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Resonant CCR algebras and a universal short-time correction to frequency conversion

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Abstract

We formulate lossless, resonant multiphoton frequency conversion of arbitrary stoichiometry $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{Z}^d_{>0}$ inside the polynomial canonical commutation relation (CCR) $*$-algebra. Using the $\mathbb{Z}^d$-grading induced by the quantum Manley–Rowe conserved charges, the interaction Hamiltonian has grade zero and decomposes in the vacuum GNS Fock representation into finite-dimensional charge sectors, making essential self-adjointness elementary. Berezin–Wick symbols recover the classical coupled-mode flow as $\hbar \to 0$ while retaining exact quantum corrections in the Heisenberg generator. For coherent pump inputs and an initially empty output mode, we prove that quantum and classical output intensities agree at $O(t^2)$ and separate universally at fourth order, $J_\hbar(t) - J_{\mathrm{cl}}(t) = -\frac{|g|^4}{3} D_{\mathbf{r}}(x;\hbar)\, t^4 + O(t^6)$, with $D_{\mathbf{r}}(x;\hbar) > 0$ for every nonlinear process ($\sum_j r_j \geq 2$). Uniform-in-time remainder bounds extend this deficit to classical mixtures such as thermal light, and we interpret its phenomenology via spontaneous multi-photon back-conversion and discrete photon depletion.

1 Introduction

Resonant frequency conversion—the coherent combination of $r_1, \ldots, r_d$ photons from d pump modes into a single photon of an output mode—is a cornerstone of nonlinear optics, encompassing sum-frequency generation (SFG, $\mathbf{r} = (1,1)$), second-harmonic generation (SHG, $\mathbf{r} = (2)$), and higher-order harmonic or wave-mixing processes. In the classical limit, lossless conversion is governed by polynomial coupled-mode equations constrained by the Manley–Rowe power relations [1]. In the quantum regime, the corresponding interaction Hamiltonians are unbounded polynomials of degree $R+1 \geq 3$ in bosonic creation and annihilation operators. Early quantum-optical studies established finite-block reductions for the trilinear Hamiltonian [3] and short-time perturbative expansions for coherent-state SHG [5]; more recent work has developed analytic Heisenberg-picture and diagrammatic treatments of pump depletion [7, 9] alongside experimental and architectural realizations in trapped ions and mesoscopic nonlinear photonics [6, 8]. Two structural questions remain central: first, how to organize the algebra, Hilbert-space domain theory, and classical limit without conflating distinct structures; and second, how the earliest observable departure from classical coupled-mode dynamics behaves across general nonlinear stoichiometries.

The purpose of this paper is to give a clean, self-contained mathematical framework for finite-mode resonant conversion and to establish a universal short-time quantum correction. The framework cleanly separates three ingredients:

1. **Kinematics:** the unital polynomial CCR $*$-algebra $\mathcal{A}_\hbar$ and its vacuum GNS representation on bosonic Fock space.

2. **Dynamics:** a Hermitian interaction Hamiltonian H of grade zero with respect to the $\mathbb{Z}^d$-grading determined by the quantum Manley–Rowe charges, which reduces H to a direct sum of finite Hermitian matrices and makes essential self-adjointness immediate.

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3. **Classical approximation:** Berezin–Wick symbols [4] equipped with their associative star product $\star_\hbar$, whose $\hbar \to 0$ limit yields the classical Poisson algebra and global coupled-mode flow.

Within this setting, we compare the quantum scaled output intensity $J_\hbar(t)$, defined by $\langle \Psi_\alpha | U(t)^* a_0^\dagger a_0 U(t) | \Psi_\alpha \rangle$, with the classical output $J_{\mathrm{cl}}(t) = |z_0(t)|^2$ starting from coherent pump inputs $|\alpha_1, \ldots, \alpha_d\rangle$ [2] and an empty output mode $|0\rangle_0$. Our main result (Theorem 1) shows that while leading-order conversion agrees identically ($J_\hbar(t) = J_{\mathrm{cl}}(t) + O(t^4)$), every nonlinear process ($R = \sum_{j=1}^d r_j \geq 2$) exhibits a strictly negative quartic correction:

$$J_\hbar(t) - J_{\mathrm{cl}}(t) = -\frac{|g|^4}{3} D_{\mathbf{r}}(x;\hbar)\, t^4 + O(t^6), \qquad D_{\mathbf{r}}(x;\hbar) > 0,$$

where $x_j = |\alpha_j|^2$ are the initial classical intensities. In contrast, linear frequency conversion ($R = 1$) has $D_{(1)} \equiv 0$ and $J_\hbar(t) = J_{\mathrm{cl}}(t)$ for all t.

While short-time $O(t^4)$ expansions for specific quadratic processes such as SHG and SFG are classic in quantum optics [5, 9], the contribution of this paper is threefold: (i) a self-contained algebraic and operator-domain foundation in which the Manley–Rowe grading guarantees essential self-adjointness at arbitrary polynomial degree $R + 1$ (in contrast to non-resonant higher-order bosonic operators [10]); (ii) a unified closed-form deficit theorem for arbitrary multi-mode stoichiometry $\mathbf{r}$ accompanied by global-in-time polynomial remainder bounds that rigorously justify extending the expansion to unbounded classical mixtures such as thermal light (Corollary 1); and (iii) a transparent physical phenomenology (Section 6) showing two complementary mechanisms behind the deficit. In the Heisenberg and Wick-symbol picture, the deficit is literally the non-classical excess of the pump commutator $\hbar^{-1}[A, A^\dagger]$ (with $A = \prod_j a_j^{r_j}$), representing *spontaneous (vacuum-seeded) multi-photon back-conversion* of newly generated output photons back into the pump modes. In the Fock-sector picture, it arises from the asymmetry between discrete pump depletion on the upward transition and bosonic enhancement on the downward transition, which averages over coherent inputs to the non-Gaussian excess of the transition-rate variance $\mathrm{Var}(F_{\mathbf{p}})$.

Conventions. Throughout, $\mathbb{N} = \{0, 1, 2, \ldots\}$ denotes the nonnegative integers and $\mathbb{Z}_{>0} = \{1, 2, \ldots\}$ the positive integers. For multi-indices $\mathbf{u} = (u_0, \ldots, u_d) \in \mathbb{N}^{d+1}$ (or $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$), we use standard notation $|\mathbf{u}| = \sum_j u_j$, $\mathbf{u}! = \prod_j u_j!$, $z^{\mathbf{u}} = \prod_j z_j^{u_j}$, and $\mathbf{k} \leq \mathbf{r}$ componentwise. For $n \in \mathbb{N}$ and $r \in \mathbb{N}$, the falling factorial is

$$n^{\underline{r}} = n(n-1)\cdots(n-r+1), \qquad n^{\underline{0}} = 1,$$

with $n^{\underline{r}} = 0$ whenever $n < r$.

2 Resonant CCR algebras, Manley–Rowe grading, and GNS Fock space

We consider $d+1$ bosonic modes in a lossless, perfectly resonant interaction picture, with all spatial overlap integrals and conventional symmetry factors absorbed into a single complex coupling constant $g \in \mathbb{C}$. Retaining $\hbar > 0$ makes the classical limit transparent; setting $\hbar = 1$ recovers standard photon-number units.

Definition 1 (Polynomial CCR algebra)**.** For $\hbar > 0$, let $\mathcal{A}_\hbar$ be the unital associative $*$-algebra over $\mathbb{C}$ generated by $a_0, \ldots, a_d$ and their adjoints $a_0^\dagger, \ldots, a_d^\dagger$ (with involution $a_j^* = a_j^\dagger$), subject to the canonical commutation relations

$$[a_j, a_k^\dagger] = \hbar \delta_{jk} 1, \qquad [a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0, \qquad 0 \leq j, k \leq d. \tag{1}$$

Note that $\mathcal{A}_\hbar$ is the unbounded polynomial CCR $*$-algebra, not the bounded Weyl C^*-algebra. Mode 0 designates the output mode and modes $1, \dots, d$ designate the input (pump) modes. Fix a stoichiometry vector $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{Z}^d_{>0}$ and set

$$R = \sum_{j=1}^{d} r_j, \qquad A_{\mathbf{r}} = \prod_{j=1}^{d} a_j^{r_j}, \qquad X_{\mathbf{r}} = a_0^\dagger A_{\mathbf{r}} = a_0^\dagger \prod_{j=1}^{d} a_j^{r_j}, \qquad H = gX_{\mathbf{r}} + \overline{g} X_{\mathbf{r}}^\dagger. \tag{2}$$

The cases $\mathbf{r} = (1,1)$ and $\mathbf{r} = (2)$ describe nondegenerate sum-frequency generation (SFG) and degenerate second-harmonic generation (SHG), respectively, while $\mathbf{r} = (1)$ ($d = 1$) is linear frequency conversion (a two-mode beam splitter).

Define the mode number operators and the quantum Manley–Rowe conserved charges [1] by

$$N_j = \hbar^{-1} a_j^\dagger a_j \quad (0 \le j \le d), \qquad Q_j = N_j + r_j N_0 \quad (1 \le j \le d). \tag{3}$$

From (1) one checks immediately that $[N_k, a_k^\dagger] = a_k^\dagger$ and $[N_k, a_k] = -a_k$, while $[N_k, a_\ell] = [N_k, a_\ell^\dagger] = 0$ for $k \neq \ell$. Consequently,

$$[Q_j, a_k^\dagger] = \delta_{jk} a_k^\dagger \quad (1 \le k \le d), \qquad [Q_j, a_0^\dagger] = r_j a_0^\dagger,$$

and $[Q_j, a_k] = -[Q_j, a_k^\dagger]^*$ for all $0 \le k \le d$.

We therefore equip $\mathcal{A}_\hbar$ with a $\mathbb{Z}^d$-grading by assigning degrees to the generators:

$$\deg(a_j^\dagger) = e_j \quad (1 \le j \le d), \qquad \deg(a_0^\dagger) = \mathbf{r}, \qquad \deg(a_j) = -\deg(a_j^\dagger) \quad (0 \le j \le d), \tag{4}$$

where $e_1, \dots, e_d$ is the standard basis of $\mathbb{Z}^d$. Because each defining commutator $[a_j, a_k^\dagger] - \hbar\delta_{jk}1$ is homogeneous of degree 0 (when $j = k$, $\deg(a_j) + \deg(a_j^\dagger) = 0 = \deg(1)$; when $j \neq k$, the relation is $[a_j, a_k^\dagger] = 0$), the grading descends to $\mathcal{A}_\hbar$ and satisfies

$$\mathcal{A}_\hbar = \bigoplus_{\gamma \in \mathbb{Z}^d} \mathcal{A}_{\hbar,\gamma}, \qquad \mathcal{A}_{\hbar,\gamma}\mathcal{A}_{\hbar,\eta} \subseteq \mathcal{A}_{\hbar,\gamma+\eta}, \qquad (\mathcal{A}_{\hbar,\gamma})^* = \mathcal{A}_{\hbar,-\gamma}.$$

Since each ad_{Q_j} is a derivation of $\mathcal{A}_\hbar$, the homogeneous components are characterized intrinsically by the Manley–Rowe charges:

$$\mathcal{A}_{\hbar,\gamma} = \{B \in \mathcal{A}_\hbar : [Q_j, B] = \gamma_j B \text{ for all } 1 \le j \le d\}.$$

In particular,

$$\deg(X_{\mathbf{r}}) = \mathbf{r} - \sum_{j=1}^{d} r_j e_j = 0, \qquad \deg(X_{\mathbf{r}}^\dagger) = 0, \qquad [Q_j, H] = 0 \quad (1 \le j \le d).$$

Remark 1 (Resonance as a charge-zero interaction). The interaction $X_{\mathbf{r}}$ does *not* have grade zero under the full $\mathbb{Z}^{d+1}$-grading generated by the individual number operators $(N_0, \dots, N_d)$: its occupation change is $(+1, -r_1, \dots, -r_d)$. Rather, resonance is expressed by H having grade zero under the reduced $\mathbb{Z}^d$-grading generated by $(Q_1, \dots, Q_d)$, rather than by defining an associative multiplication of optical modes themselves. Moreover, each homogeneous component $\mathcal{A}_{\hbar,\gamma}$ is infinite-dimensional as a vector space (for instance, $\mathcal{A}_{\hbar,0}$ contains all polynomials in $N_0, \dots, N_d$); finite-dimensionality appears in the simultaneous charge eigenspaces of the Fock representation, not in the abstract algebra.

Vacuum GNS representation

By repeated application of the commutation relation $a_j a_j^\dagger = a_j^\dagger a_j + \hbar 1$, every element of $\mathcal{A}_\hbar$ can be written as a finite linear combination of normal-ordered monomials $(a^\dagger)^{\mathbf{u}} a^{\mathbf{v}}$ ($\mathbf{u}, \mathbf{v} \in \mathbb{N}^{d+1}$). That these monomials are linearly independent–hence form a vector-space basis of $\mathcal{A}_\hbar$–follows from the standard Poincaré–Birkhoff–Witt theorem for the Heisenberg Lie algebra (or directly from the faithful Bargmann–Fock representation on $\mathbb{C}[w_0, \dots, w_d]$ where $a_j^\dagger$ acts as multiplication by w_j and a_j as $\hbar\partial_{w_j}$).

Define the linear functional $\varphi\colon \mathcal{A}_\hbar \to \mathbb{C}$ on the normal-ordered basis by

$$\varphi\big((a^\dagger)^{\mathbf{u}} a^{\mathbf{v}}\big) = \begin{cases} 1, & \mathbf{u} = \mathbf{v} = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Proposition 1 (Vacuum GNS representation)**.** *The functional φ is a state on $\mathcal{A}_\hbar$: $\varphi(1) = 1$ and for any $B = \sum_{\mathbf{u},\mathbf{v}} c_{\mathbf{uv}} (a^\dagger)^{\mathbf{u}} a^{\mathbf{v}} \in \mathcal{A}_\hbar$,*

$$\varphi(B^* B) = \sum_{\mathbf{u} \in \mathbb{N}^{d+1}} \hbar^{|\mathbf{u}|} \mathbf{u}! \, |c_{\mathbf{u}0}|^2 \geq 0. \tag{6}$$

Its GNS Hilbert space is the bosonic Fock space $\mathcal{F}$ with orthonormal basis

$$|\mathbf{n}\rangle = \frac{[(a^\dagger)^{\mathbf{n}}]}{\sqrt{\hbar^{|\mathbf{n}|}\mathbf{n}!}}, \qquad \mathbf{n} \in \mathbb{N}^{d+1}, \tag{7}$$

and vacuum vector $\Omega = [1] = |\mathbf{0}\rangle$. The GNS representation acts on the common dense invariant domain $\mathcal{D}_{\mathrm{fin}} = \mathrm{span}\{|\mathbf{n}\rangle : \mathbf{n} \in \mathbb{N}^{d+1}\}$ via

$$a_j^\dagger |\mathbf{n}\rangle = \sqrt{\hbar(n_j + 1)}\, |\mathbf{n} + e_j\rangle, \qquad a_j |\mathbf{n}\rangle = \sqrt{\hbar n_j}\, |\mathbf{n} - e_j\rangle, \tag{8}$$

where the right-hand side of the second formula is 0 when $n_j = 0$.

Proof. Write $B^* = \sum_{\mathbf{u}',\mathbf{v}'} \overline{c_{\mathbf{u}'\mathbf{v}'}} (a^\dagger)^{\mathbf{v}'} a^{\mathbf{u}'}$, so that

$$B^* B = \sum_{\mathbf{u},\mathbf{v},\mathbf{u}',\mathbf{v}'} \overline{c_{\mathbf{u}'\mathbf{v}'}}\, c_{\mathbf{uv}}\, (a^\dagger)^{\mathbf{v}'} a^{\mathbf{u}'} (a^\dagger)^{\mathbf{u}} a^{\mathbf{v}}.$$

When $a^{\mathbf{u}'}(a^\dagger)^{\mathbf{u}}$ is brought into normal order, multiplying on the left by $(a^\dagger)^{\mathbf{v}'}$ and on the right by $a^{\mathbf{v}}$ preserves normal order and leaves every resulting monomial with at least $\mathbf{v}'$ creation factors and at least $\mathbf{v}$ annihilation factors. By (5), $\varphi\big((a^\dagger)^{\mathbf{v}'} a^{\mathbf{u}'} (a^\dagger)^{\mathbf{u}} a^{\mathbf{v}}\big) = 0$ unless $\mathbf{v}' = \mathbf{v} = 0$.

It remains to evaluate $\varphi\big(a^{\mathbf{u}'}(a^\dagger)^{\mathbf{u}}\big)$. Because distinct modes commute, this factors across modes:

$$\varphi\big(a^{\mathbf{u}'}(a^\dagger)^{\mathbf{u}}\big) = \prod_{j=0}^{d} \varphi\big(a_j^{u_j'} (a_j^\dagger)^{u_j}\big).$$

For a single mode, induction on $m \geq 0$ using $[a, (a^\dagger)^m] = \hbar m (a^\dagger)^{m-1}$ gives $a(a^\dagger)^m = (a^\dagger)^m a + \hbar m (a^\dagger)^{m-1}$. Applying a repeatedly n times and evaluating φ shows that the constant term in the normal ordering of $a^n (a^\dagger)^m$ vanishes unless $n = m$, in which case it equals $\hbar^m m!$. Thus $\varphi\big(a^{\mathbf{u}'}(a^\dagger)^{\mathbf{u}}\big) = \delta_{\mathbf{u},\mathbf{u}'} \hbar^{|\mathbf{u}|} \mathbf{u}!$, which proves (6).

The GNS left ideal of null elements is therefore

$$\mathcal{N}_\varphi = \{B \in \mathcal{A}_\hbar : \varphi(B^* B) = 0\} = \{B \in \mathcal{A}_\hbar : c_{\mathbf{u}0} = 0 \text{ for all } \mathbf{u}\} = \sum_{j=0}^{d} \mathcal{A}_\hbar a_j.$$

Hence the quotient $\mathcal{A}_\hbar/\mathcal{N}_\varphi$ is spanned by the cosets $[(a^\dagger)^{\mathbf{n}}]$ ($\mathbf{n} \in \mathbb{N}^{d+1}$), with inner product $\langle[(a^\dagger)^{\mathbf{m}}], [(a^\dagger)^{\mathbf{n}}]\rangle = \varphi(a^{\mathbf{m}}(a^\dagger)^{\mathbf{n}}) = \delta_{\mathbf{m},\mathbf{n}}\hbar^{|\mathbf{n}|}\mathbf{n}!$. Normalizing gives the orthonormal basis (7), whose linear span is $\mathcal{D}_{\text{fin}} = \mathcal{A}_\hbar/\mathcal{N}_\varphi$ and whose completion is $\mathcal{F}$.

Finally, left multiplication on $\mathcal{A}_\hbar/\mathcal{N}_\varphi$ gives

$$a_j^\dagger|\mathbf{n}\rangle = \frac{[(a^\dagger)^{\mathbf{n}+e_j}]}{\sqrt{\hbar^{|\mathbf{n}|}\mathbf{n}!}} = \sqrt{\hbar(n_j+1)}\,|\mathbf{n}+e_j\rangle,$$

and, since $a_j(a^\dagger)^{\mathbf{n}} - \hbar n_j(a^\dagger)^{\mathbf{n}-e_j} = (a^\dagger)^{\mathbf{n}}a_j \in \mathcal{N}_\varphi$ (for $n_j \geq 1$, and $a_j(a^\dagger)^{\mathbf{n}} \in \mathcal{N}_\varphi$ for $n_j = 0$),

$$a_j|\mathbf{n}\rangle = \frac{\hbar n_j[(a^\dagger)^{\mathbf{n}-e_j}]}{\sqrt{\hbar^{|\mathbf{n}|}\mathbf{n}!}} = \sqrt{\hbar n_j}\,|\mathbf{n}-e_j\rangle,$$

establishing (8). □

Note that the GNS generators $a_j, a_j^\dagger$ are unbounded operators on $\mathcal{D}_{\text{fin}}$ rather than elements of $B(\mathcal{F})$. Likewise, the algebra identity $1 \in \mathcal{A}_\hbar$ and the Fock vacuum vector $\Omega = [1] \in \mathcal{F}$ (characterized by $a_j\Omega = 0$) are distinct objects: in particular, $H\Omega = 0$, so the vacuum supplies the representation, not the pumps; nontrivial conversion requires an excited initial state.

3 Finite charge sectors and essential self-adjointness

On the Fock basis (7), each number operator acts diagonally as $N_j|\mathbf{n}\rangle = n_j|\mathbf{n}\rangle$, so the conserved charges $Q_j = N_j + r_jN_0$ ($1 \leq j \leq d$) act simultaneously with integer eigenvalues $p_j = n_j + r_jn_0 \in \mathbb{N}$. Conversely, specifying $\mathbf{p} = (p_1, \ldots, p_d) \in \mathbb{N}^d$ and the output occupation $k = n_0 \in \mathbb{N}$ uniquely determines $n_j = p_j - r_jk$, which is nonnegative for all j if and only if

$$0 \leq k \leq L_{\mathbf{p}}, \qquad L_{\mathbf{p}} := \min_{1\leq j\leq d}\left\lfloor \frac{p_j}{r_j} \right\rfloor.$$

For each $\mathbf{p} \in \mathbb{N}^d$, define the simultaneous charge eigenspace

$$\mathcal{H}_{\mathbf{p}} = \text{span}\left\{ |k;\mathbf{p}\rangle := |k\rangle_0 \bigotimes_{j=1}^{d} |p_j - r_jk\rangle_j : 0 \leq k \leq L_{\mathbf{p}} \right\}.$$

Because every Fock basis vector $|\mathbf{n}\rangle$ belongs to a unique sector $\mathcal{H}_{\mathbf{p}}$ (with $p_j = n_j + r_jn_0$), Fock space decomposes into an orthogonal Hilbert-space direct sum of finite-dimensional subspaces:

$$\mathcal{F} = \bigoplus_{\mathbf{p}\in\mathbb{N}^d} \mathcal{H}_{\mathbf{p}}, \qquad \dim \mathcal{H}_{\mathbf{p}} = L_{\mathbf{p}} + 1 < \infty. \tag{9}$$

Crucially, because each $\mathcal{H}_{\mathbf{p}}$ is spanned by finitely many Fock states, the dense domain $\mathcal{D}_{\text{fin}}$ coincides precisely with the algebraic direct sum $\bigoplus_{\mathbf{p}\in\mathbb{N}^d}^{\text{alg}} \mathcal{H}_{\mathbf{p}}$ of vectors supported on finitely many charge sectors.

Let us compute the matrix of H on $\mathcal{H}_{\mathbf{p}}$. By (8), applying $a_j^{r_j}$ to $|n_j\rangle_j$ yields $\hbar^{r_j/2}\sqrt{n_j^{\underline{r_j}}}\,|n_j - r_j\rangle_j$ (vanishing if $n_j < r_j$), while $a_0^\dagger|k\rangle_0 = \sqrt{\hbar(k+1)}\,|k+1\rangle_0$. Hence $X_{\mathbf{r}}$ preserves each $\mathcal{H}_{\mathbf{p}}$ and acts as a finite weighted shift:

$$X_{\mathbf{r}}|k;\mathbf{p}\rangle = \hbar^{(R+1)/2}b_k|k+1;\mathbf{p}\rangle \quad (0 \leq k < L_{\mathbf{p}}), \qquad X_{\mathbf{r}}|L_{\mathbf{p}};\mathbf{p}\rangle = 0, \tag{10}$$

with nonnegative weights $b_k = b_k(\mathbf{p})$ given by

$$b_k^2 = (k+1)\prod_{j=1}^{d}(p_j - r_jk)^{\underline{r_j}}. \tag{11}$$

Note that for $k = L_{\mathbf{p}}$, at least one j satisfies $p_j - r_j L_{\mathbf{p}} < r_j$, so formula (11) automatically gives $b_{L_{\mathbf{p}}} = 0$. Taking adjoints,

$$X_{\mathbf{r}}^{\dagger}|k;\mathbf{p}\rangle = \hbar^{(R+1)/2} b_{k-1}|k-1;\mathbf{p}\rangle \quad (1 \le k \le L_{\mathbf{p}}), \qquad X_{\mathbf{r}}^{\dagger}|0;\mathbf{p}\rangle = 0.$$

Consequently, on every charge sector $\mathcal{H}_{\mathbf{p}}$, the restriction $H_{\mathbf{p}} := H|_{\mathcal{H}_{\mathbf{p}}}$ is an $(L_{\mathbf{p}}+1)\times(L_{\mathbf{p}}+1)$ Hermitian tridiagonal matrix with zero diagonal, subdiagonal entries $g\hbar^{(R+1)/2} b_k$, and superdiagonal entries $\overline{g}\hbar^{(R+1)/2} b_k$.

Proposition 2 (Essential self-adjointness and sector reduction)**.** *The Hamiltonian H is essentially self-adjoint on $\mathcal{D}_{\mathrm{fin}}$. Its unique self-adjoint closure $\overline{H}$ generates a strongly continuous unitary group*

$$U(t) = e^{-it\overline{H}/\hbar} = \bigoplus_{\mathbf{p}\in\mathbb{N}^d} e^{-itH_{\mathbf{p}}/\hbar}, \qquad t \in \mathbb{R},$$

which preserves every charge sector $\mathcal{H}_{\mathbf{p}}$. More generally, the same conclusion holds for any symmetric grade-zero polynomial $S \in \mathcal{A}_{\hbar,0}$ ($S^* = S$).

Proof. Let $S \in \mathcal{A}_{\hbar,0}$ be symmetric ($S^* = S$); in particular, this covers $S = H$. Since $[Q_j, S] = 0$ for all $1 \le j \le d$, any vector $\psi \in \mathcal{H}_{\mathbf{p}}$ satisfies $Q_j(S\psi) = S(Q_j\psi) = p_j S\psi$, so $S(\mathcal{H}_{\mathbf{p}}) \subseteq \mathcal{H}_{\mathbf{p}}$. Because $\dim \mathcal{H}_{\mathbf{p}} < \infty$ and S is symmetric on $\mathcal{D}_{\mathrm{fin}}$, each restriction $S_{\mathbf{p}} := S|_{\mathcal{H}_{\mathbf{p}}}$ is a bounded self-adjoint operator on $\mathcal{H}_{\mathbf{p}}$.

Define the direct-sum operator $S_{\oplus} := \bigoplus_{\mathbf{p}\in\mathbb{N}^d} S_{\mathbf{p}}$ on the domain

$$\mathrm{Dom}(S_{\oplus}) = \left\{ \psi = \bigoplus_{\mathbf{p}\in\mathbb{N}^d} \psi_{\mathbf{p}} \in \mathcal{F} : \sum_{\mathbf{p}\in\mathbb{N}^d} \|S_{\mathbf{p}}\psi_{\mathbf{p}}\|^2 < \infty \right\}, \qquad (S_{\oplus}\psi)_{\mathbf{p}} = S_{\mathbf{p}}\psi_{\mathbf{p}}.$$

Clearly $S_{\oplus}$ is symmetric and extends $S|_{\mathcal{D}_{\mathrm{fin}}}$. To check that $S_{\oplus}$ is self-adjoint, let $\eta \in \mathrm{Dom}(S_{\oplus}^*)$ and let $\phi \in \mathcal{H}_{\mathbf{p}} \subset \mathrm{Dom}(S_{\oplus})$. Then

$$\langle \eta_{\mathbf{p}}, S_{\mathbf{p}}\phi\rangle = \langle \eta, S_{\oplus}\phi\rangle = \langle S_{\oplus}^*\eta, \phi\rangle = \langle (S_{\oplus}^*\eta)_{\mathbf{p}}, \phi\rangle.$$

Because $S_{\mathbf{p}}$ is self-adjoint on $\mathcal{H}_{\mathbf{p}}$, we obtain $(S_{\oplus}^*\eta)_{\mathbf{p}} = S_{\mathbf{p}}\eta_{\mathbf{p}}$ for every $\mathbf{p} \in \mathbb{N}^d$. Consequently, $\sum_{\mathbf{p}} \|S_{\mathbf{p}}\eta_{\mathbf{p}}\|^2 = \|S_{\oplus}^*\eta\|^2 < \infty$, so $\eta \in \mathrm{Dom}(S_{\oplus})$ and $S_{\oplus}^*\eta = S_{\oplus}\eta$. Thus $S_{\oplus}$ is self-adjoint.

Finally, for any $\psi \in \mathrm{Dom}(S_{\oplus})$ and any finite subset $F \subset \mathbb{N}^d$, the truncation $\psi_F := \bigoplus_{\mathbf{p}\in F} \psi_{\mathbf{p}}$ lies in $\bigoplus_{\mathbf{p}}^{\mathrm{alg}} \mathcal{H}_{\mathbf{p}} = \mathcal{D}_{\mathrm{fin}}$, and its graph-norm distance to ψ is

$$\|\psi - \psi_F\|^2 + \|S_{\oplus}(\psi - \psi_F)\|^2 = \sum_{\mathbf{p}\notin F} \big(\|\psi_{\mathbf{p}}\|^2 + \|S_{\mathbf{p}}\psi_{\mathbf{p}}\|^2\big),$$

which can be made arbitrarily small by choosing F sufficiently large. Therefore $\mathcal{D}_{\mathrm{fin}}$ is a core for $S_{\oplus}$, i.e., $\overline{S|_{\mathcal{D}_{\mathrm{fin}}}} = S_{\oplus}$ is self-adjoint. By functional calculus on orthogonal direct sums, $e^{-it\overline{H}/\hbar} = \bigoplus_{\mathbf{p}\in\mathbb{N}^d} e^{-itH_{\mathbf{p}}/\hbar}$. □

Thus the Manley–Rowe grading does more than catalog selection rules (as in the classic finite-block diagonalization of the trilinear model [3]): it supplies a direct, completely elementary domain proof for nonlinear bosonic Hamiltonians of arbitrarily high polynomial degree, including extensions with detuning (ΔN_0) or Kerr self- and cross-phase modulation ($\sum_{j,k} \chi_{jk} N_j N_k$), all of which have grade zero. This stands in sharp contrast to non-resonant higher-order bosonic interactions such as $\xi(a^{\dagger})^k + \bar{\xi}a^k$ ($k \ge 3$), which lack finite-dimensional invariant sectors and fail to be essentially self-adjoint on $\mathcal{D}_{\mathrm{fin}}$ unless regularized by sufficiently strong number-operator terms [10].

4 Berezin–Wick symbols and the classical limit

To connect the quantum algebra $\mathcal{A}_\hbar$ to classical coupled-mode dynamics, we use the coherent-state (Berezin–Wick) symbol calculus [2, 4]. For $z = (z_0, z_1, \dots, z_d) \in \mathbb{C}^{d+1}$, the normalized coherent state in $\mathcal{F}$ is

$$|z\rangle = e^{-\|z\|^2/(2\hbar)} \sum_{\mathbf{n}\in\mathbb{N}^{d+1}} \frac{z^{\mathbf{n}}}{\sqrt{\hbar^{|\mathbf{n}|}\mathbf{n}!}}\,|\mathbf{n}\rangle, \qquad \|z\|^2 = \sum_{j=0}^{d} |z_j|^2. \tag{12}$$

From (8) one verifies immediately that $|z\rangle$ belongs to the domain of every polynomial in $a_j, a_j^\dagger$ (since $\sum_{\mathbf{n}} (|\mathbf{n}|+1)^m |z|^{2\mathbf{n}}/(\hbar^{|\mathbf{n}|}\mathbf{n}!) < \infty$ for all $m \in \mathbb{N}$), that $\langle z|z\rangle = 1$, and that

$$a_j|z\rangle = z_j|z\rangle, \qquad \langle z|a_j^\dagger = \bar{z}_j\langle z|, \qquad 0 \le j \le d.$$

The **Wick symbol** of an element $B \in \mathcal{A}_\hbar$ is the polynomial function on $\mathbb{C}^{d+1}$ defined by

$$\sigma_\hbar(B)(z,\bar{z}) = \langle z|B|z\rangle.$$

On the normal-ordered basis,

$$\sigma_\hbar\big((a^\dagger)^{\mathbf{u}} a^{\mathbf{v}}\big)(z,\bar{z}) = \bar{z}^{\mathbf{u}} z^{\mathbf{v}}.$$

Hence $\sigma_\hbar$ is a linear isomorphism from $\mathcal{A}_\hbar$ onto the space $\mathbb{C}[z,\bar{z}]$ of polynomials in $z_0, \dots, z_d$ and $\bar{z}_0, \dots, \bar{z}_d$, sending B^* to $\overline{\sigma_\hbar(B)}$. However, $\sigma_\hbar$ is *not* a homomorphism for pointwise multiplication; instead, operator multiplication pulls back to the Berezin–Wick star product.

Proposition 3 (Wick star product and exact Heisenberg generator)**.** *For all $B, C \in \mathcal{A}_\hbar$,*

$$\sigma_\hbar(BC) = \sigma_\hbar(B) \star_\hbar \sigma_\hbar(C), \tag{13}$$

where for $f, k \in \mathbb{C}[z,\bar{z}]$ the finite sum

$$(f \star_\hbar k)(z,\bar{z}) = \sum_{\boldsymbol{\nu}\in\mathbb{N}^{d+1}} \frac{\hbar^{|\boldsymbol{\nu}|}}{\boldsymbol{\nu}!} (\partial_z^{\boldsymbol{\nu}} f)\,(\partial_{\bar{z}}^{\boldsymbol{\nu}} k) \tag{14}$$

defines an associative product on $\mathbb{C}[z,\bar{z}]$. For the resonant Hamiltonian (2), with classical symbol

$$h(z,\bar{z}) = \sigma_\hbar(H)(z,\bar{z}) = g\bar{z}_0 z^{\mathbf{r}} + \overline{g} z_0 \bar{z}^{\mathbf{r}} \qquad \Big(z^{\mathbf{r}} := \prod_{j=1}^{d} z_j^{r_j}\Big), \tag{15}$$

and Poisson bracket

$$\{f,k\} = -i \sum_{j=0}^{d} \big(\partial_{z_j} f\, \partial_{\bar{z}_j} k - \partial_{\bar{z}_j} f\, \partial_{z_j} k\big), \tag{16}$$

the Wick symbol of the Heisenberg generator $\mathcal{L}_\hbar f := \sigma_\hbar\big(\frac{i}{\hbar}[H,B]\big)$ (where $f = \sigma_\hbar(B)$) is given exactly by

$$\mathcal{L}_\hbar f = \frac{i}{\hbar}(h \star_\hbar f - f \star_\hbar h) = \{f,h\} + i \sum_{\substack{0 \le \boldsymbol{\nu} \le \mathbf{r} \\ |\boldsymbol{\nu}| \ge 2}} \hbar^{|\boldsymbol{\nu}|-1} \binom{\mathbf{r}}{\boldsymbol{\nu}} \left(g\bar{z}_0 z^{\mathbf{r}-\boldsymbol{\nu}} \partial_{\bar{z}}^{\boldsymbol{\nu}} f - \overline{g} z_0 \bar{z}^{\mathbf{r}-\boldsymbol{\nu}} \partial_z^{\boldsymbol{\nu}} f\right), \tag{17}$$

where $\binom{\mathbf{r}}{\boldsymbol{\nu}} = \prod_{j=1}^{d} \binom{r_j}{\nu_j}$ and the derivatives act on the pump variables $z_1, \dots, z_d$.

Proof. By bilinearity it suffices to verify (13) on normal-ordered monomials $B = (a^\dagger)^{\mathbf{u}} a^{\mathbf{v}}$ and $C = (a^\dagger)^{\mathbf{u}'} a^{\mathbf{v}'}$. For a single mode, induction on m and n using $[a, a^\dagger] = \hbar 1$ gives the standard Wick normal-ordering identity

$$a^m (a^\dagger)^n = \sum_{\nu=0}^{\min(m,n)} \frac{\hbar^\nu}{\nu!}\, m^{\underline{\nu}}\, n^{\underline{\nu}}\, (a^\dagger)^{n-\nu} a^{m-\nu}. \tag{18}$$

(Indeed, for $m = 1$ this is $a(a^\dagger)^n = (a^\dagger)^n a + \hbar n (a^\dagger)^{n-1}$, and multiplying on the left by a advances m to $m+1$ via Pascal's identity for $\binom{m}{\nu}$.) Taking the product of (18) across all $d+1$ commuting modes yields

$$BC = (a^\dagger)^{\mathbf{u}}\, a^{\mathbf{v}} (a^\dagger)^{\mathbf{u}'}\, a^{\mathbf{v}'} = \sum_{\boldsymbol{\nu} \in \mathbb{N}^{d+1}} \frac{\hbar^{|\boldsymbol{\nu}|}}{\boldsymbol{\nu}!}\, \mathbf{v}^{\underline{\boldsymbol{\nu}}}\, (\mathbf{u}')^{\underline{\boldsymbol{\nu}}}\, (a^\dagger)^{\mathbf{u}+\mathbf{u}'-\boldsymbol{\nu}} a^{\mathbf{v}+\mathbf{v}'-\boldsymbol{\nu}},$$

where terms with $\nu_j > \min(v_j, u'_j)$ vanish automatically. Applying $\sigma_\hbar$ and observing that $\partial_z^{\boldsymbol{\nu}} (\bar{z}^{\mathbf{u}} z^{\mathbf{v}}) = \mathbf{v}^{\underline{\boldsymbol{\nu}}} \bar{z}^{\mathbf{u}} z^{\mathbf{v}-\boldsymbol{\nu}}$ while $\partial_{\bar{z}}^{\boldsymbol{\nu}} (\bar{z}^{\mathbf{u}'} z^{\mathbf{v}'}) = (\mathbf{u}')^{\underline{\boldsymbol{\nu}}} \bar{z}^{\mathbf{u}'-\boldsymbol{\nu}} z^{\mathbf{v}'}$ gives

$$\sigma_\hbar(BC) = \sum_{\boldsymbol{\nu} \in \mathbb{N}^{d+1}} \frac{\hbar^{|\boldsymbol{\nu}|}}{\boldsymbol{\nu}!} \big(\partial_z^{\boldsymbol{\nu}} (\bar{z}^{\mathbf{u}} z^{\mathbf{v}})\big) \big(\partial_{\bar{z}}^{\boldsymbol{\nu}} (\bar{z}^{\mathbf{u}'} z^{\mathbf{v}'})\big) = \sigma_\hbar(B) \star_\hbar \sigma_\hbar(C).$$

For (17), expanding $h \star_\hbar f - f \star_\hbar h$ using (14) cancels the $\boldsymbol{\nu} = 0$ term $hf - fh = 0$. The $|\boldsymbol{\nu}| = 1$ terms contribute

$$\frac{i}{\hbar} \cdot \hbar \sum_{j=0}^{d} \big(\partial_{z_j} h\, \partial_{\bar{z}_j} f - \partial_{z_j} f\, \partial_{\bar{z}_j} h\big) = -i \sum_{j=0}^{d} \big(\partial_{z_j} f\, \partial_{\bar{z}_j} h - \partial_{\bar{z}_j} f\, \partial_{z_j} h\big) = \{f, h\}.$$

For $|\boldsymbol{\nu}| \geq 2$, the term $g \bar{z}_0 z^{\mathbf{r}}$ in h has no z_0-dependence and $\partial_z^{\boldsymbol{\nu}} (z^{\mathbf{r}}) = \mathbf{r}^{\underline{\boldsymbol{\nu}}} z^{\mathbf{r}-\boldsymbol{\nu}}$ for $0 \leq \boldsymbol{\nu} \leq \mathbf{r}$ (with $\nu_0 = 0$), while $\overline{g} z_0 \bar{z}^{\mathbf{r}}$ is linear in z_0 and independent of $z_1, \ldots, z_d$, so all its z-derivatives of order ≥ 2 vanish. Similarly, the only $\bar{z}$-derivatives of h of order ≥ 2 come from $\overline{g} z_0 \bar{z}^{\mathbf{r}}$. Since $\mathbf{r}^{\underline{\boldsymbol{\nu}}} / \boldsymbol{\nu}! = \binom{\mathbf{r}}{\boldsymbol{\nu}}$, substituting into $\frac{i}{\hbar}(h \star_\hbar f - f \star_\hbar h)$ gives (17). □

For the two quadratic optical processes ($R = 2$), the quantum correction in (17) has a single $O(\hbar)$ differential term:

$$\text{SFG } \mathbf{r} = (1,1): \qquad \mathcal{L}_\hbar f = \{f, h\} + i\hbar \big(g \bar{z}_0\, \partial_{\bar{z}_1} \partial_{\bar{z}_2} f - \overline{g} z_0\, \partial_{z_1} \partial_{z_2} f\big), \tag{19}$$

$$\text{SHG } \mathbf{r} = (2): \qquad \mathcal{L}_\hbar f = \{f, h\} + i\hbar \big(g \bar{z}_0\, \partial_{\bar{z}_1}^2 f - \overline{g} z_0\, \partial_{z_1}^2 f\big). \tag{20}$$

For linear conversion ($\mathbf{r} = (1)$, $R = 1$), the sum over $|\boldsymbol{\nu}| \geq 2$ is empty and $\mathcal{L}_\hbar f = \{f, h\}$ holds *identically* without $\hbar$-corrections.

Taking the $\hbar \to 0$ limit of the Wick calculus recovers the classical commutative algebra $\mathbb{C}[z, \bar{z}]$ with Poisson bracket (16) and Hamilton's coupled-mode equations:

$$\dot{z}_j = \{z_j, h\} = -i\, \partial_{\bar{z}_j} h, \qquad 0 \leq j \leq d, \tag{21}$$

explicitly

$$\dot{z}_0 = -ig \prod_{j=1}^{d} z_j^{r_j}, \qquad \dot{z}_j = -i\overline{g}\, r_j z_0 \bar{z}_j^{r_j - 1} \prod_{m \neq j} \bar{z}_m^{r_m} \quad (1 \leq j \leq d).$$

Along any solution of (21), the classical Manley–Rowe invariants

$$q_j = |z_j|^2 + r_j |z_0|^2 \qquad (1 \leq j \leq d) \tag{22}$$

and the Hamiltonian $h(z, \bar{z})$ are conserved: indeed, $\{q_j, h\} = 0$ follows from the same calculation as $[Q_j, H] = 0$. Since each $q_j \geq 0$ is constant, every trajectory remains inside the compact region $\{z \in \mathbb{C}^{d+1} : |z_j|^2 + r_j |z_0|^2 = q_j\}$, so the smooth polynomial vector field (21) generates a *global* real-analytic flow on $\mathbb{C}^{d+1}$. Note also that while $\mathcal{L}_\hbar$ acts within the polynomial algebra $\mathbb{C}[z, \bar{z}]$, the finite-time quantum Heisenberg evolution $A \mapsto U(t)^* A U(t)$ does not preserve $\mathcal{A}_\hbar$ whenever $R \geq 2$.

5 A universal quartic quantum deficit

We now compare the quantum and classical output intensities starting from coherent pump inputs and an initially empty output mode. For $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{C}^d$, set

$$|\Psi_\alpha\rangle = |0\rangle_0 \bigotimes_{j=1}^{d} |\alpha_j\rangle_j, \qquad x_j = |\alpha_j|^2 \geq 0 \quad (1 \leq j \leq d),$$

where each $|\alpha_j\rangle_j$ is the single-mode coherent state (12) with parameter α_j. Define the quantum scaled output intensity

$$J_\hbar(t) = \|a_0 U(t)|\Psi_\alpha\rangle\|^2 = \langle \Psi_\alpha | U(t)^* a_0^\dagger a_0 U(t) | \Psi_\alpha \rangle \tag{23}$$

and the classical output intensity

$$J_{\mathrm{cl}}(t) = |z_0(t)|^2, \tag{24}$$

where $z(t)$ solves (21) with initial data $z_0(0) = 0$ and $z_j(0) = \alpha_j$ $(1 \leq j \leq d)$. Note that $J_\hbar(t)$ is finite for all $t \in \mathbb{R}$: expanding $|\Psi_\alpha\rangle = \bigoplus_{\mathbf{p} \in \mathbb{N}^d} c_{\mathbf{p}} |0; \mathbf{p}\rangle$ gives weights

$$w_{\mathbf{p}} := |c_{\mathbf{p}}|^2 = \prod_{j=1}^{d} e^{-x_j/\hbar} \frac{(x_j/\hbar)^{p_j}}{p_j!}, \tag{25}$$

and since $a_0^\dagger a_0 |_{\mathcal{H}_{\mathbf{p}}} = \hbar N_0 |_{\mathcal{H}_{\mathbf{p}}} \leq \hbar L_{\mathbf{p}} I_{\mathcal{H}_{\mathbf{p}}} \leq \hbar p_1 I_{\mathcal{H}_{\mathbf{p}}}$, we have $U(t)|\Psi_\alpha\rangle \in \mathrm{Dom}(a_0)$ and

$$J_\hbar(t) = \sum_{\mathbf{p} \in \mathbb{N}^d} w_{\mathbf{p}} J_{\hbar,\mathbf{p}}(t), \qquad J_{\hbar,\mathbf{p}}(t) := \langle 0; \mathbf{p} | e^{itH_{\mathbf{p}}/\hbar} \, (\hbar N_0) \, e^{-itH_{\mathbf{p}}/\hbar} | 0; \mathbf{p} \rangle \leq \hbar p_1, \tag{26}$$

with $\sum_{\mathbf{p}} w_{\mathbf{p}} \hbar p_1 = x_1 < \infty$.

For multi-indices $0 \leq \mathbf{k} \leq \mathbf{r}$, define the combinatorial coefficients

$$C_{\mathbf{r}}(\mathbf{k}) = \prod_{j=1}^{d} \binom{r_j}{k_j}^2 k_j!, \qquad x^{\boldsymbol{\ell}} = \prod_{j=1}^{d} x_j^{\ell_j}. \tag{27}$$

We first record the elementary combinatorial and Poisson-moment identities needed for the proof.

Lemma 1 (Falling-factorial overlap identity and Poisson moments)**.** *For all $p, r \in \mathbb{N}$,*

$$(p^{\underline{r}})^2 = \sum_{k=0}^{r} \binom{r}{k}^2 k! \, p^{\underline{2r-k}}, \qquad p^{\underline{r}} (p-r)^{\underline{r}} = p^{\underline{2r}}. \tag{28}$$

Moreover, if $P \sim \mathrm{Poisson}(\mu)$ with $\mu \geq 0$, then for every $m \in \mathbb{N}$,

$$\mathbb{E}[P^{\underline{m}}] = \mu^m. \tag{29}$$

Proof. The second identity in (28) is immediate from the definition of the falling factorial (with both sides equal to 0 when $p < 2r$). For the first identity, interpret $p^{\underline{r}}$ as the number of injective maps from $\{1, \dots, r\}$ into $\{1, \dots, p\}$. Then $(p^{\underline{r}})^2$ counts all ordered pairs (f_1, f_2) of such maps. Partition these pairs by the size $k = |\mathrm{im}(f_1) \cap \mathrm{im}(f_2)| \in \{0, \dots, r\}$ of their image overlap. To construct a pair with overlap size k: choose the k-element subset of $\{1, \dots, r\}$ mapped by f_1 to the overlap ($\binom{r}{k}$ ways), choose the k-element subset mapped by f_2 to the overlap ($\binom{r}{k}$ ways), choose the bijection matching those two subsets ($k!$ ways), and assign $2r - k$ distinct values from $\{1, \dots, p\}$ to the resulting $2r - k$ distinct slots ($p^{\underline{2r-k}}$ ways, which vanishes automatically if $p < 2r - k$). Summing over $k = 0, \dots, r$ yields the first identity in (28).

For (29), if $\mu = 0$ then $P = 0$ almost surely and both sides equal $\delta_{m,0}$. If $\mu > 0$,

$$\mathbb{E}[P^{\underline{m}}] = \sum_{p=m}^{\infty} e^{-\mu} \frac{\mu^p}{p!} \frac{p!}{(p-m)!} = \mu^m \sum_{q=0}^{\infty} e^{-\mu} \frac{\mu^q}{q!} = \mu^m. \qquad \square$$

Theorem 1 (Universal quartic quantum deficit)**.** *Fix $\hbar > 0$, $g \in \mathbb{C}$, and initial pump intensities $x = (x_1, \dots, x_d) \in [0, \infty)^d$. Then*

$$J_\hbar(t) - J_{\mathrm{cl}}(t) = -\frac{|g|^4}{3} D_{\mathbf{r}}(x; \hbar)\, t^4 + O(t^6) \qquad (t \to 0), \tag{30}$$

where

$$D_{\mathbf{r}}(x; \hbar) = \sum_{\substack{0 \le \mathbf{k} \le \mathbf{r} \\ |\mathbf{k}| \ge 2}} C_{\mathbf{r}}(\mathbf{k})\, \hbar^{|\mathbf{k}|-1}\, x^{2\mathbf{r}-\mathbf{k}}. \tag{31}$$

More precisely, there exist polynomials $Q_{\mathrm{q}}(x; \hbar, |g|)$ and $Q_{\mathrm{cl}}(x; |g|)$ in $x_1, \dots, x_d$ (with nonnegative coefficients, depending only on $\mathbf{r}$, $\hbar$, and $|g|$) such that for all $t \in \mathbb{R}$,

$$\left| \big(J_\hbar(t) - J_{\mathrm{cl}}(t)\big) + \frac{|g|^4}{3} D_{\mathbf{r}}(x; \hbar)\, t^4 \right| \le \big(Q_{\mathrm{q}}(x; \hbar, |g|) + Q_{\mathrm{cl}}(x; |g|)\big)\, t^6. \tag{32}$$

In particular, whenever $R = \sum_{j=1}^d r_j \ge 2$, all $x_j > 0$, and $g \neq 0$, we have $D_{\mathbf{r}}(x; \hbar) > 0$, and hence

$$J_\hbar(t) < J_{\mathrm{cl}}(t)$$

for all sufficiently small nonzero t.

Proof. We establish the quantum expansion, the classical expansion, and their uniform-in-t sixth-order remainder bounds in four steps.

Step 1: Charge-sector expansion and evenness. Fix $\mathbf{p} \in \mathbb{N}^d$ and write $M_{\mathbf{p}} := H_{\mathbf{p}}/\hbar$ and $A_{\mathbf{p}} := \hbar N_0|_{\mathcal{H}_{\mathbf{p}}}$, so that $J_{\hbar,\mathbf{p}}(t) = \langle 0; \mathbf{p}| e^{itM_{\mathbf{p}}} A_{\mathbf{p}} e^{-itM_{\mathbf{p}}} |0; \mathbf{p}\rangle$. First observe that $J_{\hbar,\mathbf{p}}(t)$ is an even entire function of t. Indeed, the unitary output-phase reflection $V_{\mathbf{p}} := (-1)^{N_0}|_{\mathcal{H}_{\mathbf{p}}}$ satisfies $V_{\mathbf{p}}|k; \mathbf{p}\rangle = (-1)^k |k; \mathbf{p}\rangle$, hence $V_{\mathbf{p}} X_{\mathbf{r}} V_{\mathbf{p}}^* = -X_{\mathbf{r}}$ and $V_{\mathbf{p}} M_{\mathbf{p}} V_{\mathbf{p}}^* = -M_{\mathbf{p}}$, while $V_{\mathbf{p}} A_{\mathbf{p}} V_{\mathbf{p}}^* = A_{\mathbf{p}}$ and $V_{\mathbf{p}}|0; \mathbf{p}\rangle = |0; \mathbf{p}\rangle$. Therefore $V_{\mathbf{p}} e^{-itM_{\mathbf{p}}} V_{\mathbf{p}}^* = e^{itM_{\mathbf{p}}}$, which gives $J_{\hbar,\mathbf{p}}(-t) = J_{\hbar,\mathbf{p}}(t)$ for all $t \in \mathbb{R}$. In particular, all odd derivatives of $J_{\hbar,\mathbf{p}}$ vanish at $t = 0$.

Define

$$F_{\mathbf{p}} := \prod_{j=1}^d p_j^{\underline{r_j}}, \qquad B_{\mathbf{p}} := \prod_{j=1}^d (p_j - r_j)^{\underline{r_j}}.$$

By (11), the first two squared shift weights are $b_0^2 = F_{\mathbf{p}}$ and $b_1^2 = 2B_{\mathbf{p}}$. Let $s := \hbar^{(R-1)/2}$, so the off-diagonal entries of $M_{\mathbf{p}}$ are gsb_k and $\overline{g}sb_k$. We compute the Taylor coefficients of $J_{\hbar,\mathbf{p}}(t)$ through t^4, checking all three cases for $L_{\mathbf{p}}$:

- If $L_{\mathbf{p}} = 0$ (inactive sector), then $\dim \mathcal{H}_{\mathbf{p}} = 1$, $M_{\mathbf{p}} = 0$, and $J_{\hbar,\mathbf{p}}(t) \equiv 0$. Since $p_j < r_j$ for some j, we also have $F_{\mathbf{p}} = 0$.

- If $L_{\mathbf{p}} = 1$ (two-level sector), then $p_j - r_j < r_j$ for some j, so $b_1 = B_{\mathbf{p}} = 0$, and $M_{\mathbf{p}}$ is a 2×2 matrix with off-diagonal gsb_0. Thus

  $$J_{\hbar,\mathbf{p}}(t) = \hbar \sin^2(|g| s b_0 t) = \hbar |g|^2 s^2 b_0^2 t^2 - \frac{1}{3}\hbar |g|^4 s^4 b_0^4 t^4 + O_{\mathbf{p}}(t^6).$$

  Since $\hbar s^2 = \hbar^R$, $\hbar s^4 = \hbar^{2R-1}$, $b_0^2 = F_{\mathbf{p}}$, and $B_{\mathbf{p}} = 0$, this agrees with (33) below.

- If $L_{\mathbf{p}} \ge 2$, expand $|\psi_{\mathbf{p}}(t)\rangle = e^{-itM_{\mathbf{p}}}|0; \mathbf{p}\rangle = \sum_{k=0}^{L_{\mathbf{p}}} c_k(t)|k; \mathbf{p}\rangle$. Because $M_{\mathbf{p}}$ changes k by ± 1, we have $c_k(t) = O_{\mathbf{p}}(t^k)$, so $|c_k(t)|^2 = O_{\mathbf{p}}(t^6)$ for all $k \ge 3$. Computing $M_{\mathbf{p}}^m|0; \mathbf{p}\rangle$ for $m = 1, 2, 3$ directly from (10):

  $$\begin{aligned} M_{\mathbf{p}}|0; \mathbf{p}\rangle &= gsb_0|1; \mathbf{p}\rangle, \\ M_{\mathbf{p}}^2|0; \mathbf{p}\rangle &= |g|^2 s^2 b_0^2 |0; \mathbf{p}\rangle + g^2 s^2 b_0 b_1 |2; \mathbf{p}\rangle, \\ M_{\mathbf{p}}^3|0; \mathbf{p}\rangle &= g|g|^2 s^3 b_0 (b_0^2 + b_1^2)|1; \mathbf{p}\rangle + g^3 s^3 b_0 b_1 b_2 |3; \mathbf{p}\rangle \end{aligned}$$

(with the last term absent if $L_{\mathbf{p}} = 2$). Therefore

$$c_1(t) = -itgsb_0 + \frac{it^3}{6} g|g|^2 s^3 b_0 (b_0^2 + b_1^2) + O_{\mathbf{p}}(t^5),$$
$$c_2(t) = -\frac{t^2}{2} g^2 s^2 b_0 b_1 + O_{\mathbf{p}}(t^4).$$

Taking squared moduli gives

$$|c_1(t)|^2 = |g|^2 s^2 b_0^2 t^2 - \frac{1}{3}|g|^4 s^4 b_0^2 (b_0^2 + b_1^2) t^4 + O_{\mathbf{p}}(t^6),$$
$$|c_2(t)|^2 = \frac{1}{4}|g|^4 s^4 b_0^2 b_1^2 t^4 + O_{\mathbf{p}}(t^6).$$

Hence $J_{\hbar,\mathbf{p}}(t) = \hbar\big(|c_1(t)|^2 + 2|c_2(t)|^2\big) + O_{\mathbf{p}}(t^6)$ has t^4 coefficient

$$\hbar|g|^4 s^4 b_0^2 \left(-\frac{1}{3}b_0^2 - \frac{1}{3}b_1^2 + \frac{1}{2}b_1^2\right) = \frac{\hbar^{2R-1}|g|^4}{3} b_0^2 \left(\frac{1}{2}b_1^2 - b_0^2\right) = \frac{\hbar^{2R-1}|g|^4}{3} F_{\mathbf{p}}(B_{\mathbf{p}} - F_{\mathbf{p}}).$$

In all three cases ($L_{\mathbf{p}} \geq 0$), the Taylor polynomial of $J_{\hbar,\mathbf{p}}(t)$ through order 5 is

$$T_{\hbar,\mathbf{p}}(t) = \hbar^R |g|^2 F_{\mathbf{p}} t^2 + \frac{\hbar^{2R-1}|g|^4}{3} F_{\mathbf{p}}(B_{\mathbf{p}} - F_{\mathbf{p}}) t^4. \tag{33}$$

Step 2: Uniform-in-time sector remainder bound and Poisson averaging. For every $s \in \mathbb{R}$, differentiating $J_{\hbar,\mathbf{p}}(s) = \langle 0;\mathbf{p}|e^{isM_{\mathbf{p}}} A_{\mathbf{p}} e^{-isM_{\mathbf{p}}}|0;\mathbf{p}\rangle$ six times yields

$$J^{(6)}_{\hbar,\mathbf{p}}(s) = \langle 0;\mathbf{p}|e^{isM_{\mathbf{p}}} \big(\mathrm{ad}^6_{iM_{\mathbf{p}}}(A_{\mathbf{p}})\big) e^{-isM_{\mathbf{p}}}|0;\mathbf{p}\rangle.$$

Since $e^{-isM_{\mathbf{p}}}$ is unitary and $\|\,\mathrm{ad}_{iM_{\mathbf{p}}}(Y)\|_{\mathrm{op}} \leq 2\|M_{\mathbf{p}}\|_{\mathrm{op}}\|Y\|_{\mathrm{op}}$, we obtain the global bound

$$\sup_{s\in\mathbb{R}} \big|J^{(6)}_{\hbar,\mathbf{p}}(s)\big| \leq \big(2\|M_{\mathbf{p}}\|_{\mathrm{op}}\big)^6 \|A_{\mathbf{p}}\|_{\mathrm{op}}.$$

Now $\|A_{\mathbf{p}}\|_{\mathrm{op}} = \hbar L_{\mathbf{p}} \leq \hbar p_1$. Moreover, $M_{\mathbf{p}} = g\hbar^{-1} X_{\mathbf{r}}|_{\mathcal{H}_{\mathbf{p}}} + \overline{g}\hbar^{-1} X^{\dagger}_{\mathbf{r}}|_{\mathcal{H}_{\mathbf{p}}}$ is the sum of a weighted shift and its adjoint, so by the triangle inequality

$$\|M_{\mathbf{p}}\|_{\mathrm{op}} \leq 2|g|\hbar^{(R-1)/2} \max_{0\leq k<L_{\mathbf{p}}} b_k \leq 2|g|\hbar^{(R-1)/2} \sqrt{(p_1+1)\prod_{j=1}^{d} p_j^{r_j}}.$$

Consequently, Taylor's theorem with Lagrange remainder at $t = 0$ gives, for every $\mathbf{p} \in \mathbb{N}^d$ and all $t \in \mathbb{R}$,

$$\big|J_{\hbar,\mathbf{p}}(t) - T_{\hbar,\mathbf{p}}(t)\big| \leq \frac{t^6}{6!}\big(2\|M_{\mathbf{p}}\|_{\mathrm{op}}\big)^6 \|A_{\mathbf{p}}\|_{\mathrm{op}} \leq \frac{4^6|g|^6\hbar^{3R-2}}{6!}\, p_1(p_1+1)^3 \left(\prod_{j=1}^{d} p_j^{3r_j}\right) t^6. \tag{34}$$

We now average (33) and (34) over the independent Poisson variables $P_j \sim \mathrm{Poisson}(\mu_j)$ with $\mu_j = x_j/\hbar$ from (25)–(26). Since every polynomial in $(P_1,\dots,P_d)$ has expectation equal to a polynomial in $(\mu_1,\dots,\mu_d)$ with nonnegative coefficients (each monomial P_j^m is a nonnegative integer combination of falling factorials $P_j^{\underline{\ell}}$, whose expectations are μ_j^ℓ by Lemma 1), averaging (34) yields a polynomial $Q_{\mathrm{q}}(x;\hbar,|g|)$ in x with nonnegative coefficients such that

$$\big|J_{\hbar}(t) - \mathbb{E}[T_{\hbar,\mathbf{P}}(t)]\big| \leq Q_{\mathrm{q}}(x;\hbar,|g|)\, t^6 \qquad \text{for all } t \in \mathbb{R}.$$

To evaluate $\mathbb{E}[T_{\hbar,\mathbf{P}}(t)]$, Lemma 1 and independence of $P_1, \dots, P_d$ give

$$\mathbb{E}[F_{\mathbf{P}}] = \prod_{j=1}^{d} \mathbb{E}[P_j^{r_j}] = \mu^{\mathbf{r}} = \hbar^{-R} x^{\mathbf{r}},$$

$$\mathbb{E}[F_{\mathbf{P}} B_{\mathbf{P}}] = \prod_{j=1}^{d} \mathbb{E}[P_j^{2r_j}] = \mu^{2\mathbf{r}} = \hbar^{-2R} x^{2\mathbf{r}} = \big(\mathbb{E}[F_{\mathbf{P}}]\big)^2,$$

$$\mathbb{E}[F_{\mathbf{P}}^2] = \prod_{j=1}^{d} \sum_{k_j=0}^{r_j} \binom{r_j}{k_j}^2 k_j!\, \mathbb{E}[P_j^{2r_j-k_j}] = \sum_{0\le \mathbf{k}\le \mathbf{r}} C_{\mathbf{r}}(\mathbf{k})\, \mu^{2\mathbf{r}-\mathbf{k}}.$$

Notice that the $\mathbf{k}=0$ term of $\mathbb{E}[F_{\mathbf{P}}^2]$ is $C_{\mathbf{r}}(0)\mu^{2\mathbf{r}} = \mu^{2\mathbf{r}}$, which cancels $\mathbb{E}[F_{\mathbf{P}} B_{\mathbf{P}}]$ identically! Thus

$$\mathbb{E}\big[F_{\mathbf{P}}(B_{\mathbf{P}} - F_{\mathbf{P}})\big] = -\operatorname{Var}(F_{\mathbf{P}}) = -\sum_{\substack{0\le \mathbf{k}\le \mathbf{r} \\ |\mathbf{k}|\ge 1}} C_{\mathbf{r}}(\mathbf{k})\, \hbar^{|\mathbf{k}|-2R}\, x^{2\mathbf{r}-\mathbf{k}}. \tag{35}$$

Substituting into $\mathbb{E}[T_{\hbar,\mathbf{P}}(t)]$ gives the exact quantum expansion

$$J_\hbar(t) = |g|^2 x^{\mathbf{r}} t^2 - \frac{|g|^4}{3} \sum_{\substack{0\le \mathbf{k}\le \mathbf{r} \\ |\mathbf{k}|\ge 1}} C_{\mathbf{r}}(\mathbf{k})\, \hbar^{|\mathbf{k}|-1}\, x^{2\mathbf{r}-\mathbf{k}} t^4 + R_{\mathrm{q}}(t), \qquad |R_{\mathrm{q}}(t)| \le Q_{\mathrm{q}}(x;\hbar,|g|)\, t^6. \tag{36}$$

Step 3: Classical expansion, evenness, and global remainder bound. Consider the classical solution $z(t)$ of (21) with $z_0(0) = 0$ and $z_j(0) = \alpha_j$ $(1 \le j \le d)$. First, $J_{\mathrm{cl}}(t) = |z_0(t)|^2$ is even in t: defining $\tilde{z}_0(t) := -z_0(-t)$ and $\tilde{z}_j(t) := z_j(-t)$ $(1 \le j \le d)$, a direct substitution into (21) shows that $\tilde{z}(t)$ solves the same ODE system with the same initial data $\tilde{z}_0(0) = 0$ and $\tilde{z}_j(0) = \alpha_j$. By uniqueness of solutions, $\tilde{z}(t) = z(t)$, so $z_0(-t) = -z_0(t)$ and $J_{\mathrm{cl}}(-t) = J_{\mathrm{cl}}(t)$ for all $t \in \mathbb{R}$.

Next, set $w(t) := g\bar{z}_0(t) \prod_{j=1}^{d} z_j(t)^{r_j}$. From (15) and (21),

$$h = w + \bar{w} = 2\,\mathrm{Re}\, w, \qquad \dot{J}_{\mathrm{cl}} = \dot{z}_0\bar{z}_0 + z_0\dot{\bar{z}}_0 = -iw + i\bar{w} = 2\,\mathrm{Im}\, w.$$

Hence $h^2 + \dot{J}_{\mathrm{cl}}^2 = 4|w|^2 = 4|g|^2 J_{\mathrm{cl}} \prod_{j=1}^{d} |z_j|^{2r_j}$. Because $h \equiv h(z(0)) = 0$ and $|z_j(t)|^2 = x_j - r_j J_{\mathrm{cl}}(t)$ by (22), the classical output intensity satisfies the first integral

$$\dot{J}_{\mathrm{cl}}^2 = 4|g|^2 P(J_{\mathrm{cl}}), \qquad P(J) := J \prod_{j=1}^{d} (x_j - r_j J)^{r_j}. \tag{37}$$

Computing $\ddot{J}_{\mathrm{cl}} = \{\{|z_0|^2, h\}, h\}$ directly from (16)–(21) (valid for all $t \in \mathbb{R}$, with no need to divide by $\dot{J}_{\mathrm{cl}}$) yields

$$\ddot{J}_{\mathrm{cl}} = 2|g|^2 \prod_{j=1}^{d} |z_j|^{2r_j} - 2|g|^2 |z_0|^2 \sum_{j=1}^{d} r_j^2 |z_j|^{2(r_j-1)} \prod_{m\ne j} |z_m|^{2r_m} = 2|g|^2 P'(J_{\mathrm{cl}}). \tag{38}$$

Expanding $P(J) = x^{\mathbf{r}} J - \left(\sum_{j=1}^{d} r_j^2 x^{\mathbf{r}-e_j}\right) J^2 + O(J^3)$ gives

$$P'(0) = x^{\mathbf{r}}, \qquad P''(0) = -2\sum_{j=1}^{d} r_j^2 x^{\mathbf{r}-e_j}.$$

Since $J_{\mathrm{cl}}(0) = \dot{J}_{\mathrm{cl}}(0) = 0$, evaluating (38) and its second derivative

$$J_{\mathrm{cl}}^{(4)} = 2|g|^2\big(P'''(J_{\mathrm{cl}})\dot{J}_{\mathrm{cl}}^2 + P''(J_{\mathrm{cl}})\ddot{J}_{\mathrm{cl}}\big) = 4|g|^4\big(2P'''(J_{\mathrm{cl}})P(J_{\mathrm{cl}}) + P''(J_{\mathrm{cl}})P'(J_{\mathrm{cl}})\big) \tag{39}$$

at $t = 0$ gives $\ddot{J}_{\rm cl}(0) = 2|g|^2 x^{\mathbf{r}}$ and $J_{\rm cl}^{(4)}(0) = 4|g|^4 P''(0)P'(0) = -8|g|^4 \sum_{j=1}^d r_j^2 x^{2\mathbf{r}-e_j}$.

Furthermore, writing $G_4(J) := 4|g|^4\big(2P'''(J)P(J) + P''(J)P'(J)\big)$ in (39) and differentiating twice more using (37)–(38) shows that

$$J_{\rm cl}^{(6)}(s) = 4|g|^2 G_4''(J_{\rm cl}(s))P(J_{\rm cl}(s)) + 2|g|^2 G_4'(J_{\rm cl}(s))P'(J_{\rm cl}(s))$$

is a fixed polynomial in $|g|^2$, $x_1, \dots, x_d$, and $J_{\rm cl}(s)$. Because $0 \le J_{\rm cl}(s) \le x_1$ for all $s \in \mathbb{R}$ (by (22) and $r_1 \ge 1$), replacing each monomial in $J_{\rm cl}(s)$ by the corresponding power of x_1 bounds $\sup_{s\in\mathbb{R}} |J_{\rm cl}^{(6)}(s)|/6!$ by a polynomial $Q_{\rm cl}(x;|g|)$ in x with nonnegative coefficients. Since $J_{\rm cl}$ is even, Taylor's theorem gives

$$J_{\rm cl}(t) = |g|^2 x^{\mathbf{r}} t^2 - \frac{|g|^4}{3}\sum_{j=1}^d r_j^2 x^{2\mathbf{r}-e_j} t^4 + R_{\rm cl}(t), \qquad |R_{\rm cl}(t)| \le Q_{\rm cl}(x;|g|)\, t^6 \tag{40}$$

for all $t \in \mathbb{R}$.

Step 4: Subtraction and strict positivity. For $|\mathbf{k}| = 1$, we have $\mathbf{k} = e_j$ for some $1 \le j \le d$, so $C_{\mathbf{r}}(e_j) = \binom{r_j}{1}^2 1! = r_j^2$ and $\hbar^{|\mathbf{k}|-1} = \hbar^0 = 1$. Thus the fourth-order terms in (40) coincide *precisely* with the $|\mathbf{k}| = 1$ terms in (36). Subtracting (40) from (36) proves (30)–(32).

Finally, every coefficient $C_{\mathbf{r}}(\mathbf{k})\hbar^{|\mathbf{k}|-1}$ in (31) is strictly positive. If $R = \sum_{j=1}^d r_j \ge 2$, the index set $\{0 \le \mathbf{k} \le \mathbf{r} : |\mathbf{k}| \ge 2\}$ is nonempty (for instance, it contains $\mathbf{k} = \mathbf{r}$). Hence when all $x_j > 0$ and $g \ne 0$, we have $D_{\mathbf{r}}(x;\hbar) > 0$, so $J_\hbar(t) < J_{\rm cl}(t)$ for all sufficiently small nonzero t. □

Remark 2 (The linear threshold $R = 1$). When $d = 1$ and $r_1 = 1$, the interaction $H = g a_0^\dagger a_1 + \overline{g} a_0 a_1^\dagger$ describes linear frequency translation (a beam splitter). The sum defining $D_{(1)}(x;\hbar)$ is empty ($D_{(1)} \equiv 0$), and in fact $a_0(t) = \cos(|g|t)\, a_0 - i\frac{g}{|g|}\sin(|g|t)\, a_1$, giving $J_\hbar(t) = x_1 \sin^2(|g|t) = J_{\rm cl}(t)$ identically for all $t \in \mathbb{R}$. The quartic deficit is thus a strictly *nonlinear* ($R \ge 2$) quantum phenomenon.

6 Phenomenology of the quartic deficit

Theorem 1 provides an explicit algebraic formula, but its physical meaning becomes transparent when examined from two complementary perspectives: operator dynamics in the Heisenberg picture and discrete photon counting in Fock space.

Spontaneous multi-photon back-conversion in the Heisenberg picture

Let $A_{\mathbf{r}} = \prod_{j=1}^d a_j^{r_j}$ be the pump annihilation monomial from (2), so $X_{\mathbf{r}} = a_0^\dagger A_{\mathbf{r}}$. Computing the first two Heisenberg time derivatives of the output intensity $a_0^\dagger a_0$ directly from (1) gives

$$\frac{i}{\hbar}[H, a_0^\dagger a_0] = -i\big(g X_{\mathbf{r}} - \overline{g} X_{\mathbf{r}}^\dagger\big),$$

and hence the exact operator acceleration law

$$\left(\frac{i}{\hbar}\right)^2 [H,[H,a_0^\dagger a_0]] = \frac{2|g|^2}{\hbar}[X_{\mathbf{r}}^\dagger, X_{\mathbf{r}}] = 2|g|^2 A_{\mathbf{r}}^\dagger A_{\mathbf{r}} - \frac{2|g|^2}{\hbar}\,(a_0^\dagger a_0)\,[A_{\mathbf{r}}, A_{\mathbf{r}}^\dagger], \tag{41}$$

where we used $X_{\mathbf{r}}^\dagger X_{\mathbf{r}} = (a_0^\dagger a_0 + \hbar) A_{\mathbf{r}}^\dagger A_{\mathbf{r}}$ and $X_{\mathbf{r}} X_{\mathbf{r}}^\dagger = (a_0^\dagger a_0) A_{\mathbf{r}} A_{\mathbf{r}}^\dagger$.

Every qualitative feature of Theorem 1 is visible in (41):

1. **Exact $O(t^2)$ agreement.** The upward driving term $2|g|^2 A_{\mathbf{r}}^\dagger A_{\mathbf{r}}$ is already normal-ordered, so its coherent-state expectation equals $2|g|^2 x^{\mathbf{r}}$ with zero commutator correction. Because the output starts empty ($a_0|\Psi_\alpha\rangle = 0$), the second term vanishes at $t = 0$. Thus initial up-conversion proceeds at the exact classical rate: $\ddot{J}_\hbar(0) = \ddot{J}_{\rm cl}(0) = 2|g|^2 x^{\mathbf{r}}$.

2. **Enhanced restoring force at $O(t^4)$.** Only *after* output photons have been generated ($a_0^\dagger a_0 \sim t^2$) does the back-conversion restoring term $-\frac{2|g|^2}{\hbar}(a_0^\dagger a_0)[A_{\mathbf{r}}, A_{\mathbf{r}}^\dagger]$ activate. By Proposition 3, the Wick symbol of the restoring coefficient $\frac{1}{\hbar}[A_{\mathbf{r}}, A_{\mathbf{r}}^\dagger]$ is

$$\sigma_\hbar\left(\frac{1}{\hbar}[A_{\mathbf{r}}, A_{\mathbf{r}}^\dagger]\right)(z,\bar z) = \frac{1}{\hbar}\big(z^{\mathbf{r}} \star_\hbar \bar z^{\mathbf{r}} - |z|^{2\mathbf{r}}\big) = \underbrace{\sum_{j=1}^{d} r_j^2 |z|^{2(\mathbf{r}-e_j)}}_{\substack{i\{z^{\mathbf{r}},\bar z^{\mathbf{r}}\} \\ \text{(classical depletion)}}} + \underbrace{\sum_{\substack{0\le \mathbf{k}\le \mathbf{r} \\ |\mathbf{k}|\ge 2}} C_{\mathbf{r}}(\mathbf{k})\,\hbar^{|\mathbf{k}|-1}\,|z|^{2(\mathbf{r}-\mathbf{k})}}_{\substack{\widetilde D_{\mathbf{r}}(|z|^2;\hbar) \\ \text{(spontaneous back-conversion)}}}, \tag{42}$$

where $D_{\mathbf{r}}(x;\hbar) = x^{\mathbf{r}}\,\widetilde D_{\mathbf{r}}(x;\hbar)$.

The index $|\mathbf{k}|$ counts how many pump creation and annihilation operators contract against the vacuum when normal-ordering $A_{\mathbf{r}}A_{\mathbf{r}}^\dagger$. Physically, when an output photon reconverts into R pump photons, each factor $a_j a_j^\dagger = a_j^\dagger a_j + \hbar$ splits into a stimulated part ($a_j^\dagger a_j$) and a spontaneous vacuum part ($\hbar$).

- The **single-contraction terms** ($|\mathbf{k}| = 1$) carry a single power of $\hbar$, which cancels the $1/\hbar$ prefactor in Heisenberg's equation; they survive the $\hbar \to 0$ limit as the classical Poisson bracket $i\{z^{\mathbf{r}}, \bar z^{\mathbf{r}}\}$, reproducing classical stimulated back-conversion and pump depletion in (38).
- The **multi-contraction terms** ($|\mathbf{k}| \ge 2$) carry positive powers $\hbar^{|\mathbf{k}|-1}$ and represent **simultaneous spontaneous (vacuum-seeded) down-conversion** of the newly created output photon back into the pump modes–ranging from partially spontaneous processes ($|\mathbf{k}| = 2$) up to fully spontaneous R-photon parametric down-conversion into the vacuum ($|\mathbf{k}| = R$, contributing $\mathbf{r}!\,\hbar^{R-1}$).

Because every contraction coefficient in (42) is strictly positive, vacuum fluctuations in the pump modes *always* stiffen the restoring force pulling energy back out of mode 0. When $g \neq 0$ and all $x_j > 0$, dividing (30) by the leading conversion signal $|g|^2 x^{\mathbf{r}} t^2$ expresses the fractional quantum deficit directly in terms of $\widetilde D_{\mathbf{r}}$:

$$\frac{J_\hbar(t) - J_{\mathrm{cl}}(t)}{|g|^2 x^{\mathbf{r}} t^2} = -\frac{|g|^2}{3}\,\widetilde D_{\mathbf{r}}(x;\hbar)\,t^2 + O(t^4) = -\frac{|g|^2}{3}\left[\sigma_\hbar\left(\frac{1}{\hbar}[A_{\mathbf{r}}, A_{\mathbf{r}}^\dagger]\right) - i\{z^{\mathbf{r}}, \bar z^{\mathbf{r}}\}\right]_{|z_j|^2 = x_j} t^2 + O(t^4). \tag{43}$$

Discrete photon depletion and variance in Fock space

In the Fock-sector proof of Theorem 1, the same deficit appears in a complementary guise. Within a fixed charge sector $\mathcal{H}_{\mathbf{p}}$, the t^4 coefficient (33) is proportional to $F_{\mathbf{p}}(B_{\mathbf{p}} - F_{\mathbf{p}}) \le 0$, reflecting the competition between two second-step transitions out of the one-photon state $|1;\mathbf{p}\rangle$:

- Stepping *upward* to $|2;\mathbf{p}\rangle$ must remove a second batch of $\mathbf{r}$ photons from an *already-depleted* pump containing $p_j - r_j$ photons, giving rate factor $B_{\mathbf{p}} = \prod_j (p_j - r_j)^{\underline{r_j}}$ (which vanishes outright in two-level sectors $L_{\mathbf{p}} = 1$, where discrete photon exhaustion blocks $|2;\mathbf{p}\rangle$ completely).
- Stepping *downward* back to $|0;\mathbf{p}\rangle$ creates $\mathbf{r}$ photons into the pump and enjoys bosonic enhancement back up to p_j, giving the larger rate factor $F_{\mathbf{p}} = \prod_j p_j^{\underline{r_j}} \ge B_{\mathbf{p}}$.

Averaging over the Poisson photon statistics of coherent pumps turns $\mathbb{E}[F_{\mathbf{P}}(B_{\mathbf{P}} - F_{\mathbf{P}})]$ into $-\operatorname{Var}(F_{\mathbf{P}})$ in (35). The $|\mathbf{k}| = 1$ part of $\operatorname{Var}(F_{\mathbf{P}})$ matches the linearized (delta-method) Poisson variance $\sum_j (\partial_{\mu_j}\mu^{\mathbf{r}})^2 \operatorname{Var}(P_j) = \sum_j r_j^2 \mu^{2\mathbf{r}-e_j}$, which is classical pump depletion; the $|\mathbf{k}| \ge 2$ remainder $D_{\mathbf{r}}(x;\hbar)$ is precisely the non-Gaussian, multiphoton excess of $\operatorname{Var}(F_{\mathbf{P}})$.

Three optical examples and scaling regimes

Write $\bar{n}_j = x_j/\hbar$ for the mean initial photon number in pump mode j.

1. Sum-frequency generation, $\mathbf{r} = (1,1)$. Writing $x = |\alpha_1|^2$ and $y = |\alpha_2|^2$, Theorem 1 gives

$$J_\hbar(t) = |g|^2 xy\, t^2 - \frac{|g|^4}{3} xy(x+y+\hbar)\, t^4 + O(t^6), \qquad J_{\mathrm{cl}}(t) = |g|^2 xy\, t^2 - \frac{|g|^4}{3} xy(x+y)\, t^4 + O(t^6),$$

and therefore

$$\boxed{J_\hbar(t) - J_{\mathrm{cl}}(t) = -\frac{\hbar|g|^4}{3}\, xy\, t^4 + O(t^6).} \tag{44}$$

Here $\sigma_\hbar(\hbar^{-1}[a_1 a_2, a_1^\dagger a_2^\dagger]) = x + y + \hbar$: the $+\hbar$ term is spontaneous parametric down-conversion (SPDC) of an output photon into a signal–idler pair. The ratio of the quantum deficit to the classical t^4 pump-depletion term is

$$\frac{\text{quantum deficit}}{\text{classical depletion}} = \frac{\hbar}{x+y} = \frac{1}{\bar{n}_1 + \bar{n}_2}.$$

Three regimes stand out immediately:

- *Macroscopic pumps ($\bar{n}_1 + \bar{n}_2 \gg 1$):* the deficit is suppressed by $O\big((\bar{n}_1 + \bar{n}_2)^{-1}\big)$ relative to classical depletion, quantifying convergence to the coupled-mode equations.
- *Undepleted-pump limit ($\bar{n}_1 \gg 1$, $\bar{n}_2 \sim 1$):* even when mode 2 is at the single-photon level, the relative deficit remains small ($\approx 1/\bar{n}_1$), because fixing the bright pump mode 1 reduces H to an effective linear beam splitter ($R_{\mathrm{eff}} = 1$) between modes 2 and 0. Both pump modes must be weak for SPDC back-conversion to rival classical depletion.
- *Few-photon regime ($\bar{n}_1, \bar{n}_2 \lesssim 1$):* the quantum deficit matches or exceeds classical pump depletion (at $\bar{n}_1 = \bar{n}_2 = \frac{1}{2}$, it doubles the t^4 turnover), making the effect directly relevant to trapped-ion simulations of the trilinear Hamiltonian [6], cavity or circuit QED converters, and mesoscopic nanophotonic platforms operating beyond the undepleted-pump approximation [7, 8].

2. Second-harmonic generation, $\mathbf{r} = (2)$. With $H = g a_0^\dagger a_1^2 + \overline{g} a_0 (a_1^\dagger)^2$ and $x = |\alpha_1|^2$ (cf. short-time perturbative and elementary-process treatments in [5, 9]), Theorem 1 gives

$$J_\hbar(t) = |g|^2 x^2 t^2 - \frac{|g|^4}{3}(4x^3 + 2\hbar x^2)\, t^4 + O(t^6),$$

so that

$$\boxed{J_\hbar(t) - J_{\mathrm{cl}}(t) = -\frac{2\hbar|g|^4}{3}\, x^2 t^4 + O(t^6).} \tag{45}$$

Comparing (45) with (44) at $x = y$ reveals an extra factor of 2: SHG is *not* obtained merely by identifying the two input amplitudes in the three-mode SFG formula. The factor of $2 = 2!$ is a bosonic indistinguishability enhancement in $\hbar^{-1}[a_1^2, (a_1^\dagger)^2] = 4a_1^\dagger a_1 + 2\hbar$: spontaneous down-conversion of a second-harmonic photon creates two identical photons into the *same* fundamental mode. Relative to classical pump depletion, the quantum deficit has ratio $2\hbar x^2/(4x^3) = 1/(2\bar{n}_1)$. Note also that for both SFG and SHG, the fractional deficit (43) is completely independent of the pump intensities: $-\frac{\hbar|g|^2}{3}t^2$ for SFG and $-\frac{2\hbar|g|^2}{3}t^2$ for SHG.

3. Third-harmonic generation, $\mathbf{r} = (3)$. For $R \geq 3$, the deficit polynomial $D_{\mathbf{r}}(x;\hbar)$ exhibits multiple orders in $\hbar$. For $\mathbf{r} = (3)$,

$$D_{(3)}(x;\hbar) = 18\hbar x^4 + 6\hbar^2 x^3, \qquad J_\hbar(t) - J_{\mathrm{cl}}(t) = -|g|^4\big(6\hbar x^4 + 2\hbar^2 x^3\big)\, t^4 + O(t^6).$$

Here $18\hbar x^4$ arises from partially spontaneous back-conversion ($|\mathbf{k}| = 2$: two spontaneous and one stimulated fundamental photon, with $C_{(3)}(2) = \binom{3}{2}^2 2! = 18$), while $6\hbar^2 x^3$ arises from purely spontaneous three-photon down-conversion into the vacuum ($|\mathbf{k}| = 3$, with $C_{(3)}(3) = 3! = 6$). For bright pumps ($\bar{n}_1 \gg 1$) the $O(\hbar)$ partially spontaneous channel dominates; for $\bar{n}_1 < 1/3$ the $O(\hbar^2)$ purely spontaneous channel dominates.

Robustness to classical pump fluctuations and thermal light

The global-in-time polynomial remainder bound (32) in Theorem 1 implies that the quartic deficit cannot be washed out by classical pump noise, amplitude fluctuations, or inter-mode intensity correlations–even for unbounded classical distributions such as thermal light.

Corollary 1 (Classical mixtures of coherent inputs)**.** *Let $\rho_{\mathrm{in}} = \int_{\mathbb{C}^d} |\Psi_\alpha\rangle\langle\Psi_\alpha|\, d\nu(\alpha)$ be a classical probability mixture of coherent pump states (with the output mode in vacuum), where ν is a Borel probability measure on $\mathbb{C}^d$ whose intensity random vector $X = (|\alpha_1|^2, \dots, |\alpha_d|^2)$ satisfies $\mathbb{E}[X_j^{3R+4}] < \infty$ for every $1 \le j \le d$. Let $J_\hbar(t) := \mathrm{Tr}\big(U(t)\rho_{\mathrm{in}}U(t)^* a_0^\dagger a_0\big)$ and let $\mathbb{E}[J_{\mathrm{cl}}(t)] := \int_{\mathbb{C}^d} J_{\mathrm{cl}}(t;\alpha)\, d\nu(\alpha)$ be the classical ensemble average with the* same *initial amplitude distribution ν. Then for fixed $\hbar > 0$,*

$$J_\hbar(t) - \mathbb{E}[J_{\mathrm{cl}}(t)] = -\frac{|g|^4}{3}\,\mathbb{E}\big[D_{\mathbf{r}}(X;\hbar)\big]\, t^4 + O(t^6) \qquad (t \to 0). \tag{46}$$

If $R \ge 2$, $g \ne 0$, and $\nu\big(\{\alpha \in \mathbb{C}^d : \alpha_1 \cdots \alpha_d \ne 0\}\big) > 0$, then $\mathbb{E}[D_{\mathbf{r}}(X;\hbar)] > 0$ and $J_\hbar(t) < \mathbb{E}[J_{\mathrm{cl}}(t)]$ for all sufficiently small nonzero t.

Proof. By Tonelli's theorem and the bound $0 \le J_{\hbar,\mathbf{p}}(t) \le \hbar p_1$, the trace $J_\hbar(t) = \int_{\mathbb{C}^d} J_\hbar(t;\alpha)\, d\nu(\alpha)$ is finite for every $t \in \mathbb{R}$ (bounded by $\mathbb{E}[X_1] < \infty$). From (34) and Step 3 of the proof of Theorem 1, the polynomials $Q_{\mathrm{q}}(x;\hbar,|g|)$ and $Q_{\mathrm{cl}}(x;|g|)$ in (32) have total degree in x at most $3R+4$ (specifically $3R+4$ for Q_{q} and $3R-2$ for Q_{cl}), while $D_{\mathbf{r}}(x;\hbar)$ has degree at most $2R-2$. By Hölder's inequality (or the elementary bound $X^{\boldsymbol{\ell}} \le 1 + \sum_{j=1}^d X_j^{|\boldsymbol{\ell}|}$), the hypothesis $\mathbb{E}[X_j^{3R+4}] < \infty$ ensures that every monomial of total degree at most $3R+4$ in X is ν-integrable. Integrating (32) against $d\nu(\alpha)$ therefore yields

$$\left|\big(J_\hbar(t) - \mathbb{E}[J_{\mathrm{cl}}(t)]\big) + \frac{|g|^4}{3}\,\mathbb{E}\big[D_{\mathbf{r}}(X;\hbar)\big]\, t^4\right| \le \mathbb{E}\big[Q_{\mathrm{q}}(X;\hbar,|g|) + Q_{\mathrm{cl}}(X;|g|)\big]\, t^6$$

for all $t \in \mathbb{R}$, proving (46). Under the stated positivity conditions, $D_{\mathbf{r}}(X;\hbar) > 0$ on a set of positive ν-measure and $D_{\mathbf{r}}(X;\hbar) \ge 0$ everywhere, so $\mathbb{E}[D_{\mathbf{r}}(X;\hbar)] > 0$. □

As a concrete illustration, consider thermal pump modes [2], whose Glauber–Sudarshan P-representation is a Gaussian mixture of coherent states with independent exponentially distributed intensities $X_j \sim \mathrm{Exp}(1/\bar{x}_j)$ (satisfying $\mathbb{E}[X_j^m] = m!\,\bar{x}_j^m$):

- For **SFG** with independent thermal pumps of mean intensities $\bar{x}, \bar{y}$, $\mathbb{E}[XY] = \bar{x}\bar{y}$, so the quartic deficit $-\frac{\hbar|g|^4}{3}\bar{x}\bar{y}\,t^4$ is *identical* to that of coherent pumps with the same mean intensities.
- For **SHG** with a thermal pump of mean intensity $\bar{x}$, photon bunching gives $\mathbb{E}[X^2] = 2\bar{x}^2$, so the quartic deficit doubles to $-\frac{4\hbar|g|^4}{3}\bar{x}^2 t^4$–matching the same factor of 2 that doubles the leading t^2 conversion rate, leaving the fractional deficit $-\frac{2\hbar|g|^2}{3}t^2$ invariant across all classical pump statistics.

Short-time universality versus late-time dephasing

Theorem 1 and Corollary 1 are strictly local-in-time statements, not global ordering inequalities over later conversion cycles. At longer times, the quantum and classical dynamics diverge qualitatively: in nondegenerate SFG with $x \neq y$, for instance, the classical trajectory $J_{\mathrm{cl}}(t)$ executes periodic Jacobi-elliptic oscillations that return periodically to $J_{\mathrm{cl}} = 0$, whereas the quantum expectation $J_\hbar(t) = \sum_{\mathbf{p}} w_{\mathbf{p}} J_{\hbar,\mathbf{p}}(t)$ averages finite-matrix oscillations across charge sectors with incommensurate eigenfrequencies. Because these sector oscillations dephase (exhibiting collapse and partial revival), $J_\hbar(t)$ settles near a strictly positive plateau and therefore *exceeds* $J_{\mathrm{cl}}(t)$ whenever the classical cycle swings back near zero. The universal deficit is thus a signature of the initial conversion window before multi-sector dephasing sets in.

7 Structural boundaries of the resonant algebraic model

The construction developed here keeps three distinct layers cleanly separated:

1. **Kinematics:** the polynomial CCR $*$-algebra $\mathcal{A}_\hbar$ and its vacuum GNS representation on Fock space $\mathcal{F}$.

2. **Dynamics:** a symmetric interaction Hamiltonian $H \in \mathcal{A}_{\hbar,0}$ of grade zero under the Manley–Rowe charge grading, whose finite-dimensional charge sectors guarantee essential self-adjointness.

3. **Classical approximation:** the Berezin–Wick star-product calculus and its pointwise $\hbar \to 0$ limit, rather than treating Wick symbols as multiplicative homomorphisms.

At the same time, it is important to record what this finite-mode algebraic framework does *not* derive on its own. Spatial mode profiles and nonlinear susceptibilities enter exclusively through the coupling constant g–schematically, for a $\chi^{(2)}$ medium,

$$g \propto \int \chi^{(2)}(r)\, u_1(r) u_2(r) \overline{u_0(r)}\, dr,$$

rather than through the multiplication table of an abstract mode algebra. Frequency resonance, spatial phase matching, group-velocity dispersion, propagation (eikonal or paraxial equations), and optical losses all require additional physical modeling beyond the Manley–Rowe grading.

What the framework *does* provide is a rigorous, transparent foundation in which finite-dimensional charge sectors make nonlinear bosonic dynamics elementary, the classical coupled-mode equations emerge cleanly as the $\hbar \to 0$ limit of Wick-symbol calculus, and a universal, parameter-free short-time quantum deficit separates every nonlinear frequency-conversion process from its classical counterpart.

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