Revenue optimality of the straight-jacket auction for independent uniform goods

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Abstract

A seller facing a single buyer with private values for several goods can screen the buyer with lotteries over bundles. We prove that, for one additive buyer whose values for at most 82 goods are independently uniform on the unit interval, the deterministic straight-jacket auction maximizes expected revenue among all incentive-compatible and individually rational mechanisms, including randomized mechanisms. In arbitrary dimension the same conclusion holds whenever the recursive prices satisfy their volume equations and the full sequence, after replacing each price by the minimum of its suffix, is concave. The geometric result behind this implication characterizes when a symmetric polymatroid polytope maximizes volume minus a weighted sum of coordinate projection volumes over all its compact subsets. The test checks only the polytopes associated with suffixes of the ordered generator list. Its proof constructs a bounded-divergence field in a boundary collar. Exact integer sign certificates establish concavity through 82 goods and certify longer price prefixes in specified ranges.

Keywords: multidimensional mechanism design, revenue maximization, straight-jacket auction, polymatroids, anisotropic perimeter

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1 Introduction.

A seller with several goods can use allocation probabilities to screen a buyer along several dimensions of private information [11, 8, 9, 2, 6]. Randomization enlarges the set of incentive-compatible menus and can improve revenue even for two independent and identically distributed values [5]. Deterministic optimality is therefore a substantive property of the prior. Independent uniform values give a natural setting in which to ask whether a price for each bundle size can match the best revenue available from lotteries [3].

Giannakopoulos and Koutsoupias [4, Theorem 4.2 and Conjecture 4.3] introduced the straight-jacket auction, proved its global optimality through six goods, and conjectured optimality in every dimension. Their deficiency approach gives a dimension-free sufficient condition for comparing the menu to randomized mechanisms and a slice condition for maximum deficiency [4, Lemma 5.6 and Theorem 4.9]. Joswig, Klimm, and Spitz [7] identified the generalized permutahedra [10] in the problem, gave a product decomposition of allocation regions and a clipping algorithm for the prices, and computed prices and revenues through twelve goods, with certified numerical bounds through ten. Their results on critical prices concern deterministic schedules.

The main geometric result here is an exact suffix criterion for maximum deficiency over arbitrary compact subsets of a symmetric polymatroid polytope. It evaluates the anisotropic perimeter penalty at which the reflected polytope becomes a minimizer. We construct a field in a collar by combining fields from lower-dimensional suffix bodies. Equality of their normal traces with a reference field gives a bounded distributional divergence across the interfaces. The criterion then verifies the sold-coordinate inequalities needed for the revenue comparison whenever the complete normalized price sequence is concave. An exact sign computation establishes that concavity in every dimension through 82. This endpoint is the extent of the present full-menu certificate.

1.1 Model and results.

Let $m \geq 1$, $[m] = \{1, \dots, m\}$, and $d = 1/(m+1)$. The risk-neutral buyer has additive values x uniformly distributed on $[0,1]^m$. A direct mechanism specifies measurable marginal allocation probabilities $Q(x) \in [0,1]^m$ and an integrable expected payment $\tau(x)$. It is feasible when its utility $u(x) = x \cdot Q(x) - \tau(x)$ satisfies $u(x) \geq 0$ and $u(x) \geq x \cdot Q(z) - \tau(z)$ for all x, z. Expected revenue is the integral of τ over the cube.

The straight-jacket auction offers each nonempty bundle I at $q_{|I|}$ and the empty bundle at zero. The raw prices are chosen in order by

$$\left| \left\{ x \in [0,1]^r : \sum_{i \in I} x_i \leq q_{|I|} \text{ for all } \varnothing \neq I \subseteq [r] \right\} \right| = 1 - rd, \qquad 1 \leq r \leq m. \tag{1}$$

The normalization through r has $p_0 = 0$ and $p_j = \min_{j \leq k \leq r} q_k$ for $1 \leq j \leq r$. Let $\Delta_j = p_j - p_{j-1}$. We call the prefix concave if $\Delta_1 \geq \cdots \geq \Delta_r \geq 0$. A lowered price at size j is attained at some larger raw size. Any bundle at the lowered price is dominated by a raw bundle containing it and sold at that same price whenever all coordinates are positive. Thus normalization preserves utility everywhere and revenue almost everywhere. The no-sale constraints are also preserved, since a lowered subset cap follows by extending the subset to a size attaining the minimum.

Theorem 1 (Revenue optimality)**.** *For each $1 \leq m \leq 82$, the raw prices in* (1) *exist uniquely, the full normalization is concave, and the straight-jacket auction maximizes expected revenue over all feasible direct mechanisms, including lotteries. For any $m \geq 1$, a sequence satisfying* (1) *yields the same revenue conclusion if its full normalization is concave.*

For a nonincreasing nonnegative list $a = (a_1, \dots, a_n)$, put $S_k(a) = \sum_{i=1}^{k} a_i$ and

$$K(a) = \left\{ y \in \mathbb{R}^n_{\geq 0} : \sum_{i \in I} y_i \leq S_{|I|}(a) \text{ for all } \varnothing \neq I \subseteq [n] \right\}.$$

For compact $H \subseteq \mathbb{R}^n_{\geq 0}$ and $\kappa > 0$, let $D_\kappa(H) = |H| - \kappa \sum_{i=1}^{n} |\pi_i H|$, where π_i deletes coordinate i and volumes use their ambient dimensions. Write $D = D_1$, $V(a) = |K(a)|$, and use $a_{\leq k}$ and $a_{>k}$ for the prefix and suffix at k. The zero-dimensional point $K(())$ has volume and deficiency one. The empty set has volume zero in every dimension.

Theorem 2 (Suffix criterion)**.** *Let $a_1 \geq \cdots \geq a_n > 0$ and $\kappa > 0$. Then $D_\kappa(H) \leq D_\kappa(K(a))$ for every compact $H \subseteq K(a)$ if and only if $D_\kappa(K(a_{>k})) \geq 0$ for every $0 \leq k < n$.*

For example, $K(4, 1/2)$ is a square of side four with a corner triangle of leg $7/2$ removed. Its deficiency is $15/8$, whereas its singleton suffix has deficiency $-1/2$.

Theorem 3 (Price continuation)**.** *Fix $1 \le n < m$. If $m \le 82$, or $n \le 77$, or $78 \le n \le 94$ and $m \ge n + 6$, the raw prices through $n + 1$ exist uniquely and every normalized prefix through $n + 1$ is concave. With normalization through n, one has $q_{n+1} < p_n + \Delta_n$.*

Section 2 proves the suffix criterion. Section 3 uses it to compare revenues, and Section 4 reduces continuation to a facet comparison. Section 5 proves the stated ranges with exact arithmetic. The continuation result concerns prefixes when $m > 82$.

2 The suffix criterion.

2.1 Polytopes and compression.

The polytope $K(a)$ is the downward closure of the convex hull $P(a)$ of the permutations of a. Indeed, to bound a linear functional on $K(a)$, drop its nonpositive coefficients and order the positive ones as $v_1 \ge \cdots \ge v_j > 0$. Set $v_{j+1} = 0$. Summation by parts bounds the functional by $\sum_{i=1}^{j}(v_i - v_{i+1})S_i(a) = \sum_{i=1}^{j} v_i a_i$, attained by placing the first j generators on those coordinates and setting the others to zero. This proves equality of support functions. Each coordinate projection is $K(a_{\le n-1})$, since inserting a zero preserves the bounds. The section at $y_1 = a_1$ is $K(a_{>1})$, since the caps involving coordinate one reduce to suffix caps and imply the other caps.

A downset contains $[0, y]$ whenever it contains y. Following the compression argument of [4, Lemma 5.3], for a compact $H \subseteq K(a)$, compress fibers in one coordinate by replacing each nonempty fiber by the closed interval starting at zero of its length. Fiber length is upper semicontinuous on the projected base by compactness and outer regularity on the line. The compressed set is therefore compact and is contained in $K(a)$. Its volume and the projection deleting the compressed coordinate are preserved. For each other projection, the length of a compressed projected fiber is the supremum of the original fiber lengths over the deleted coordinate. It is bounded by the measure of the union of those fibers. Compression in a new coordinate preserves already acquired downward closure, since the fibers in question are nested. Compressing in every coordinate thus gives a compact downset with deficiency at least $D_\kappa(H)$.

Define $C(a) = \{x \in \mathbb{R}^n : (|x_1|, \dots, |x_n|) \in K(a)\}$. Its support function is $\sum_i a_i |v|_{(i)}$, where $|v|_{(i)}$ are the decreasingly ordered absolute coordinates. In particular $C(a) = C(a - R\mathbf{1}) + R[-1, 1]^n$ for $0 < R < a_n$. This is the rolling Wulff shape condition of [1, Definition 2.7] for the surface tension $\|\nu\|_1$. Write $\mathcal{P}(E) = \sum_i |D_i \mathbf{1}_E|(\mathbb{R}^n)$ for the corresponding anisotropic perimeter. Slicing a convex body in coordinate directions gives $\mathcal{P}(E) = 2\sum_i |\pi_i E|$, because almost every fiber over the projection is a nondegenerate interval. For the reflection $H^{\pm}$ of a compact downset, the same fiber calculation gives $|H^{\pm}| = 2^n |H|$ and $\mathcal{P}(H^{\pm}) \le 2^n \sum_i |\pi_i H|$, allowing zero-length fibers. In the convex case $H = K(a)$ this perimeter bound is an equality.

2.2 Fields and the collar.

For a positive ordered list a, set

$$M(a) = \max_{0 \le k < n} \frac{\mathcal{P}(C(a_{>k}))}{|C(a_{>k})|}. \tag{2}$$

We prove by induction that $C(a)$ minimizes $\mathcal{P}(E) - \mu|E|$ among its measurable finite-perimeter subsets for $\mu \ge M(a)$. Necessity is established below. In dimension one $M(a) = 1/a_1$, and any positive-volume subset has perimeter at least two, proving sufficiency there. Volume and projection volumes are continuous for compact convex sets under Hausdorff convergence. Outer neighborhoods

bound them from above, and for a full-dimensional limit, small homothetic contractions around an interior point are eventually contained in the approximants by support-function convergence. This gives the lower bound and proves continuity of M on positive lists.

We will use a field consequence of the anisotropic variational theory of **(author?)** [1]. If a convex body E minimizes at $\mu > 0$, then for any $\eta > \mu$ there is a global field Z with $\|Z\|_\infty \le 1$ componentwise, $0 \le \operatorname{div} Z \le \eta$, $\operatorname{div} Z = 0$ off E, and

$$\int_E \operatorname{div} Z = \mathcal{P}(E). \tag{3}$$

In dimension one use the field x/A clipped to $[-1, 1]$ on the interval $[-A, A]$. In higher dimension take $\lambda > \eta$ and datum $\mathbf{1}_E$ in Proposition 4.1 of [1]. The minimizer u lies in $[0, 1]$ and vanishes off E. Its superlevel sets for $0 < t < 1 - \eta/\lambda$ are E up to null sets, by that proposition and Lemma 4.3(ii), since the coefficients $\lambda(1 - t)$ exceed μ. Negating the field in Lemma 3.2(a) gives $\operatorname{div} Z = \lambda(\mathbf{1}_E - u)$. This gives the divergence bounds, and the calibration identity in the proof of Proposition 4.1 combined with (2.8) gives (3). The normal trace on a polytope facet is bounded above by $\|\nu\|_1$, by convolution with a positive mollifier and integration by parts near the facet, or directly by the trace bound in Theorem 2.10 of [1]. Equality of total flux in (3) forces that bound to be attained on every facet almost everywhere. If a Lipschitz field R has the same normal trace, the integration-by-parts identity with compactly supported tests gives

$$\operatorname{div}(\mathbf{1}_E(Z - R)) = \mathbf{1}_E \operatorname{div}(Z - R) \tag{4}$$

as distributions, with the right-hand side extended by zero.

Assume the induction hypothesis in dimensions less than n. On a collar of $\partial C(a)$ define

$$\rho(x) = \max_{\varnothing \ne I \subseteq [n],\, \sigma \in \{-1,1\}^I} \frac{\sum_{i \in I} \sigma_i x_i - S_{|I|}(a)}{|I|}. \tag{5}$$

For $|t| < a_n$, the sublevel body at t is $C(a + t\mathbf{1})$ by the signed subset caps. Equivalently ρ is signed distance in the cube norm on this collar, by the support function for positive t and by testing inclusion of an interior cube for negative t.

Almost everywhere the maximizing pair (I, σ) is unique. In its region put $k = |I|$. For fixed x_I, the height $t = \rho(x)$ equals the affine expression for that pair. The fiber in the complementary variables is either empty or the interior of

$$E_{k,t} = C(a_{k+1} + t, \dots, a_n + t). \tag{6}$$

Indeed the active signed coordinates must satisfy their proper-subset caps strictly, and equality on I reduces each signed superset bound to a tail cap. Conversely a mixed subset containing h active coordinates and ℓ other coordinates has sum at most $S_h(a) + S_{k+\ell}(a) - S_k(a) + (h + \ell)t$, which is at most its cap since the generators decrease. The bound is strict for a proper nonempty active part or a nonempty tail part in the fiber interior. Changing an active sign makes the sum strictly smaller, since subtraction of the cap on $I \setminus \{i\}$ gives $\sigma_i x_i \ge a_k + t > 0$. These facts also give the description when the tail is the dimension-zero point.

Fix $\varepsilon > 0$. By continuity choose $0 < r < a_n$ so that for $1 \le k < n$ and $|t| \le r$, $M(a_{>k} + t\mathbf{1}) < M(a) + \varepsilon$. By induction and (3), each tail has a saturated field $Z_{k,t}$ with divergence in $[0, M(a) + \varepsilon]$. These fields can be selected measurably. On a fixed open cube containing the tails compactly, use the product of the bounded weak-* balls for the field and its divergence. This is compact metrizable by separability of L^1. The constraints of distributional divergence, vanishing of divergence off the

tail, and total flux (3) define a closed graph, using L^1 continuity of the indicators and continuity of perimeter. At each scale, take a finite cover by closed balls of radius 2^{-j} and choose the first ball at that scale whose intersection with the earlier choices and the feasible fiber is nonempty. For a fixed history, nonemptiness is closed in t by compactness. The nested intersections select a Borel point. The weak-* measurable pairs admit bounded jointly measurable representatives on the product: integration against finite sums of tensor products defines bounded functionals on product L^1, and a countable dense family of tests in the tail variables identifies the resulting slices for almost every t.

On $|\rho| < r$, take $R_i(x)$ to be $x_i/(a_n + \rho(x))$ clipped to $[-1, 1]$. This is Lipschitz. In an active coordinate it equals the prescribed sign σ_i. On a tail at fixed t it has saturated trace: each coordinate in a tight signed subset in (6) has signed magnitude at least $a_n + t$, by subtracting the cap with that coordinate deleted. On each unique-pair region replace the tail components of R by $Z_{k,t}$, keeping the active components of R. Call the resulting field N.

To compute its divergence, write N as R plus one correction supported on each unique-pair region. For fixed x_I, the correction is supported on $E_{k,t}$ and has only tail components. By (4) its tail divergence after extension by zero is the extension of the interior divergence, with no interface measure. Fubini applies even for the exceptional parameters in the measurable selection, since t is a nonconstant affine function of x_I. Testing against a smooth function on the collar and summing the corrections thus gives an absolutely continuous ambient divergence. Inside the regions the active components are locally constant, so the resulting divergence equals that of $Z_{k,t}$, or zero if the region has no tail. We obtain

$$\|N\|_\infty \le 1, \qquad 0 \le \operatorname{div} N \le M(a) + \varepsilon \qquad (|\rho| < r). \tag{7}$$

At the relative interior of each facet of $C(a)$ the maximizing signed pair is unique and persists on a neighborhood. The components of N on the support of that facet normal equal its signs throughout the neighborhood, almost everywhere.

2.3 The perimeter comparison and necessity.

Write $C = C(a)$. The coefficients for which C minimizes form a closed upper ray, by comparison of affine objectives, and this ray is nonempty by the rolling cube condition and [1, Lemma 4.3(iv)]. Suppose its left endpoint μ_* exceeds $M(a)$. Choose $\varepsilon > 0$ so that $M(a) + \varepsilon < \mu_*$, and take $\mu_j \uparrow \mu_*$ from below with $\mu_j > \mathcal{P}(C)/|C|$. Since this last ratio bounds the Cheeger threshold from above, [1, Theorem 6.3] gives convex minimizers $C_j \subseteq C$, unique up to null sets at these coefficients and at μ_*. Their volumes are bounded away from zero because the objective at C tends to a strictly negative number. Compactness of closed convex sets in the Hausdorff metric and continuity of volume and perimeter show that every subsequential limit minimizes at μ_*. By uniqueness it equals C up to null sets, hence equals C as a full-dimensional closed convex body. Thus $C_j \to C$. Uniform convergence of support functions places the core $\{\rho \le -r/2\}$ in C_j for large j: each point of this core has its cube of radius $r/2$ in C.

For large j, extend N by zero away from the collar and mollify it with a nonnegative approximate identity. On $C \setminus C_j$ the smoothed divergence is bounded by $M(a) + \varepsilon$ for sufficiently small smoothing radius. The smoothed flux on ∂C_j is at most $\mathcal{P}(C_j)$, by the componentwise bound, while the flux on ∂C tends to $\mathcal{P}(C)$ by the facet signs and dominated convergence. The divergence theorem on the two convex bodies therefore gives

$$\mathcal{P}(C) - \mathcal{P}(C_j) \le (M(a) + \varepsilon)|C \setminus C_j|. \tag{8}$$

This follows the shell comparison of [1, Proposition 7.1]. Since $\mu_j < \mu_*$, the shell has positive volume.

For j with $\mu_j > M(a) + \varepsilon$, (8) gives a strictly smaller objective at C than at C_j, a contradiction. This proves sufficiency in the induction.

The relations for reflection show that the suffix conditions of Theorem 2 are equivalent to $M(a) \le 1/\kappa$. Apply the perimeter comparison at $\mu = 1/\kappa$ to the reflection of a compressed competitor. For a compact downset H this yields

$$-2^n \mu D_\kappa(K(a)) \le \mathcal{P}(H^{\pm}) - \mu |H^{\pm}| \le -2^n \mu D_\kappa(H).$$

Compression proves sufficiency for arbitrary compact competitors.

For necessity, use the slice argument of [4, Lemma 5.6]. Let a compact downset L maximize D_κ over its compact subsets, and suppose its section L_t at first coordinate $t > 0$ is nonempty. As $u \uparrow t$, the sections contain L_t and converge to it in Hausdorff distance by compactness. Volume is continuous here, including after projection: interior points of the limit eventually belong to the approaching downsets, points outside the limit are eventually excluded, and the boundary has volume zero. For the latter fact, away from the coordinate planes every line parallel to the all-ones vector meets the boundary at most once, since a positive point strictly below a point of a downset in every coordinate is interior. If a compact $B \subseteq L_t$ had larger deficiency than L_t, the same strict improvement would hold against every L_u with $t - \delta < u < t$ for small $\delta > 0$. Replace the sections in that open interval by B, retaining the original endpoint sections. This is a compact subset of L. Fubini gives a strictly positive change in the volume minus the projection terms that retain the first coordinate, and the projection deleting the first coordinate can only decrease. This contradicts maximality. Therefore L_t maximizes deficiency among its compact subsets. Iterating at the largest generator of the successive suffixes of $K(a)$ proves necessity by comparison with the empty set at each stage. This also proves sharpness of (2): if $C(a)$ minimizes at $\mu > 0$, the reflection inequality implies maximum deficiency at $\kappa = 1/\mu$ and hence $\mu \ge M(a)$. For $\mu \le 0$ the empty set has strictly smaller objective than the full body.

3 From deficiency to revenue.

Work with a concave normalization through an offered size $s \le m$, keeping d fixed. Set $a_i = (1-\Delta_i)/d$ for $1 \le i \le s$ and $K_j = K(a_j, \dots, a_1)$. The generators satisfy $1 \le a_1 \le \cdots \le a_s \le 1/d$, since $p_1 \le q_1 = 1-d$. Consider the normalized menu restricted to $r \le s$ available coordinates. The product decomposition below is the decomposition of Joswig, Klimm, and Spitz [7] in these coordinates.

A bundle S of size $j > 0$ maximizes utility exactly when

$$\begin{aligned} \sum_{i \in I} x_i &\ge p_j - p_{j-|I|} \quad (I \subseteq S), \\ \sum_{i \in H} x_i &\le p_{j+|H|} - p_j \quad (H \subseteq [r] \setminus S). \end{aligned} \tag{9}$$

Necessity follows by comparing with subsets and supersets. For a competitor B, put $h = |S \setminus B|$ and $\ell = |B \setminus S|$. The added values are at most $p_{j+\ell} - p_j \le p_{j-h+\ell} - p_{j-h}$ by concavity, and the lost values are at least $p_j - p_{j-h}$. Combining the inequalities proves sufficiency. Reflecting and scaling the sold coordinates as $y_i = (1 - x_i)/d$ gives precisely K_j. The bound $y_i \le 1/d$ is implied by $a_j \le 1/d$. The unsold factor, denoted $A_{j,r}$, is given by the second line of (9). Its volume is zero if $j < r$ and $p_{j+1} = p_j$, and one for the dimension-zero factor.

On $x_i = 1$ every maximizing bundle contains i: adding i raises utility by at least $1 - p_1 > 0$ if it was absent. Ties of distinct bundles have zero volume in the cube and on each upper face, where the eligible bundles have distinct gradients in the free coordinates. On a sold-coordinate upper face,

$y_i = 0$, and the zero section of the sold factor is its projection by downward closure. Accordingly the cell's volume minus d times its total upper-face measure is

$$d^j |A_{j,r}| D(K_j). \tag{10}$$

Suppose the raw price equations through s hold. The following induction extends the face balance in [4, Lemma 6.5] to clipped schedules. Call r active if $r = s$ or $p_{r+1} > p_r$. At active $r < s$, concavity implies that all increments through $r + 1$ are positive. Then $p_i = q_i$ for $i \le r$, as $p_i < p_{i+1}$ rules out a later suffix minimizer. At $r = s$, normalization preserves the no-sale volume. Thus the no-sale volume is $1 - rd$ at active r, and is at most $1 - rd$ otherwise because the normalized caps can only decrease. Summing (10) gives sale volume minus rd. In increasing order of r, the smaller active sizes have already been shown to have zero deficiency and the smaller inactive sizes have zero unsold volume. The remaining term is $d^r D(K_r)$. This proves $D(K_r) \ge 0$ at all sizes, with equality at active sizes. The suffixes of the list defining K_j give $K_1, \dots, K_j$, so Theorem 2 proves

$$D(H) \le 0 \qquad \text{for every compact } H \subseteq K_j \text{ at active size } j. \tag{11}$$

Take $s = m$, let u_* be the posted menu utility, and let u be any feasible utility. Incentive compatibility bounds its increment on an increasing coordinate segment of length h between hQ_i at the two endpoints. This gives monotonicity and the coordinate Lipschitz bound one. Where the coordinate derivative exists, the forward and backward segment inequalities at that point give $\partial_i u = Q_i$. Coordinatewise integration by parts yields

$$\mathrm{Rev}(u) = \sum_{i=1}^{m} \int_{x_i=1} u \, \mathrm{d}x_{-i} - d^{-1} \int_{[0,1]^m} u \, \mathrm{d}x. \tag{12}$$

In an active cell fix the unsold coordinates. The function $g = u - u_*$ is continuous and coordinatewise nondecreasing in the reflected sold coordinates, since u_* has slope one in each original sold coordinate. Every sublevel set of g in K_j is a compact downset. By (11), its volume is at most the sum of its zero-section measures. The full body has equality in this comparison, so each strict superlevel set has the reverse inequality. Integrate over levels from the minimum to the maximum of g, using $D(K_j) = 0$ to cancel the constant minimum. This gives

$$\int_{K_j} g(y) \, \mathrm{d}y \;\ge\; \sum_{i=1}^{j} \int_{K_j \cap \{y_i = 0\}} g \, \mathrm{d}y_{-i}. \tag{13}$$

On returning to the original coordinates, both sides carry the factor d^{j-1} in (12). Integration over the unsold factor proves that the cell contributes at most zero to $\mathrm{Rev}(u) - \mathrm{Rev}(u_*)$. Inactive cells have zero volume and upper-face measure by (9) and the upper-face eligibility rule. The no-sale region has no upper-face measure and $u - u_* = u \ge 0$ there. Summing proves the arbitrary-dimension implication of Theorem 1, by the deficiency-to-revenue argument of [4, Theorem 4.9] in this normalization.

4 The next price.

Proposition 4 (Facet test)**.** *Suppose $1 \le n < m$ and the raw price equations through n hold with a concave normalization through n. Put $b_i = (1 - \Delta_{n+1-i})/d$ and $c = (b_1, b)$, and let $T(a)$ be the $(|a| - 1)$-dimensional volume of $P(a)$ divided by $\sqrt{|a|}$, with $T((a_1)) = 1$. The next raw price exists uniquely, and the following comparisons are equivalent, also with all three inequalities strict:*

$$q_{n+1} \le p_n + \Delta_n, \qquad D(K(c)) \le 0, \qquad \frac{S_{n+1}(c)}{n+1} T(c) \le V(c_{\le n}). \tag{14}$$

Whenever the comparisons hold, the normalization through $n+1$ is concave.

Proof. First offer only sizes through n on $[0,1]^{n+1}$. On an upper face almost every winning bundle includes the fixed coordinate, since the singleton beats the empty bundle and a bundle omitting that coordinate can improve by a swap. For a winning j-bundle fix its unsold coordinates. Its sold slice is upward closed by the utility comparisons and is contained, after reflection and scaling, in K_j, by comparisons with its subsets. A size $j < n$ with $p_{j+1} = p_j$ contributes zero volume and upper-face measure because adding a positive unsold coordinate strictly improves utility. At all other sizes, (11) bounds the deficiency of the reflected slice by zero. The sold-face measures are its projections by downward closure. Integrating over the unsold values and summing therefore bounds the prefix sale volume by $(n+1)d$.

For a fixed size-n bundle let its one unsold value be $t \in (1-d, 1)$. Its sold slice has positive volume because every point with all sold values greater than t prefers this bundle. Indeed swaps favor it among size-n bundles, and the best smaller bundle is a subset whose omitted values exceed the corresponding increments. Every winning point in the slice has all sold values at least t, again by swapping. The reflected slice has each coordinate width at most $(1-t)/d < 1$. Its positive volume is strictly less than any one projection volume. This gives a strict integrated inequality for sale volume. The no-sale set of the prefix therefore has volume greater than $1 - (n+1)d > 0$.

The new full-bundle cap intersects this compact convex no-sale set with a total-sum halfspace. The resulting volume is continuous in the cap, since level hyperplanes have zero volume, and strictly increases between zero and the maximum total sum on the set. For strictness, any interval of sums between these endpoints contains a sum of an interior point by convexity and full dimension, and hence an open subset. The target determines a unique root. The no-sale set used here is the raw prefix no-sale set by normalization.

Now try the full-bundle price $p_n + \Delta_n$. Its extension is concave. In (10) all proper cells contribute zero: formerly active sizes have zero deficiency, including size n, and other sizes have zero unsold volume. The sale volume minus $(n+1)d$ equals $d^{n+1} D(K(c))$. Monotonicity around the root yields the first equivalence in (14), including strictness. A root between p_n and $p_n + \Delta_n$ respects the increment bound. A smaller root clips all earlier normalized caps at that root and appends a flat endpoint. Clipping lowers the increment at the crossing and sets the subsequent increments to zero, preserving concavity.

For a positive ordered list a of length l, the equality face of $K(a)$ for a size-k cap is $P(a_{\le k}) \times K(a_{>k})$. The prefix is forced to have its full total, the supersets give the tail caps, and the mixed bounds follow from decreasing generators. Decomposing into pyramids from the origin, the height and face area are $S_k(a)/\sqrt{k}$ and $\sqrt{k} T(a_{\le k}) V(a_{>k})$, respectively. Coordinate faces have height zero and redundant faces have zero facet area. Subtract the corresponding sum for the l projections, each $K(a_{\le l-1})$. Using $l\binom{l-1}{k} = (l-k)\binom{l}{k}$ gives

$$D(K(a)) = \frac{1}{l} \sum_{k=1}^{l} \binom{l}{k} S_k(a) T(a_{\le k}) D(K(a_{>k})) - V(a_{\le l-1}). \tag{15}$$

For $l = 1$ the formula is $a_1 - 1$. In (15) for c, the first term vanishes because $D(K(b)) = 0$. If the old prefix has no zero increments, every proper nonempty suffix has zero deficiency. If there are $s \geq 1$ final zero increments, then $T(c_{\leq k}) = 0$ for $2 \leq k \leq s + 1$, since those prefixes are constant. For $s + 2 \leq k \leq n$, the remaining suffix has active length $j = n + 1 - k$ with $p_j < p_{j+1}$, so it has zero deficiency. In both cases the only surviving summand is the full-sum facet term, proving the last equivalence. □

5 Exact continuation certificates.

5.1 Volume recurrences and cap derivatives.

The generalized permutahedron volume formulas of [7] admit a factorial-scaled recurrence suitable for integer arithmetic. Set $\widehat{T}(a) = (l-1)!T(a)$ and $\widehat{V}(a) = l!V(a)$ for length l, with $\widehat{T}((a_1)) = \widehat{V}(()) = 1$. For ordered nonnegative generators,

$$\widehat{T}(a) = \sum_{k=1}^{l-1} \binom{l-2}{k-1} \left[\binom{l}{k} S_k(a) - \binom{l-1}{k-1} S_l(a) \right] \widehat{T}(a_{\leq k}) \widehat{T}(a_{>k}) \quad (l \geq 2), \tag{16}$$

$$\widehat{V}(a) = \sum_{k=1}^{l} \binom{l}{k} \binom{l-1}{k-1} S_k(a) \widehat{T}(a_{\leq k}) \widehat{V}(a_{>k}). \tag{17}$$

The second formula is the pyramid sum from Section 4. For the first, $P(a)$ is the full-sum section of $K(a)$, by the downward-closure description. Its size-k equality face is $P(a_{\leq k}) \times P(a_{>k})$ by the same cap argument. In the sum hyperplane, the distance from the barycenter $(S_l(a)/l)\mathbf{1}$ to that face is $(S_k(a) - kS_l(a)/l)/\sqrt{k(l-k)/l}$. The face area is $\sqrt{k(l-k)}T(a_{\leq k})T(a_{>k})$. The pyramid volume formula, normalized by $\sqrt{l}$ and multiplied by $(l-1)!$, gives (16). If all entries are identical, $P(a)$ is a point and both sides vanish for $l \geq 2$. Otherwise the transpositions of unequal entries give directions spanning the sum-zero space, so the formula uses the full dimension $l - 1$.

For a differentiable path of positive ordered lists, let s_k be the derivative of $S_k(a)$. The change in the volume when a cap for a particular k-subset moves is its cap speed times $T(a_{\leq k})V(a_{>k})$. To see this, slice the strip between the old and new cap planes. Adjacent sections converge in area to the equality face. Outer neighborhoods give the upper bound, including for a lower-dimensional face. For a face of positive area, a point in its relative interior satisfying the other inequalities strictly allows a small move beyond the cap when that inequality alone is relaxed. The hull with such a point gives the lower bound for outer sections, and an interior point gives it for inner sections. Intersections of strips for distinct caps have volume $O(h^2)$ for a step of size h, by integration in the two independent normal directions in a fixed bounded box. Thus the contributions add, even for caps moving in different directions. Subtract the derivatives for the l projections, whose caps are the same up through $l - 1$. This gives

$$\frac{\mathrm{d}}{\mathrm{d}t} D(K(a)) = \sum_{k=1}^{l} \binom{l}{k} s_k \, T(a_{\leq k}) D(K(a_{>k})). \tag{18}$$

The recurrences ensure differentiability along linear paths in the ordered cone, including repeated entries. In particular, if every nonempty suffix of a positive tail b has nonnegative deficiency, $D(K(A, b))$ is strictly increasing for $A \geq b_1$. Each term of (18) is nonnegative, and the full-sum term $T((A, b))$ is positive for $A > b_1$. If all tail deficiencies vanish, the derivative for $A > b_1$ equals that term.

5.2 Balanced generators and transfer.

Start with $\alpha_1 = 1$, and write $\sigma_j = \sum_{i=1}^{j} \alpha_i$, $\sigma_0 = 0$. At length $j \geq 2$ put $b = (\alpha_{j-1}, \ldots, \alpha_1)$. If $D(K(b_1, b)) < 0$, there is a unique $\alpha_j > b_1$ for which $D(K(\alpha_j, b)) = 0$. Indeed (18) gives strict increase, and for $A > j^2$ the pyramid from the last-coordinate projection to Ae_j has volume A/j times one projection volume. It lies in $K(A, b)$, so its volume exceeds the sum of the j equal projection volumes. Continuity gives existence.

If $m \geq n$ and $\alpha_n < m + 1$, the balanced generators through length n yield

$$q_j = j - \frac{\sigma_j}{m+1}, \qquad 1 \leq j \leq n. \tag{19}$$

The proposed increments are positive and decreasing. Compute sale volume minus jd in dimension j using (10): all the sold factors have zero deficiency by the generator equations. The proposed prices therefore solve (1) and are unique by the first price equation and Proposition 4 applied inductively. At any transferred prefix $j < m$, the strict duplication sign $D(K(\alpha_j, \alpha_j, \alpha_{j-1}, \ldots, \alpha_1)) < 0$ yields strict continuation by (14). This applies also at the last transferred prefix, where the next root may be obtained by clipping.

5.3 Certified signs for generator totals.

To bound the error of using approximate generators, fix length n and a trial total S. Approximate each earlier σ_i by s_i with $|s_i - \sigma_i| \leq w$, $s_0 = 0$, and $s_1 = 1$. The true and approximate lists have leading entries $S - \sigma_{n-1}$ and $S - s_{n-1}$ and tails formed from successive differences of their respective totals in reverse order. Suppose both lists are ordered, positive, and bounded by M, with $0 < w \leq 1/(4n)$. Along their linear interpolation the full cap is fixed and proper caps move at speed at most w. Each prefix sum of a suffix of length l differs from the true one by at most $2w$, by subtracting its two endpoint totals. True prefix sums are at least one, so the interpolated suffix body is between the $(1 - 2w)$ and $(1 + 2w)$ dilates of the exact suffix body. This also bounds its projections. Since the exact suffix has zero deficiency, the absolute deficiency of the interpolated suffix is at most

$$M^l\big((1 + 2w)^l - (1 - 2w)^l\big) \leq 8lwM^l.$$

The inequality uses $(1 + 2w)^{l-1} \leq 2$, by the binomial expansion and $2nw \leq 1/2$. Deleting one coordinate from the permutahedron in its sum hyperplane scales area by $1/\sqrt{k}$, and the projected prefix of size k lies in a cube of side M. Thus $T(a_{\leq k}) \leq M^{k-1}$. Integrating (18) along the interpolation shows that the error in the full deficiency is bounded by

$$8n\, 2^n M^{n-1} w^2. \tag{20}$$

Here is the integer test used in the computational supplement. On the grid $B = 2^{320}$, let $W = 2^{78}$. Store brackets for $B\sigma_i$ of width at most W, initially exact for $i = 0, 1$. Take their integer midpoints rounded down and form the integer tail g from their reversed successive differences. For an integer trial leading coordinate v, the common total is $(v + J_{n-1})/B$, where J_{n-1} is the preceding midpoint. The interval $g_1 + 3W \leq v \leq 2nB - W$ ensures the leading exact entry is at least the previous exact one, since its error is at most W/B and the preceding generator's error is at most $2W/B$. The earlier exact entries are ordered by induction, the approximate tail order is checked, and all entries are at most $2n$ by the previous upper bounds and the trial interval. With $g^- = g_{\leq n-2}$, compute

$$\begin{aligned} f(v) &= \widehat{V}(v, g) - n^2 B\, \widehat{V}(v, g^-), \\ E &= 8n\, 2^n (2n)^{n-1} n!\, W^2 B^{n-2}. \end{aligned} \tag{21}$$

Here $f(v)$ is $n!B^n$ times the approximate deficiency, so $f(v) \pm E$ bound the true scaled deficiency by (20). The program verifies $f(v) < -E$ at $v = g_1 + 3W$. By monotonicity this certifies the strict duplication sign at the true previous leading generator. A right endpoint with $f(v) > E$ is verified, first testing the smaller of $2nB - W$ and the left endpoint plus $4B$, then $2nB - W$ if needed.

To locate a narrow bracket, search by the sign of f within these endpoints until the bracket has width at most $W/16$. Expand each side by $W/4$ within the original interval and check that the resulting endpoints have $f < -E$ and $f > E$, respectively, and width at most W. Only these last two certified signs are used for the narrow bracket. Adding J_{n-1} gives the new total bracket. Subtract the old total upper endpoint from the new lower endpoint, and the old lower endpoint from the new upper endpoint, to enclose $B\alpha_n$. Check that the new lower generator endpoint exceeds the previous upper generator endpoint and that the new upper generator endpoint is at most $2nB$. These tests certify the order, the error hypotheses for the next stage, and all transfer ceilings quoted below.

5.4 Direct price brackets and range coverage.

The checker uses (19) while $m + 1$ strictly exceeds the certified upper bound on the next generator. Price brackets obtained from total brackets are rounded outwards to the grid. After transfer ends, let the current stage be r, and use the vectors of earlier upper and earlier lower raw endpoints in two searches. In each search normalize the earlier caps by suffix minima. Verify that the resulting caps $b_1, \dots, b_{r-1}$ have nonnegative, nonincreasing increments and $b_1 < B$ in integer units. Put $b_0 = 0$. For trial v between zero and $2b_{r-1} - b_{r-2}$, use $c_j = \min(b_j, v)$ for $j < r$, and $c_r = v$. Their list of increments ∇c is ordered and nonnegative, by the clipping rule. The singleton bound means the unit cube adds no constraint after rescaling. The exact no-sale residual is

$$(m + 1)\widehat{V}(\nabla c) - (m + 1 - r)r!B^r. \tag{22}$$

It is negative at zero, and the checker verifies it is strictly positive at the extrapolated cap in both searches. Search down to adjacent grid points with residual nonpositive at the lower endpoint and strictly positive at the upper endpoint. The sign searches propose a grid point from an integer Newton step whenever its slope is positive, constrained to an interior grid point, and use a midpoint at every ninth iteration. Only the residual sign determines the replacement endpoint. The slope proposes a point and is never used as a sign bound.

Keep the lower endpoint from the search with earlier upper caps and the upper endpoint from the one with earlier lower caps. Monotonicity in the earlier caps makes these valid bounds for the true root, with a strict upper bound. If L_i, U_i are the raw endpoints and $L_0 = U_0 = 0$, the check

$$U_r \le \max(L_{r-1},\, 2L_{r-1} - U_{r-2}) \tag{23}$$

proves strict extrapolation, since the true extrapolation in grid units is the larger of Bq_{r-1} and $B(2q_{r-1} - q_{r-2})$ at $r \ge 3$, and equals $2Bq_1$ at $r = 2$. Proposition 4 now preserves prefix concavity. As a separate final check, the program identifies each suffix minimum by separating its upper endpoint strictly from every other lower endpoint in that suffix. It assigns a zero increment when consecutive minimizing indices agree. For all other increments it subtracts the appropriate opposing endpoints. Each lower increment bound is checked against the next upper bound, ending with zero.

The integer tests succeed at generator lengths $2, \dots, 95$, certifying strict duplication at lengths $1, \dots, 94$. The ceilings of the generator upper bounds are $n + 6$ for $58 \le n \le 76$ and $n + 7$ for $77 \le n \le 95$. Every exact generator lies strictly below its upper bound. Thus for $m \ge 83$, $\alpha_n \le \alpha_{77} < 84 \le m + 1$ when $n \le 77$. If $78 \le n \le 94$ and $m \ge n + 6$, then $\alpha_n < n + 7 \le m + 1$.

Table 1: Coverage of the exact price and generator tests.

Range	Transferred roots	Direct price signs
$1 \le m \le 4$	All	None
$5 \le m \le 81$	Initial segment by (19)	Remaining sizes
$m = 82$	Sizes 1 through 76	Sizes 77 through 82
$m \ge 83,\ n \le 77$	Through length n	Next sign by duplication
$78 \le n \le 94,\ m \ge n + 6$	Through length n	Next sign by duplication

Transfer and the strict duplication sign give the next root and concavity through it in both cases. For $m \le 82$ the checker verifies the whole sequence, using the generator transfer followed by the direct price signs wherever needed, as in Table 1. This proves Theorem 3 and completes Theorem 1.

The facet test also specifies the inequality needed to continue outside these ranges. For example, the comparison at $m = 83, n = 78$ is open within these results. When a prefix ends in a flat segment, continuation requires the next raw price to fall strictly below that segment's price for the strict facet test. The exact suffix characterization leaves these comparisons as questions about the recursive prices.

Computational reproducibility.

The accompanying source `check_ranges.cpp` implements the integer tests in Sections 5.3 and 5.4. It uses a C++17 compiler and the GNU Multiple Precision Arithmetic Library. Running `check_ranges 1 82` certifies the generator ceilings, the price signs, and the final normalized increments. The first and last price verdicts report one transferred root at $m = 1$, and 76 transferred and six direct roots at $m = 82$. The output concludes with `ALL CHECKS PASSED`. The computations use generated integers and require no external data.

A Extremal totals for suffix constraints.

Proposition 5. *Suppose $0 < \alpha_1 < \cdots < \alpha_j$ are balanced generators as in Section 5.2, allowing $j = 0$. For $j \ge 1$, the minimum total of a positive ordered list of length j with all nonempty suffix deficiencies nonnegative is σ_j. At total $S \ge \sigma_j$, the list $(S - \sigma_{j-1}, \alpha_{j-1}, \ldots, \alpha_1)$ maximizes deficiency among positive ordered lists with all proper nonempty suffix deficiencies nonnegative.*

Let $l > j$, $A \ge \theta > 0$, and let $a = (A^{[l-j-1]}, \theta, \alpha_j, \ldots, \alpha_1)$ be ordered, where $A^{[k]}$ denotes k repetitions. If $D(K(a)) = 0$ and all proper nonempty suffix deficiencies are nonnegative, then a has minimum total among positive ordered lists of length l bounded by A with all nonempty suffix deficiencies nonnegative.

Proof. For the first minimum, induct on j, starting with the interval threshold one. A feasible list of smaller total can be dilated by $\lambda > 1$ to total σ_j. In dimension i dilation changes deficiency to $\lambda^{i-1}(D(K) + (\lambda - 1)|K|)$. Hence the rescaled list has nonnegative suffix deficiencies and strictly positive full deficiency. By induction, its suffix of length $i < j$ has total at least σ_i. Its prefix caps are therefore bounded by the corresponding caps of $(\alpha_j, \ldots, \alpha_1)$, whose maximum subset deficiency is zero by Theorem 2. This is a contradiction. At fixed total $S \ge \sigma_j$, the same cap comparison gives containment in $K(S - \sigma_{j-1}, \alpha_{j-1}, \ldots, \alpha_1)$. The latter body has nonnegative full deficiency by (18) or the interval formula, and its proper nonempty suffix deficiencies vanish. Theorem 2 gives the asserted maximum.

For the fixed-cap minimum, a competitor of total at most the total of a lies inside $K(a)$ if its proper suffixes have nonnegative deficiency. For prefix lengths up to $l-j-1$ use the common bound A on each generator. For the remaining prefix lengths, the first part bounds the complementary suffix of length i below by σ_i, including $i=0$. Thus each prefix cap is at most that of a, and Theorem 2 bounds the competitor's full deficiency by zero. If its total is strictly smaller, interpolate toward the constant list $(A,\dots,A)$ for a small time so the total is still smaller. Every cap moves outward or stays fixed. By induction on suffix length and (18), the deficiencies of all suffixes remain nondecreasing, with a strictly positive contribution at the full cap of any nonconstant suffix whose total increases. For a constant suffix of length i with entries t, the deficiency is $t^{i-1}(t-i)$, which increases for $t \ge i$ as t increases. Consequently the perturbed competitor has positive full deficiency, a contradiction. □

B A duplication inequality in low dimension.

For a nonempty ordered positive list a of length k, let $a^- = a_{\le k-1}$ and $R(a) = V(a)/(kV(a^-))$. This ratio is homogeneous of degree one.

Corollary 6. *If b has length $1 \le n \le 6$ and $R(b) \le R(s)$ for every proper nonempty suffix s of b, then $R((b_1, b)) < R(b)$.*

Proof. For $n=1$, duplicating gives a square and halves the ratio. For $n=2$, rescale so $R(b)=1$. Then $b_2 \ge 1$, and the square minus its corner triangle has area $b_1^2 - (b_1-b_2)^2/2 = 2b_1$. Therefore $b_1^2 - (b_1-1)^2/2 \le 2b_1$, giving $b_1 \le 1+\sqrt{2} < 3$. Every fiber of the duplicated body has length at most b_1, so the new ratio is at most $b_1/3$.

For $3 \le n \le 6$ put $c=(b_1,b)$, let s_j be the suffix of length j, put $x_i = b_i - b_{i+1}$ for $i<n$, $x_n = b_n$, and set $S = \sum_i x_i$. We have $R(a) = \widehat{V}(a)/(k^2\widehat{V}(a^-))$ for a list of length k, with positive denominators. Define the integer polynomials

$$F_n = (n+1)^2\widehat{V}(c^-)\widehat{V}(b) - n^2\widehat{V}(b^-)\widehat{V}(c),$$
$$G_{n,j} = n^2\widehat{V}(b^-)\widehat{V}(s_j) - j^2\widehat{V}(b)\widehat{V}(s_j^-).$$

The hypothesis gives $G_{n,j} \ge 0$ and the asserted comparison is $F_n > 0$. With $h_3 = h_4 = 0$ and $h_5 = h_6 = 1$, the multiplier data in the supplement give nonnegative integer polynomials $P_{n,j}$ such that

$$H_n = S^{h_n}F_n - \sum_{j=1}^{n-1} P_{n,j}G_{n,j}$$

has nonnegative coefficients. Expansion by (16)–(17) verifies this for all monomials of degree $2n+h_n$, including absent monomials with coefficient zero. At $x_n^{2n+h_n}$ the coefficients are $144, 2880, 86400, 86400$ for $n=3,4,5,6$, respectively. Thus $H_n > 0$ as $x_n > 0$, proving $F_n > 0$. The supplement includes exact polynomial verification of these signs. □

Data and code availability

The supplementary archive `supplement.zip` contains the exact-arithmetic C++17 certificate `check_ranges.cpp`, the symbolic duplication verifier `check_duplication.py`, and reproduction instructions.

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