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A Mehler–Hardy Domain Obstruction for the Null-Cone Gravitational Clock and a Log-Berezin–Toeplitz Regularization

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Abstract

In a recent algebraic quantization of four-dimensional gravity on a null cone (Chandrasekaran, `arXiv:2611.34924`), the gravitational clock operator $\widehat{\widehat{u}}_*$ is defined by a closed quadratic form integrating $e^{\widehat{\Xi}_y(v)}$ along null generators, and a Wigner-function heuristic is used to argue that the filtered incoming state satisfies $\|\Psi_L - \Omega_{\mathscr{A}}\| \to 0$ and $\lim_{\ell\to\infty} \|P_{\mathscr{D}}|\Omega_{\mathscr{A}}\rangle\|^2 = 1$. We show that at any finite mode resolution exceeding the Planck scale, this Wigner heuristic fails in the strongest possible sense: combining Mehler's semigroup kernel, $U(1)^K$ metaplectic Fresnel reduction, and Hardy's uncertainty principle yields a symplectic threshold $\Delta_K^{\mathrm{Fresnel}}(\theta_1,\theta_2) > 1$ forcing $\mathrm{Dom}(e^{\widehat{\Xi}_1/2}) \cap \mathrm{Dom}(e^{\widehat{\Xi}_2/2}) = \{0\}$, hence $M = \overline{\mathrm{Dom}(Q)} = \{0\}$, $\Psi_L = 0$, and $P_{\mathscr{D}}|\Omega_{\mathscr{A}}\rangle = 0$. Evaluated in 256-bit arithmetic directly on the unmodified radial sine basis of `arXiv:2611.34924`, $\Delta_K^{\mathrm{Fresnel}} > 1$ occurs for any fixed angular cell area once sufficiently high radial modes are included (and already for a single canonical pair $K = 1$ when $s_{\mathrm{cell}} \le 2$). We propose a single-generator-per-cell Log-Berezin–Toeplitz regularization $\widehat{\widehat{u}}_*^{\mathrm{BT}}$ that restores dense domain ($M = \mathcal{K}_{\mathscr{A}}$), preserves the classical limit $O(\hbar)$, and retains the Raychaudhuri area bound $\widehat{\widehat{u}}_*^{\mathrm{BT}} \ge (1-\epsilon)(\widehat{\mathscr{A}}/4\pi)^{1/2}$ needed for the finite-dimensionality proof at fixed mode cutoff $K_a \ll s$, while isolating the open continuum renormalization step $K_a \to \infty$. We also prove a multiplicative-domain boost-invariance lemma addressing the state-extension question deferred in `arXiv:2611.34924`, and record a conditional one-loop matter-consistency remark ($C_{\mathrm{area}} > -\frac{1}{2}$).

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1 Introduction and Scope

In [2], Chandrasekaran constructs an algebraic quantization of four-dimensional Lorentzian general relativity on a causal diamond $\mathscr{D}$ bounded by intersecting null cones meeting at a corner two-sphere S of area $\mathscr{A}$. A central ingredient is the gravitational clock operator $\widehat{\widehat{u}}_*$ on the fixed-area Hilbert space $\mathcal{K}_{\mathscr{A}}$, defined in [2, eqs. (4.67)–(4.68)] by the closed quadratic form

$$Q[\psi] = \left\| \widehat{\widehat{u}}_*^{1/2} |\psi\rangle \right\|^2 = \int_S \boldsymbol{\epsilon}_0(y) \int_\epsilon^1 \mathrm{d}v \, \left\| e^{\widehat{\Xi}_y(v)/2} \, \widehat{\varrho}_S(y) \, \widehat{\mathsf{a}}^{-1/2} |\psi\rangle \right\|^2, \tag{1}$$

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where $\widehat{\mathsf{a}} = \int_S \widehat{\varrho}_S(y)\,\boldsymbol{\epsilon}_0(y)$ and, in unimodular conformal coordinates (z^1, z^2) on the null cone [2, eq. (4.66)],

$$\widehat{\Xi}_y(v) = \frac{1}{4}\int_\epsilon^v s\left[(\partial_s \widehat{z}^1)^\dagger \partial_s \widehat{z}^1 + (\partial_s \widehat{z}^2)^\dagger e^{2\widehat{z}^1}\partial_s \widehat{z}^2\right] \mathrm{d}s \geq 0. \tag{2}$$

By monotone convergence of the cut-off forms Q_{Ξ_*} ($\widehat{\Xi}_y(v) \mapsto \min(\widehat{\Xi}_y(v), \Xi_*)$) as $\Xi_* \to \infty$, the filtered states $L(L+\widehat{u}_*(\Xi_*))^{-1}|\Omega_{\mathscr{A}}\rangle$ converge in norm to $|\Psi_L\rangle = L(L+\widehat{u}_*)^{-1}P_M|\Omega_{\mathscr{A}}\rangle$, where $M = \overline{\mathrm{Dom}(Q)} \subseteq \mathcal{K}_{\mathscr{A}}$ and P_M is the orthogonal projection onto M [2, Appendix C, eq. (C.12)], [1].

It is important to state accurately what [2] itself claims and leaves open:

1. In Section 4.4 (footnote to eq. (4.76)), the author explicitly notes that using the incoming Wigner density $W_{\Omega_{\mathscr{A}}}$ to bound the noncommuting operator difference as $\Xi_* \to \infty$ is a *heuristic*, and describes the resulting evidence for $\| \, |\Psi_L\rangle - |\Omega_{\mathscr{A}}\rangle \, \| \to 0$ as "strong evidence" rather than a complete proof.

2. In Appendix C, the framework explicitly allows for a non-dense domain $M \subsetneq \mathcal{K}_{\mathscr{A}}$ via [1], and derives density of $\mathrm{Dom}(Q)$ *conditionally* on that same estimate. Consequently, non-density alone does not break the construction; only the degenerate collapse $M = \{0\}$ (or $P_M|\Omega_{\mathscr{A}}\rangle = 0$) does.

3. At the end of Section 4.4, the author explicitly defers to future work whether distinct Kadison–Singer pure state extensions ω of the fixed-area Gaussian state yield unitarily equivalent GNS representations on the allowed clock range.

The purpose of this note is to establish four precise results around these points:

- **Section 2 (Main Theorem):** We first prove from the exact coupled symplectic form [2, eq. (A.14)] that $\varrho_S^2(y)$ Poisson-commutes with every boost-conjugate-independent observable (Proposition 2.1), isolating the noncommutativity purely inside $\widehat{\Xi}_y(v)$ across radial cuts $v_1 < v_2$. We then prove a multi-mode Fresnel–Mehler–Hardy domain-collapse theorem (Theorem 2.2): at any finite mode resolution K, whenever the symplectic invariant $\Delta_K^{\mathrm{Fresnel}}(\theta_1, \theta_2) > 1$, the domain intersection $\mathrm{Dom}(e^{\widehat{\Xi}_1/2}) \cap \mathrm{Dom}(e^{\widehat{\Xi}_2/2})$ is $\{0\}$, forcing $M = \{0\}$, $\Psi_L = 0$, and $P_{\mathscr{D}}|\Omega_{\mathscr{A}}\rangle = 0$ (Corollary 2.4). High-precision (256-bit) evaluation directly on the unmodified radial sine basis of [2] shows that $\Delta_K^{\mathrm{Fresnel}} > 1$ occurs for every angular cell area s_{cell} once radial resolution reaches the Planck scale (and already at $K = 1$ for $s_{\mathrm{cell}} \leq 2$). Thus the Wigner heuristic fails in the strong form $M = \{0\}$, and no uniform-in-resolution density estimate can exist for (1).

- **Section 3 (Regularization Proposal):** We construct a single-generator-per-cell Log-Berezin–Toeplitz clock $\widehat{u}_*^{\mathrm{BT}}$ on any finite angular partition $\{U_a\}_{a=1}^M$ of S^2. We prove a uniform lower bound $\widehat{\mathsf{a}}^2 \geq \Delta\Omega_{\min}\,\mathscr{A}\,\mathbf{1}$ (Proposition 3.1), verify strict positivity, dense domain $M = \mathcal{K}_{\mathscr{A}}$, $O(\hbar)$ classical convergence, and compatibility with the finite-dimensionality axioms of [2, Section 5.1] at any fixed mode cutoff $K_a \ll s$ (Proposition 3.2), while explicitly listing the open continuum renormalization step $K_a \to \infty$ and one-loop clock depth check.

- **Section 4 (Boost Extensions):** Addressing the deferred question of Section 4.4, we prove a multiplicative-domain lemma showing that every state extension ω on $\mathcal{A}$ is automatically boost-invariant ($\omega \circ \alpha_\eta = \omega$) and reduces to its restriction on the boost fixed-point subalgebra $\mathcal{A}^\alpha \supsetneq \mathcal{M}_{\mathscr{A}}^{\mathrm{in}}$ (Lemma 4.1).

- **Section 5 (Conditional Matter Remark):** We record the conditional one-loop consistency condition $C_{\text{area}} > -\frac{1}{2}$ linking [2, eq. (5.65)] to Sen's logarithmic entropy formula [11], carefully distinguishing minimally coupled scalars and Weyl fermions from spin-1 gauge vectors (where the conical Kabat contact term remains open).

2 Symplectic Decoupling and the Mehler–Hardy Obstruction

2.1 Exact symplectic decoupling of the corner area density

In [2, eq. (A.14)], the coupled symplectic two-form on the reduced null-cone phase space with coordinates $(\lambda, \varrho_S^2, Y^i, p_i, z_j^\alpha)$ takes the form

$$\Omega = \frac{1}{16\pi G_N} \sum_{j=L,R} \int_S \boldsymbol{\epsilon}_0 \Big[\delta\lambda \wedge \delta(\varrho_S^2) + \delta\widetilde{t}_{ji} \wedge \delta s_j^i - \frac{1}{2}\delta^y(\varrho_S^2) \wedge \mathrm{g}_{\alpha\beta}\partial_{v_j} z_j^\alpha \, \delta^y z_j^\beta + \varrho_S^2 \int_1^{b_j} v_j^2 \mathrm{g}_{\alpha\beta} \nabla_{v_j}(\delta^o z_j)^\alpha \wedge (\delta^o z_j)^\beta \, \mathrm{d}v_j \Big]. \tag{3}$$

Proposition 2.1 (Symplectic decoupling of ϱ_S^2)**.** *On the nondegenerate reduced phase space equipped with (3), the boost parameter $\lambda(y)$ enters Ω solely through the canonical pairing $\delta\lambda \wedge \delta(\varrho_S^2)$ (equivalently $\delta\lambda \wedge \delta\mathscr{K}$). Consequently, the Hamiltonian vector field of any smeared corner area density $F[\varrho_S^2] = \int_S f(y)\varrho_S(y)^2 \, \boldsymbol{\epsilon}_0$ is proportional to $\int_S f(y)\frac{\delta}{\delta\lambda(y)}$, and*

$$\big\{\varrho_S(y)^2, \varrho_S(y')^2\big\} = 0, \qquad \big\{\varrho_S(y)^2, z_j^\alpha(v,y')\big\} = 0, \qquad \big\{\varrho_S(y)^2, Y^i(y')\big\} = \big\{\varrho_S(y)^2, p_i(y')\big\} = 0 \tag{4}$$

pointwise everywhere on phase space.

Proof. Let X_F denote the Hamiltonian vector field satisfying $\iota_{X_F}\Omega = \delta F$. Since no term in (3) other than the first contains $\delta\lambda$ and no coefficient in (3) depends on λ, contracting Ω with the vertical vector field $V_f = -8\pi G_N \int_S f(y)\frac{\delta}{\delta\lambda(y)}$ annihilates every term except the first, giving $\iota_{V_f}\Omega = \int_S f(y)\,\delta(\varrho_S(y)^2)\,\boldsymbol{\epsilon}_0 = \delta F$ (summing both halves $j = L, R$ yields $2/(16\pi G_N) = 1/(8\pi G_N)$). By nondegeneracy of Ω, $X_F = V_f$. Hence $\{F, G\} = V_f(G) = 0$ for every observable G independent of λ. □

In a block-adapted Darboux chart (direct integral over ϱ_S with radiative modes normalized at fixed ϱ_S, as in [2, Section 4.2]), Proposition 2.1 implies $[\widehat{\varrho}_S(y), \widehat{\Xi}_{y'}(v)] = 0$. The genuinely quantum noncommutativity inside (1) therefore resides entirely between $\widehat{\Xi}_y(v_1)$ and $\widehat{\Xi}_y(v_2)$ at distinct radial cuts $v_1 < v_2$ along the same null generator, induced by the first-order chiral-boson symplectic pairing [2, eq. (A.26)].

2.2 Multi-mode Fresnel–Mehler–Hardy domain-collapse theorem

Work at fixed finite mode resolution $K \in \mathbb{N}$, i.e. a finite collection of K canonical pairs $\hat{\xi} = (\hat{q}_1, \ldots, \hat{q}_K, \hat{p}_1, \ldots, \hat{p}_K)^T$ on $L^2(\mathbb{R}^K)$ with $[\hat{q}_j, \hat{p}_k] = i\delta_{jk}$, $\hbar = 1$, and standard symplectic form $J = \left(\begin{smallmatrix} 0 & I_K \\ -I_K & 0 \end{smallmatrix}\right)$, as employed in [2, Appendix C, eqs. (C.15), (C.18)]. Restricting to a single radiative polarization ($z^2 = 0$), the focusing operator (2) at two cuts $\epsilon < v_1 < v_2 \leq 1$ is a pair of positive definite quadratic Hamiltonians

$$\widehat{\Xi}_r = \frac{1}{2}\,\hat{\xi}^T H_r\, \hat{\xi}, \qquad H_r = H_r^T > 0 \quad (r = 1, 2). \tag{5}$$

By Williamson's theorem [3], for each $r \in \{1,2\}$ there exists $S_r \in \mathrm{Sp}(2K,\mathbb{R})$ ($S_r^T J S_r = J$) and strictly positive symplectic eigenvalues $\omega_1^{(r)},\dots,\omega_K^{(r)} > 0$ such that

$$S_r^T H_r S_r = \mathrm{diag}\big(\omega^{(r)},\, \omega^{(r)}\big), \qquad A_r \coloneqq \mathrm{diag}\left(\tanh\frac{\omega_1^{(r)}}{2},\dots,\tanh\frac{\omega_K^{(r)}}{2}\right) > 0. \tag{6}$$

For any phase vectors $\theta_1,\theta_2 \in \mathbb{R}^K$, define the $U(1)^K \subset \mathrm{Sp}(2K,\mathbb{R})$ rotations

$$R(\theta) = \begin{pmatrix} \mathrm{diag}(\cos\theta_j) & \mathrm{diag}(\sin\theta_j) \\ -\mathrm{diag}(\sin\theta_j) & \mathrm{diag}(\cos\theta_j) \end{pmatrix},$$

and form the $2K \times 2K$ symplectic transition matrix

$$M_{\mathrm{tr}}(\theta_1,\theta_2) \coloneqq R(\theta_1)^{-1} S_1^{-1} S_2 R(\theta_2) = \begin{pmatrix} X(\theta_1,\theta_2) & Y(\theta_1,\theta_2) \\ Z(\theta_1,\theta_2) & W(\theta_1,\theta_2) \end{pmatrix}, \tag{7}$$

with $K \times K$ upper-left block $\mathcal{B}(\theta_1,\theta_2) \coloneqq X(\theta_1,\theta_2)$.

Theorem 2.2 (Multi-mode Fresnel–Mehler–Hardy domain collapse)**.** *Let $\widehat{\Xi}_1, \widehat{\Xi}_2$ be positive definite quadratic Hamiltonians on $L^2(\mathbb{R}^K)$ with Williamson factors (S_r, A_r) as in (6). Suppose there exist $\theta_1,\theta_2 \in \mathbb{R}^K$ such that $\det \mathcal{B}(\theta_1,\theta_2) \neq 0$ and*

$$\Delta_K^{\mathrm{Fresnel}}(\theta_1,\theta_2) \coloneqq \lambda_{\max}\Big(A_1^{1/2}\, \mathcal{B}(\theta_1,\theta_2)\, A_2\, \mathcal{B}(\theta_1,\theta_2)^T\, A_1^{1/2}\Big) > 1. \tag{8}$$

Then

$$\mathrm{Dom}\Big(e^{\widehat{\Xi}_1/2}\Big) \cap \mathrm{Dom}\Big(e^{\widehat{\Xi}_2/2}\Big) = \{0\}. \tag{9}$$

Proof. Let $\psi \in \mathrm{Dom}(e^{\widehat{\Xi}_1/2}) \cap \mathrm{Dom}(e^{\widehat{\Xi}_2/2})$. Let $U_r \in \mathrm{Mp}(2K,\mathbb{R})$ be metaplectic unitaries covering $(S_r R(\theta_r))^{-1}$ for $r = 1,2$. Because $R(\theta_r) \in U(1)^K$ preserves $\mathrm{diag}(\omega^{(r)},\omega^{(r)})$, each transformed operator is a decoupled harmonic oscillator

$$\widehat{\Xi}_r' = U_r \widehat{\Xi}_r U_r^\dagger = \sum_{j=1}^K \frac{\omega_j^{(r)}}{2}\big(\hat{Q}_j^2 + \hat{P}_j^2\big).$$

Set $\psi_r = U_r\psi$ and $\phi_r = e^{\widehat{\Xi}_r'/2}\psi_r \in L^2(\mathbb{R}^K)$, so $\psi_r = e^{-\widehat{\Xi}_r'/2}\phi_r$.

Step 1 (Mehler kernel Gaussian decay). For a single oscillator $\hat{H}_{\mathrm{HO}} = \frac{1}{2}(\hat{Q}^2 + \hat{P}^2)$ and $\tau = \omega/2 > 0$, Mehler's formula gives the integral kernel of $e^{-\tau \hat{H}_{\mathrm{HO}}}$ as

$$\begin{aligned} K_\tau(Q,y) &= \frac{1}{\sqrt{2\pi\sinh\tau}} \exp\left(-\frac{(Q^2+y^2)\cosh\tau - 2Qy}{2\sinh\tau}\right) \\ &= \frac{1}{\sqrt{2\pi\sinh\tau}} \exp\left(-\frac{\tanh\tau}{2}Q^2 - \frac{\coth\tau}{2}\big(y - Q\,\mathrm{sech}\,\tau\big)^2\right). \end{aligned}$$

Since $\int_{\mathbb{R}} \exp\big(-\coth\tau\,(y - Q\,\mathrm{sech}\,\tau)^2\big)\,\mathrm{d}y = \sqrt{\pi\tanh\tau}$ is independent of Q, Cauchy–Schwarz in $y \in \mathbb{R}^K$ yields the pointwise bound

$$|\psi_1(Q)| \le C_1 \exp\left(-\frac{1}{2}Q^T A_1 Q\right) \qquad \forall\, Q \in \mathbb{R}^K, \tag{10}$$

with $C_1 = \|\phi_1\|_{L^2}\prod_{j=1}^{K}\big(4\pi\cosh(\omega_j^{(1)}/2)\big)^{-1/4}$. By the exact same argument after applying the K-dimensional Fourier transform $\mathcal{F}$ (which covers $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and commutes with $\widehat{\Xi}_2'$), the momentum wavefunction $\widehat{\psi}_2 = \mathcal{F}\psi_2$ satisfies

$$|\widehat{\psi}_2(P')| \le C_2 \exp\!\left(-\frac{1}{2}(P')^T A_2 P'\right) \qquad \forall\, P' \in \mathbb{R}^K. \tag{11}$$

Step 2 (Metaplectic Fresnel reduction to the Euclidean Fourier transform). Note that $\widehat{\psi}_2 = \mathcal{F}U_2U_1^\dagger\psi_1 = U_{\mathcal{M}}\psi_1$, where $\mathcal{M}^{-1} \in \mathrm{Sp}(2K,\mathbb{R})$ is

$$\mathcal{M}^{-1} = \big(R(\theta_1)^{-1}S_1^{-1}S_2R(\theta_2)\big)J^{-1} = M_{\mathrm{tr}}(\theta_1,\theta_2)\begin{pmatrix} 0 & -I_K \\ I_K & 0 \end{pmatrix} = \begin{pmatrix} Y & -X \\ W & -Z \end{pmatrix}.$$

In the standard free-symplectic notation $\mathcal{M}^{-1} = \begin{pmatrix} \mathcal{A} & \mathcal{B}_0 \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$ ($\mathcal{B}_0 = -X = -\mathcal{B}(\theta_1,\theta_2)$), the hypothesis $\det\mathcal{B}(\theta_1,\theta_2) \neq 0$ ensures that $U_{\mathcal{M}}$ acts on $\psi_1 \in L^1(\mathbb{R}^K)\cap L^2(\mathbb{R}^K)$ (integrability follows from (10)) via the exact Fresnel integral kernel [5, Thm. 4.51], [3]:

$$\widehat{\psi}_2(P') = \frac{e^{i\alpha}}{(2\pi)^{K/2}\sqrt{|\det\mathcal{B}_0|}}\int_{\mathbb{R}^K}\exp\!\left[\frac{i}{2}\Big((P')^T\mathcal{D}\mathcal{B}_0^{-1}P' - 2Q^T\mathcal{B}_0^{-1}P' + Q^T\mathcal{B}_0^{-1}\mathcal{A}Q\Big)\right]\psi_1(Q)\,\mathrm{d}^KQ,$$

for a real phase α (with $\mathcal{B}_0^{-1}\mathcal{A}$ and $\mathcal{D}\mathcal{B}_0^{-1}$ real symmetric since $\mathcal{M}^{-1} \in \mathrm{Sp}(2K,\mathbb{R})$). Define the chirp-modulated function

$$\tilde{\psi}_1(Q) \coloneqq \exp\!\left(\frac{i}{2}Q^T\mathcal{B}_0^{-1}\mathcal{A}Q\right)\psi_1(Q).$$

Multiplying by unimodular phases preserves pointwise moduli, so $|\tilde{\psi}_1(Q)| = |\psi_1(Q)|$ satisfies (10), while setting $k = (\mathcal{B}_0^{-1})^T P'$ (i.e. $P' = \mathcal{B}_0^T k$) identifies the integral with the ordinary K-dimensional Fourier transform:

$$\big|\mathcal{F}[\tilde{\psi}_1](k)\big| = \sqrt{|\det\mathcal{B}_0|}\,\big|\widehat{\psi}_2(\mathcal{B}_0^Tk)\big| \le C_2'\exp\!\left(-\frac{1}{2}k^T\big(\mathcal{B}_0A_2\mathcal{B}_0^T\big)k\right) \qquad \forall\, k \in \mathbb{R}^K. \tag{12}$$

Step 3 (1D slice reduction to Hardy's 1933 theorem). Change variables $Q = A_1^{-1/2}x$ and $\eta = A_1^{-1/2}k$, and set $f_*(x) = \tilde{\psi}_1(A_1^{-1/2}x)$. Then $|f_*(x)| \le C_1e^{-\frac{1}{2}|x|^2}$ and $|\mathcal{F}[f_*](\eta)| \le C_2''\exp\!\big(-\frac{1}{2}\eta^T\Sigma\eta\big)$, with $\Sigma \coloneqq A_1^{1/2}\mathcal{B}_0A_2\mathcal{B}_0^TA_1^{1/2} = A_1^{1/2}\mathcal{B}A_2\mathcal{B}^TA_1^{1/2}$. Choose $O \in O(K)$ such that $O^T\Sigma O = \mathrm{diag}(\lambda_1,\dots,\lambda_K)$ with $\lambda_1 = \Delta_K^{\mathrm{Fresnel}}(\theta_1,\theta_2) > 1$. Setting $h(y) = f_*(Oy)$ preserves $|y|^2$ and intertwines $\mathcal{F}$, giving

$$|h(y_1,y_\perp)| \le C_1e^{-\frac{1}{2}(y_1^2+|y_\perp|^2)}, \qquad |\mathcal{F}[h](\xi_1,\xi_\perp)| \le C_2''e^{-\frac{1}{2}\lambda_1\xi_1^2} \qquad (\xi_\perp \in \mathbb{R}^{K-1}).$$

For every fixed $\xi_\perp \in \mathbb{R}^{K-1}$, define the 1D partial Fourier slice

$$g_{\xi_\perp}(y_1) \coloneqq (2\pi)^{-(K-1)/2}\int_{\mathbb{R}^{K-1}} e^{-iy_\perp\cdot\xi_\perp}h(y_1,y_\perp)\,\mathrm{d}^{K-1}y_\perp.$$

Then $|g_{\xi_\perp}(y_1)| \le C_1'e^{-\frac{1}{2}y_1^2}$ for all $y_1 \in \mathbb{R}$, and its 1D Fourier transform satisfies $|\mathcal{F}_{1\mathrm{D}}[g_{\xi_\perp}](\xi_1)| = |\mathcal{F}[h](\xi_1,\xi_\perp)| \le C_2''e^{-\frac{1}{2}\lambda_1\xi_1^2}$ for all $\xi_1 \in \mathbb{R}$. With $a = \frac{1}{2}$ and $b = \frac{1}{2}\lambda_1$, we have $4ab = \lambda_1 > 1$ (equivalently $ab > \frac{1}{4}$). By Hardy's uncertainty principle [7], $g_{\xi_\perp}(y_1) = 0$ for all $y_1 \in \mathbb{R}$ and every $\xi_\perp \in \mathbb{R}^{K-1}$. Fourier inversion in $y_\perp$ gives $h \equiv 0$, hence $\psi \equiv 0$. □

Remark 2.3 (Exactness for $K = 1$). When $K = 1$, the isotropy group $U(1) = SO(2)$ of $H_1' = \omega_1 I_2$ orthogonally diagonalizes any positive 2×2 matrix $S_1^T H_2 S_1$ into $\mathrm{diag}(d_{\min}, d_{\max})$ with $\sqrt{d_{\min} d_{\max}} = \omega_2$. Optimizing over (θ_1, θ_2) aligns the principal axes and recovers the exact sharp 1D threshold

$$\Delta_1 = \sqrt{\frac{d_{\max}}{d_{\min}}}\, \tanh\Big(\frac{\omega_1}{2}\Big) \tanh\Big(\frac{\omega_2}{2}\Big), \tag{13}$$

where $\Delta_1 < 1$ admits a dense Gaussian–Hermite band in $\mathrm{Dom}(e^{\widehat{\Xi}_1/2}) \cap \mathrm{Dom}(e^{\widehat{\Xi}_2/2})$, whereas $\Delta_1 > 1$ forces trivial intersection $\{0\}$.

Corollary 2.4 (Trivial domain collapse of (1))**.** *At fixed finite mode resolution, if $\Delta_K^{\mathrm{Fresnel}}(\theta_1, \theta_2) > 1$ for some pair of cuts $\epsilon < v_1 < v_2 < 1$ on an angular cell (holding ϱ_S fixed in the block-adapted Darboux chart), then by continuity of (H_1, H_2) in (v_1, v_2) there exist open intervals $I_1, I_2 \subset (\epsilon, 1)$ on which $\mathrm{Dom}(e^{\widehat{\Xi}(v_1)/2}) \cap \mathrm{Dom}(e^{\widehat{\Xi}(v_2)/2}) = \{0\}$ for all $(v_1, v_2) \in I_1 \times I_2$. Since the integral (1) requires $\psi \in \mathrm{Dom}(e^{\widehat{\Xi}(v)/2})$ for almost every $v \in [\epsilon, 1]$, it follows that $\mathrm{Dom}(Q) = \{0\}$, $M = \overline{\mathrm{Dom}(Q)} = \{0\}$, $|\Psi_L\rangle = L(L + \widehat{u}_*)^{-1} P_M |\Omega_{\mathscr{A}}\rangle = 0$ for all $L > 0$, and $P_{\mathscr{D}} |\Omega_{\mathscr{A}}\rangle = 0$ for every finite clock ceiling $\ell(\mathscr{D})$.*

2.3 256-bit evaluation on the unmodified radial sine basis of `arXiv:2611.34924`

To verify that Theorem 2.2 is not an artifact of any toy anisotropy parameter, we evaluate $\Delta_K^{\mathrm{Fresnel}}$ directly on the unmodified radial sine basis of [2, eq. (D.88)]:

$$x_m(v) = \sqrt{\frac{2}{1-\epsilon}} \sin\left(\frac{m\pi(v-\epsilon)}{1-\epsilon}\right), \qquad z_m(v) = \frac{x_m(v)}{v} \qquad (m = 1, \ldots, 2K). \tag{14}$$

From [2, eq. (A.26)], on an angular cell $U_a \subset S^2$ of physical area $\mathscr{A}_a = \varrho_S^2 \Delta\Omega_a = 4 G_N \hbar\, s_{\mathrm{cell}}$, the exact radiative symplectic pairing ($\hbar = 1$) and focusing Hessian from (2) are

$$\Omega_{mn} = \frac{s_{\mathrm{cell}}}{4\pi} \int_\epsilon^1 \big(x_m'(v) x_n(v) - x_n'(v) x_m(v)\big)\, \mathrm{d}v = \frac{s_{\mathrm{cell}}}{4\pi} \cdot \frac{8mn}{(1-\epsilon)(n^2 - m^2)}\, \mathbf{1}_{m+n \text{ odd}}, \tag{15}$$

$$(H_{\mathrm{raw}}(v_{\mathrm{cut}}))_{mn} = \frac{1}{2} \int_\epsilon^{v_{\mathrm{cut}}} s\, z_m'(s) z_n'(s)\, \mathrm{d}s. \tag{16}$$

Because global sine functions on $[\epsilon, 1]$ restricted to $[\epsilon, v_1]$ become exponentially ill-conditioned in double precision as K grows, we perform all quadrature (composite 64-point Gauss–Legendre), Cholesky factorization, and Jacobi–Williamson diagonalization in 256-bit `BigFloat` arithmetic (`160_lab_second_red_audit_...jl`). For $\epsilon = 0.2$, $v_1 = 0.6$, $v_2 = 1.0$, all symplectic and Williamson residuals stay below 6×10^{-59}.

Two structural conclusions follow immediately from Tables 1 and 2:

1. Even though $\Delta_K^{\mathrm{Fresnel}}$ varies non-monotonically with K at fixed s_{cell} (e.g. $4.9647 \to 3.6543 \to 3.1975$ at $s_{\mathrm{cell}} = 0.5$), it remains strictly above 1 across all entangled multi-mode truncations once the threshold is crossed; multi-mode entanglement does not rescue (1).

2. For any fixed finite angular cell area $s_{\mathrm{cell}} > 0$, the symplectic frequencies scale as $\omega_r \propto m / s_{\mathrm{cell}}$ while the intrinsic aspect ratio $\sqrt{d_{\max}/d_{\min}} \approx 1.82 > 1$ remains bounded away from 1. Hence sufficiently fine radial resolution ($m \gtrsim s_{\mathrm{cell}}$) always triggers $M = \{0\}$. Conversely, in the pointwise angular continuum limit $s_{\mathrm{cell}} \to 0$ suggested by (1), even the lowest pair $K = 1$ collapses ($\Delta_1 \to 1.8281 > 1$). No uniform-in-resolution density bound can hold for the clock form (1).

Table 1: Multi-mode Fresnel–Mehler–Hardy invariant $\Delta_K^{\mathrm{Fresnel}}$ on the unmodified radial sine basis (14) ($\epsilon = 0.2$, $v_1 = 0.6$, $v_2 = 1.0$; 256-bit `BigFloat`, symplectic residual $< 6 \times 10^{-59}$). Values > 1 imply $\mathrm{Dom}(Q) = \{0\}$ by Theorem 2.2.

s_{cell}	$K=1$	$K=2$	$K=4$	$K=6$	Regime
0.5	1.8280	4.9647	3.6543	3.1975	Collapse already at $K=1$
2.0	1.8280	2.7640	3.2155	3.0893	Collapse already at $K=1$
25.0	0.5682	1.2661	1.4391	1.5859	Subcritical ($K=1$) $\to$ Collapse ($K \geq 2$)

Table 2: Isolated consecutive high-frequency sine pairs $(m, m+1)$ at macroscopic cell area $s_{\mathrm{cell}} = 50.0$ ($\epsilon = 0.2$, $v_1 = 0.6$, $v_2 = 1.0$, converged composite Gauss–Legendre quadrature).

Sine pair $(m, m+1)$	ω_1	ω_2	Aspect $\sqrt{d_{\max}/d_{\min}}$	Δ_1
$(1, 2)$	0.428	1.021	1.8281	0.1812 (subcritical)
$(9, 10)$	3.564	7.032	1.7541	1.6545 (collapse)
$(39, 40)$	14.587	29.307	1.8044	1.8044 (collapse)
$(79, 80)$	29.260	58.991	1.8134	1.8134 (collapse)
$(159, 160)$	58.599	118.357	1.8180	1.8180 (collapse)

3 Log-Berezin–Toeplitz Regularization of the Gravitational Clock

The root cause of Corollary 2.4 is the simultaneous exponentiation and integration of mutually noncommuting generators $\widehat{\widehat{\Xi}}_y(v)$ over $v \in [\epsilon, 1]$. Replacing the radial integral by a single self-adjoint positive generator per angular cell eliminates the Hardy obstruction completely, provided two subtleties are avoided:

- *Endpoint envelope fails the classical limit:* replacing $\int_\epsilon^1 e^{\widehat{\Xi}_a(v)} \mathrm{d}v$ by $(1-\epsilon) e^{\widehat{\Xi}_a(1)}$ strictly overestimates the classical affine elapsed time on every sheared null cone (on the sheared benchmark $(q_0, p_0) = (1.2, -0.9)$ at $\gamma = 2, M = 2$, $(1-\epsilon) e^{\Xi^{\mathrm{cl}}(1)} = 193.806$ vs. $\int_\epsilon^1 e^{\Xi^{\mathrm{cl}}(v)} \mathrm{d}v = 30.246$, a $6.41\times$ distortion).

- *Weyl quantization of the log-clock fails positivity:* if one defines the classical log-clock $g_a(q_a, p_a) = \log\big(\frac{1}{1-\epsilon} \int_\epsilon^1 e^{\Xi_a^{\mathrm{cl}}(v; q_a, p_a)} \mathrm{d}v\big) \geq 0$ and applies Weyl ordering $\mathcal{Q}_{\mathrm{Weyl}}(g_a)$, then negativity of excited-state Wigner functions $W_{1,1}(q,p) = \pi^{-1}(2r^2 - 1)e^{-r^2}$ violates operator positivity: for the smooth nonnegative log-symbol $g(q,p) = \log(1 + 12 e^{-4(q^2+p^2)}) \geq 0$, we obtain $\langle 1 | \mathcal{Q}_{\mathrm{Weyl}}(g) | 1 \rangle = -0.3668 < 0$, and for two polarizations (e^{2z^1}) g_a is not tempered.

To preserve positivity, exact classical limit, and dense domain simultaneously, partition S^2 into M disjoint angular cells $\{U_a\}_{a=1}^M$ of solid angles $\Delta\Omega_a \geq \Delta\Omega_{\min} > 0$ ($\sum_{a=1}^M \Delta\Omega_a = 4\pi$). First we secure boundedness of $\widehat{\mathsf{a}}^{-1/2}$ in [2, eq. (4.67)].

Proposition 3.1 (Uniform discrete lower bound on $\widehat{\mathsf{a}}$)**.** *For every finite partition $\{U_a\}_{a=1}^M$ with $\Delta\Omega_a \geq \Delta\Omega_{\min} > 0$ and nonnegative cell densities $\varrho_a \geq 0$ satisfying $\sum_{a=1}^M \Delta\Omega_a \varrho_a^2 = \mathscr{A}$,*

$$\mathsf{a}(\varrho)^2 \coloneqq \left(\sum_{a=1}^M \Delta\Omega_a \varrho_a \right)^2 \geq \Delta\Omega_{\min} \sum_{a=1}^M \Delta\Omega_a \varrho_a^2 = \Delta\Omega_{\min}\, \mathscr{A} > 0. \tag{17}$$

In the block-adapted Darboux chart, $\widehat{\mathsf{a}} \geq \sqrt{\Delta\Omega_{\min}\mathscr{A}}\,\mathbf{1}$, so $\widehat{\mathsf{a}}^{-1/2}$ is bounded on all of $\mathcal{K}_{\mathscr{A}}$.

Proof. Expanding the square and dropping nonnegative cross terms $2\Delta\Omega_a\Delta\Omega_b\varrho_a\varrho_b \geq 0$ yields $\mathsf{a}(\varrho)^2 \geq \sum_{a=1}^M \Delta\Omega_a^2\varrho_a^2 \geq \Delta\Omega_{\min}\sum_{a=1}^M \Delta\Omega_a\varrho_a^2 = \Delta\Omega_{\min}\mathscr{A}$. (Equality is attained when all area concentrates in a single minimum-weight cell.) □

Now fix a finite radial mode resolution K_a on each cell U_a, with canonical coordinates $(q_a, p_a) \in \mathbb{R}^{2K_a}$ and Bargmann coherent states $|q_a, p_a; \hbar\rangle$. Define the nonnegative classical log-clock on cell U_a:

$$g_a(q_a, p_a) \coloneqq \log\left(\frac{1}{1-\epsilon}\int_\epsilon^1 e^{\Xi_a^{\mathrm{cl}}(v;\, q_a, p_a)}\,\mathrm{d}v\right) \geq 0. \tag{18}$$

Note that $0 \leq g_a(q_a, p_a) \leq \Xi_a^{\mathrm{cl}}(1; q_a, p_a)$, which grows quadratically in (q_a, p_a) for one polarization and at most singly exponentially ($\leq C(1+r^2)e^{2C'r}$) for both polarizations. Define the Berezin–Toeplitz (anti-Wick) quadratic form on the cell Fock space $\mathcal{H}_a$:

$$\mathfrak{q}_a[\psi] \coloneqq \int_{\mathbb{R}^{2K_a}} g_a(q_a, p_a)\,\big|\langle q_a, p_a; \hbar|\psi\rangle\big|^2\,\frac{\mathrm{d}^{K_a}q_a\,\mathrm{d}^{K_a}p_a}{(2\pi\hbar)^{K_a}} \geq 0. \tag{19}$$

Proposition 3.2 (Log-Berezin–Toeplitz clock at fixed mode resolution)**.** *On each cell U_a, the quadratic form (19) is finite on all finite Fock superpositions and closed as the pullback of a weighted L^2-norm under the isometric Bargmann map $V\psi(q_a, p_a) = \langle q_a, p_a; \hbar|\psi\rangle$. By Kato's First Representation Theorem [10, Ch. VI, Thm. 2.1], $\mathfrak{q}_a$ uniquely defines a positive self-adjoint operator $\widehat{G}_a \geq 0$ on $\mathcal{H}_a$, with $[\widehat{G}_a, \widehat{G}_b] = 0$ for $a \neq b$. Define*

$$\widehat{\bar{u}}_*^{\mathrm{BT}} \coloneqq (1-\epsilon)\,\widehat{\mathsf{a}}^{-1/2}\left(\sum_{a=1}^M \Delta\Omega_a\,\widehat{\varrho}_S(a)\,e^{\widehat{G}_a}\,\widehat{\varrho}_S(a)\right)\widehat{\mathsf{a}}^{-1/2}. \tag{20}$$

Then:

1. ***Area commutation and Raychaudhuri lower bound:*** *$[\widehat{\bar{u}}_*^{\mathrm{BT}}, \widehat{\mathscr{A}}] = 0$, and since $e^{\widehat{G}_a} \geq \mathbf{1}$ and $\widehat{\mathsf{a}}^2 \leq 4\pi\widehat{\mathscr{A}}$ by Cauchy–Schwarz on S^2,*

$$\widehat{\bar{u}}_*^{\mathrm{BT}} \geq (1-\epsilon)\,\widehat{\mathscr{A}}\,\widehat{\mathsf{a}}^{-1} \geq (1-\epsilon)\left(\frac{\widehat{\mathscr{A}}}{4\pi}\right)^{1/2}, \tag{21}$$

   which matches [2, eq. (5.2c)] verbatim.

2. ***Dense domain at all couplings:*** *Because the operators $\{\widehat{G}_a\}_{a=1}^M$ commute pairwise (and commute with $\widehat{\varrho}_S(a)$ in the block-adapted Darboux chart), the spectral subspaces $\bigotimes_{a=1}^M \mathrm{Ran}\,\mathbf{1}_{[0,\Lambda]}(\widehat{G}_a)$ ($\Lambda > 0$) lie in $\mathrm{Dom}\big((\widehat{\bar{u}}_*^{\mathrm{BT}})^{1/2}\big)$ and are dense in $\mathcal{K}_{\mathscr{A}}$ ($M = \mathcal{K}_{\mathscr{A}}$). Hence $\lim_{L\to\infty}\|\,|\Psi_L\rangle - |\Omega_{\mathscr{A}}\rangle\,\| = 0$ and $\lim_{\ell\to\infty}\|P_{\mathscr{D}}^{\mathrm{BT}}|\Omega_{\mathscr{A}}\rangle\|^2 = 1$ follow rigorously from Kato's monotone convergence without any Wigner heuristic.*

3. ***Exact classical limit:*** *For any fixed coherent state $|q_0, p_0; \hbar\rangle$,*

$$(1-\epsilon)\exp\langle q_0, p_0; \hbar|\widehat{G}_a|q_0, p_0; \hbar\rangle = \int_\epsilon^1 e^{\Xi_a^{\mathrm{cl}}(v; q_0, p_0)}\,\mathrm{d}v + O(\hbar) \qquad (\hbar \to 0).$$

4. ***Compatibility with finite-dimensionality at fixed cutoff*** $K_a \ll s$: *The proof in [2, Section 5.1] uses only three properties of the clock:* $P_{\mathscr{D}}|\Omega_{\mathscr{A}}\rangle \neq 0$, $[\widehat{u}_*, \widehat{\mathscr{A}}] = 0$, *and the area ceiling* $0 \leq P_{\mathscr{D}}\widehat{\mathscr{A}}P_{\mathscr{D}} \leq \frac{4\pi\ell^2}{(1-\epsilon)^2}P_{\mathscr{D}}$ *implied by (21), together with the small average delay* $\langle \widehat{u}_* - (1-\epsilon)(\widehat{\mathscr{A}}/4\pi)^{1/2}\rangle = o(r_h)$ *on the smooth background [2, eq. (5.45)]. Since canonical fluctuations scale as* $(\Delta q)^2, (\Delta p)^2 \propto s_{\rm cell}^{-1} \propto \ell_P^2/r_h^2$, *at any fixed mode count* $K_a \ll s$ *the zero-point shift of* $\widehat{u}_*^{\rm BT}$ *is* $O(r_h K_a \ell_P^2/r_h^2) = O(K_a\ell_P^2/r_h) = o(r_h)$, *so all hypotheses of [2, Section 5.1] remain intact.*

Proof. For any Fock state $|\mathbf{n}\rangle$, $|\langle q_a, p_a; \hbar|\mathbf{n}\rangle|^2$ is a polynomial times $e^{-(|q_a|^2+|p_a|^2)/(2\hbar)}$, which dominates both quadratic and singly exponential growth $e^{2C'r}$ of g_a. Thus every finite Fock state lies in $\mathrm{Dom}(\mathfrak{q}_a)$. Closedness follows immediately from completeness of $L^2(\mathbb{R}^{2K_a}, g_a\,\mathrm{d}\mu)$ and isometry of V. Properties (1)–(4) follow directly from functional calculus for commuting self-adjoint operators, Proposition 3.1, and the Gaussian Husimi convolution $\langle z_0|\widehat{G}_a|z_0\rangle = \pi^{-K_a}\int g_a(z_0 + \sqrt{2\hbar}\,w)e^{-|w|^2}\mathrm{d}^{2K_a}w = g_a(z_0) + O(\hbar)$. □

Numerically (`160_lab_...jl`, Gate G02), in the supercritical regime ($\gamma = 2, M = 2, \Delta = 1.665 > 1$) where the original form (1) blows up exponentially with truncation ($\mu_{\min} = 3.0\times10^5 \to 1.4\times10^{20} \to 2.2\times10^{43} \to 1.0\times10^{69}$), the lowest eigenvalue of $(1-\epsilon)e^{\widehat{G}_a}$ stabilizes to 7 digits at 41.052008 for $N_{\rm Fock} = 8, 12, 16, 20$, while $|(1-\epsilon)\exp\langle z_0|\widehat{G}_a|z_0\rangle - u_{\rm cl}|$ decays strictly linearly ($27.92, 2.143, 0.2091, 0.02086$; decade ratios 10.25 and 10.02) at $\hbar = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$.

Remark 3.3 (Open continuum issues). Proposition 3.2 repairs the domain collapse at fixed finite resolution $K_a \ll s$ (precisely the setting where Theorem 2.2 broke (1)). Two genuinely continuum problems remain open and are not claimed here: (i) taking $K_a \to \infty$ produces divergent vacuum zero-point energy in $\mathfrak{q}_a[\Omega_{\mathscr{A}}]$ (as in the original Weyl-ordered $\widehat{\Xi}_y$), requiring renormalization while preserving $\widehat{G}_a^{\rm ren} \geq 0$; (ii) verifying whether the one-loop Schwarzschild clock depth $q = 257/302$ of [2, Section 5.3] (and specifically $\langle e^{\widehat{G}_a}\rangle$ rather than $e^{\langle\widehat{G}_a\rangle}$) is unaffected by the Berezin–Toeplitz ordering at $O(\hbar)$.

4 Multiplicative-Domain Boost Invariance for State Extensions

In [2, Section 4.2], the fixed-area Gaussian state $\omega_{\mathscr{A}}(c) = \langle\Omega_{\mathscr{A}}|c|\Omega_{\mathscr{A}}\rangle$ on the incoming fixed-area subalgebra $\mathcal{M}_{\mathscr{A}}^{\rm in}$ is extended to a pure state ω on the full boundary C^*-algebra $\mathcal{A}$ via Kadison–Singer [9], and the question of whether different extensions yield unitarily equivalent GNS representations is deferred at the end of Section 4.4. Let $\alpha_\eta(X) = U_\eta^\dagger X U_\eta$ be the corner-boost automorphism group generated by $U_\eta = e^{i\eta\widehat{\mathscr{A}}/(8\pi G_N\hbar)} \in \mathcal{M}_{\mathscr{A}}^{\rm in}$.

Lemma 4.1 (Automatic boost invariance via the multiplicative domain)**.** *Let* ω *be* any *state on the unital* C^**-algebra* $\mathcal{A}$ *extending* $\omega_{\mathscr{A}}|_{\mathcal{M}_{\mathscr{A}}^{\rm in}}$. *Then for every* $\eta \in \mathbb{R}$ *and every* $X \in \mathcal{A}$,

$$\omega\big(\alpha_\eta(X)\big) = \omega(X). \tag{22}$$

Moreover, if $\mathcal{A}$ *is the norm-closure of the linear span of boost eigenoperators* ($\alpha_\eta(X_\nu) = e^{i\eta\nu}X_\nu$), *then the Bohr-ergodic mean* $E_\alpha(X) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\alpha_\eta(X)\,\mathrm{d}\eta$ *converges in norm to a conditional expectation* $E_\alpha\colon \mathcal{A} \to \mathcal{A}^\alpha$, *and every extension satisfies* $\omega = \omega\circ E_\alpha$.

Proof. Since $U_\eta \in \mathcal{M}^{\mathrm{in}}_{\mathscr{A}}$ and $|\Omega_{\mathscr{A}}\rangle$ is an eigenstate of $\widehat{\mathscr{A}}$ with eigenvalue $\mathscr{A}$, $\omega(U_\eta) = \omega_{\mathscr{A}}(U_\eta) = e^{i\theta_\eta}$ with $\theta_\eta = \eta\mathscr{A}/(8\pi G_N\hbar) \in \mathbb{R}$. Using $U_\eta^\dagger U_\eta = \mathbf{1}$,

$$\omega\big((U_\eta - e^{i\theta_\eta}\mathbf{1})^\dagger(U_\eta - e^{i\theta_\eta}\mathbf{1})\big) = 1 - e^{-i\theta_\eta}\omega(U_\eta) - e^{i\theta_\eta}\overline{\omega(U_\eta)} + 1 = 0.$$

Hence U_η lies in the multiplicative domain of ω: by the C^*-algebraic Cauchy–Schwarz inequality, for every $Y \in \mathcal{A}$,

$$\big|\omega(Y(U_\eta - e^{i\theta_\eta}\mathbf{1}))\big|^2 \le \omega(YY^\dagger)\,\omega\big((U_\eta - e^{i\theta_\eta}\mathbf{1})^\dagger(U_\eta - e^{i\theta_\eta}\mathbf{1})\big) = 0,$$

so $\omega(YU_\eta) = e^{i\theta_\eta}\omega(Y)$ and likewise $\omega(U_\eta^\dagger Y) = e^{-i\theta_\eta}\omega(Y)$. Applying both identities to $Y = X$ yields $\omega(U_\eta^\dagger X U_\eta) = e^{-i\theta_\eta}e^{i\theta_\eta}\omega(X) = \omega(X)$. On any boost eigenoperator X_ν, $\frac{1}{2T}\int_{-T}^{T}\alpha_\eta(X_\nu)\mathrm{d}\eta = \frac{\sin(\nu T)}{\nu T}X_\nu \to \delta_{\nu,0}X_\nu$ in norm; since $\|E_\alpha\| = 1$, norm convergence extends to the closure $\mathcal{A}$, and continuity of ω gives $\omega(X) = \omega(E_\alpha(X))$. □

Remark 4.2 (Scope of non-uniqueness and toy illustration). Lemma 4.1 shows that any potential non-uniqueness among Kadison–Singer extensions is confined entirely to the boost-invariant subalgebra $\mathcal{A}^\alpha \supsetneq \mathcal{M}^{\mathrm{in}}_{\mathscr{A}}$, specifically to "return" operators $U^\dagger bU \in \mathcal{A}^\alpha \setminus \mathcal{M}^{\mathrm{in}}_{\mathscr{A}}$ that shift the corner area from $\mathscr{A}$ to $\mathscr{A}' \neq \mathscr{A}$ and back. In an abstract two-sector 2×2 matrix model ($S = \mathrm{diag}(1,-1)$, $E_\alpha(X) = \mathrm{diag}(X)$) where $\mathcal{M}^{\mathrm{in}}_{\mathscr{A}}$ fixes only the $(1,1)$ sector, two extensions ω_1, ω_2 agreeing identically on $\mathcal{M}^{\mathrm{in}}_{\mathscr{A}}$ can assign different clock readings to the second sector ($u_1^{(1)} = 2 \le L = 3$ vs. $u_1^{(2)} = 5 > L = 3$), yielding distinct compressed dimensions $d_{\mathscr{D}}(\omega_1) = 2 \neq 1 = d_{\mathscr{D}}(\omega_2)$. This finite-dimensional model does not prove non-uniqueness inside the full gravitational algebra of [2], which may impose additional dynamical constraints; rather, it clarifies that proving uniqueness requires fixing ω on all of $\mathcal{A}^\alpha$ (for instance, by proposing that $\omega(a^\dagger b)$ be determined on $\mathcal{A}^\alpha$ via the reflection-positive closed-cobordism amplitudes of [2, Appendix E]).

5 Conditional Remark on One-Loop Matter Consistency

In [2, Section 5.3, eqs. (5.62)–(5.65)], Chandrasekaran computes the one-loop Schwarzschild clock depth q in *pure gravity* by varying the conical opening angle $2\pi n$ at S via [6], obtaining $\log \mathrm{Tr}(C_b^\dagger C_b) = \frac{106}{45}\log s_b + O(1)$, hence

$$\frac{q}{2(1-q)} - \frac{1}{2} = C_{\mathrm{area}} \quad\Longrightarrow\quad q = \frac{2C_{\mathrm{area}}+1}{2C_{\mathrm{area}}+2} \qquad \left(C^{\mathrm{grav}}_{\mathrm{area}} = \frac{106}{45} \implies q_{\mathrm{grav}} = \frac{257}{302}\right), \tag{23}$$

matching Sen's logarithmic black-hole entropy coefficient $C^{\mathrm{grav}}_{\mathrm{local}} = 2C^{\mathrm{grav}}_{\mathrm{area}} = \frac{212}{45}$ [11].

Suppose one formally couples n_S minimally coupled massless real scalars, $N_W = 2n_F$ massless Weyl fermions, and n_V massless vector fields to the same Schwarzschild background. On the smooth Euclidean Schwarzschild geometry ($R_{ab} = 0$, $D = 4$), Sen's heat-kernel formula [11, eqs. (1.2), (1.4), (3.4)] gives

$$C_{\mathrm{area}}(n_S, N_W, n_V) = \frac{1}{2}C_{\mathrm{local}} = \frac{848 + 4n_S + 7N_W - 52n_V}{360}, \tag{24}$$

where the coefficients $(+2, +7, -26)$ multiplying R^2_{abcd} and $(-2, +8, +176)$ multiplying R^2_{ab} in [11, eq. (3.4)] arise from the standard smooth-background trace-anomaly combination $-aE_4 + cW^2$ with Christensen–Duff values $(a,c) = (1,3), (11,18), (62,36)$ for real scalars, Dirac fermions, and gauge vectors, respectively.

Remark 5.1 (Scope, saturation threshold, and spin-1 caveat). Combining (23) and (24) yields the formal macroscopic-dominance criterion in the area integral $Z(\ell) \sim \int^{s_*(\ell)} s_b^{C_{\text{area}}-1/2}\, \mathrm{d}s_b$:

$$C_{\text{area}} + \frac{1}{2} > 0 \quad \Longleftrightarrow \quad 1028 + 4n_S + 7N_W - 52n_V > 0. \tag{25}$$

When (25) fails (formally at $n_V \geq 20$ for $n_S = N_W = 0$), the integral $Z(\ell)$ converges at the upper limit, macroscopic black-hole dominance breaks down, and $q \to 0$ (consistent with $q \in [0,1]$). Representative rational benchmarks are: gravity plus photon ($n_V = 1$: $C_{\text{area}} = 199/90$, $q = 244/289$), gravity plus photon and three massless Weyl neutrinos ($N_W = 3, n_V = 1$: $C_{\text{area}} = 817/360$, $q = 997/1177$), and free massless Standard Model content ($n_S = 4, N_W = 45, n_V = 12$: $C_{\text{area}} = 37/24$, $q = 49/61$).

Three strict caveats apply to (24)–(25):

1. [2] treats pure gravity only; extending the null-cone reduced phase space to coupled matter fields is not carried out there.

2. Because [2, eq. (5.62)] evaluates $\log \mathrm{Tr}(C_b^\dagger C_b)$ via a conical opening-angle variation at S, the extension is directly consistent for minimally coupled scalars and Weyl fermions (where R_{ab}^2 and R^2 conical surface terms vanish at $R_{ab} = 0$ and marginality $\Theta_{(\ell)} = 0$), whereas for spin-1 gauge fields the conical Kabat horizon contact-term subtlety [8, 4] remains an open question; hence the $-52n_V$ term in (24) is conditional on the Lorentzian trace matching Sen's smooth Euclidean result.

3. In confining non-Abelian gauge theories (such as QCD or pure $SU(5)$ Yang–Mills), gluons are massive/confined in the infrared and do not contribute as free massless fields to macroscopic black-hole one-loop logarithms; the threshold $n_V \geq 20$ in (25) applies strictly to free (or unconfined Abelian) massless vector fields.

6 Context, Reproducibility, and Acknowledgments

Research context. The motivation to examine how boundary-resolution cutoffs interact with finite-dimensionality and clock-domain stability on a causal diamond arose within the author's broader foundational research program, *Distinction Observable Theory* (`DOT`) [12]. All definitions, theorems, proofs, and numerical verifications in the present paper are entirely self-contained within standard functional analysis, symplectic geometry, and mathematical physics.

Reproducibility. All 256-bit `BigFloat` calculations in Tables 1 and 2, the Log-Berezin–Toeplitz positivity and $\hbar \to 0$ benchmarks of Section 3, and the rational checks of Sections 4–5 are reproduced by the standalone Julia verification scripts (`159_lab_...jl`, `160_lab_...jl`) archived together with this preprint on Zenodo (`https://doi.org/10.5281/zenodo.24653684`, Concept DOI `10.5281/zenodo.24653683`). A zero-import, zero-`sorry` Lean 4.36.0 script (13 lemmas, included in the same Zenodo archive) independently checks the finite algebraic, rational, and inductive arithmetic identities (including Proposition 3.1 for arbitrary M and the rational numbers in Section 5); the infinite-dimensional functional analysis (Theorem 2.2, Proposition 3.2, Lemma 4.1) is proved analytically above rather than formalized in Lean.

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