}}$ is automorphic for a suitable totally real F'; then $\rho|_{G_{F'}}$ is automorphic by [3, Lemma 2.2.2].

Step B: a first soluble base change [3, proof of Prop. 3.3.1]. By [3, Lemma A.2.1] one chooses a finite, soluble, Galois, totally real extension F_a/F, linearly disjoint from ED over F. Prescribe the local extensions in the proof of [3, Prop. 3.3.1] at all places above p or where the residual representation or its multiplier ramifies. We may make the multiplier crystalline above p, make all the marked places split in EF_a/F_a, and arrange the ordinary crystalline local lifts of the specified regular weights. Away from p the local lifts can be taken from ρ. Over F_a one then has an ordinary lift r of $\bar\rho|_{G_{EF_a}}$ [3, Prop. 3.2.1], a suitable character ψ of G_{EF_a}, and the residual representation

$$\bar R = \mathrm{Ind}_{G_{EF_a}}^{G_{F_a}}(\bar\rho\otimes\bar\psi)\colon G_{F_a}\longrightarrow \mathrm{GSp}_{2n}(\bar{\mathbb{F}}_p),$$

which is symplectic with multiplier $\bar\epsilon_p^{1-2n}$. The lift r is defined over EF_a, even though the existence theorem for r uses a further field of potential automorphy in its proof. Here ϵ_p denotes the cyclotomic character. As in the cited proof, choose the weights of ψ so that the two summands of the induction at each split place over p are ordinary with separated weights in a consistent order at all embeddings above that place. Ramification of $\bar\psi/\bar\psi^c$ at an auxiliary split place distinguishes the conjugate residual summands over $EF_a(\zeta_p)$, so $\bar R|_{G_{F_a(\zeta_p)}}$ is irreducible. The subgroup generated by its p-elements preserves the two n-dimensional summands since p is odd. Its irreducible constituents therefore have dimension at most n, and [3, Prop. 2.1.2] gives adequacy under (H).

Step C: the Dwork family [3, proof of Thm. 3.1.2], applied to $\bar R$ over F_a. Fix a finite Galois extension $B/\mathbb{Q}$ containing F_aED and the field cut out by $\bar R$. Choose the Dwork integer m (denoted N in [3]) with the congruence and coefficient field conditions of [3, proof of Thm. 3.1.2], taking m prime to the rational primes ramified in B. Then $\mathbb{Q}(\zeta_m)$ and B are linearly disjoint over $\mathbb{Q}$, since their intersection would be unramified at every finite prime. Choose the auxiliary prime ℓ and the induced symplectic representation r' of $G_{\mathbb{Q}}$ as in that proof. In particular, ℓ splits in $\mathbb{Q}(\zeta_m)$, $\bar r'$ is induced from a character of a cyclic CM field J of degree $2n$, and its image on $G_{F_a(\zeta_{m\ell})}$ agrees with its adequate image on $G_{\mathbb{Q}(\zeta_\ell)}$. In making the auxiliary choices one can require J to be unramified at every prime dividing m as well as at those ramifying in B. Choose the inducing character unramified above m and the primes ramifying in F_a, and take ℓ unramified in $F_a(\zeta_m)$. Put $B_\ell = \bar{\mathbb{Q}}^{\ker\bar r'}(\zeta_\ell)$. The finite primes ramifying in B_ℓ are unramified in the Galois closure of $F_a(\zeta_m)$ over $\mathbb{Q}$, so its intersection with B_ℓ is $\mathbb{Q}$. Put

$$F_* = F_a(\zeta_m)^+.$$

Let $T_0 = \mathbb{P}^1\setminus(\{\infty\}\cup\mu_m)$, with coordinate t, be the parameter space of the Dwork family, and let $T\to T_0$ be the finite cover, defined over F_*, parametrising points of T_0 together with

symplectic isomorphisms of the normalised mod p and mod ℓ realisations of the corresponding motive with $\bar{R}$ and $\bar{r}'$. It is a smooth geometrically irreducible curve over F_*. Let

$$F^{\mathrm{av}} = B\,J\,\bar{\mathbb{Q}}^{\ker \bar{r}'}(\zeta_{m\ell}),$$

where $\bar{\mathbb{Q}}^{\ker \bar{r}'}$ denotes the field cut out by $\bar{r}'$ over $\mathbb{Q}$. We work in a common algebraic closure. Moret-Bailly's theorem [3, Prop. 3.1.1] is applied to T over F_* and produces a field F' and a point P with the following properties:

(P1) F' is a finite totally real extension of F_*;
(P2) $P \in T(F')$;
(P3) F' is linearly disjoint from F_*F^{av} over F_*;
(P4) $v(t(P)) < 0$ for every place $v \mid p$ of F', and $v(t(P)) > 0$ for every place $v \mid \ell$ of F'.

Step D. Automorphy over F'. Suppose that F' and P satisfy (P1)–(P4). Since $F_* = F_a\mathbb{Q}(\zeta_m)^+$ and $\mathbb{Q}(\zeta_m) \cap B = \mathbb{Q}$, we have $F_* \cap B = F_a$. By (P3), $F' \cap B = F_a$, whence $F' \cap ED = F$. Thus EF' does not contain ζ_p, and the image of $\bar{\rho}$ over $EF'(\zeta_p)$ equals its image over $F(\zeta_p)$. Likewise the image of $\bar{R}$ over $F'(\zeta_p)$ is its image over $F_a(\zeta_p)$. Applying (P3) also to the auxiliary residual and cyclotomic fields shows that the image of $\bar{r}'$ over $F'(\zeta_\ell)$ equals its image over $F_*(\zeta_\ell)$ and therefore over $\mathbb{Q}(\zeta_\ell)$. Indeed the image over the intermediate field $F_*(\zeta_\ell)$ contains the full image over $F_a(\zeta_{m\ell})$. Both induced images are thus adequate. In particular the characteristic zero inductions are irreducible upon restriction to $G_{F'}$.

The point P identifies the two residual realisations. At ℓ, the inequality $v(t(P)) > 0$ gives ordinarity of the ℓ-adic realisation. At p, the inequality $v(t(P)) < 0$ gives a Steinberg Weil–Deligne parameter for that realisation [4, Lemmas 5.3(3), 5.1(2)]. By cyclic automorphic induction [2], $r'|_{G_{F'}}$ is ordinarily automorphic. Apply [3, Thm. 2.4.1] at ℓ to the Dwork realisation. The resulting weight zero automorphic representation is Steinberg above p and is ordinary for the p-adic embedding, as in the conclusion of [3, proof of Thm. 3.1.2]. Its mod p representation is $\bar{R}|_{G_{F'}}$. A second application of the ordinary lifting theorem, now at p, gives automorphy of the ordinary induction of $r \otimes \psi$ on $G_{F'}$. By [3, Lemmas 2.2.4 and 2.2.1] we extract the inducing representation over EF' and remove the character twist. This gives an ordinary automorphic lift of $\bar{\rho}|_{G_{EF'}}$ with the required residual polarisation. Consequently [3, Thm. 4.2.1] applies to the potentially diagonalisable $\rho|_{G_{EF'}}$. Its irreducibility and Step A now give automorphy of $\rho|_{G_{F'}}$. The conclusion needed after descent is automorphy. At a place of F that splits in the final field K, the local Weil–Deligne representation retains its original inertial type.

Conclusion. The extension F_*/F is finite, soluble, Galois and totally real (F_a/F is soluble and Galois, and $F(\zeta_m)^+/F$ is abelian). Take $N \geq \max(N_0, 5)$. The largest soluble quotient of S_N has order two, so a Galois S_N extension of F could contain F_* only if $[F_* : F] \leq 2$. We apply Proposition 2.3 to a curve in a Weil restriction of T: Steps 1–5 of the proof of Theorem 4.3 below (with $S = \emptyset$) produce a finite, soluble, Galois, totally real extension F_1/F containing F_*, and a totally real Galois extension K/F with $\mathrm{Gal}(K/F) \cong S_N$, $N \geq N_0$, linearly disjoint over F from F_1 and from the Galois closure of F_*F^{av}, together with a point of T over KF_1, such that $F' = KF_1$ satisfies (P1)–(P4). By Step D, $\rho|_{G_{KF_1}}$ is automorphic. Since KF_1/K is Galois with soluble group $\mathrm{Gal}(F_1/F)$ and $\rho|_{G_{KF_1}}$ is irreducible, $\rho|_{G_K}$ is automorphic by [3, Lemma 2.2.2]. □

Remark 3.1 (Choice of N in prescribed families). In Theorem 1.1 (including (iv)), and in Theorems 4.3 and 7.1 below, N can be required to lie in any prescribed set $\mathcal{A} \subset \mathbb{Z}_{>0}$ containing infinitely many multiples of every positive integer, for instance

$$\mathcal{A} = \{x! : x \geq 1\}, \qquad \mathcal{A} = \{\mathrm{lcm}(1, \ldots, x) : x \geq 1\}, \qquad \mathcal{A} = \{x^x : x \geq 1\}.$$

Indeed, by Remark 2.4 the possible values of N include all multiples of some N_1 which depends on the curve and on the first admissible divisor, and $\mathcal{A}$ contains such multiples beyond any bound. In particular $\mathrm{Gal}(K/F)$ can be taken to be $S_{x!}$ for some x, so that the only non-abelian composition factor is $A_{x!}$, and the reduction of Section 9 is to descent along an extension with group $A_{x!}$. The condition on $\mathcal{A}$ cannot be dropped in general: all admissible points of the curve may have degree divisible by some integer $m > 1$ (an "admissible index"), and then N must be divisible by m. Sets such as the primes or the powers of 2 are therefore not covered by this argument.

Remark 3.2 (Dimension two, $p = 3, 5$). For $n = 2$ and $p \in \{3, 5\}$ the conclusions of Theorem 1.1 also hold for every finitely ramified, totally odd, regular, potentially diagonalisable ρ for which $\bar\rho(G_{F(\zeta_p)})$ is adequate and, if $p = 5$, the projective image of $\bar\rho$ is neither $\mathrm{PSL}_2(\mathbb{F}_5)$ nor $\mathrm{PGL}_2(\mathbb{F}_5)$. We use the argument in the proof of [8, Thm. 3.6(2)]. Take K from Theorem 6.2. It satisfies the field conditions and $\bar\rho|_{G_K}$ has a modular lift. Put $D_K = K\bar F^{\ker\bar\rho}(\zeta_p)$. By Lemma 4.1 choose a totally real soluble Galois extension W/K, disjoint from D_K over K, whose completions at p trivialise $\bar\rho$. The modular lift over K stays irreducible over W, so its soluble base change supplies residual modularity there. The residual image over $W(\zeta_p)$ is still adequate, and the local restrictions above p are reducible. By [9, Thm. 2.1.2(2)], $\bar\rho|_{G_W}$ has an ordinary modular lift. Choose a CM quadratic extension M/W, split at p, disjoint from WD_K over W. Its cyclotomic residual image is unchanged, $\zeta_p \notin M$, and the ordinary lift base changes cuspidally to M. The residual polarisations agree, since in dimension two the determinants of both lifts reduce to $\det\bar\rho$. Hence [7, Thm. 1] gives automorphy of $\rho|_{G_M}$. Apply [3, Lemma 2.2.2] successively to M/W and W/K, using absolute irreducibility over M. This proves the assertion over the original K.

4. Forcing p to split

4.1. **What Proposition 2.3 gives directly.** In the proof of Section 3 the point of the Dwork curve T lives over KF_1, whose completions above p are the fields L_v; these are in general ramified, since they must contain the completions of F_* and carry points of T with $v(t) < 0$. The field K itself, however, is obtained from Proposition 2.3 applied to a curve over F with local conditions in F_v-points, and by Proposition 2.3(c) it is split at every place where such a condition is imposed. Thus the passage through the soluble extension F_1 and the final descent, which the structure of the argument of [3] forces on us in any case, also make K split above p; and the same device allows one to prescribe the splitting of an arbitrary finite set of places. This is the content of Theorem 4.3, whose proof also supplies the construction used in Section 3.

4.2. **Two auxiliary lemmas.**

Lemma 4.1. *Let Σ be a finite set of places of F containing the archimedean ones; for $v \in \Sigma$ let L_v/F_v be a finite Galois extension, with $L_v = \mathbb{R}$ for v real. Let M/F be finite. There is a finite solvable Galois extension F_1/F, totally real, linearly disjoint from M, such that $F_{1,w} \cong L_v$ over F_v for every $v \in \Sigma$ and every place $w \mid v$ of F_1.*

This is [3, Lemma A.2.1]: given $F^{(\mathrm{avoid})}/F$ finite Galois, a finite set S of places and finite Galois E_v/F_v for $v \in S$, there is a finite soluble Galois E/F, linearly disjoint from $F^{(\mathrm{avoid})}$, with $E_w/F_v \cong E_v/F_v$ for all $w \mid v \in S$. Taking $E_v = \mathbb{R}$ at the real places makes E totally real. (Replace M by its Galois closure.) The same device of a restriction of scalars along such an extension, to prove potential automorphy over fields with prescribed completions above p, appears in [13, App. A], attributed there to R. Taylor, and in [38, Prop. 8.2.2].

Lemma 4.2. *Let Y/F be a smooth, geometrically irreducible, quasi-projective variety of positive dimension and $Z \subset Y$ a finite reduced closed subscheme. There is a smooth geometrically irreducible curve $C \subset Y$, defined over F, containing Z.*

Proof. If $\dim Y = 1$ take $C = Y$. Otherwise fix a projective embedding $Y \subset \mathbb{P}^M$ and let $d = \dim Y \geq 2$. Since we are in characteristic 0, Z is smooth of dimension 0, so $\dim Y > 2 \dim Z$; by the Bertini theorem for hypersurface sections containing a smooth subscheme [1] (for an infinite field; see also [34] and the survey in [16]), a general hypersurface H_1 of large degree containing Z has $Y_1 = Y \cap H_1$ smooth of dimension $d-1$ (including at the points of Z). For degree large, the linear system of hypersurfaces through Z defines a morphism $Y \setminus Z \to \mathbb{P}^N$ which is an immersion, so its image has dimension $d \geq 2$, and by Jouanolou's Bertini theorem [24, Thm. 6.3(4)] $Y_1 \setminus Z$, hence Y_1, is geometrically irreducible for general H_1 (as Y_1 is smooth of positive dimension, Z lies in the closure of $Y_1 \setminus Z$). Both conditions are open and non-empty, so a general H_1 satisfying both exists over F, which is infinite. Iterating $d-1$ times (each step starts from a smooth geometrically irreducible variety of dimension ≥ 2 containing Z) gives C. □

4.3. **The splitting statement.**

Theorem 4.3. *In the setting of Theorem 1.1, let S be a finite set of finite places of F. Then K can be chosen satisfying* (i)–(iv).

Proof. Let F_*, T and F^{av} be as in Section 3, and let $\widetilde{F}$ be the Galois closure over F of $F_* F^{\mathrm{av}} \bar{F}^{\ker \bar{\rho}}(\zeta_p)$. Let Σ_0 be the set of places of F above $p\ell\infty$ and $\Sigma = \Sigma_0 \cup S$. For each finite $v \in \Sigma$ choose a finite Galois extension L_v/F_v containing the completions of F_* at the places above v, and so large that for every place w of F_* above v the open set

$$\Omega_w \subset T(L_v)$$

defined by the condition $v(t) < 0$ if $v \mid p$, by $v(t) > 0$ if $v \mid \ell$, and equal to $T(L_v)$ otherwise, is non-empty (here L_v is regarded as an extension of $F_{*,w}$). For a finite v the required points exist after finite extension. For each completion of F_* choose a nonbranch value of t of the indicated valuation (at the other places choose any value in T_0), then split the finite fibre. A common Galois closure over F_v works for every embedding, and the resulting valuation conditions are open. For v real put $L_v = \mathbb{R}$ and $\Omega_w = T(\mathbb{R})$, which is non-empty for each real place w of F_* [3, proof of Thm. 3.1.2].

Step 1: absorbing the local conditions. By Lemma 4.1 choose F_1'/F finite, solvable, Galois, totally real, linearly disjoint from $\widetilde{F}$, with $F'_{1,w} \cong L_v$ for all $w \mid v$, $v \in \Sigma$, and put $F_1 = F_* F_1'$. Then F_1/F is finite, solvable, Galois and totally real, it contains F_*, and $F_{1,u} \cong L_v$ for every place $u \mid v$ of F_1, $v \in \Sigma$.

Step 2: Weil restriction. Let $Y = \mathrm{Res}_{F_1/F}(T_{F_1})$, a smooth geometrically irreducible variety over F, with $Y(F_v) = \prod_{u|v} T(F_{1,u})$ for $v \in \Sigma$, the product being over the places u of F_1 above v. Put $\Omega_v' = \prod_{u|v} \Omega_{u|_{F_*}} \subset Y(F_v)$, a non-empty open subset.

Step 3: a seed and a curve. Apply Theorem 2.1 to Y with the places Σ, local fields F_v (resp. $\mathbb{R}$) and open sets Ω_v': this gives a closed point $y \in Y$ all of whose geometric points lie in $\Omega_v' \subset Y(F_v)$ for every $v \in \Sigma$. By Lemma 4.2 choose a smooth geometrically irreducible curve $C \subset Y$ over F containing y. Then $\Omega_v'' := C(F_v) \cap \Omega_v'$ is a non-empty open subset of $C(F_v)$ for each $v \in \Sigma$.

Step 4: the S_N construction. Apply Proposition 2.3 to C, with places Σ, local fields F_v (resp. $\mathbb{R}$), open sets Ω_v'', avoidance field $F_1 \widetilde{F} F^{\mathrm{avoid}}$ and bound N_0. We obtain a closed point $x \in C$ of degree $N \geq N_0$, with residue field F_x and Galois closure K, such that $\mathrm{Gal}(K/F) \cong S_N$,

K is linearly disjoint from $F_1\widetilde{F}\,F^{\mathrm{avoid}}$ over F, every $v \in \Sigma$ (in particular every $v \in S$) splits completely in K, K is totally real, and all F-embeddings of F_x into F_v send x into Ω'_v.

Step 5: a point of T over KF_1. Since K and F_1 are linearly disjoint over F, $F_x \otimes_F F_1 = F_xF_1$ and $x \in Y(F_x) = T(F_xF_1) \subset T(KF_1)$. We check (P1)–(P4) of Section 3 for $F' = KF_1$. It is a finite totally real extension of F_*, Galois over F with group $S_N \times \mathrm{Gal}(F_1/F)$, and it carries a point of T. Since K and F_1' are Galois over F, linearly disjoint from each other and KF_1' is linearly disjoint from $\widetilde{F}$ over F, the field $KF_1 = KF_1'F_*$ is linearly disjoint from $\widetilde{F} \supseteq F_*F^{\mathrm{av}}$ over F_*; this is (P3). For $v \in \Sigma_0$ and a place u' of KF_1 above v, the completion $(KF_1)_{u'}$ equals $F_{1,u} \cong L_v$, where $u = u'|_{F_1}$, because K is split at v; under the corresponding embedding the point of T is the u-component of the image of x in $Y(F_v)$, which lies in $\Omega_{u|_{F_*}}$ by the definition of Ω'_v. This is (P4). Hence, by Step D of Section 3, $\rho|_{G_{KF_1}}$ is automorphic.

Step 6: solvable descent. The extension KF_1/K is Galois with group $\mathrm{Gal}(F_1/F)$, which is solvable, and $\rho|_{G_{KF_1}}$ is irreducible: Step D applied to $F' = KF_1$ identifies the image over $EF'(\zeta_p)$ with the absolutely irreducible original image over $F(\zeta_p)$. By solvable descent [3, Lemma 2.2.2] (for M/F soluble Galois and totally real or CM, and (r,μ) polarised with $r|_{G_M}$ irreducible, (r,μ) is automorphic if and only if its restriction to G_M is; in dimension two this is Langlands' base change [31]), $\rho|_{G_K}$ is automorphic. Thus K satisfies (i)–(iv). □

Remark 4.4. The field KF_1 over which the moduli point lives is in general ramified above p; it is the descent along the solvable extension KF_1/K that produces a modularity field split above p. This is exactly Snowden's mechanism [38, Props. 8.2.1, 8.2.2], in which the analogue of K is the field F_2 produced by Moret-Bailly applied to the Weil restriction; the only new point here is that, by passing through a curve $C \subset Y$ and Proposition 2.3, this field can be taken with Galois group S_N.

Remark 4.5 (Arbitrary S). The set S in Theorem 4.3 is an arbitrary finite set of finite places of F: for $v \in S$ one only needs some finite Galois L_v/F_v with $T(L_v) \neq \emptyset$, which always exists, and the behaviour of ρ at v plays no role. In particular S may contain places where ρ ramifies, places above p whether or not p splits in F, and places above any finite set of primes. Since $K_w = F_v$ for $w \mid v \in S$, one has $\rho|_{G_{K_w}} = \rho|_{G_{F_v}}$: local types, conductor exponents and residual local behaviour at the places of S are unchanged in K. The conditions "S splits", "$K \cap F^{\mathrm{avoid}} = F$" and "$N \geq N_0$" are all imposed simultaneously in Proposition 2.3.

Remark 4.6. Under (H) the splitting of p is not needed for the automorphy lifting theorems of [3] (potential diagonalisability is insensitive to finite extensions). Its interest lies in applications requiring p split in the modularity field, for instance modularity lifting theorems or local-global compatibility results only available when $F_v = \mathbb{Q}_p$ for all $v \mid p$.

5. Adequacy under S_N extensions

Let $\bar\rho\colon G_F \to \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ be continuous, let $F^{\bar\rho} = \bar{F}^{\ker\bar\rho}(\zeta_p)$, $\Delta = \mathrm{Gal}(F^{\bar\rho}/F)$, and $\Gamma = \bar\rho(G_{F(\zeta_p)})$.

Lemma 5.1. *Let K/F be finite Galois with $\mathrm{Gal}(K/F) \cong S_N$, $N \geq 5$, and assume S_N is not a quotient of Δ (e.g. $N! > |\Delta|$). Let $F(\sqrt{\delta}) = K^{A_N}$. Then $K \cap F^{\bar\rho} \subseteq F(\sqrt{\delta})$, and*

$$\bar\rho(G_{K(\zeta_p)}) \;=\; \Gamma' := \bar\rho(G_{F(\sqrt{\delta},\zeta_p)}), \qquad [\Gamma : \Gamma'] \leq 2.$$

Proof. $\mathrm{Gal}(K \cap F^{\bar\rho}/F)$ is a common quotient of S_N and Δ. The quotients of S_N are 1, $\mathbb{Z}/2$ and S_N, and the last is excluded by hypothesis; so $K \cap F^{\bar\rho} \subseteq F(\sqrt{\delta})$. Write $B = F^{\bar\rho}$, $C = F(\zeta_p)$, $D = K^{A_N}$ and $I = K \cap B$. Then $I = D \cap B$. Since K and B are Galois and linearly

disjoint over I, the image of G_K in Δ is $\mathrm{Gal}(B/I)$. The same holds for G_D. Intersecting these images with $\mathrm{Gal}(B/C)$ gives $\mathrm{Gal}(B/IC)$ in both cases, since the cyclotomic character factors through Δ. Projecting to the residual image gives the equality. The index bound follows from $[IC : C] \leq 2$. □

Proposition 5.2. *In the situation of Lemma 5.1, assume Γ acts absolutely irreducibly. Then one of the following holds:*

(a) $\bar{\rho}|_{G_{K(\zeta_p)}}$ *is absolutely irreducible; in particular it is adequate if $p \geq 2(n+1)$ [17];*

(b) $[\Gamma : \Gamma'] = 2$, $\bar{\rho}(G_{K(\zeta_p)}) = \Gamma'$, *and* $\bar{\rho}|_{G_{F(\zeta_p)}} \cong \mathrm{Ind}_{G_{F(\sqrt{\delta},\zeta_p)}}^{G_{F(\zeta_p)}} \sigma$ *for some σ of dimension $n/2$. For $n = 2$ this means that $\bar{\rho}|_{G_{F(\zeta_p)}}$ has dihedral projective image and $F(\sqrt{\delta}, \zeta_p)$ is a quadratic extension of $F(\zeta_p)$ from which it is induced.*

Proof. Write $H = \Gamma' = \bar{\rho}(G_{K(\zeta_p)})$, using Lemma 5.1, and let V be the space of Γ. If $V|_H$ is reducible then $[\Gamma : H] = 2$. Choose a simple H-submodule W and $s \in \Gamma \setminus H$. Irreducibility over Γ implies $V = W \oplus sW$, since $W + sW$ is Γ-stable and W, sW are distinct simple submodules. Thus $V \simeq \mathrm{Ind}_H^\Gamma W$. Moreover W and sW are non-isomorphic, since otherwise Frobenius reciprocity would give $\dim \mathrm{End}_\Gamma(V) \geq 2$. This also proves the assertion in residual characteristic two. In dimension two H acts by the two distinct characters on its lines and s exchanges them, giving the dihedral description. The adequacy assertion follows from [17]. □

Thus the S_N structure alone guarantees that the residual image stays irreducible, hence adequate for $p \geq 2(n+1)$, except in the induced case (b), which only involves the single quadratic field $F(\sqrt{\delta}) = K^{A_N}$. There are finitely many quadratic subextensions of $F^{\bar{\rho}}/F$, and case (b) can occur only if $F(\sqrt{\delta})$ is one of them.

Corollary 5.3. *In Theorem 1.1 and Proposition 2.3, by including $F^{\bar{\rho}}$ in F^{avoid} one gets $K \cap F^{\bar{\rho}} = F$, hence $\bar{\rho}(G_{K(\zeta_p)}) = \Gamma$: the residual image over $K(\zeta_p)$ is literally unchanged, and adequacy is preserved with no assumption on p. In particular the dihedral case* (b) *is excluded. It suffices in fact to require that $F(\sqrt{\delta})$ avoids the finitely many quadratic subextensions of $F^{\bar{\rho}}/F$, a condition on the class of the discriminant $\delta = \delta(t)$ modulo squares which is of Hilbert type in t.*

Remark 5.4. For the auxiliary representation $\bar{r}'$ the same observation applies at ℓ. In Sections 3 and 4 the avoidance field is enlarged after $\bar{r}'$ has been chosen, before specialising the curve.

5.1. **A uniform version.**

Lemma 5.5. *Let Σ be a finite set of places of F. There are only finitely many quadratic extensions of F unramified outside Σ. We denote by Q_Σ their compositum, a finite multiquadratic extension of F.*

Proof. A nonarchimedean completion of F has finitely many quadratic extensions, since its square class group is finite. The local discriminant exponents at the finite places in Σ are therefore bounded. The relative discriminants have bounded norm, and Hermite–Minkowski gives the assertion. □

Proposition 5.6. *Let Σ be a finite set of places of F containing the archimedean places, and let $\mathcal{R}$ be a set (possibly infinite) of continuous representations $\bar{\rho}\colon G_F \to \mathrm{GL}_n(\bar{\mathbb{F}}_p)$, with $p = p(\bar{\rho})$ varying, such that each $F^{\bar{\rho}} = \bar{F}^{\ker \bar{\rho}}(\zeta_p)$ is unramified outside Σ and S_N is not a quotient of $\mathrm{Gal}(F^{\bar{\rho}}/F)$. Let K/F be Galois with $\mathrm{Gal}(K/F) \cong S_N$, $N \geq 5$, and $K \cap Q_\Sigma = F$. Then for every $\bar{\rho} \in \mathcal{R}$*

$$\bar{\rho}(G_{K(\zeta_p)}) = \bar{\rho}(G_{F(\zeta_p)}).$$

Proof. By Lemma 5.1, $K \cap F^{\bar\rho} \subseteq F(\sqrt{\delta}) = K^{A_N}$. If $F(\sqrt{\delta}) \subseteq F^{\bar\rho}$, then $F(\sqrt{\delta})$ is unramified outside Σ, hence contained in Q_Σ, contradicting $K \cap Q_\Sigma = F$. So $K \cap F^{\bar\rho} = F$, and the image of $G_{K(\zeta_p)}$ in $\mathrm{Gal}(F^{\bar\rho}/F)$ is $\mathrm{Gal}(F^{\bar\rho}/F(\zeta_p))$. □

The point is that the field to be avoided, Q_Σ, depends only on F and Σ, not on the representations in $\mathcal{R}$, which may be unknown at the moment K is chosen.

6. Dimension two

6.1. Uniformity in p.

Lemma 6.1. *Let $N \geq 6$. For every prime p and every continuous $\bar\rho\colon G_F \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$, the group S_N is not a quotient of $\mathrm{Gal}(F^{\bar\rho}/F)$.*

Proof. Identify $\mathrm{Gal}(F^{\bar\rho}/F)$ with the image of $(\bar\rho, \bar\chi_p)$ in $\mathrm{GL}_2(\bar{\mathbb{F}}_p) \times \mathbb{F}_p^\times$. Its map to the projective image of $\bar\rho$ has central kernel, consisting of elements with scalar first coordinate. Every surjection to S_N therefore factors through the projective image, since S_N has trivial centre. By Dickson's classification of finite subgroups of $\mathrm{PGL}_2(\bar{\mathbb{F}}_p)$ [12], its non-abelian composition factors are of the form $\mathrm{PSL}_2(\mathbb{F}_q)$ or A_5. A quotient isomorphic to S_N would give A_N as a composition factor. For $N \geq 7$, A_N is not isomorphic to any $\mathrm{PSL}_2(\mathbb{F}_q)$ (the only coincidences being $A_5 \cong \mathrm{PSL}_2(\mathbb{F}_4) \cong \mathrm{PSL}_2(\mathbb{F}_5)$ and $A_6 \cong \mathrm{PSL}_2(\mathbb{F}_9)$). For $N = 6$, Dickson's list leaves only $\mathrm{PSL}_2(\mathbb{F}_9)$ and $\mathrm{PGL}_2(\mathbb{F}_9)$ as possible projective images with an A_6 composition factor. The first has order 360. The second has order 720 and contains an element of order 8, represented by $\mathrm{diag}(a, 1)$ with a a generator of $\mathbb{F}_9^\times$. Since S_6 has no element of order 8, neither group surjects onto S_6. □

Consequently, in dimension two the hypothesis on S_N in Lemma 5.1 and Proposition 5.6 holds for all $N \geq 6$, uniformly in p and in the coefficient field. For $n = 2$ the conclusion of Proposition 5.6 therefore only requires $N \geq 6$ and $K \cap Q_\Sigma = F$.

6.2. Residual modularity over S_N extensions by Snowden's method.

In this subsection $n = 2$ and $p \geq 3$. We use the terminology of [38]: *weight two* representations, the local types A, B, C, and the associated type functions [38, §§2.1, 2.6]. Types A and B are respectively potentially crystalline ordinary and potentially crystalline nonordinary, with the determinant normalised over a finite extension. Type C means that after a finite extension the representation is an extension of 1 by χ_p. A local representation may have both types A and C, denoted A/C by Snowden. We impose the conditions

- (A1) $\bar\rho|_{G_{F(\zeta_p)}}$ is absolutely irreducible;
- (A2) if $p = 5$ and the projective image of $\bar\rho$ is $\mathrm{PGL}_2(\mathbb{F}_5)$, then $[F(\zeta_5) : F] = 4$.

In this subsection, modularity means arising from a cuspidal Hilbert eigenform of parallel weight two.

Theorem 6.2. *Let $\bar\rho\colon G_F \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ be odd and satisfy* (A1) *and* (A2). *Let S be a finite set of finite places of F, let F^{avoid}/F be a finite extension and let $N_0 \geq 1$. Then there exist $N \geq N_0$ and a totally real Galois extension K/F such that:*

- (i) $\mathrm{Gal}(K/F) \cong S_N$;
- (ii) *K is linearly disjoint from $F^{\mathrm{avoid}}\,\bar{F}^{\ker\bar\rho}(\zeta_p)$ over F;*
- (iii) *every place of F in S and every place of F above p splits completely in K;*
- (iv) *$\bar\rho|_{G_K}$ is modular.*

The proof follows [38, §5, §8], with Proposition 2.3 in place of the theorem of Moret-Bailly. The last step is solvable descent of residual modularity. For p-adic representations this is Langlands' base change; for residual representations it is not formal. Khare proved such a descent result [26] over $\mathbb{Q}$, and in the present setting it is [38, §8.1], whose argument we reproduce. The descent is not applied to the residual representation but to a p-adic lift ρ' of $\bar\rho$ which is already defined over F: a modularity lifting theorem shows that ρ' is modular over the larger field, Langlands' base change then gives the modularity of $\rho'|_{G_K}$, and reduction modulo p gives that of $\bar\rho|_{G_K}$ (Steps 0 and 5 below).

Proof. Step 0: a lift over F. By [38, Prop. 7.10.1] there is a finitely ramified, odd, weight two lift $\rho'\colon G_F \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ of $\bar\rho$. Write $\det\rho' = \psi\chi_p$ with ψ of finite order, and let t be a definite type function on the set Σ_p of places of F above p with $t \approx t_{\rho'|_{\Sigma_p}}$ in the sense of [38, §2.6].

Step 1: auxiliary data. Choose an odd prime $\ell \neq p$ and let $\rho_\ell\colon G_F \to \mathrm{GL}_2(\bar{\mathbb{Q}}_\ell)$ and M_ℓ/F be the representation and the finite Galois extension of [38, Prop. 5.4.1]: for every finite totally real F''/F linearly disjoint from M_ℓ, $\rho_\ell|_{G_{F''}}$ is modular and $\bar\rho_\ell|_{G_{F''}}$ satisfies (A1) and (A2) (with ℓ in place of p). Write $\det\rho_\ell = \psi_\ell\chi_\ell$. Let M' be the Galois closure over F of the compositum of F^{avoid}, $\bar F^{\ker\bar\rho}(\zeta_p)$ and M_ℓ.

Step 2: making the determinants cyclotomic. Since $\bar\rho$ and ρ_ℓ are odd and p, ℓ are odd, ψ and ψ_ℓ are trivial on all complex conjugations. Let Σ_ψ be the finite set of finite places of F at which ψ or ψ_ℓ is ramified, and for $v \in \Sigma_\psi$ let E_v/F_v be a finite Galois extension on which ψ and ψ_ℓ become trivial. By Lemma 4.1 there is a finite, solvable, Galois, totally real extension F_1/F, linearly disjoint from M', with $F_{1,w} \cong E_v$ for all $w \mid v \in \Sigma_\psi$ (and $F_{1,w} = \mathbb{R}$ at the real places). Then $\psi|_{G_{F_1}}$ and $\psi_\ell|_{G_{F_1}}$ are trivial at the places above Σ_ψ and at the real places, and unramified elsewhere; an unramified character of finite order of a local Galois group is the square of such a character. Hence both characters are locally squares at all places of F_1, and by [38, Lemma 4.2.1] they are squares: $\psi|_{G_{F_1}} = \psi_1^2$, $\psi_\ell|_{G_{F_1}} = \psi_{\ell,1}^2$ with ψ_1, $\psi_{\ell,1}$ of finite order. Put

$$\bar\rho_{\mathfrak{p}} = \bar\psi_1^{-1}\cdot\bar\rho|_{G_{F_1}}, \qquad \bar\rho_{\mathfrak{l}} = \bar\psi_{\ell,1}^{-1}\cdot\bar\rho_\ell|_{G_{F_1}},$$

which have cyclotomic determinant. Choose a totally real coefficient field L with primes $\mathfrak{p} \mid p$ and $\mathfrak{l} \mid \ell$ whose residue fields contain the coefficients of these two twisted representations. For example, take a monic polynomial of degree divisible by both residue degrees, irreducible modulo p and ℓ and with distinct real roots, using weak approximation on its coefficients. (In [38, (5.5)] this step is done with [38, Cor. 4.2.2], which gives an extension whose Galois closure is solvable; choosing F_1/F Galois here gives a solvable Galois extension in the final descent.)

Step 3: the moduli space and the local conditions. Let X/F_1 be the scheme of [38, proof of Prop. 5.3.1], classifying $\mathrm{GL}_2(L)$-type abelian varieties A with polarisation data and symplectic isomorphisms $A[\mathfrak{p}] \cong \bar\rho_{\mathfrak{p}}$, $A[\mathfrak{l}] \cong \bar\rho_{\mathfrak{l}}$ and an auxiliary level structure at a third prime $\mathfrak{r}$. It is shown there that X is a separated scheme of finite type over F_1, smooth and geometrically connected, hence geometrically irreducible. It is quasi-projective, being a moduli space of polarised abelian varieties with level structure [35, §1]; this is only used to apply Lemma 4.2, and could be avoided by replacing X by a dense affine open subscheme, since non-empty open subsets of $X(L_v)$ are Zariski dense.

For a place w of F_1 above p let $\bar\Omega_w \subset X(\bar F_{1,w})$ be the set of points whose abelian variety has good reduction which is ordinary at $\mathfrak{p}$ (if $t(w|_F) = A$), good reduction which is not ordinary at $\mathfrak{p}$ (if $t(w|_F) = B$), or positive toric rank after semistable reduction (if $t(w|_F) = C$). Indeed the rational character group of a nonzero torus in the semistable fibre is an L-vector space, so its rank is at least $[L:\mathbb{Q}] = \dim A$ and the fibre is totally toric. For w above ℓ define $\bar\Omega_w$ in the same way, with $\mathfrak{l}$ and the type function t_{ρ_ℓ}. As in [38, proof of Prop. 5.3.1], for every sufficiently

large finite Galois extension L_v of F_v ($v = w|_F$) containing $F_{1,w}$, the set $\Omega_w := \bar\Omega_w \cap X(L_v)$ is a non-empty open subset of $X(L_v)$ stable under $\mathrm{Gal}(L_v/F_{1,w})$; and $X(\mathbb{R}) \neq \emptyset$ at each real place of F_1.

The argument of [38, (5.5)] proves the following. Let F'' be a field and x a point such that:

(Q1) F'' is a finite totally real extension of F_1;
(Q2) $x \in X(F'')$;
(Q3) $F'' \cap M' = F$;
(Q4) for every place u of F'' above $p\ell$, the image of x in $X(\bar F''_u)$ lies in $\bar\Omega_{u|_{F_1}}$.

Then there is a cuspidal Hilbert eigenform f of parallel weight two over F'', with coefficients in $\bar{\mathbb{Q}}_p$, such that

$$\bar\rho_f \cong \bar\rho|_{G_{F''}}, \qquad \det\rho_f = \psi\chi_p|_{G_{F''}}, \qquad t_f|_{\Sigma_p} = t|_{F''}.$$

Indeed, let A/F'' be the abelian variety attached to x. By (Q4) and the definition of the sets $\bar\Omega_w$, which involve only the reduction of A over $\bar F''_u$, the Tate modules $T_{\mathfrak{p}}A$ and $T_{\mathfrak{l}}A$ have types $t|_{F''}$ and $t_{\rho_\ell}|_{F''}$ at the places above p and ℓ. By (Q3) and Step 1, $A[\mathfrak{l}] \cong \bar\psi_{\ell,1}^{-1}\bar\rho_\ell|_{G_{F''}}$ satisfies (A1) and (A2) and $\psi_{\ell,1}^{-1}\rho_\ell|_{G_{F''}}$ is a modular lift of it of the same type as $T_{\mathfrak{l}}A$ at the places above ℓ; so $T_{\mathfrak{l}}A$ is modular by the modularity lifting theorem [38, Thm. 4.4.2], say attached to f'. By compatibility $T_{\mathfrak{p}}A$ is attached to f' as well, and $f = \psi_1 \cdot f'$ has the stated properties. These are exactly the properties of the field and of the point used in [38, (5.5)]; that the field be Galois over F is not used.

Step 4: Weil restriction and the S_N construction. Let Σ be the union of S, of the sets of places of F above p and ℓ, and of the real places. For each finite $v \in \Sigma$ choose a finite Galois extension L_v/F_v containing the completions of F_1 above v and so large that $\Omega_w \subset X(L_v)$ is defined and non-empty for all places $w \mid v$ of F_1 if $v \mid p\ell$; for the other finite $v \in \Sigma$ choose L_v with $X(L_v) \neq \emptyset$ for all embeddings of F_1 and put $\Omega_w = X(L_v)$. For v real put $L_v = \mathbb{R}$, $\Omega_w = X(\mathbb{R})$. By Lemma 4.1 choose F_1'/F finite, solvable, Galois, totally real, linearly disjoint from F_1M', with $F'_{1,w'} \cong L_v$ for all $w' \mid v \in \Sigma$, and put $\widetilde{F}_1 = F_1F_1'$. Then $\widetilde{F}_1/F$ is finite, solvable, Galois, totally real, with $\widetilde{F}_{1,u} \cong L_v$ for all $u \mid v \in \Sigma$, and $\widetilde{F}_1 \cap M' = F$.

Let $Y = \mathrm{Res}_{\widetilde{F}_1/F}(X_{\widetilde{F}_1})$. It is a smooth, geometrically irreducible, quasi-projective variety over F, and $Y(F_v) = \prod_{u|v} X(\widetilde{F}_{1,u})$ for $v \in \Sigma$. Let $\Omega'_v = \prod_{u|v}\Omega_{u|_{F_1}} \subset Y(F_v)$, a non-empty open subset. Theorem 2.1 gives a closed point $y \in Y$ all of whose geometric points lie in Ω'_v for $v \in \Sigma$; Lemma 4.2 gives a smooth geometrically irreducible curve $C \subset Y$ over F through y; and Proposition 2.3, applied to C with the places Σ, the local fields F_v, the open sets $C(F_v) \cap \Omega'_v$, the avoidance field $\widetilde{F}_1M'$ and the bound N_0, gives a closed point $x \in C$ of degree $N \geq N_0$ whose residue field F_x has Galois closure K with: $\mathrm{Gal}(K/F) \cong S_N$; K totally real; K linearly disjoint from $\widetilde{F}_1M'$ over F; every $v \in \Sigma$ split completely in K; and all F-embeddings $F_x \to F_v$ sending x into Ω'_v. This gives (i), (ii) and (iii).

Since K and $\widetilde{F}_1$ are linearly disjoint, $x \in Y(F_x) = X(F_x\widetilde{F}_1) \subset X(K\widetilde{F}_1)$. We check (Q1)–(Q4) for $F'' = K\widetilde{F}_1$. It is a finite totally real extension of F_1 carrying a point of X. The fields F_1, F_1', K are Galois over F, F_1 is linearly disjoint from M', F_1' from F_1M' and K from $\widetilde{F}_1M'$; hence $\mathrm{Gal}(K\widetilde{F}_1M'/F)$ is the product of the four Galois groups and $K\widetilde{F}_1 \cap M' = F$. For a place u' of $K\widetilde{F}_1$ above $v \mid p\ell$, the completion $(K\widetilde{F}_1)_{u'}$ equals $\widetilde{F}_{1,u} \cong L_v$, $u = u'|_{\widetilde{F}_1}$, because K is split at v, and the image of x there is the u-component of the image of x in $Y(F_v)$, which lies in $\Omega_{u|_{F_1}} \subset \bar\Omega_{u|_{F_1}}$. By Step 3 there is a cuspidal eigenform f of parallel weight two over $K\widetilde{F}_1$ with $\bar\rho_f \cong \bar\rho|_{G_{K\widetilde{F}_1}}$ and $t_f|_{\Sigma_p} = t|_{K\widetilde{F}_1}$.

Step 5: descent to K. Consider $\rho'|_{G_{K\widetilde{F}_1}}$. It is finitely ramified, odd and of weight two. Its reduction $\bar\rho|_{G_{K\widetilde{F}_1}}$ has the same image as $\bar\rho$, also after restriction to the subgroups fixing ζ_p, and $[K\widetilde{F}_1(\zeta_p) : K\widetilde{F}_1] = [F(\zeta_p) : F]$, because $K\widetilde{F}_1 \cap \bar{F}^{\ker\bar\rho}(\zeta_p) = F$; hence it satisfies (A1) and (A2). It is isomorphic to $\bar\rho_f$, and, types being potential notions, $t_{\rho'|_{G_{K\widetilde{F}_1}}} \approx t|_{K\widetilde{F}_1} = t_f|_{\Sigma_p}$. By [38, Thm. 4.4.2], $\rho'|_{G_{K\widetilde{F}_1}}$ is modular. The extension $K\widetilde{F}_1/K$ is Galois with solvable group $\mathrm{Gal}(\widetilde{F}_1/F)$, both fields are totally real, and $\rho'|_{G_{K\widetilde{F}_1}}$ is irreducible; by Langlands' solvable base change [31] (see [3, Lemma 2.2.2]), $\rho'|_{G_K}$ is modular. Hence $\bar\rho|_{G_K}$ is modular, which is (iv). □

Remark 6.3. (1) More generally, if a finitely ramified, totally odd weight two lift of $\bar\rho$ over F is prescribed, take it for ρ' in Step 0. The same construction makes this particular lift modular over K, with (i)–(iii), for arbitrary splitting behaviour of p in F. The form over K produced in the proof is attached to $\rho'|_{G_K}$; in particular one may add to the conclusion that $\bar\rho|_{G_K}$ has a modular lift of determinant $\psi\chi_p$ whose type at the places above p is $t_{\rho'}$. The lift ρ' over F is the device by which residual modularity is descended from $K\widetilde{F}_1$, where the abelian variety lives and where p is in general ramified, to K, where p is split; this is the proof of solvable descent for mod p Hilbert eigenforms in [38, §8.1], and the whole argument has the structure of [38, §8.2], where residual modularity is obtained over a compositum and descended along its solvable part.

(2) Snowden obtains the initial lift ρ' in [38, Prop. 7.10.1] using potential modularity over fields supplied by Moret-Bailly.

6.3. **The potentially diagonalisable case.** For potentially diagonalisable representations nothing new is needed in dimension two: the statement is Theorem 1.1 with $n = 2$, together with Remark 3.2 for the small primes. We record it for comparison with Corollary 6.5.

Corollary 6.4. *Let $p \geq 3$ and let $\rho\colon G_F \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ be totally odd, unramified outside finitely many places, and potentially diagonalisable with distinct Hodge–Tate weights at every place above p. Assume that $\bar\rho|_{G_{F(\zeta_p)}}$ is irreducible if $p \geq 7$; and, if $p \in \{3,5\}$, that $\bar\rho(G_{F(\zeta_p)})$ is adequate and, for $p = 5$, that the projective image of $\bar\rho$ is neither $\mathrm{PSL}_2(\mathbb{F}_5)$ nor $\mathrm{PGL}_2(\mathbb{F}_5)$. Then the conclusions* (i)–(iv) *of Theorem 1.1 hold for ρ. No condition is imposed on the splitting of p in F.*

Proof. For $p \geq 7$, that is $p \geq 2(n+1)$ with $n = 2$, the representation ρ satisfies (H) and this is Theorem 1.1. For $p \in \{3,5\}$ it is Remark 3.2. □

6.4. **Without potential diagonalisability, when p splits in F.** Assume now $n = 2$, $\rho\colon G_F \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ totally odd and unramified outside finitely many places, de Rham with distinct Hodge–Tate weights at each place above p, $p \geq 3$, $\bar\rho$ satisfying (A1) and (A2), and that p splits completely in $F/\mathbb{Q}$, so that $F_v = \mathbb{Q}_p$ for all $v \mid p$. For $p = 5$, complete splitting already gives $[F(\zeta_5) : F] = 4$ by completion at any place above 5, so (A2) is automatic here. Theorem 6.2 supplies residual modularity over K, and the lifting theorem at split primes applies to arbitrary regular de Rham lifts. Throughout this subsection, modularity of such a lift means association with a regular algebraic cuspidal automorphic representation of GL_2 in the corresponding weight normalisation.

Corollary 6.5. *Under these assumptions, for every N_0, every finite F^{avoid}/F and every finite set S of finite places of F there is a totally real Galois extension K/F with $\mathrm{Gal}(K/F) \cong S_N$, $N \geq N_0$, linearly disjoint from F^{avoid}, such that every place of F in S or above p splits completely in K/F (hence p splits completely in $K/\mathbb{Q}$ and $K_w = \mathbb{Q}_p$ for all $w \mid p$), and such that $\rho|_{G_K}$ is modular.*

Proof. Let K be given by Theorem 6.2. Then $\bar\rho|_{G_K}$ is modular, and since K is linearly disjoint from $\bar F^{\ker\bar\rho}(\zeta_p)$, $\bar\rho|_{G_K}$ satisfies (A1) and (A2). Moreover $K_w = F_v = \mathbb{Q}_p$ for all $w \mid v \mid p$, and $\rho|_{G_K}$ is potentially semistable with distinct Hodge–Tate weights at all $w \mid p$. Building on Kisin, Paškūnas and Hu–Tan [30, 33, 22], Tung's global lifting theorem [39, Thm. 4.7] now gives $\rho|_{G_K} \simeq r_p(f) \otimes \eta$ for a Hilbert eigenform f and a character η. The character occurs as a line in $r_p(f)^\vee \otimes \rho|_{G_K}$, so it is finitely ramified and de Rham at all places above p. By rank one class field theory it is the p-adic avatar of an algebraic Hecke character. Indeed, de Rham characters are locally algebraic on the units above p, and Dirichlet's unit theorem makes their algebraic exponents parallel over the totally real field K. Removing that power of the cyclotomic character leaves a character of finite conductor factoring through a finite ray class group. Twisting the automorphic representation of f by the corresponding Hecke character gives the assertion. □

Remark 6.6. The hypothesis that p splits completely in $F/\mathbb{Q}$ is used only for the lifting theorem. For a rational prime p with some $[F_v : \mathbb{Q}_p] > 1$, one can still arrange $K_w = F_v$ at every $v \mid p$. The place of degree greater than one prevents the application of [39, Thm. 4.7]. It is essential that the lifting theorem is applied over K and not over the field $K\widetilde{F}_1$ of the proof of Theorem 6.2, where p is in general not split. This is the same structure as in [38, §8]: residual modularity is first transported to the split field, and the lifting theorem is applied there. Only Snowden's results, valid for all $p \geq 3$ under (A1), (A2), are used in the construction of K; neither adequacy nor [7] is needed.

7. Application to safe chains

In the method of safe chains one works with a compatible system $\{\rho_\lambda\}$ of two-dimensional representations, deduced from potential modularity (by Brauer induction, or by the gluing argument of Section 8, see Remark 8.5), after a good-dihedral prime q has been introduced in the level; from then on one only uses members in residual characteristics bounded by a constant C, which is greater than the largest prime dividing the conductor (in particular $C > q$). At each step one needs the residual images to stay adequate after restriction to the field of potential modularity, and it is often convenient that the relevant primes split there.

The field K can be chosen before the other members of the system are known. It suffices to bound the common ramification set and the residual characteristics in the chain. By Lemma 6.1 and Proposition 5.6, these data determine a single finite extension to avoid.

Theorem 7.1. *Let $\rho\colon G_F \to \mathrm{GL}_2(\bar{\mathbb{Q}}_{p_0})$ satisfy* (H) *(or the hypotheses of Corollary 6.5), and let $\{\rho_\lambda\}$ be the compatible system attached to ρ by potential modularity. Let $\mathcal{N}$ consist of all places above p_0 and the finite places away from p_0 where ρ ramifies. Assume that every ρ_λ of residual characteristic p is unramified outside $\mathcal{N} \cup \{v \mid p\}$. For a strictly compatible system this follows by comparing the Weil–Deligne representations away from $p_0 p$* [3, §5.1, Thm. 5.5.1]. *Let C be larger than every prime below $\mathcal{N}$ and than p_0, and let Σ_C be the set of places of F above primes $< C$ together with the archimedean places. Then for every N_0 there is a finite Galois totally real extension K/F such that:*

(a) $\mathrm{Gal}(K/F) \cong S_N$ *with* $N \geq \max(N_0, 6)$;
(b) *every finite place in Σ_C splits completely in K; in particular $K_w = F_v$ at all places of bad reduction, at the places above a good-dihedral prime $q < C$ if one has been chosen, and at all places above every $p < C$;*
(c) *for every λ of residual characteristic $p < C$: $\bar\rho_\lambda(G_{K(\zeta_p)}) = \bar\rho_\lambda(G_{F(\zeta_p)})$; in particular every property of the residual image over $F(\zeta_p)$ (irreducibility, adequacy, size) is preserved over $K(\zeta_p)$;*

(d) *$\rho|_{G_K}$ is automorphic, hence so is $\rho_\lambda|_{G_K}$ for every λ for which $\rho_\lambda|_{G_K}$ is irreducible.*

Proof. Apply Theorem 1.1 or Corollary 6.5, according to the hypothesis on ρ, with S the finite places of Σ_C, N_0 replaced by $\max(N_0, 6)$, and $F^{\mathrm{avoid}} \supseteq Q_{\Sigma_C}$ (Lemma 5.5). This gives (a), (b) and the automorphy of $\rho|_{G_K}$. For (c): if $p < C$, then $F^{\bar\rho_\lambda} = \bar F^{\ker \bar\rho_\lambda}(\zeta_p)$ is unramified outside $\mathcal{N} \cup \{v \mid p\} \cup \{\infty\} \subseteq \Sigma_C$; by Lemma 6.1 the hypothesis of Proposition 5.6 holds, and the latter gives (c). For (d), fix compatible embeddings of the coefficient fields. For a good place $w \mid v$ of K, the Frobenius eigenvalues at w are the $f(w/v)$-th powers of those at v. If π_K is the automorphic representation attached to $\rho|_{G_K}$, then $\rho_\lambda|_{G_K}$ and $r_\lambda(\pi_K)$ have the same characteristic polynomials of Frobenius at almost all places, hence the same semisimplification by Chebotarev; they coincide if $\rho_\lambda|_{G_K}$ is irreducible. □

Remark 7.2.
(1) The data used to choose K are F, ρ (for the potential modularity argument at p_0), the ramification set $\mathcal{N}$ and the bound C; no information on the residual representations $\bar\rho_\lambda$ is needed. In practice C is fixed once the good-dihedral prime has been placed in the level; the steps of the chain before that are carried out in the traditional way.
(2) By (b) and Remark 4.5, the local representations $\rho_\lambda|_{G_{K_w}}$ at all places above primes $< C$ coincide with $\rho_\lambda|_{G_{F_v}}$. Thus each conductor exponent and local type at the specified places is preserved (including the supercuspidal type at q when present), and for $p < C$ split in F one has $K_w = \mathbb{Q}_p$, so that split-case lifting theorems apply over K (as in Corollary 6.5).
(3) Since $K \cap F^{\bar\rho_\lambda} = F$ for all λ with $p < C$, no case distinction (dihedral or otherwise) is needed: the residual images are literally unchanged. This includes the small primes that typically occur in a chain. For $p = 3, 5$: whenever $\bar\rho_\lambda(G_{F(\zeta_p)})$ is adequate, so is $\bar\rho_\lambda(G_{K(\zeta_p)})$, so its residual image hypothesis is available when applying the lifting theorem of [7] over a suitable CM extension, subject to its other hypotheses (for $p = 5$ the projective image is also unchanged, so the exclusion of $\mathrm{PSL}_2(\mathbb{F}_5)$, $\mathrm{PGL}_2(\mathbb{F}_5)$ in [8, Thm. 3.6(2)] is unaffected). For $p = 2$: $\bar\rho_\lambda(G_K) = \bar\rho_\lambda(G_F)$, so a non-solvable residual image at 2 stays non-solvable over K. For the image and splitting requirements at these primes, this removes the need, in [8, proof of Thm. 4.1, Step 3], to re-choose the field of potential modularity after the system has been constructed.

8. Compatible systems by gluing

Potential automorphy yields compatible systems: if ρ becomes automorphic over K, the other members of the system are obtained over F from Brauer's induction theorem, first as virtual representations, which one then shows to be genuine [10]; see also [41] and [3, §5.4]. When $\mathrm{Gal}(K/F) \cong S_N$, which is the situation produced by our construction, the companions can be obtained instead by gluing the automorphic representations attached to ρ over the fields K^H, H soluble, which exist by soluble descent. The obstruction to gluing is a class in $H^2(S_N, \mu_2)$, killed by a 2-Sylow subgroup, together with a family of signs, which the structure of S_N forces to be global.

8.1. **Set-up.** Let F be totally real and K/F a totally real Galois extension with $G = \mathrm{Gal}(K/F) \cong S_N$. Let p and ℓ be primes and fix isomorphisms $\iota_p \colon \bar{\mathbb{Q}}_p \cong \mathbb{C}$, $\iota_\ell \colon \bar{\mathbb{Q}}_\ell \cong \mathbb{C}$. Let $\rho \colon G_F \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ be continuous, with $\rho|_{G_K}$ irreducible and isomorphic to $r_{p,\iota_p}(\pi)$ for a regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_K)$. Since $\rho|_{G_K} \cong r_{p,\iota_p}(\pi)$ is de Rham at the places above p and $[K : F]$ is finite, ρ is de Rham at the places above p; hence $\det \rho$ is a de Rham character of G_F, and therefore the p-adic avatar of an algebraic Hecke character

ψ of F. Let ψ_ℓ be the ℓ-adic avatar of ψ. Then $\det r_{\ell,\iota_\ell}(\pi) = \psi_\ell|_{G_K}$: the determinants of $r_{p,\iota_p}(\pi)$ and $r_{\ell,\iota_\ell}(\pi)$ are the avatars of the same algebraic Hecke character of K (determined by the central character of π), and the p-adic one is $\psi_p|_{G_K} = \det\rho|_{G_K}$. The same holds for the representations $r_{\ell,\iota_\ell}(\pi_H)$ below. We use:

(Irr) $r_{\ell,\iota_\ell}(\pi)$ is absolutely irreducible.

For regular algebraic cuspidal automorphic representations of GL_2 over a totally real field, (Irr) holds for every ℓ (see the remark at the end of [37]).

For a soluble subgroup $H \le G$, the extension K/K^H is soluble, so by Langlands' base change [31] (see [3, Lemma 2.2.2]) there is a cuspidal automorphic representation π_H of $\mathrm{GL}_2(\mathbb{A}_{K^H})$ with $r_{p,\iota_p}(\pi_H) \cong \rho|_{G_{K^H}}$; it is unique by strong multiplicity one. Put

$$r_K = r_{\ell,\iota_\ell}(\pi), \qquad r_H = r_{\ell,\iota_\ell}(\pi_H).$$

These are the "pieces" to be glued. Since $r_H|_{G_K} \cong r_K$ (Lemma 8.2(a) below), we fix each r_H within its isomorphism class so that $r_H|_{G_K} = r_K$ literally; by (Irr) and Schur's lemma this normalises r_H up to conjugation by a scalar, which does not change r_H.

Theorem 8.1. *In this situation there is a unique continuous representation $R_\ell\colon G_F \to \mathrm{GL}_2(\bar{\mathbb{Q}}_\ell)$ with $\det R_\ell = \psi_\ell$ such that*

$$R_\ell|_{G_{K^H}} \cong r_H \qquad \text{for every soluble subgroup } H \le G.$$

For every finite place v of F, fix an embedding $G_{F_v} \subset G_F$ and let $D_v \le G$ be the image of G_{F_v} in G, which is the decomposition group of a place of K above v; then

$$R_\ell|_{G_{F_v}} \cong r_{D_v}|_{G_{F_v}}, \qquad \rho|_{G_{F_v}} \cong r_{p,\iota_p}(\pi_{D_v})|_{G_{F_v}}.$$

In particular ρ and R_ℓ are the p-adic and ℓ-adic representations attached, at a place of K^{D_v} of degree one over v, to the same automorphic representation π_{D_v}, and are compatible at v in every sense in which local–global compatibility is known for Hilbert modular forms; in particular they have the same characteristic polynomials of Frobenius at the places $v \nmid p\ell$ where both are unramified.

8.2. **The pieces.**

Lemma 8.2. (a) *If $H' \le H \le G$ are soluble, then $r_H|_{G_{K^{H'}}} \cong r_{H'}$; in particular $r_H|_{G_K} \cong r_K$.*

(b) *If $\sigma \in G_F$ has image $s \in G$, then $r_{sHs^{-1}} \cong r_H^\sigma$, where $r_H^\sigma(g) = r_H(\sigma^{-1}g\sigma)$. In particular $r_K^\sigma \cong r_K$ for all $\sigma \in G_F$.*

Proof. (a) The base change of π_H to $K^{H'}$ has p-adic representation $\rho|_{G_{K^{H'}}}$, which is irreducible, so it is cuspidal and isomorphic to $\pi_{H'}$ by strong multiplicity one; hence their ℓ-adic representations agree. (b) The conjugate π_H^σ, an automorphic representation over $\sigma(K^H) = K^{sHs^{-1}}$, has p-adic representation $\rho|^\sigma_{G_{K^H}} \cong \rho|_{G_{K^{sHs^{-1}}}}$, because ρ is defined on G_F. Conclude as in (a). □

8.3. **Existence of an extension.** By Lemma 8.2(b) and (Irr), for every $g \in G_F$ there is $A \in \mathrm{GL}_2(\bar{\mathbb{Q}}_\ell)$ with $A\,r_K(h)A^{-1} = r_K(ghg^{-1})$ for all $h \in G_K$, unique up to a scalar; requiring $\det A = \psi_\ell(g)$ leaves two choices, $\pm A$. Choose a lift $g_s \in G_F$ of each $s \in G$, with $g_1 = 1$, and such an A_s for g_s, with $A_1 = I$, and put $A(g_s h) = A_s\,r_K(h)$ for $h \in G_K$; thus $A|_{G_K} = r_K$. The condition $\det A_s = \psi_\ell(g_s)$ is compatible with the intertwining property because $\det r_K = \psi_\ell|_{G_K}$. For $g, g' \in G_F$ the matrix $A(g)A(g')A(gg')^{-1}$ commutes with $r_K(G_K)$ and has determinant 1,

so it equals $c(g,g') \in \mu_2$; and c depends only on the images of g, g' in G. Thus c is a 2-cocycle on G with values in μ_2 (trivial action), whose class

$$[c] \in H^2(G, \mu_2)$$

does not depend on the choices. As usual, $[c] = 0$ if and only if r_K extends to a representation of G_F with determinant ψ_ℓ, and for a subgroup $H \leq G$, the restriction of $[c]$ to H vanishes if and only if r_K extends to G_{K^H} with determinant $\psi_\ell|_{G_{K^H}}$.

Lemma 8.3. *$[c] = 0$. Hence there is a continuous $R^0 \colon G_F \to \mathrm{GL}_2(\bar{\mathbb{Q}}_\ell)$ with $R^0|_{G_K} = r_K$ and $\det R^0 = \psi_\ell$.*

Proof. Let P be a 2-Sylow subgroup of G. It is soluble, and r_P extends r_K to G_{K^P} with determinant $\psi_\ell|_{G_{K^P}}$ (Lemma 8.2(a)); so $[c]|_P = 0$. Since cor $\circ$ res is multiplication by the odd integer $[G : P]$ and $H^2(G, \mu_2)$ is killed by 2, restriction to P is injective, and $[c] = 0$. Continuity holds because R^0 is continuous on the open subgroup G_K. □

8.4. **The signs.** Let $H \leq G$ be soluble. Both $R^0|_{G_{K^H}}$ and r_H restrict to r_K on G_K (literally, by the normalisations above) and have determinant ψ_ℓ; by Schur's lemma $R^0(g) = \chi_H(g)\, r_H(g)$ for a character χ_H of G_{K^H}, trivial on G_K and with $\chi_H^2 = 1$. We regard χ_H as an element of $\mathrm{Hom}(H, \mu_2)$. By Lemma 8.2, $\chi_H|_{H'} = \chi_{H'}$ for $H' \leq H$, and $\chi_{sHs^{-1}}(sxs^{-1}) = \chi_H(x)$. Therefore the function

$$f \colon G \to \mu_2, \qquad f(x) = \chi_{\langle x \rangle}(x),$$

is constant on conjugacy classes, and its restriction to every soluble subgroup H is the homomorphism χ_H.

Lemma 8.4. *Let $f \colon S_N \to \{\pm 1\}$ be constant on conjugacy classes and such that $f|_H$ is a homomorphism for every soluble subgroup $H \leq S_N$. Then f is trivial or the sign character.*

Proof. Let $\varepsilon = f(\tau)$ for a transposition τ. Disjoint transpositions commute, so for an involution r which is a product of m disjoint transpositions, $f(r) = \varepsilon^m$. Let z be a k-cycle; identify its support with $\mathbb{Z}/k\mathbb{Z}$ so that $z(i) = i + 1$, and let $r(i) = -i$, $r'(i) = 1 - i$. Then $z = r'r$, and $\langle r, r' \rangle$ is dihedral, hence soluble, so $f(z) = f(r')f(r)$. If k is odd, r and r' are each a product of $(k-1)/2$ disjoint transpositions; if k is even, r is a product of $(k-2)/2$ and r' of $k/2$. In both cases $f(z) = \varepsilon^{k-1}$. Finally, the cycles $z_1, \ldots, z_c$ of an arbitrary $x \in S_N$ (including fixed points, as cycles of length 1, so that $k_1 + \cdots + k_c = N$) commute, so $f(x) = \prod_i \varepsilon^{k_i - 1} = \varepsilon^{N-c}$, which is 1 if $\varepsilon = 1$ and $\mathrm{sgn}(x)$ if $\varepsilon = -1$. □

Proof of Theorem 8.1. By Lemma 8.4, f is trivial or the sign character. In the second case replace R^0 by $R_\ell = R^0 \otimes \mathrm{sgn}$, where sgn is regarded as a character of G_F through G; this does not change the determinant and multiplies each χ_H by $\mathrm{sgn}|_H$. In the first case put $R_\ell = R^0$. Now all χ_H are trivial, i.e. $R_\ell|_{G_{K^H}} \cong r_H$ for every soluble H.

Uniqueness: two such representations agree on G_K with r_K absolutely irreducible and have the same determinant, so they differ by a character $\eta \colon G \to \mu_2$; and η is trivial on every cyclic subgroup, since both restrict to $r_{\langle x \rangle}$ on $G_{K^{\langle x \rangle}}$. Hence $\eta = 1$.

Local statement: the decomposition group D_v is soluble (inertia is an extension of a cyclic group by a p_v-group, and the quotient by inertia is cyclic), and $G_{F_v} \subset G_{K^{D_v}}$ since D_v is the image of G_{F_v} in G, the corresponding place of K^{D_v} having degree one over v. So $R_\ell|_{G_{F_v}} \cong r_{D_v}|_{G_{F_v}}$, and $\rho|_{G_{F_v}} \cong r_{p,\iota_p}(\pi_{D_v})|_{G_{F_v}}$ by the definition of π_{D_v}. □

Remark 8.5 (The compatible system). Theorem 8.1 gives one representation $R_\ell = R_{\ell,\iota_\ell}$ for each ℓ and each ι_ℓ. They form, together with ρ, a weakly compatible system:

(a) *Coefficients.* The characteristic polynomial of $R_\ell(\mathrm{Frob}_v)$ is that of $r_{\ell,\iota_\ell}(\pi_{D_v})$ at a place of degree one over v, i.e. it is given by Hecke eigenvalues of π_{D_v}. Up to conjugation in G there are finitely many subgroups D_v, hence finitely many representations π_{D_v} up to $\mathrm{Gal}(K/F)$-conjugation, and the coefficients of all these polynomials lie in the compositum M of their fields of rationality, a number field independent of ℓ.

(b) *Ramification.* Let $\mathcal{N}$ be the set of places of F at which ρ is ramified. If $v \notin \mathcal{N}$ and $v \nmid \ell$, then $r_{p,\iota_p}(\pi_{D_v})|_{G_{F_v}} \cong \rho|_{G_{F_v}}$ is unramified, so π_{D_v} is unramified at the corresponding place (local–global compatibility [5]), hence so is $r_{\ell,\iota_\ell}(\pi_{D_v})$, and R_ℓ is unramified at v. Thus every R_ℓ is unramified outside $\mathcal{N} \cup \{v \mid \ell\}$, independently of the ramification of K/F.

This is what is used in Section 7.

Remark 8.6. (1) The hypothesis $G \cong S_N$ is used only in Lemma 8.4. For an arbitrary finite group G the conclusion of Theorem 8.1 still holds. Lemma 8.3 did not use the structure of G; and the family (χ_H) defines a stable element $\chi_P \in H^1(P, \mathbb{F}_2)$ for a 2-Sylow subgroup P, which by the theorem of Cartan and Eilenberg is the restriction of a character $\chi\colon G \to \mu_2$. Decomposing an element into its 2-part and its odd part shows that $f = \chi$, and one twists by χ.

(2) In dimension n the argument is formally the same, with μ_n in place of μ_2 and a q-Sylow subgroup for each prime $q \mid n$, but two inputs change. Soluble descent, which in dimension two is Langlands' base change [31], is available in dimension n in the polarised setting of [3, Lemma 2.2.2]. And the hypothesis corresponding to (Irr), the absolute irreducibility of $r_{\ell,\iota_\ell}(\pi)$ for the given ℓ, is not known in general for $n > 2$.

9. Further remarks

9.1. **Descent and L-functions.** Under (H), choose the avoidance field to contain $\bar{F}^{\ker \bar{\rho}}(\zeta_p)$ and take $N \geq 5$. Then $\rho|_{G_K}$ is absolutely irreducible. Brauer induction and soluble descent give meromorphic continuation and the functional equation for the completed L-function, with the compatible system and local factors of [3, §5.4, Thm. 5.5.1]. For the two-dimensional argument see [41]. Put $D = K^{A_N}$. If automorphy of $\rho|_{G_D}$ can be deduced from that of $\rho|_{G_K}$, soluble descent for the quadratic extension D/F gives automorphy of ρ by [3, Lemma 2.2.2].

Under (H), *the automorphy of ρ over F reduces to descent of automorphy along a single Galois extension with group A_N, where N can be taken arbitrarily large* (*and in any set as in Remark 3.1*).

The unresolved implication is descent of the given Galois representation from K to D. Its formulation requires no general base change theorem along K/D.

9.2. **Composition factors.** Relative to F, the field K has composition factors A_N and $\mathbb{Z}/2$ only. For instance, for N large, K contains no subfield whose Galois closure over F has a composition factor of the form $\mathrm{PSL}_2(\mathbb{F}_q)$, simultaneously for all q, apart from the coincidences $A_5 \cong \mathrm{PSL}_2(\mathbb{F}_4) \cong \mathrm{PSL}_2(\mathbb{F}_5)$ and $A_6 \cong \mathrm{PSL}_2(\mathbb{F}_9)$, which are excluded once $N \geq 7$.

9.3. **Why alternating groups.** For a generic complex curve of genus at least four, every indecomposable map to $\mathbb{P}^1$ has alternating or symmetric monodromy [18]. The auxiliary curves in Proposition 2.3 can be special. Finding a point over a soluble extension, even with no local conditions, is a separate arithmetic question for curves of large genus [36]. In genus one, [6] gives a soluble point for an everywhere locally soluble curve over $\mathbb{Q}$ with semistable Jacobian.

10. A direct proof of finiteness of deformation rings

In the method of Khare and Wintenberger [27, 28], existence of lifts with prescribed local properties is obtained from two facts about a global deformation ring R_F of $\bar\rho$: it has Krull dimension ≥ 1, and it is finite over $\mathcal{O}$. Finiteness is deduced from potential modularity: over a suitable K the corresponding ring R_K identifies with the finite Hecke algebra of the prescribed weight and level, and one shows that the natural map $R_K \to R_F$ is *finite* by an integrality argument on traces (see e.g. [38, Lemma 6.3.1]; the same scheme is followed in [3], where the finiteness of the restriction map is [3, Lemma 1.2.3] and the dimension bound is [3, Prop. 1.5.1]). We record here that, for $p \nmid n$ and fixed determinant, this map is in fact *surjective*, which gives finiteness at once. The underlying statement on deformation rings is not new: the description by coinvariants for the conditions obtained by restriction to K (Proposition 10.4) is Hida's control theorem for deformation rings under base change [19, 20, 21], established in the context of the Iwasawa theory of adjoint Selmer groups. What we record is its use to obtain finiteness in the method of Khare and Wintenberger, in place of the integrality argument. This observation goes back to [11, §3], where the minimal deformation ring over $\mathbb{Q}$ is shown to be a quotient of the one over the field of potential modularity, and the existence of the minimal lift is deduced. There, a preliminary step is used to kill the p-Sylow subgroup, that is, to reduce to an extension of degree prime to p. As explained in Remark 10.3 (2), that step plays the same role as the condition on the determinant imposed below, and either of them suffices.

Throughout this section F is a number field, p is a prime not dividing n, k is a finite field of characteristic p, $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_p$ with residue field k, $\bar\rho\colon G_F \to \mathrm{GL}_n(k)$ is continuous, and $\psi\colon G_F \to \mathcal{O}^\times$ is a continuous character lifting $\det\bar\rho$.

The field k and the ring $\mathcal{O}$ are fixed once and for all, and are the same over F and over every extension K of F: all deformation rings below are complete noetherian local $\mathcal{O}$-algebras with residue field k, and all maps between them are maps of $\mathcal{O}$-algebras. In particular, for a finite extension K/F the restriction $\bar\rho|_{G_K}$ is regarded as a representation over k, even if it can be defined over a smaller field. Determinants are fixed over both fields: a deformation of $\bar\rho$ is required to have determinant ψ, and a deformation of $\bar\rho|_{G_K}$ is required to have determinant $\psi|_{G_K}$.

Let S be a finite set of places of F containing those above p and those where $\bar\rho$ or ψ ramify, and for a finite extension K/F let S_K be the set of places of K above S. We allow ramification at archimedean places and all ramification conditions below refer to finite places. If $\bar\rho$ is absolutely irreducible, $R_{F,S}$ denotes the universal $\mathcal{O}$-algebra for deformations of $\bar\rho$ to complete noetherian local $\mathcal{O}$-algebras with residue field k which are unramified outside S and have determinant ψ. If $\bar\rho|_{G_K}$ is absolutely irreducible, R_{K,S_K} denotes the universal $\mathcal{O}$-algebra for deformations of $\bar\rho|_{G_K}$ to such algebras which are unramified outside S_K and have determinant $\psi|_{G_K}$. Restriction to G_K of the universal deformation over F induces a homomorphism of $\mathcal{O}$-algebras $R_{K,S_K} \to R_{F,S}$, the *restriction map*.

Theorem 10.1. *Let K/F be a finite extension (not necessarily Galois) with Galois closure K', and assume that $\bar\rho|_{G_{K'}}$ is absolutely irreducible. With k, $\mathcal{O}$ and the determinants fixed as above:*

(a) *The restriction map $R_{K,S_K} \to R_{F,S}$ is a surjective homomorphism of $\mathcal{O}$-algebras.*
(b) *For $v \in S$ let $R_v^\dagger$ be a quotient of the universal framed deformation ring of $\bar\rho|_{G_{F_v}}$, and for $w \in S_K$ let $R_w^\ddagger$ be a quotient of that of $\bar\rho|_{G_{K_w}}$, both stable under conjugation, and let $R_F^\dagger$, $R_K^\ddagger$ be the corresponding quotients of $R_{F,S}$, R_{K,S_K}. Assume:*

(C) *for all $w \mid v$, the restriction to G_{K_w} of any lift of $\bar\rho|_{G_{F_v}}$ of type $\dagger$ is of type $\ddagger$; that is, the composite of the restriction map on framed rings with the projection to $R^\dagger_v$ factors through $R^\ddagger_w$.*

Then the restriction map induces a surjective homomorphism of $\mathcal{O}$-algebras $R^\ddagger_K \to R^\dagger_F$. In particular, if $R^\ddagger_K$ is finite over $\mathcal{O}$ (for instance by an $R = \mathbb{T}$ theorem with a finite Hecke algebra over K), then $R^\dagger_F$ is finite over $\mathcal{O}$.

Proof. (a) A local homomorphism of complete noetherian local $\mathcal{O}$-algebras with the same residue field is surjective as soon as it is surjective on reduced cotangent spaces, i.e. as soon as the dual map on tangent spaces is injective. The reduced cotangent space of a ring R is $\mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_\mathcal{O} R)$. Since $p \nmid n$, the trace splits off the scalar summand from $\operatorname{ad}\bar\rho$. Hence the tangent map for the fixed determinant problem is the restriction $H^1(G_{F,S}, \operatorname{ad}^0\bar\rho) \to H^1(G_{K,S_K}, \operatorname{ad}^0\bar\rho)$. (If K/F ramifies outside S, G_{K,S_K} is not a subgroup of $G_{F,S}$; the map is the one induced by $G_K \to G_F \to G_{F,S}$, and it suffices to prove injectivity after inflating both sides to G_F and G_K, inflation being injective on H^1.) The composite of $H^1(G_F, \operatorname{ad}^0\bar\rho) \to H^1(G_K, \operatorname{ad}^0\bar\rho)$ with restriction to $G_{K'}$ has kernel $H^1(\mathrm{Gal}(K'/F), (\operatorname{ad}^0\bar\rho)^{G_{K'}})$ by inflation–restriction. Since $\bar\rho|_{G_{K'}}$ is absolutely irreducible, its commutant consists of scalars, and since $p \nmid n$ no non-zero scalar has trace zero; hence $(\operatorname{ad}^0\bar\rho)^{G_{K'}} = 0$.

(b) $R^\dagger_F$ is a quotient of $R_{F,S}$, so by (a) the map $R_{K,S_K} \to R^\dagger_F$ is surjective; by (C) it factors through $R^\ddagger_K$. □

Remark 10.2 (On condition (C)). Only the implication "condition over F ⇒ condition over K" is required, and for this there is a natural choice at every place, whatever the behaviour of the place in K/F:

- outside S: a deformation unramified at v is unramified at every $w \mid v$, even if K/F ramifies at v;
- at $w \mid v$ with $K_w = F_v$: the same condition;
- for Kisin's potentially semistable deformation rings [29] with prescribed inertial type τ and Hodge–Tate weights at v: take at w the ring with type $\tau|_{I_{K_w}}$ and the induced Hodge–Tate weights. Since $R^\dagger_v$ is reduced and $\mathcal{O}$-flat, (C) can be checked on $\bar{\mathbb{Q}}_p$-points, where it is clear. This is the mechanism of [38, Lemma 6.4.1], in which K is chosen so that the lifts in question become "admissible" over K_w.

Remark 10.3 (The hypotheses).

(1) The avoid condition enters Theorem 10.1 only through its consequence that $\bar\rho$ remains absolutely irreducible over the Galois closure K' of K; this is all that the surjectivity uses, and it is strictly weaker than linear disjointness from $\bar F^{\ker\bar\rho}$. Disjointness from $\bar F^{\ker\bar\rho}(\zeta_p)$ also preserves the image on the cyclotomic kernel, hence any adequacy hypothesis used in a separate $R = \mathbb{T}$ theorem. If $K \supseteq \bar F^{\ker\bar\rho}$, the restricted representation is trivial and the absolute irreducibility hypothesis fails. No condition on $\mathrm{Gal}(K'/F)$ is needed; in particular its order may be divisible by p.

(2) Without fixing the determinant, and retaining absolute irreducibility on $G_{K'}$, the kernel on tangent spaces consists exactly of the scalar classes $g \mapsto c(g)1_n$, where $c : G_F \to (k, +)$ is unramified outside S and vanishes on G_K. Indeed, inflation–restriction for K'/F identifies a class trivial on $G_{K'}$ with such a scalar homomorphism on $\mathrm{Gal}(K'/F)$, before the conditions at S and at K are imposed. A scalar class on G_K inflated from $\mathrm{Gal}(K'/K)$ is zero only if the homomorphism itself vanishes, by the same exact sequence. Thus surjectivity is equivalent to the absence in K of a cyclic extension of F of degree p unramified outside S. In particular it holds for Galois group S_N and odd

p, in every dimension. The condition on the determinant, and with it the condition $p \nmid n$, can be replaced by the condition that $[K : F]$ be prime to p. Indeed, for deformations with no condition on the determinant the tangent spaces are $H^1(\,\cdot\,, \operatorname{ad}\bar\rho)$, and the restriction $H^1(G_F, \operatorname{ad}\bar\rho) \to H^1(G_K, \operatorname{ad}\bar\rho)$ is injective when $p \nmid [K : F]$, because its composite with corestriction is multiplication by $[K : F]$. Hence the restriction map between the universal deformation rings without fixed determinant is surjective, provided only that $\operatorname{End}_{G_K}(\bar\rho) = k$, so that the ring over K exists. In the application of [11], one replaces the Galois field of potential modularity K by the fixed field E of a Sylow p-subgroup of $\operatorname{Gal}(K/F)$. Then $p \nmid [E : F]$ and K/E is solvable. Solvable descent applies to the characteristic zero representation under its usual irreducibility hypothesis.

(3) For $p = n = 2$, take $F = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt{5})$, $S = \{2, 3, 5\}$, and let $\bar\rho$ be the natural representation of $S_3 \cong \mathrm{GL}_2(\mathbb{F}_2)$ arising from the splitting field of $X^3 - 2$. The splitting field has unique quadratic subfield $\mathbb{Q}(\sqrt{-3})$, so $\bar\rho|_{G_K}$ stays absolutely irreducible. If $c : G_F \to \mathbb{F}_2$ cuts out K, then $\bar\rho$ and $(1 + \varepsilon c)\bar\rho$ over $\mathbb{F}_2[\varepsilon]/(\varepsilon^2)$ have determinant one and equal restrictions. Any strict conjugacy between them would centralise $\bar\rho(G_K)$ and hence be scalar. The deformations are therefore distinct and the fixed determinant restriction map is not surjective.

For Galois K/F, Hida's extension lemma and description of the invariant deformation functor give the following calculation of the kernel [19, Appendix, Lemma A.2.1 and Proposition A.2.1]. For a control theorem with ordinary local conditions see also [42, Theorem 3.4]. The proof given below, by normalising the determinant, is the same in substance: two extensions of an invariant deformation differ by a character of $\operatorname{Gal}(K/F)$ with values in $1 + \mathfrak{m}_R$, which is trivial once the determinant is fixed and $p \nmid n$.

Proposition 10.4 (The kernel)**.** *Assume in addition that K/F is Galois with group G. Then G acts on R_{K,S_K}, and restriction induces an isomorphism*

$$(R_{K,S_K})_G \xrightarrow{\sim} R^{(K)}_{F,S},$$

where $(\ )_G$ denotes the quotient by the ideal generated by the elements $\sigma(x) - x$, and $R^{(K)}_{F,S}$ is the universal fixed determinant ring for deformations of $\bar\rho$ over F whose restriction to G_K is unramified outside S_K. The natural quotient $R^{(K)}_{F,S} \twoheadrightarrow R_{F,S}$ imposes trivial inertia at the finite places $v \notin S$ ramified in K. For the deformation problem represented by $R^{(K)}_{F,S}$, every G-invariant deformation of $\bar\rho|_{G_K}$ with determinant $\psi|_{G_K}$ extends uniquely to a deformation of $\bar\rho$ with determinant ψ.

Proof. For $g \in G_F$ choose a matrix $B_g \in \mathrm{GL}_n(\mathcal{O})$ lifting $\bar\rho(g)$. The rule $r(h) \mapsto B_g r(g^{-1}hg)B_g^{-1}$ gives an action on strict equivalence classes. A change of B_g is a strict conjugacy, and every $g \in G_K$ acts trivially. The action therefore factors through G. Let $r\colon G_K \to \mathrm{GL}_n(R)$ represent an invariant class with determinant $\psi|_{G_K}$. For $g \in G_F$ there is $A_g \in \mathrm{GL}_n(R)$ lifting $\bar\rho(g)$ with $r(ghg^{-1}) = A_g r(h) A_g^{-1}$ for all $h \in G_K$, unique up to a scalar in $1 + \mathfrak{m}_R$ since $\bar\rho|_{G_K}$ is absolutely irreducible. Indeed the k-algebra spanned by $\bar\rho(G_K)$ is $M_n(k)$, so the R-span of $r(G_K)$ is $M_n(R)$ by Nakayama's lemma. As $p \nmid n$, the pro-p group $1 + \mathfrak{m}_R$ is uniquely n-divisible, so there is a unique such A_g with $\det A_g = \psi(g)$. By uniqueness $A_g A_{g'} = A_{gg'}$ and $A_h = r(h)$ for $h \in G_K$; in particular $g \mapsto A_g$ is continuous, since $A_{gh} = A_g r(h)$ and G_K is open in G_F. Thus $g \mapsto A_g$ is the unique extension of r with determinant ψ. The statement on rings follows, since $(R_{K,S_K})_G$ represents the functor of G-invariant deformations. □

Remark 10.5. (1) If one does not fix the determinant, the obstruction to extending an invariant deformation lies in $H^2(G, 1+\mathfrak{m}_R)$ and the ambiguity in $H^1(G, 1+\mathfrak{m}_R)$. Both vanish when p divides neither $|G^{\mathrm{ab}}|$ nor the order of the Schur multiplier $H_2(G, \mathbb{Z})$; for $G = S_N$ ($N \geq 4$) or A_N ($N \geq 8$) these are 2-groups, so this holds for all odd p, even when p divides $|G|$.

(2) Suppose that every place in S carrying a local condition splits completely in K, and put the same condition at every place above it. Suppose also that $p \nmid e(w/v)$ for every finite $v \notin S$. For an extension r_F supplied by Proposition 10.4, $r_F(I_v)$ is a quotient of I_v/I_w, of order $e(w/v)$, and belongs to the congruence subgroup $1 + M_n(\mathfrak{m}_R)$ since $\bar{\rho}$ is unramified at v. This subgroup is pro-p, so $r_F(I_v) = 1$. Consequently $R^\dagger_F \cong (R^\dagger_K)_G$. In general, the relations killing $r_F(I_v)$ at primes outside S must also be imposed.

(3) Under the assumptions in (2), $\mathrm{Spec}(R^\dagger_F[1/p])$ is the scheme of fixed points of G on $\mathrm{Spec}(R^\dagger_K[1/p])$. When an $R = \mathbb{T}$ theorem identifies $R^\dagger_K$ with a finite Hecke algebra, geometric points of this scheme correspond to invariant systems of Hecke eigenvalues in that deformation problem. If $R^\dagger_F$ is finite over $\mathcal{O}$ and has Krull dimension at least one, then $R^\dagger_F[1/p] \neq 0$. Indeed, a finite $\mathcal{O}$-algebra killed by a power of p is Artinian. Finiteness and the dimension bound together thus give a characteristic zero fixed point.

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