Completeness of the Bethe Ansatz through Intersection Numbers in $\mathbb{P}^N$

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Abstract

We develop an approach based on intersection numbers to study the completeness of the Bethe ansatz for periodic spin-1/2 XXX chains. Completeness of the Bethe ansatz predicts that the number of solutions to the Bethe equations—after removing collisions, identifying permutations, and appropriately regularizing singular roots—equals $\binom{L}{N} - \binom{L}{N-1}$, where L is the length of the chain and N is the number of flipped spins. We propose a counting strategy that starts from the original Bethe equations and introduces a family of weighted polynomial systems as corrections. For each system, every root of the Bethe equations is assigned a local index given by its intersection multiplicity. Our aim is for the weighted sum of these local indices to equal $1/N!$ at physical roots and to vanish otherwise; globally, the weighted sum of projective intersection numbers then yields the desired completeness count. We establish several parts of this program conditionally: the vanishing of weights at roots with collisions among nonsingular coordinates follows from a distinct-root assumption for the Q-polynomial in Baxter's TQ relation, and configurations with unequal numbers of roots at $\mathrm{i}/2$ and $-\mathrm{i}/2$ are ruled out by comparing vanishing orders in the TQ relation. For configurations containing balanced singular pairs, we use an inhomogeneity deformation to interpret the local weight geometrically as a normalized count of nearby exceptional irreducible components with distinct coordinates and no site-pinned singular pairs. It remains an open question why such exceptional components exist only when a root contains at most one copy of each singular value $\mathrm{i}/2$ and $-\mathrm{i}/2$. Explicit local calculations for $N \leq 4$ and an extension of the projective counting construction to higher-spin XXX_S chains are provided in the appendices.

1 Background: the XXX Bethe ansatz

The Bethe ansatz originated in Bethe's 1931 study of the one-dimensional Heisenberg chain [4], where he constructed eigenfunctions using an explicit ansatz for their coordinate dependence. The method and its extensions have since become central tools in the theory of quantum integrable systems, including the XXX, XXZ, and XYZ spin chains and the related six- and eight-vertex models of statistical mechanics. Standard accounts of this broader setting include Baxter's monograph [2] and the treatment of the quantum inverse scattering method by Korepin, Bogoliubov, and Izergin [14].

Consider the periodic spin-1/2 XXX chain of length L, with state space $\mathcal{H} = (\mathbb{C}^2)^{\otimes L}$. Up to an additive constant and overall scale, its Hamiltonian is

$$H = \frac{1}{4}\sum_{\ell=1}^{L}(\boldsymbol{\sigma}_\ell \cdot \boldsymbol{\sigma}_{\ell+1} - 1), \qquad \boldsymbol{\sigma}_{L+1} = \boldsymbol{\sigma}_1.$$

The sector with N down (flipped) spins has dimension $\binom{L}{N}$. The state space carries a natural $\mathfrak{sl}_2$ action that commutes with H, and the highest-weight subspace in the sector with N down spins

has dimension

$$d(L,N)=\binom{L}{N}-\binom{L}{N-1}. \tag{1}$$

The algebraic Bethe ansatz constructs candidate eigenvectors

$$B(\lambda_1)\cdots B(\lambda_N)|0\rangle$$

from rapidities $\lambda_1,\dots,\lambda_N$ satisfying the Bethe equations

$$\left(\frac{\lambda_k+\mathrm{i}/2}{\lambda_k-\mathrm{i}/2}\right)^L=\prod_{j\neq k}\frac{\lambda_k-\lambda_j+\mathrm{i}}{\lambda_k-\lambda_j-\mathrm{i}},\qquad k=1,\dots,N. \tag{2}$$

Throughout this paper we assume $L\geq 2N$.

It is convenient to rescale the parameters by setting

$$\lambda_j=\frac{\mathrm{i}}{2}x_j,\qquad x_j=\frac{2\lambda_j}{\mathrm{i}},$$

so that the singular rapidities $\lambda_j=\pm\mathrm{i}/2$ correspond to $x_j=\mp 1$. Define the *Bethe polynomial*

$$P(x_1,\dots,x_N)=(x_1+1)^L\prod_{j\neq 1}(x_1-x_j-2)-(x_1-1)^L\prod_{j\neq 1}(x_1-x_j+2), \tag{3}$$

which is symmetric in its last $N-1$ arguments, and set

$$P_i(x)=P(x_i,x_1,\dots,\widehat{x_i},\dots,x_N).$$

The polynomial form of the periodic Bethe equations is then $P_i(x)=0$ for $i=1,\dots,N$.

Physical roots are required to have pairwise distinct coordinates; repeated coordinates are called *collisions*. Since the Bethe creation operators $B(\lambda_j)$ commute, we identify root configurations that differ by a permutation of coordinates. Roots containing both 1 and -1, which up to permutation may be labeled with $x_1=1$ and $x_2=-1$, are called *singular*. When the remaining coordinates $x_3,\dots,x_N$ are distinct and different from ± 1, physical singular roots must satisfy the additional regularization condition

$$\left[-\prod_{j=3}^{N}\frac{x_j+1}{x_j-1}\right]^L=1; \tag{4}$$

see [17, 18] for a detailed discussion of regularization procedures.

By completeness of the Bethe ansatz, we mean that the Bethe vectors obtained from physical roots, with singular roots appropriately regularized, span the highest-weight subspace of dimension $d(L,N)$. Early spectral results include the infinite-chain Plancherel theorem of Babbitt and Thomas [1] and the two-down-spin analysis of Koma and Ezawa [13]. Baxter [3] examined exceptional roots and completeness for the six- and eight-vertex models, while algebro-geometric approaches were pursued by Langlands and Saint-Aubin [15] and through the quotient-ring counting methods of Jiang and Zhang [10].

For the homogeneous spin-1/2 XXX chain, Mukhin, Tarasov, and Varchenko [16, Section 12] proved completeness in the polynomial-kernel (Wronskian) formulation. The precise relation to physical solutions of the Bethe equations was further clarified by the regularization criteria of Nepomechie and Wang [17, 18], homotopy-continuation root counts by Hao, Nepomechie, and Sommese [7], and the characterization by Granet and Jacobsen of physical distinct-root configurations through polynomial companions and two zero-remainder conditions [6].

Closely related in spirit are the combinatorial completeness counts developed by Kirillov [11] and Kirillov and Liskova [12]. For the spin-1/2 XXX chain, the corresponding combinatorial identity takes the form

$$\sum_{\substack{m_1,m_2,\ldots\geq 0\\ \sum_{j\geq 1} j\, m_j = N}} \prod_{j\geq 1} \binom{v_j + m_j}{m_j} = d(L,N), \qquad v_j = L - 2\sum_{k\geq 1} \min(j,k) m_k.$$

Here m_j records the number of strings of length j, v_j is the associated vacancy number, and the binomial factors count admissible integer quantum numbers attached to the strings. The string hypothesis models Bethe roots as clusters with approximately equal real parts and equally spaced imaginary parts. Interpreting these combinatorial labels as a complete parametrization of physical roots requires caution, however: finite-chain roots can deviate substantially from ideal strings, so the combinatorial identity alone does not establish completeness. In particular, the generalized XXZ analysis of Kirillov and Liskova relies on the string hypothesis [12].

This note seeks an intersection-theoretic explanation of the completeness count $d(L,N)$. Rather than organizing Bethe roots into strings, we start directly from the polynomial system $P_1 = \cdots = P_N = 0$ and remove collision diagonals via inclusion–exclusion (Möbius inversion on the partition lattice). We then subtract the excess multiplicity arising along the locus of a single singular pair $(1,-1)$. Because this subtraction overcorrects configurations containing multiple singular pairs, we introduce higher-order corrections supported on loci with r fixed singular pairs for $0 \leq r \leq \lfloor N/2 \rfloor$. Taking projective intersection numbers leads to the alternating binomial identity

$$\boxed{\sum_{r=0}^{\lfloor N/2\rfloor} (-1)^r \binom{L}{r} \binom{L+N-2r-2}{N-2r} = d(L,N).} \tag{5}$$

This construction expresses the target dimension $d(L,N)$ as a weighted sum of global intersection numbers in $\mathbb{P}^N$ and reduces completeness in the root picture to local multiplicity identities and boundary vanishing. We emphasize that the conjectural local and boundary identifications formulated below concern this direct intersection-theoretic counting scheme; completeness of the XXX Bethe algebra in the polynomial formulation is already proved in [16].

2 The counting strategy

Throughout, N denotes the total number of Bethe roots. For an auxiliary system with fixed singular roots, M denotes the number of free coordinates and m the number of free partition blocks. In the local component analysis of Section 5, k denotes the number of copies of each singular value ± 1, r the number of prescribed singular pairs, and s the total number of site-pinned pairs.

Assume that the common zero locus of $P_1 = \cdots = P_N = 0$ in $\mathbb{C}^N$ is finite, and write

$$X = \{x^1, \ldots, x^{|X|}\} \subseteq \mathbb{C}^N$$

for this set of ordered roots. Assign to each $x^i \in X$ the target weight

$$w_i = \begin{cases} 1/N!, & x^i \text{ is a physical root,} \\ 0, & \text{otherwise.} \end{cases}$$

Since every physical root has pairwise distinct coordinates, its S_N-orbit consists of $N!$ ordered tuples in X. The desired completeness statement is therefore

$$\sum_{i=1}^{|X|} w_i = d(L,N). \tag{6}$$

We work with formal $\mathbb{Q}$-linear combinations of points (zero-cycles with rational coefficients):

$$Z = \sum_i w_i(Z)[x^i], \qquad w_i(Z) \in \mathbb{Q},$$

where $[x^i]$ denotes the formal generator associated with x^i. We write $\deg Z = \sum_i w_i(Z)$ for the total affine weight of Z. For any map ϕ of point sets, the pushforward is defined linearly by

$$\phi_*\left(\sum_p w_p[p]\right) = \sum_p w_p[\phi(p)],$$

summing weights whenever several points share the same image. In particular, pushing forward under the quotient map from ordered tuples to unordered configurations sums the weights across each permutation orbit. We write $\mathrm{Sym}^M(\mathbb{A}^1)$ for the space of unordered configurations of M complex numbers, allowing repetitions.

For a zero-dimensional ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_N]$, we define its global colength (total number of roots counted with multiplicity) and its local multiplicity at an isolated point $p \in \mathbb{C}^N$, respectively, by

$$\ell(I) = \dim_{\mathbb{C}}(\mathbb{C}[x_1, \ldots, x_N]/I),$$
$$\ell_p(I) = \dim_{\mathbb{C}}(\mathbb{C}[x_1, \ldots, x_N]/I)_{\mathfrak{m}_p},$$

where $\mathfrak{m}_p$ is the maximal ideal at p, with $\ell_p(I) = 0$ if $p \notin V(I)$. The associated formal cycle is

$$Z(I) = \sum_{p \in V(I)} \ell_p(I)[p], \qquad \deg Z(I) = \ell(I).$$

Our strategy is to construct zero-dimensional complete-intersection ideals $I_j \subseteq \mathbb{C}[x_1, \ldots, x_N]$, each generated by N polynomials with $V(I_j) \subseteq X$, together with rational weights $w_j^{\mathrm{aux}} \in \mathbb{Q}$. Setting $Z_j = Z(I_j)$ and writing

$$Z_j = \sum_{i=1}^{|X|} l_{ij}[x^i], \qquad l_{ij} = \ell_{x^i}(I_j),$$

we require that the weighted formal sum

$$Z_{\mathrm{Bethe}} = \sum_j w_j^{\mathrm{aux}} Z_j \tag{7}$$

coincide with the target cycle $\sum_i w_i[x^i]$, or equivalently that the pointwise identity

$$\sum_j w_j^{\mathrm{aux}} l_{ij} = w_i \qquad \text{for every } i = 1, \ldots, |X| \tag{8}$$

hold. Taking global degrees then yields

$$\sum_i w_i = \sum_j w_j^{\mathrm{aux}} \ell(I_j), \qquad \ell(I_j) = \deg Z_j = \sum_i l_{ij}. \tag{9}$$

The explicit polynomial generators of I_j allow us to compute their projective intersection numbers via Bézout's theorem after compactifying $\mathbb{A}^N$ in $\mathbb{P}^N$ (Section 2.3).

The construction proceeds in two stages:

1. **Depermutation:** To eliminate collision diagonals and account for coordinate permutations, we apply Möbius inversion on the partition lattice.

2. **Singular-locus correction:** We construct auxiliary cycles supported on loci with fixed singular pairs $(1, -1)$ and take an alternating sum to cancel excess multiplicities at singular roots, aiming to achieve the pointwise weight identity (8) and consequently the global count (6).

Concretely, we build the auxiliary ideals from the Bethe polynomials P_i as follows. Let $A, B \subseteq [N]$ be disjoint index sets, with either $A = B = \varnothing$ or both A and B nonempty. Let $\pi = \{S_1, \ldots, S_m\}$ be a set partition of the remaining indices $R = [N] \setminus (A \cup B)$, and choose a representative $r_\alpha \in S_\alpha$ for each block. For $t \in R$, define the reduced Bethe polynomial by

$$P_t^{\mathrm{red}}(x_R) := \frac{P_t(x)|_{x_a=1\ (a\in A),\, x_b=-1\ (b\in B)}}{(x_t+1)^{|A|}(x_t-1)^{|B|}}. \tag{10}$$

Because $L \geq 2N$, both terms in the numerator of P_t are divisible by $(x_t+1)^{|A|}(x_t-1)^{|B|}$ upon specializing $x_a = 1$ $(a \in A)$ and $x_b = -1$ $(b \in B)$, so P_t^{red} is a polynomial in the free variables x_R. When $A = B = \varnothing$, we simply set $P_t^{\mathrm{red}} = P_t$. Suppressing the dependence of P_t^{red} on A and B in the notation, we define the auxiliary ideal

$$\begin{aligned} I_{A,B;\pi} = \big(&x_a - 1 \ (a \in A), \quad x_b + 1 \ (b \in B), \\ &P_{r_\alpha}^{\mathrm{red}} \ (1 \leq \alpha \leq m), \\ &x_t - x_{r_\alpha} \quad (1 \leq \alpha \leq m,\ t \in S_\alpha \setminus \{r_\alpha\})\big). \end{aligned} \tag{11}$$

The zero locus $V(I_{A,B;\pi})$ is contained in the subset of X where the coordinates in A are pinned to 1, those in B are pinned to -1, and coordinates within each block of π coincide. Moreover, the ideal $I_{A,B;\pi}$ is independent of the choice of block representatives r_α. Assuming $I_{A,B;\pi}$ is zero-dimensional, we set

$$Z_{A,B;\pi} = Z(I_{A,B;\pi})$$

and abbreviate $Z_{\varnothing,\varnothing;\pi}$ to Z_π. Our goal is to determine explicit rational coefficients $w^{\mathrm{aux}}_{A,B;\pi}$ such that

$$Z_{\mathrm{Bethe}} = \sum_{A,B,\pi} w^{\mathrm{aux}}_{A,B;\pi} Z_{A,B;\pi}.$$

2.1 Depermutation

The combinatorial mechanism for sieving out coordinate collisions is Möbius inversion on the partition lattice Π_N; see [19, Section 3.7]. To recall the set-theoretic prototype, let Ω be a finite set and let $S \subseteq \Omega^N$ be invariant under coordinate permutations. For each partition $\pi \in \Pi_N$ of $[N]$, let $S_\pi \subseteq S$ consist of all tuples satisfying $x_i = x_j$ whenever i and j belong to the same block of π (coordinates in distinct blocks may still coincide). Ordering Π_N by refinement, with minimal element $\hat{0}$ given by the partition into singletons, Möbius inversion shows that the number of S_N-orbits in S with pairwise distinct coordinates is

$$\frac{1}{N!} \sum_{\pi \in \Pi_N} \mu(\hat{0}, \pi)\, \#S_\pi, \qquad \mu(\hat{0}, \pi) = \prod_{B \in \pi} (-1)^{|B|-1}(|B|-1)!.$$

We lift this sieve from finite sets to formal cycles recording local intersection multiplicities. Let $F(x_1, \ldots, x_N)$ be a polynomial symmetric in its last $N-1$ arguments, and set

$$F_i(x) = F(x_i, x_1, \ldots, \widehat{x_i}, \ldots, x_N).$$

For a partition $\pi = \{S_1, \ldots, S_m\} \in \Pi_N$ with chosen representatives $r_\alpha \in S_\alpha$, define

$$I_{F;\pi} = (F_{r_\alpha},\ x_t - x_{r_\alpha} : 1 \leq \alpha \leq m,\ t \in S_\alpha \setminus \{r_\alpha\})$$

and $Z_{F;\pi} = Z(I_{F;\pi})$.

Definition 2.1 (Depermutation sum)**.** The *depermutation sum* associated with F is the formal cycle

$$Z_F^{\mathrm{dep}} = \frac{1}{N!} \sum_{\pi \in \Pi_N} \mu(\hat{0}, \pi) Z_{F;\pi}. \tag{12}$$

Returning to the Bethe polynomial $F = P$, the depermutation sum Z_P^{dep} requires further correction along the singular locus. To formulate this correction, we apply depermutation to the reduced systems with r fixed singular pairs. For $0 \le r \le \lfloor N/2 \rfloor$, let $A, B \subseteq [N]$ be disjoint subsets with $|A| = |B| = r$, and put $R = [N] \setminus (A \cup B)$ and $M = |R| = N - 2r$. Define

$$\begin{aligned} Z_{A,B}^{\mathrm{dep}} &= \frac{1}{M!} \sum_{\pi \in \Pi(R)} \mu(\hat{0}, \pi) Z_{A,B;\pi}, \\ Z_r^{\mathrm{dep}} &= \frac{(r!)^2 (N-2r)!}{N!} \sum_{\substack{A,B \subseteq [N],\ A \cap B = \varnothing \\ |A| = |B| = r}} Z_{A,B}^{\mathrm{dep}}, \end{aligned} \tag{13}$$

where $\Pi(R)$ denotes the partition lattice on the M free labels, so that depermutation acts solely on the free coordinates. In particular, $Z_0^{\mathrm{dep}} = Z_P^{\mathrm{dep}}$.

Conjecture 2.2 (Alternating completeness formula)**.** *Assume $L \ge 2N$ and that all reduced partition ideals $I_{A,B;\pi}$ are zero-dimensional. Then the alternating formal sum*

$$\boxed{\tilde{Z} = \sum_{r=0}^{\lfloor N/2 \rfloor} (-1)^r \binom{L}{r} Z_r^{\mathrm{dep}}} \tag{14}$$

equals $Z_{\mathrm{Bethe}} = \sum_{i=1}^{|X|} w_i [x^i]$.

The projective intersection number of $\tilde{Z}$ is evaluated in Section 2.3. Identifying that projective count with the affine degree $\deg Z_{\mathrm{Bethe}}$ requires the boundary-vanishing and local-multiplicity properties outlined in Section 2.4.

Remark 2.3 (Sums of symmetric functions over physical Bethe roots)**.** Many physical applications require summing symmetric functions over all physical Bethe roots rather than merely counting solutions. Because Conjecture 2.2 is formulated as an equality of formal cycles, it extends immediately to such trace formulas. Let $h \in \mathbb{C}[x_1, \dots, x_N]^{S_N}$ be a symmetric polynomial. If Conjecture 2.2 holds, then

$$\sum_{[p] \text{ physical}} h(p) = \sum_{r=0}^{\lfloor N/2 \rfloor} (-1)^r \binom{L}{r} \sum_{p \in X} w_r(p) h(p), \tag{15}$$

where $w_r(p)$ denotes the coefficient of the ordered tuple p in Z_r^{dep}, the left-hand sum runs over unordered physical root orbits $[p]$, and the right-hand sum runs over all ordered roots $p \in X$ (whose weights sum to 1 across each physical S_N-orbit). Setting $h = 1$ recovers the completeness count.

Expanding Z_r^{dep} expresses the right-hand side of (15) as a linear combination of sums over the zero-dimensional ideals $I_j = I_{A,B;\pi}$, each of which can be evaluated algebraically without solving for the individual roots via the quotient-algebra trace

$$\sum_{p \in V(I_j)} \ell_p(I_j) h(p) = \mathrm{Tr}_{A_j}(M_h), \qquad A_j = \mathbb{C}[x_1, \dots, x_N]/I_j, \qquad M_h(a) = ha.$$

The same trace formula extends to rational symmetric functions h whose denominators do not vanish on $V(I_j)$; see [10]. For small N, these quotient-algebra traces can be computed symbolically in closed form.

2.2 A generalized TQ relation and depermutation

Fix a polynomial $F(x_1, \dots, x_N)$ symmetric in its last $N-1$ arguments, and write $F_i(x) = F(x_i, x_1, \dots, \widehat{x_i}, \dots, x_N)$. Let e_j and $\hat{e}_j$ denote the jth elementary symmetric polynomials in $x_1, \dots, x_N$ and $x_2, \dots, x_N$, respectively, with $e_0 = 1$. The quotient map from ordered roots to their elementary symmetric polynomials is

$$q : \mathbb{A}^N \longrightarrow \mathrm{Sym}^N(\mathbb{A}^1) \simeq \mathbb{A}^N, \qquad x \longmapsto e(x) = (e_1(x), \dots, e_N(x)).$$

Using $e_1, \dots, e_N$ as coordinates on $\mathrm{Sym}^N(\mathbb{A}^1)$, define the universal monic polynomial

$$Q(t; e) = \sum_{j=0}^{N} (-1)^j e_j t^{N-j}.$$

We write Z^{dep} for the depermutation sum Z_F^{dep} of this system.

A generalized TQ relation rewrites the system $F_1 = \cdots = F_N = 0$ in terms of the symmetric coordinates e. Specifically, we construct a polynomial $\Phi(t; e)$ whose divisibility by $Q(t; e)$ in $\mathbb{C}[e_1, \dots, e_N][t]$ encodes the equations $F_i(x) = 0$ away from the collision locus. By the symmetry of F in $x_2, \dots, x_N$, there exists a unique polynomial g such that

$$F(x_1, \dots, x_N) = g(x_1; \hat{e}_1, \dots, \hat{e}_{N-1}).$$

Define

$$E_j(t; e) = \sum_{k=0}^{j} (-t)^k e_{j-k}, \qquad 1 \le j < N,$$
$$\Phi(t; e) = g\big(t; E_1(t; e), \dots, E_{N-1}(t; e)\big).$$

From the recurrence relations among elementary symmetric polynomials, one has $E_j(x_1; e(x)) = \hat{e}_j$, and hence by symmetry

$$\Phi(x_i; e(x)) = F_i(x), \qquad i = 1, \dots, N.$$

Whenever the coordinates $x_1, \dots, x_N$ are pairwise distinct, the condition $F_i(x) = 0$ for all i is therefore equivalent to the polynomial divisibility $Q(t; e(x)) \mid \Phi(t; e(x))$.

Since $Q(t; e)$ is monic of degree N in t, Euclidean division in $\mathbb{C}[e_1, \dots, e_N][t]$ yields unique polynomials $U(t; e)$ and $R(t; e)$ such that

$$\Phi(t; e) = U(t; e)Q(t; e) + R(t; e), \qquad R(t; e) = \sum_{j=0}^{N-1} \rho_j(e) t^j. \tag{16}$$

We define the *generalized TQ ideal* by

$$J = (\rho_0, \dots, \rho_{N-1}) \subseteq \mathbb{C}[e_1, \dots, e_N]. \tag{17}$$

Proposition 2.4 (Exclusion of simple zeros with repeated coordinates)**.** *Let $I_F = (F_1, \dots, F_N)$, and let $p \in V(I_F)$ be a simple zero, meaning that the Jacobian matrix $(\partial F_i/\partial x_j)(p)$ is invertible. If $p_a = p_b$ for some $a \neq b$, then $q(p) \notin V(J)$. In particular, when J is zero-dimensional, the local multiplicity of $q(p)$ in $Z(J)$ is zero.*

Proof. Put $e^* = q(p)$ and $u = p_a = p_b$, and consider the nonzero tangent vector $v \in \mathbb{C}^N$ with $v_a = 1$, $v_b = -1$, and $v_i = 0$ for $i \neq a, b$. Since every elementary symmetric polynomial e_j is

invariant under transposing x_a and x_b, its partial derivatives with respect to x_a and x_b agree at p, which implies $dq_p(v) = 0$.

The identity $F_i(x) = \Phi(x_i; q(x))$ holds identically on $\mathbb{A}^N$, including along the collision locus. Differentiating F_i at p in the direction v via the chain rule and using $dq_p(v) = 0$ yields

$$(dF_i)_p(v) = \partial_t \Phi(p_i; e^*) v_i.$$

Suppose toward a contradiction that $e^* \in V(J)$, so that $Q(t; e^*) \mid \Phi(t; e^*)$. Because $p_a = p_b = u$, the factor $(t-u)^2$ divides $Q(t; e^*)$ and hence also $\Phi(t; e^*)$, forcing $\partial_t \Phi(u; e^*) = 0$. Consequently $(dF_i)_p(v) = 0$ for $i = a, b$, as well as for all $i \neq a, b$ since $v_i = 0$. Thus $v \neq 0$ lies in the kernel of the Jacobian matrix $(\partial F_i / \partial x_j)(p)$, contradicting the assumption that p is a simple zero of I_F. Therefore $q(p) \notin V(J)$, and $\ell_{q(p)}(J) = 0$. □

Assuming the relevant ideals are zero-dimensional, pushing forward the depermutation sum to $\mathrm{Sym}^N(\mathbb{A}^1)$ recovers the cycle of the generalized TQ ideal:

$$q_* Z^{\mathrm{dep}} = Z(J).$$

Heuristically, this identity reflects the fact that after a generic perturbation of F making all zeros of the partition systems simple, Möbius inversion cancels all collision orbits while Proposition 2.4 removes simple collision roots from $V(J)$.

For the XXX Bethe system, we combine this relationship with the following structural hypothesis on solutions to Baxter's TQ relation.

Distinct-root assumption away from the singular values. Let $L \geq 2N$, and suppose a monic polynomial $Q(t)$ of degree N admits a polynomial companion $T(t)$ satisfying Baxter's TQ relation

$$T(t)Q(t) = (t+1)^L Q(t-2) + (t-1)^L Q(t+2). \tag{18}$$

Factor $Q(t)$ as

$$Q(t) = (t-1)^a (t+1)^b Q_{\mathrm{reg}}(t), \qquad Q_{\mathrm{reg}}(1) Q_{\mathrm{reg}}(-1) \neq 0.$$

We assume that $Q_{\mathrm{reg}}(t)$ is squarefree; equivalently, every root of $Q(t)$ outside $\{1, -1\}$ is simple, whereas arbitrary multiplicities $a, b \geq 0$ at the singular values ± 1 are permitted.

The absence of repeated roots in Q_{reg} as a consequence of the polynomiality of T is discussed in [9, Introduction, eq. (1.3) and footnote 2], where extensive numerical verification is reported while noting that a complete proof for the spin-1/2 XXX chain remains open. Here we adopt the precise statement above—allowing singular roots at ± 1—as an explicit working assumption. Combined with the pushforward relation $q_* Z^{\mathrm{dep}} = Z(J)$, this assumption implies that no root configuration with collisions outside $\{1, -1\}$ lies in the support of Z_P^{dep}. The same conclusion applies to each reduced balanced fixed-pair system: if $H(t)$ satisfies the reduced TQ relation obtained after fixing r singular pairs $(1, -1)$, then $Q(t) = (t^2-1)^r H(t)$ satisfies Baxter's original TQ relation (18).

2.3 Projective intersection numbers and the combinatorial identity

We compactify the affine space of ordered roots via the standard embedding

$$\mathbb{A}^N \hookrightarrow \mathbb{P}^N, \qquad (x_1, \ldots, x_N) \longmapsto [1 : x_1 : \cdots : x_N],$$

denoting homogeneous coordinates on $\mathbb{P}^N$ by $[z : X_1 : \cdots : X_N]$, so that the hyperplane at infinity is $z = 0$. Since the two products in $P(x_1, \ldots, x_N)$ agree at $z = 0$, the homogenization of P to degree $L + N - 2$ is

$$\widehat{P}(z, X_1, \ldots, X_N) = \frac{1}{z}\Big((X_1 + z)^L \prod_{j \neq 1} (X_1 - X_j - 2z) - (X_1 - z)^L \prod_{j \neq 1} (X_1 - X_j + 2z) \Big),$$

which is a homogeneous polynomial of degree $L+N-2$. The reduced Bethe polynomials P_t^{red} for a system with $|A|+|B|$ fixed singular coordinates and $M = N-|A|-|B|$ free coordinates homogenize similarly to degree $L+M-2$. Following Fulton [5], Bézout's theorem motivates the following global degree for each auxiliary partition system.

Definition 2.5 (Projective intersection number)**.** Put $M = N-|A|-|B|$. For a partition π of $[N]\setminus(A\cup B)$ into $m=|\pi|$ blocks, we define the projective intersection number of $Z_{A,B;\pi}$ by

$$\boxed{\deg_{\mathbb{P}^N} Z_{A,B;\pi} = (L+M-2)^m.} \tag{19}$$

We extend $\deg_{\mathbb{P}^N}$ by $\mathbb{Q}$-linearity to all formal linear combinations of the cycles $Z_{A,B;\pi}$, while retaining deg for the affine degree. To equate $\deg_{\mathbb{P}^N}\tilde{Z}$ with $\deg\tilde{Z}$, one must verify that $\tilde{Z}$ has zero net weight along the boundary hyperplane $z=0$.

Remark 2.6 (Agreement with the BKK theorem)**.** The projective degree (19) admits a natural polyhedral interpretation via the Bernstein–Khovanskii–Kushnirenko (BKK) theorem [20, Chapter 3]. Put $M = N-|A|-|B|$ and $D = L+M-2$. For $M\geq 1$, consider the ambient monomial supports of the m reduced equations upon identifying coordinates within each block of π, prior to any accidental coefficient cancellation. Let $\mathcal{N}_\alpha \subseteq \mathbb{R}^m_{\geq 0}$ denote the downward closure of the Newton polytope of block α. Because the αth reduced equation has total degree at most D and contains the uncancelled pure power $x_{r_\alpha}^D$, its polytope satisfies the sandwiching inclusions

$$[0, De_\alpha] \subseteq \mathcal{N}_\alpha \subseteq D\Delta_m, \qquad \Delta_m = \Big\{u\in\mathbb{R}^m_{\geq 0} : \sum_{\alpha=1}^m u_\alpha \leq 1\Big\}.$$

Normalizing the mixed volume on $\mathbb{R}^m$ so that $\mathrm{MV}(\Delta_m,\dots,\Delta_m)=1$, monotonicity of mixed volumes yields

$$\begin{aligned} D^m &= \mathrm{MV}([0,De_1],\dots,[0,De_m]) \\ &\leq \mathrm{MV}(\mathcal{N}_1,\dots,\mathcal{N}_m) \leq \mathrm{MV}(D\Delta_m,\dots,D\Delta_m) = D^m. \end{aligned}$$

Thus the BKK bound in $(\mathbb{C}^\times)^m$ equals $D^m = (L+M-2)^m$, matching the projective Bézout count (19).

Proposition 2.7 (Projective completeness count)**.** *For disjoint subsets $A,B\subseteq[N]$ with $|A|=|B|=r$ and $M=N-2r$, one has*

$$\deg_{\mathbb{P}^N} Z^{\mathrm{dep}}_{A,B} = \deg_{\mathbb{P}^N} Z^{\mathrm{dep}}_r = \binom{L+M-2}{M}. \tag{20}$$

Consequently,

$$\boxed{\deg_{\mathbb{P}^N}\tilde{Z} = \sum_{r=0}^{\lfloor N/2\rfloor}(-1)^r\binom{L}{r}\binom{L+N-2r-2}{N-2r} = \binom{L}{N} - \binom{L}{N-1}.} \tag{21}$$

Proof. Writing $D = L+M-2$, the classical partition-lattice identity

$$\sum_{\pi\in\Pi_M}\mu(\hat{0},\pi)D^{|\pi|} = D(D-1)\cdots(D-M+1)$$

follows from Möbius inversion applied to maps $[M]\to[D]$: there are $D^{|\pi|}$ maps constant on the blocks of π, and sieving over Π_M isolates the injective maps. Dividing by $M!$ gives

$$\deg_{\mathbb{P}^N} Z^{\mathrm{dep}}_{A,B} = \frac{1}{M!}\sum_{\pi\in\Pi_M}\mu(\hat{0},\pi)D^{|\pi|} = \binom{D}{M} = \binom{L+M-2}{M},$$

which also holds trivially for $M = 0$. Since the average defining Z_r^{dep} in (13) runs over $\binom{N}{r,r,M} = N!/((r!)^2 M!)$ pairs (A, B) of identical projective degree, $\deg_{\mathbb{P}^N} Z_r^{\mathrm{dep}} = \binom{L+M-2}{M}$ as well.

To evaluate the alternating sum in (21), we compute the product of generating functions in $\mathbb{Q}[[t]]$:

$$\left(\sum_{r=0}^{L}(-1)^r\binom{L}{r}t^{2r}\right)\left(\sum_{M\geq 0}\binom{L+M-2}{M}t^M\right) = \frac{(1-t^2)^L}{(1-t)^{L-1}}$$
$$= (1-t)(1+t)^L.$$

Extracting the coefficient of t^N on both sides establishes (21), with the convention $\binom{L}{-1} = 0$ when $N = 0$. □

Proposition 2.7 shows that the global projective intersection number of $\tilde{Z}$ automatically reproduces the highest-weight dimension $d(L, N)$ (see Appendix B for the extension of this projective identity to higher-spin XXX_S chains). Proving Conjecture 2.2 therefore reduces to showing that no net weight escapes to infinity and that the local weight of $\tilde{Z}$ at every affine root $p \in X$ matches the target physical weight $w(p)$.

2.4 Plan for a proof of Conjecture 2.2

For each affine root $p \in X$, define its exact singular index sets

$$A(p) = \{j \in [N] : p_j = 1\}, \qquad B(p) = \{j \in [N] : p_j = -1\},$$

and let $\pi(p)$ denote the equality partition of the remaining nonsingular labels $[N] \setminus (A(p) \cup B(p))$, so that two nonsingular indices belong to the same block of $\pi(p)$ if and only if their coordinates in p coincide. Write $a = |A(p)|$ and $b = |B(p)|$. Note that if either a or b is positive, the Bethe equation $P_i(p) = 0$ at that singular coordinate forces at least one coordinate to take the opposite singular value; hence either $a = b = 0$ or $a, b \geq 1$. Write $w_r(p)$ for the weight of p in Z_r^{dep}, and set $w(p) = w_i$ when $p = x^i$.

Completing the proof of Conjecture 2.2 breaks down into the following six steps:

1. **Boundary vanishing:** Show that every boundary point on the hyperplane $z = 0$ in $\mathbb{P}^N$ has zero net weight in $\tilde{Z}$, ensuring that $\deg_{\mathbb{P}^N} \tilde{Z} = \deg \tilde{Z}$.

2. **Nonsingular collisions:** Show that each Z_r^{dep} assigns zero weight to any affine configuration p having collisions outside $\{1, -1\}$, so that the support of Z_r^{dep} is contained in the locus where $\pi(p) = \hat{0}$.

3. **Unbalanced singular roots:** Show that each Z_r^{dep} assigns zero weight to any configuration with $a \neq b$, restricting its support to balanced configurations with $a = b$.

4. **Distinct nonsingular roots ($a = b = 0$, $\pi(p) = \hat{0}$):** Establish that $(P_1, \ldots, P_N)$ has local multiplicity $\ell_p = 1$ at every distinct nonsingular root p and that the associated Bethe vector is nonzero. Since only $r = 0$ contributes, p receives the target weight $1/N!$.

5. **Single singular pair ($a = b = 1$, $\pi(p) = \hat{0}$):** Here only $r = 0$ and $r = 1$ contribute, so one must verify that
$$w_0(p) - Lw_1(p) = w(p). \tag{22}$$
Writing $e_p = \ell_p(I_{A(p),B(p);\hat{0}})$ for the local multiplicity of the reduced fixed-pair system at p, the required local-multiplicity relation is
$$\ell_p(P_1, \ldots, P_N) = \begin{cases} Le_p + 1, & \text{if } p \text{ satisfies the physical regularization condition (4),} \\ Le_p, & \text{otherwise.} \end{cases} \tag{23}$$

In particular, when p is a simple root of the reduced system ($e_p = 1$), $\ell_p(P_1, \dots, P_N)$ equals $L+1$ at physical singular roots and L at nonphysical singular roots.

6. **Multiple balanced singular pairs ($a = b \geq 2$, $\pi(p) = \hat{0}$):** Because such configurations contain repeated coordinates (± 1 with multiplicity $a \geq 2$), they are nonphysical and have target weight $w(p) = 0$. One must therefore prove the alternating cancellation

$$\sum_{r=0}^{a} (-1)^r \binom{L}{r} w_r(p) = 0.$$

Reduction under the distinct-root assumption. The distinct-root assumption of Section 2.2 immediately resolves Steps 2 and 3. Indeed, combined with the pushforward identity $q_* Z^{\mathrm{dep}} = Z(J)$, it excludes all affine configurations with collisions outside $\{1, -1\}$ both for the full Bethe system ($r = 0$) and for each reduced balanced fixed-pair system ($r \geq 1$), leaving only configurations with $\pi(p) = \hat{0}$.

Furthermore, Baxter's TQ relation (18) rules out unbalanced singular configurations ($a \neq b$) by comparing vanishing orders at $t = \pm 1$. Specifically, writing $Q(t) = (t-1)^a (t+1)^b Q_{\mathrm{reg}}(t)$ with $Q_{\mathrm{reg}}(\pm 1) \neq 0$, the left-hand side $T(t)Q(t)$ of (18) vanishes to order at least a at $t = 1$. Since $L \geq 2N > a$, the term $(t-1)^L Q(t+2)$ on the right-hand side vanishes to order at least $L > a$ at $t = 1$, forcing $(t-1)^a$ to divide $(t+1)^L Q(t-2)$ and hence $Q(t-2)$, which means $(t+1)^a \mid Q(t)$ and thus $b \geq a$. Applying the symmetric argument at $t = -1$ gives $a \geq b$, so $a = b$. Because the same reasoning applies after factoring out $(t^2-1)^r$ for any $0 \leq r \leq \min(a, b)$, every unbalanced configuration has $w_r(p) = 0$ for all r.

At the boundary $z = 0$ (Step 1), we prove unconditionally in Section 3 that every common zero at infinity has at least two equal nonzero homogeneous coordinates, so the boundary support lies entirely on collision diagonals. Showing that the alternating depermutation weights cancel along these boundary collision loci requires comparing the local intersection multiplicities of the full and partitioned systems at infinity, which is verified for $N = 3$ in Appendix A and remains open in general. In the affine chart, the remaining open problems are the local multiplicity analysis for a single singular pair $a = b = 1$ (treated in the simple reduced case in Section 4 and for $N = 2$ in Appendix A) and the alternating cancellation for multiple balanced singular pairs $a = b \geq 2$ (interpreted via an inhomogeneity deformation in Section 5 and verified explicitly for $p = (1, 1, -1, -1)$ at $N = 4$ in Appendix A).

3 Roots at infinity

We now show that all projective roots of the Bethe equations on the boundary hyperplane $z = 0$ lie on collision diagonals. Using homogeneous coordinates $[z : X_1 : \cdots : X_N]$ on $\mathbb{P}^N$ with $x_i = X_i/z$ in the affine chart $z \neq 0$, the homogenization of P_i to degree $L + N - 2$ is

$$\widehat{P}_i(z, X) = \frac{1}{z}\Big((X_i + z)^L \prod_{j \neq i} (X_i - X_j - 2z) - (X_i - z)^L \prod_{j \neq i} (X_i - X_j + 2z)\Big).$$

Expanding the numerator to first order in z and setting $z = 0$ yields the boundary system

$$\widehat{P}_i(0, X) = 2X_i^{L-1}\Big(L \prod_{j \neq i} (X_i - X_j) - 2X_i \sum_{j \neq i} \prod_{h \neq i,j} (X_i - X_h)\Big) = 0, \qquad i = 1, \dots, N. \tag{24}$$

Proposition 3.1 (Collisions at infinity)**.** *Assume $L \geq 2N$ and $N \geq 1$. Every common zero $[0 : X_1 : \cdots : X_N] \in \mathbb{P}^N$ of the homogenized Bethe equations (24) satisfies*

$$X_i = X_j \neq 0 \qquad \text{for some distinct indices } i, j \in [N].$$

In particular, every boundary root lies on a collision diagonal.

Proof. Suppose toward a contradiction that the nonzero coordinates of $[0 : X_1 : \cdots : X_N]$ are pairwise distinct. Let $S = \{i \in [N] : X_i \neq 0\}$, so that $h = |S| \geq 1$. For each $i \in S$, all differences $X_i - X_j$ $(j \neq i)$ are nonzero; dividing (24) by $2X_i^{L-1}\prod_{j\neq i}(X_i - X_j)$ therefore gives

$$L = 2\sum_{j\neq i}\frac{X_i}{X_i - X_j} = 2(N-h) + 2\sum_{\substack{j\in S\\ j\neq i}}\frac{X_i}{X_i - X_j}.$$

Summing this relation over all $i \in S$ and using the pairwise identity

$$\frac{X_i}{X_i - X_j} + \frac{X_j}{X_j - X_i} = 1$$

for each unordered pair $\{i, j\} \subseteq S$ yields

$$Lh = 2h(N-h) + h(h-1), \qquad \text{or equivalently} \qquad L = 2N - h - 1 \leq 2N - 2,$$

which contradicts the assumption $L \geq 2N$. □

The exact same argument extends to the reduced systems obtained after fixing r balanced singular pairs $(1, -1)$. In that setting there are $M = N - 2r$ free coordinates, satisfying the reduced affine equations

$$\begin{aligned}(x_i+1)^{L-r}(x_i-3)^r\prod_{j\neq i}(x_i - x_j - 2)\\ -(x_i-1)^{L-r}(x_i+3)^r\prod_{j\neq i}(x_i - x_j + 2) = 0,\end{aligned}$$

where the products run over the remaining $M-1$ free coordinates. Homogenizing these equations to degree $L + M - 2$ in the projective space $\mathbb{P}^M$ of free coordinates produces the boundary system (24) with M variables and with L replaced by $L - 4r$ inside the parentheses, while the prefactor remains $2X_i^{L-1}$. Since $L \geq 2N$ implies $L - 4r \geq 2(N - 2r) = 2M$, the proof of Proposition 3.1 again forces at least two equal nonzero free coordinates at every boundary root when $M \geq 2$ (for $M = 1$ there are no boundary roots, and for $M = 0$ the free space is a point).

We note that Proposition 3.1 is unconditional and does not rely on the distinct-root assumption of Section 2.2. However, showing that all boundary roots lie on collision diagonals is weaker than proving that the depermutation sum assigns zero weight to the boundary: the homogenized systems may exhibit higher local multiplicities or even positive-dimensional excess components along $z = 0$. Proving that these boundary contributions cancel across the partition lattice remains a key step toward establishing Conjecture 2.2.

4 Exactly one singular pair: $a = b = 1$

Singular roots containing a single pair $(1, -1)$ have been extensively studied via regularization of the Bethe vector by Nepomechie and Wang [17, Section 3], including twist regularization [18, Section 2], and by Granet and Jacobsen [6, Sections 2–4] through polynomial companions and zero-remainder conditions. The local multiplicity criterion (23) follows from combining the results of [17] and [16]. Here we give a self-contained elementary proof in the case where the reduced fixed-pair system has a simple zero.

Let $N \geq 2$ and $L \geq 2N$, and let $p = (1, -1, z_3, \ldots, z_N) \in X$ have pairwise distinct coordinates. Set

$$I_{\text{full}} = (P_1, \ldots, P_N), \qquad I_{\text{red}} = I_{\{1\},\{2\};\hat{0}}.$$

Proposition 4.1 (Simple reduced singular roots)**.** *Suppose $p = (1, -1, z_3, \dots, z_N)$ is a simple zero of I_{red}, so that $e_p = \ell_p(I_{\mathrm{red}}) = 1$. Then*

$$\ell_p(I_{\mathrm{full}}) \begin{cases} \geq L+1, & \text{if } p \text{ satisfies the physical condition (4),} \\ = L, & \text{otherwise.} \end{cases}$$

Proof. Introduce local coordinates $u, v, y_3, \dots, y_N$ near p by setting $x_1 = 1 + u$, $x_2 = -1 + v$, and $x_j = y_j$ for $j \geq 3$, and define

$$C(u, y) = \prod_{j=3}^{N} (u - 1 - y_j), \qquad D(u, y) = \prod_{j=3}^{N} (u + 3 - y_j).$$

Since $z_j \neq -1$ for all $j \geq 3$, we have $C(0, z) \neq 0$. Solving $P_1 = 0$ for v in the formal power series ring near $(u, y) = (0, z)$ yields

$$v = u - \frac{4u^L D(u, y)}{(2+u)^L C(u, y) - u^L D(u, y)} = u + O(u^L).$$

Substituting this expression for v into the remaining Bethe polynomials gives $P_2 = u^L H(u, y)$ and $P_j = G_j(u, y)$ for $3 \leq j \leq N$, where H and G_j are regular at $(u, y) = (0, z)$. Evaluating at $u = 0$, we find $G_j(0, y) = (y_j^2 - 1) P_j^{\mathrm{red}}(y)$. Because $z_j \neq \pm 1$ and p is a simple zero of I_{red}, the Jacobian matrix of $(G_3, \dots, G_N)$ with respect to $(y_3, \dots, y_N)$ at $(0, z)$—whose rows are those of the reduced Jacobian scaled by the nonzero factors $z_j^2 - 1$—is invertible. By the implicit function theorem, the equations $G_3 = \cdots = G_N = 0$ determine a unique local curve $y = y(u)$ with $y(0) = z$, reducing the local quotient algebra of I_{full} at p to

$$\mathbb{C}[[u]]/(u^L h(u)), \qquad h(u) = H(u, y(u)).$$

A direct calculation at $u = 0$ gives

$$H(0, y) = 4(-1)^{N-2} \left[-\prod_{j=3}^{N} (y_j + 3) + (-1)^L \prod_{j=3}^{N} \frac{(y_j - 1)(y_j - 3)}{y_j + 1} \right].$$

Since p has pairwise distinct coordinates, it contains no exact strings other than the singular pair $(1, -1)$; in particular, $z_j \neq \pm 3$ and $z_i - z_j \neq \pm 2$ for $3 \leq i, j \leq N$ [6, Lemma 1]. Multiplying the reduced Bethe equations $P_j^{\mathrm{red}}(z) = 0$ over $j = 3, \dots, N$ therefore yields

$$\left(\prod_{j=3}^{N} \frac{z_j + 1}{z_j - 1} \right)^{L-1} = \prod_{j=3}^{N} \frac{z_j + 3}{z_j - 3}.$$

Substituting this product identity into $h(0) = H(0, z)$ shows that

$$h(0) = 0 \quad \Longleftrightarrow \quad \left[-\prod_{j=3}^{N} \frac{z_j + 1}{z_j - 1} \right]^L = 1. \tag{25}$$

Consequently, if p is nonphysical, then $h(0) \neq 0$ and $\ell_p(I_{\mathrm{full}}) = L$; if p satisfies the physical regularization condition (4), then $h(0) = 0$ and $\ell_p(I_{\mathrm{full}}) \geq L + 1$. □

5 The local index as a degree of exceptional components

Throughout this section we work with ordered roots and study balanced singular configurations $p = (1^k, -1^k, z_{2k+1}, \dots, z_N)$ with $k \geq 1$. By introducing site inhomogeneities, we deform the Bethe equations so that p splits into a collection of local branches (irreducible components over the deformation base). Under natural flatness and genericity assumptions, the local multiplicity of p in each auxiliary cycle $Z_{A,B;\pi}$ is governed by the degrees of the branches lying along specific site-endpoint hyperplanes, and the alternating sum in $\tilde{Z}$ cancels all branches that remain pinned to at least one site pair. The local weight $w_p(\tilde{Z})$ is thereby identified with the normalized degree count of *exceptional* components that avoid all site-endpoint hyperplanes.

Deformation of the Bethe equations. Introduce formal site inhomogeneity parameters $\theta_1, \dots, \theta_L$ and set

$$a_\theta(t) = \prod_{j=1}^{L}(t + \theta_j + 1), \qquad d_\theta(t) = \prod_{j=1}^{L}(t + \theta_j - 1).$$

The inhomogeneous Bethe polynomials are

$$P_i^\theta(x) = a_\theta(x_i)\prod_{j\neq i}(x_i - x_j - 2) - d_\theta(x_i)\prod_{j\neq i}(x_i - x_j + 2), \qquad i = 1, \dots, N,$$

which specialize to the homogeneous Bethe polynomials $P_i(x)$ at $\theta = 0$. Let

$$T = \operatorname{Spec}\mathbb{C}[[\theta_1, \dots, \theta_L]],$$

and work inside the relative affine space $T \times \mathbb{A}^N$ locally near $(0, p)$, where

$$p = (1^k, -1^k, z_{2k+1}, \dots, z_N), \qquad z_i \neq \pm 1, \qquad z_i \neq z_j \ (i \neq j).$$

Let $\bar{X}_p$ denote the multiset of irreducible components of the deformed Bethe scheme $P_1^\theta = \cdots = P_N^\theta = 0$ passing through $(0, p)$ that dominate T and are generically finite over T, where each such component appears in $\bar{X}_p$ with multiplicity equal to its degree over T (counted with scheme-theoretic multiplicity).

Definition 5.1 (Site labels and equality partitions of a local component)**.** Let $C \in \bar{X}_p$ be a local component through $(0, p)$ dominating T. Define its positive and negative site-label maps

$$f_C, g_C : [k] \longrightarrow [L] \cup \{\mathcal{E}\}$$

by

$$f_C(a) = \begin{cases} j, & \text{if } x_a = 1 - \theta_j \text{ identically in the function field of } C, \\ \mathcal{E}, & \text{if no such } j \in [L] \text{ exists,} \end{cases}$$

$$g_C(b) = \begin{cases} j, & \text{if } x_{k+b} = -1 - \theta_j \text{ identically in the function field of } C, \\ \mathcal{E}, & \text{if no such } j \in [L] \text{ exists.} \end{cases}$$

Here the symbol $\mathcal{E}$ designates an *exceptional* coordinate that specializes to 1 or -1 at $\theta = 0$ without being identically pinned to any site hyperplane $x = \pm 1 - \theta_j$ along C.

Put $E_C^+ = f_C^{-1}(\mathcal{E}) \subseteq [k]$ and $E_C^- = g_C^{-1}(\mathcal{E}) \subseteq [k]$. Let $\pi_1(C)$ be the partition of E_C^+ placing $a, a' \in E_C^+$ in the same block if and only if $x_a = x_{a'}$ identically on C, and define $\pi_2(C)$ on E_C^- analogously using the negative coordinates x_{k+b}. We call the component C *exceptional* if

$$f_C = g_C \equiv \mathcal{E} \qquad \text{and} \qquad \pi_1(C) = \pi_2(C) = \hat{0},$$

where $\hat{0}$ denotes the singleton partition on $[k]$.

Observe that the positive and negative coordinate clusters carry the exact same set of site labels in $[L]$:

$$\operatorname{Im} f_C \cap [L] = \operatorname{Im} g_C \cap [L].$$

Indeed, if $x_a = 1 - \theta_j$ identically on C, then $d_\theta(x_a) = 0$ while $a_\theta(x_a) \neq 0$ at the generic point of C; the deformed Bethe equation $P_a^\theta = 0$ then forces $\prod_{m \neq a}(x_a - x_m - 2) = 0$, so some coordinate x_{k+b} (which must specialize to -1 at $\theta = 0$) satisfies $x_{k+b} = -1 - \theta_j$ identically on C, and conversely.

We now express the local multiplicity of p in $Z_{A,B;\pi}$ in terms of these components. For $|A| = |B| = r$ with $0 \leq r \leq k$, symmetry allows us to take

$$A = \{1, \ldots, r\}, \qquad B = \{k+1, \ldots, k+r\}.$$

At $(0, p)$, only partitions π of $R = [N] \setminus (A \cup B)$ whose blocks do not mix the subsets $\{r+1, \ldots, k\}$, $\{k+r+1, \ldots, 2k\}$, and the singletons $\{2k+1\}, \ldots, \{N\}$ can contribute. We deform $I_{A,B;\pi}$ by pinning the ith fixed singular pair to site i:

$$x_i = 1 - \theta_i, \qquad x_{k+i} = -1 - \theta_i, \qquad 1 \leq i \leq r.$$

A component $C \in \bar{X}_p$ compatible with these r pinned pairs and the block equalities of π must satisfy:

1. $f_C(i) = g_C(i) = i$ for $1 \leq i \leq r$;
2. if $i, j \in \{r+1, \ldots, k\}$ belong to the same block of π, then $f_C(i) = f_C(j)$, and if this common value is $\mathcal{E}$, then i and j belong to the same block of $\pi_1(C)$;
3. if $k+i, k+j \in \{k+r+1, \ldots, 2k\}$ belong to the same block of π, then $g_C(i) = g_C(j)$, and if this common value is $\mathcal{E}$, then i and j belong to the same block of $\pi_2(C)$.

Algebraically, put

$$S = \mathbb{C}[[\theta_1, \ldots, \theta_L]][x_1, \ldots, x_N], \qquad R = [N] \setminus (A \cup B),$$

write $\pi = \{S_1, \ldots, S_m\}$ with representatives $r_\alpha \in S_\alpha$, and let σ_r denote the substitution $x_i = 1 - \theta_i$, $x_{k+i} = -1 - \theta_i$ $(1 \leq i \leq r)$. For $t \in R$, the deformed reduced Bethe polynomial is

$$P_t^{\theta,\mathrm{red}}(x_R) := \frac{\sigma_r(P_t^\theta)}{\prod_{j=1}^r ((x_t + \theta_j)^2 - 1)} = a_{\theta,r}(x_t) \prod_{j \in R \setminus \{t\}} (x_t - x_j - 2) - d_{\theta,r}(x_t) \prod_{j \in R \setminus \{t\}} (x_t - x_j + 2),$$

where

$$a_{\theta,r}(u) = \prod_{j=r+1}^{L} (u + \theta_j + 1) \prod_{j=1}^{r} (u + \theta_j - 3),$$
$$d_{\theta,r}(u) = \prod_{j=r+1}^{L} (u + \theta_j - 1) \prod_{j=1}^{r} (u + \theta_j + 3).$$

The deformed auxiliary ideal is then

$$\begin{aligned} \bar{I}_{A,B;\pi} = (&x_i - 1 + \theta_i,\ x_{k+i} + 1 + \theta_i \quad (1 \leq i \leq r), \\ &P_{r_\alpha}^{\theta,\mathrm{red}} \quad (1 \leq \alpha \leq m), \\ &x_t - x_{r_\alpha} \quad (1 \leq \alpha \leq m,\ t \in S_\alpha \setminus \{r_\alpha\})) \subseteq S, \end{aligned} \tag{26}$$

whose special fiber at $\theta = 0$ recovers $I_{A,B;\pi}$:

$$\frac{\bar{I}_{A,B;\pi} + (\theta_1, \ldots, \theta_L)}{(\theta_1, \ldots, \theta_L)} = I_{A,B;\pi} \quad \text{in } \mathbb{C}[x_1, \ldots, x_N].$$

To equate the special-fiber multiplicity $\ell_p(I_{A,B;\pi})$ with the sum of generic degrees over T of the compatible components, we assume that near $(0,p)$ all deformed partition schemes $\mathrm{Spec}(S/\bar{I}_{A,B;\pi})$ are finite and flat over T, that the full and reduced free-root systems have geometrically reduced generic fibers (simple generic roots), and that the free roots of each reduced system do not identically coincide with the fixed site endpoints $\pm 1-\theta_j$ $(1 \le j \le r)$.

Under these flatness and genericity assumptions, depermutation over $\Pi(R)$ retains precisely the components whose $N-2r$ free coordinates are generically pairwise distinct. For $0 \le r \le k$, with the first r pairs pinned to sites $1,\dots,r$, we introduce two local degree-weighted component counts at p:

- $D_r(p)$ denotes the degree-weighted count of components $C \in \bar{X}_p$ of the reduced fixed-pair family having generically distinct coordinates (so that $f_C(i) = g_C(i) = i$ for $1 \le i \le r$, each site label in $[L]$ appears at most once in f_C and in g_C, and $\pi_1(C) = \pi_2(C) = \hat{0}$), allowing additional site-pinned pairs among the remaining $k-r$ singular pairs.
- $\tilde{D}_r(p)$ denotes the degree-weighted count of those components in $D_r(p)$ that carry *no* additional site-pinned pairs beyond the prescribed r sites:

$$f_C(i) = g_C(i) = \begin{cases} i, & 1 \le i \le r, \\ \mathcal{E}, & r < i \le k, \end{cases} \qquad \pi_1(C) = \pi_2(C) = \hat{0}.$$

In particular, $\tilde{D}_0(p)$ is the degree-weighted number of exceptional components at p.

Because p has k coordinates equal to 1 and k coordinates equal to -1, there are $\binom{k}{r}^2$ subsets (A,B) with $A \subseteq A(p)$, $B \subseteq B(p)$, and $|A| = |B| = r$ that contribute at p, each yielding the same count $D_r(p)$ by permutation symmetry. Multiplying the depermutation prefactor $1/(N-2r)!$ by the averaging factor $(r!)^2(N-2r)!/N!$ from (13) gives

$$\boxed{w_r(p) = \frac{(r!)^2}{N!}\binom{k}{r}^2 D_r(p) = \frac{(k!)^2}{N!\,((k-r)!)^2} D_r(p), \qquad 0 \le r \le k.} \tag{27}$$

Stratifying the components counted by $D_r(p)$ according to their total number $s = \#(\mathrm{Im}\, f_C \cap [L]) \in \{r,\dots,k\}$ of site-pinned pairs yields the refinement identity

$$\boxed{D_r(p) = \sum_{s=r}^{k} \binom{L-r}{s-r}\binom{k-r}{s-r}^2 ((s-r)!)^2 \tilde{D}_s(p) = \sum_{s=r}^{k} \binom{L-r}{s-r}\left(\frac{(k-r)!}{(k-s)!}\right)^2 \tilde{D}_s(p).} \tag{28}$$

Indeed, any component in $D_r(p)$ carrying s site-pinned pairs is uniquely specified beyond the first r pinned pairs by:

1. choosing $s-r$ additional sites from the $L-r$ remaining sites in $\binom{L-r}{s-r}$ ways;
2. choosing $s-r$ of the $k-r$ remaining positive indices and $s-r$ of the $k-r$ remaining negative indices in $\binom{k-r}{s-r}^2$ ways; and
3. assigning the chosen positive and negative indices bijectively to the $s-r$ chosen sites in $((s-r)!)^2$ ways, while all remaining $k-s$ positive and negative indices carry the label $\mathcal{E}$.

Simultaneously permuting the unused sites in T and the unpinned singular coordinates (which fixes the special fiber point p) identifies each such stratum isomorphically with the standard stratum counted by $\tilde{D}_s(p)$, establishing (28).

Proposition 5.2 (Cancellation of nonexceptional components)**.** *Under the flatness, generic simplicity, and endpoint-avoidance assumptions above, the local weight of* $p = (1^k, -1^k, z_{2k+1}, \dots, z_N)$ *in* $\tilde{Z}$ *is given by*

$$\boxed{w_p(\tilde{Z}) = \sum_{r=0}^{k} (-1)^r \binom{L}{r} w_r(p) = \frac{1}{N!} \tilde{D}_0(p).} \tag{29}$$

Thus the alternating sum cancels all site-pinned components and retains only the normalized degree count of exceptional components at p*.*

Proof. Since p contains only k copies of ± 1, terms with $r > k$ vanish at p. Substituting (27) and (28) into the alternating sum and interchanging the order of summation gives

$$\begin{aligned} w_p(\tilde{Z}) &= \sum_{r=0}^{k} (-1)^r \binom{L}{r} \frac{(k!)^2}{N!\,((k-r)!)^2} D_r(p) \\ &= \frac{(k!)^2}{N!} \sum_{s=0}^{k} \frac{\tilde{D}_s(p)}{((k-s)!)^2} \sum_{r=0}^{s} (-1)^r \binom{L}{r} \binom{L-r}{s-r}. \end{aligned}$$

Using the binomial identity $\binom{L}{r}\binom{L-r}{s-r} = \binom{L}{s}\binom{s}{r}$, the coefficient of $\tilde{D}_s(p)$ simplifies to

$$\frac{(k!)^2}{N!\,((k-s)!)^2} \binom{L}{s} \sum_{r=0}^{s} (-1)^r \binom{s}{r}.$$

The inner sum equals $(1-1)^s = 0$ for every $s \geq 1$ and 1 for $s = 0$, leaving only the $s = 0$ term

$$w_p(\tilde{Z}) = \frac{\tilde{D}_0(p)}{N!}.$$

□

Proposition 5.2 shows that under the stated deformation hypotheses, the alternating sum over fixed singular pairs cancels every component lying along one or more site-endpoint hyperplanes $x_a + \theta_j - 1 = 0$ or $x_{k+b} + \theta_j + 1 = 0$ $(1 \leq a, b \leq k,\ 1 \leq j \leq L)$ and isolates $\tilde{D}_0(p)/N!$. In light of Steps 5 and 6 in Section 2.4, Conjecture 2.2 predicts that $\tilde{D}_0(p) = 1$ at physical singular roots with $k = 1$ and $\tilde{D}_0(p) = 0$ whenever $k \geq 2$ (or when $k = 1$ is nonphysical). Understanding conceptually why exceptional components exist only for $k = 1$ remains an interesting open problem.

Disclosure of AI assistance

AI tools were used to assist with writing, language editing, and locating relevant references. The author is responsible for the mathematical content, the accuracy of the references, and the final text.

A Numerical results on small N

This appendix presents explicit local calculations verifying the proposed counting formula and the cancellation of boundary contributions for small numbers of Bethe roots ($N = 2, 3, 4$). For the singular configurations $p = (1, -1)$ and $p = (1, 1, -1, -1)$, we show that their local multiplicities in $Z_{A,B;\pi}$ depend on the parity of L, while for $p = (1, 1, -1, -1)$ these parity-dependent contributions cancel in the alternating sum $\tilde{Z}$. We also show that for $N = 3$, the homogenized system has a unique root at infinity whose net weight vanishes after depermutation.

A.1 $N = 2$: The single singular pair $p = (1, -1)$

For $N = 2$ and $L \geq 4$, the two Bethe equations are $P_1(x, y) = P(x, y) = 0$ and $P_2(x, y) = P(y, x) = 0$, where

$$P(x, y) = (x + 1)^L(x - y - 2) - (x - 1)^L(x - y + 2).$$

We compute the local multiplicity at the singular pair $p = (1, -1)$ and verify that the multiplicity relation (23) holds with $e_p = 1$, even though the individual multiplicity depends on the parity of L.

Multiplicity at the singular pair. At $p = (1, -1)$, set $x = 1 + u$ and $y = -1 + v$ and work in the completed local ring $\mathbb{C}[[u, v]]$. Since $P_1(1 + u, -1 + v)$ is linear in v with a unit coefficient in $\mathbb{C}[[u]]$, solving $P_1 = 0$ for $v = v(u)$ gives

$$v - u = -\frac{4u^L}{(2 + u)^L - u^L} = -2^{2-L}u^L\left(1 - \frac{L}{2}u + O(u^2)\right).$$

Substituting $v = v(u)$ into the second equation yields

$$\begin{aligned} P_2(1 + u, -1 + v(u)) &= v(u)^L(v(u) - u - 4) - (v(u) - 2)^L(v(u) - u) \\ &= 4((-1)^L - 1)u^L - 4L(-1)^L u^{L+1} + O(u^{L+2}). \end{aligned}$$

The local multiplicity $\ell_p(P_1, P_2)$ is the colength of (P_1, P_2) in $\mathbb{C}[[u, v]]$, which after eliminating v equals the vanishing order of the series above: L for odd L and $L + 1$ for even L. Since $w_1(p) = 1/2$, subtracting the singular correction $Lw_1(p) = L/2$ and summing over the two permutations $(1, -1)$ and $(-1, 1)$ shows that the unordered singular pair contributes 1 for even L and 0 for odd L. This agrees with the physical regularization condition (4), which reduces to $(-1)^L = 1$ for $N = 2$.

A.2 $N = 3$: Roots at infinity

Let $N = 3$ and $L \geq 6$, and write $[z : X_1 : X_2 : X_3]$ for the homogeneous coordinates on $\mathbb{P}^3$. At $z = 0$, the first boundary equation in (24) reads

$$\widehat{P}_1(0, X) = 2X_1^{L-1}(L(X_1 - X_2)(X_1 - X_3) - 2X_1(2X_1 - X_2 - X_3)) = 0,$$

with $\widehat{P}_2(0, X)$ and $\widehat{P}_3(0, X)$ obtained by permuting the coordinates. By Proposition 3.1, any boundary solution has at least two equal nonzero coordinates, so up to permutation we may assume $X_1 = X_2 = a \neq 0$ and write $X_3 = b$. Then

$$\widehat{P}_1(0, a, a, b) = -4a^L(a - b) = 0,$$

which forces $b = a$. Conversely, all three boundary equations vanish identically along the triple diagonal $X_1 = X_2 = X_3$. Thus the homogenized system has a unique point at infinity, namely $q = [0 : 1 : 1 : 1]$.

We now determine the local multiplicity of q in each homogenized partition cycle $Z_{\varnothing,\varnothing;\pi}$ and verify that its net weight vanishes after depermutation. For $i \neq j$, symmetry gives the divisibility

$$\widehat{P}_i - \widehat{P}_j = (X_i - X_j)K_{ij}.$$

In the affine chart $X_1 = 1$, expanding $\widehat{P}_i$ at $z = 0$ to first order in the deviations $\delta_m = X_m - 1$ gives the linear part $-4(2\delta_i - \delta_j - \delta_k)$, where $\{i, j, k\} = \{1, 2, 3\}$. Taking the difference $\widehat{P}_i - \widehat{P}_j$

shows that $K_{ij}(q) = -12 \neq 0$, so K_{ij} is a unit in the local ring at q. Consequently, the full local ideal $(\widehat{P}_1, \widehat{P}_2, \widehat{P}_3)$ at q already contains the diagonal generators $X_1 - X_2$ and $X_1 - X_3$. For any partition $\pi \in \Pi_3$, imposing the block equalities of π and retaining one equation $\widehat{P}_{r_\alpha}$ per block therefore produces the exact same local ideal at q: the omitted equations $\widehat{P}_t$ agree with $\widehat{P}_{r_\alpha}$ modulo the block equalities, and the full local ideal already enforces $X_1 = X_2 = X_3$.

Along the triple diagonal $X_1 = X_2 = X_3 = 1$, the single remaining equation becomes

$$\widehat{P}_1(z, 1, 1, 1) = 4z((1+z)^L - (1-z)^L) = 8Lz^2 + O(z^4).$$

Thus the completed local quotient ring at q is $\mathbb{C}[[z]]/(z^2)$ for every partition $\pi \in \Pi_3$, so q has local multiplicity 2 in each projective cycle $Z_{\varnothing,\varnothing;\pi}$. Since the five partitions of [3] have Möbius coefficients $\mu(\hat{0}, \pi)$ equal to 1 for the singleton partition, -1 for each of the three two-block partitions, and 2 for the one-block partition, the depermutation weight at q is

$$\frac{1}{3!}(2 - 3 \cdot 2 + 2 \cdot 2) = 0.$$

A.3 $N = 4$: The root $p = (1, 1, -1, -1)$

Let $N = 4$ and $L \geq 8$. We compute the three local weights $w_0(p)$, $w_1(p)$, and $w_2(p)$ at the double singular pair $p = (1, 1, -1, -1)$ under the standing finiteness assumption.

No fixed pairs ($r = 0$). The only partitions $\pi \in \Pi_4$ contributing at p are the four partitions obtained by imposing either, both, or neither of the intra-cluster equalities $\delta_+ = x_1 - x_2 = 0$ and $\delta_- = x_3 - x_4 = 0$. Define the divided differences $H_+ = (P_1 - P_2)/\delta_+$ and $H_- = (P_3 - P_4)/\delta_-$. Near p, the full ideal (P_1, P_2, P_3, P_4) is generated by $P_1, P_3, \delta_+ H_+, \delta_- H_-$. By the additivity of local intersection multiplicities with respect to product generators, $\ell_p(P_1, P_3, \delta_+ H_+, \delta_- H_-)$ decomposes into four terms corresponding to choosing one factor from each product $\delta_+ H_+$ and $\delta_- H_-$. Subtracting the two single-diagonal partition contributions and adding back the double-diagonal contribution in the depermutation sum (12) cancels the three terms involving δ_+ or δ_-, leaving precisely

$$w_0(p) = \frac{1}{4!}\ell_p(P_1, P_3, H_+, H_-).$$

To compute $\ell_p(P_1, P_3, H_+, H_-)$, put $Q(t) = \prod_{i=1}^4 (t - x_i)$ and set

$$S(t) = (t+1)^L Q(t-2) + (t-1)^L Q(t+2).$$

For each of the two clusters $\{x_1, x_2\}$ and $\{x_3, x_4\}$, evaluating $S(t)$ at the first root and taking the divided difference between the two roots gives the two coefficients of the linear remainder of $S(t)$ modulo $(t - x_1)(t - x_2)$ and $(t - x_3)(t - x_4)$, respectively. Since $P_i(x) = -\frac{1}{2}S(x_i)$, the local ideal (P_1, P_3, H_+, H_-) is generated precisely by the coefficients of these two quadratic remainders. Because the quadratic factors $(t - x_1)(t - x_2)$ and $(t - x_3)(t - x_4)$ are coprime near p, the vanishing of both remainders is equivalent to the divisibility $Q(t) \mid S(t)$. Write

$$Q(t) = A(t-1)B(t+1), \qquad A(u) = u^2 - au + b, \qquad B(v) = v^2 - cv + d,$$

where a, b, c, d lie in the maximal ideal at p. The natural map from ordered roots near p to the elementary symmetric coordinates (a, b, c, d) is finite and flat of degree $2! \cdot 2! = 4$ (adjoining one root of each monic quadratic yields a free module of rank 2, with the second root fixed by their sum). Thus if M_L denotes the local colength of the remainder equations in $\mathbb{C}[[a, b, c, d]]$, then

$$\ell_p(P_1, P_3, H_+, H_-) = 4M_L, \qquad w_0(p) = \frac{4M_L}{4!} = \frac{M_L}{6}.$$

Substituting $t = 1 + u$ and $t = -1 + v$ into $S(t)$ and dividing by units in the local ring yields the two polynomial congruences

$$B(u) + u^L U(u) \equiv 0 \pmod{A(u)}, \qquad A(v) + v^L V(v) \equiv 0 \pmod{B(v)},$$

where

$$U(u) = \frac{A(u+2)B(u+4)}{(u+2)^L A(u-2)}, \qquad V(v) = \frac{A(v-4)B(v-2)}{(v-2)^L B(v+2)}.$$

By the formal implicit function theorem, the first congruence uniquely expresses (c, d) as power series in (a, b). Assign quasi-homogeneous weights by giving a, c, u, v weight 1 and b, d weight 2. Reducing u^L modulo the weighted homogeneous quadratic $A(u) = u^2 - au + b$ produces a linear remainder whose u-coefficient has weight $L - 1$ and whose constant term has weight L. Consequently,

$$c = a + O_{\mathrm{w}}(L-1), \qquad d = b + O_{\mathrm{w}}(L),$$

where $O_{\mathrm{w}}(j)$ denotes terms of quasi-homogeneous weight at least j.

More explicitly, writing $\mathrm{rem}_A(\cdot)$ for the remainder of polynomial division by $A(u)$, the first congruence gives $A(u) - B(u) = \mathrm{rem}_A\big(u^L U(u)\big)$. Substituting this into the second congruence yields $\mathrm{rem}_A\big(u^L U(u)\big) + u^L V(u) \equiv 0 \pmod{B(u)}$. Because $c - a = O_{\mathrm{w}}(L-1)$ and $d - b = O_{\mathrm{w}}(L)$, replacing B by A inside $U(u)$, $V(u)$, and the modulus alters the resulting coefficients only at weights strictly higher than the leading terms (shifting weights by at least $L - 2 > 1$). Thus the weighted initial forms of the two remaining equations in $\mathbb{C}[[a, b]]$ are given by the coefficients of $\mathrm{rem}_A\big(u^L(U(u) + V(u))\big)$ evaluated at $B = A$. Expanding $U(u)$ and $V(u)$ at $B = A$ to weight 1 gives

$$U(u) = 2^{4-L}\left(1 + \frac{5-L}{2}u - \frac{5}{4}a + O_{\mathrm{w}}(2)\right),$$
$$V(u) = (-1)^L 2^{4-L}\left(1 + \frac{L-5}{2}u + \frac{5}{4}a + O_{\mathrm{w}}(2)\right).$$

- For even L, the constant terms add, so up to the nonzero scalar 2^{5-L} the weighted initial forms are the two coefficients of $\mathrm{rem}_A(u^L)$. These initial forms are quasi-homogeneous of weights $L - 1$ and L, and their only common zero is $(a, b) = (0, 0)$ since $A(u) \mid u^L$ forces $A(u) = u^2$.

- For odd L, the constant terms in $U(u) + V(u)$ cancel, and up to a nonzero scalar the weighted initial forms are the coefficients of

  $$\mathrm{rem}_A\Big(u^L\big((5-L)u - \tfrac{5}{2}a\big)\Big),$$

  which are quasi-homogeneous of weights L and $L + 1$. Their only common zero is again $(a, b) = (0, 0)$: if $A(u) \mid u^L\big((5-L)u - \frac{5}{2}a\big)$, then $A(u)$ cannot have two distinct nonzero roots; if its roots are 0 and $a \neq 0$, evaluating the linear factor at $u = a$ gives $(\frac{5}{2} - L)a \neq 0$; and if $A(u) = (u - u_0)^2$ has a nonzero double root u_0, then $a = 2u_0$ and the linear factor at $u = u_0$ equals $-Lu_0 \neq 0$.

In both parities, the two weighted initial forms have an isolated common zero at the origin and hence form a regular sequence in $\mathbb{C}[a, b]$. By weighted Bézout's theorem, the local colength M_L coincides with the colength of the initial ideal, which for quasi-homogeneous generators of weights d_1, d_2 in variables of weights $1, 2$ equals $d_1 d_2/2$ (the value at $z = 1$ of the Hilbert series $(1 - z^{d_1})(1 - z^{d_2})/((1 - z)(1 - z^2))$). Therefore

$$M_L = \begin{cases} \binom{L}{2}, & L \text{ even}, \\ \binom{L+1}{2}, & L \text{ odd}. \end{cases}$$

One or two fixed pairs ($r = 1, 2$). After fixing one singular pair ($r = 1$), the two remaining free coordinates (x, y) satisfy the reduced system $G(x, y) = G(y, x) = 0$, where

$$G(x, y) = (x + 1)^{L-1}(x - 3)(x - y - 2) - (x - 1)^{L-1}(x + 3)(x - y + 2).$$

Let $b_L = \ell_{(1,-1)}\big(G(x, y), G(y, x)\big)$ denote the local multiplicity at $(x, y) = (1, -1)$. Setting $x = 1 + u$, $y = -1 + v$, and $\eta = u - v$, solving $G(1 + u, -1 + v) = 0$ for η gives

$$\eta = \frac{4u^{L-1}(u + 4)}{(2 + u)^{L-1}(u - 2) - u^{L-1}(u + 4)} = -2^{4-L}u^{L-1}\left(1 + \frac{5 - 2L}{4}u + O(u^2)\right).$$

Substituting $v = u - \eta$ into $G(y, x)$ yields a series in u with leading term $32u^{L-1}$ for even L and $8(2L - 5)u^L$ for odd L. Hence

$$b_L = \begin{cases} L - 1, & L \text{ even}, \\ L, & L \text{ odd}. \end{cases}$$

There are $\binom{2}{1}^2 = 4$ choices of a single fixed positive and negative index pair (A, B) compatible with $p = (1, 1, -1, -1)$. Because the two free coordinates at p are $(1, -1)$ and hence distinct, only the singleton partition contributes to $Z^{\mathrm{dep}}_{A,B}$ at p, giving $b_L/2!$ per choice prior to multiplying by the averaging factor $(1!)^2 2!/4! = 2!/4!$. Thus

$$w_1(p) = 4 \cdot \frac{2!}{4!} \cdot \frac{b_L}{2!} = \frac{b_L}{6}.$$

Finally, fixing both singular pairs ($r = 2$) leaves a single compatible index choice $A = \{1, 2\}$, $B = \{3, 4\}$ and $M = 0$ free coordinates, giving

$$w_2(p) = \frac{(2!)^2 0!}{4!} = \frac{1}{6}.$$

Alternating cancellation. Combining the three weights, the net weight of $\tilde{Z}$ at $p = (1, 1, -1, -1)$ is

$$w_p(\tilde{Z}) = w_0(p) - Lw_1(p) + \binom{L}{2} w_2(p) = \frac{1}{6}\left(M_L - Lb_L + \binom{L}{2}\right) = 0.$$

Indeed, for even L we have $M_L + \binom{L}{2} = 2\binom{L}{2} = L(L - 1) = Lb_L$, whereas for odd L we have $M_L + \binom{L}{2} = \binom{L+1}{2} + \binom{L}{2} = L^2 = Lb_L$. Thus the alternating sum cancels identically at the double singular pair $p = (1, 1, -1, -1)$ (and at all $\binom{4}{2} = 6$ of its distinct permutations) for every $L \geq 8$, even though the individual weights $w_0(p)$ and $w_1(p)$ depend on the parity of L.

B Extension to the spin-S XXX chain

In this appendix, we show how our auxiliary-ideal construction and projective intersection count extend naturally to the homogeneous periodic integrable spin-S XXX chain (XXX_S). Throughout this appendix, we fix a spin $S \in \frac{1}{2}\mathbb{Z}_{>0}$, a chain length $L \geq 2$, and a magnon number $0 \leq N \leq \lfloor SL \rfloor$. We first formulate the reduced fixed-string systems and compute their global projective intersection numbers, and then compare the resulting local indices against known physicality criteria in several concrete examples.

B.1 Bethe equations and normalization

The state space of the periodic spin-S chain of length L is $\mathcal{H}_S = (\mathbb{C}^{2S+1})^{\otimes L}$. A Bethe state with N magnons (spin-lowering excitations, with at most $2S$ excitations per site) has total spin projection $S^z = SL - N$. The spin-S Bethe equations take the form

$$\left(\frac{\lambda_i + \mathrm{i}S}{\lambda_i - \mathrm{i}S}\right)^L = \prod_{j\neq i} \frac{\lambda_i - \lambda_j + \mathrm{i}}{\lambda_i - \lambda_j - \mathrm{i}}, \qquad i = 1, \dots, N; \tag{30}$$

see [8, 9]. Under the same rescaling $\lambda_i = \mathrm{i}x_i/2$ as in the main text, clearing denominators yields the polynomial Bethe equations

$$\begin{aligned} P_i^{(S)}(x) = (x_i + 2S)^L \prod_{j\neq i}(x_i - x_j - 2) \\ - (x_i - 2S)^L \prod_{j\neq i}(x_i - x_j + 2) = 0, \qquad i = 1, \dots, N. \end{aligned} \tag{31}$$

As in the spin-1/2 case, the leading terms of degree $L + N - 1$ cancel; the coefficient of the pure power x_i^{L+N-2} is $4(SL - N + 1) \geq 4 > 0$, so each $P_i^{(S)}$ has total degree $L + N - 2$. When $S = 1/2$, $P_i^{(1/2)}$ coincides with P_i in (3). (For $N = 0$, the system is empty.)

B.2 The highest-weight counting target

Let $b_S(L, N)$ denote the dimension of the weight subspace of $\mathcal{H}_S$ with $S^z = SL - N$. Since each of the L sites can accommodate between 0 and $2S$ spin flips, $b_S(L, N)$ is given by the restricted-occupancy generating function

$$b_S(L, N) = [t^N](1 + t + \cdots + t^{2S})^L.$$

In the Clebsch–Gordan decomposition of $\mathcal{H}_S$, every irreducible $\mathfrak{sl}_2$-representation of highest weight $SL - K$ ($0 \leq K \leq N \leq \lfloor SL \rfloor$) contributes a one-dimensional subspace to the weight space $S^z = SL - N$. The multiplicity of the irreducible representation of spin $SL - N$—that is, the dimension of the highest-weight subspace with N magnons—is therefore given by the difference

$$\boxed{d_S(L, N) = b_S(L, N) - b_S(L, N - 1) = [t^N]\,(1 - t)(1 + t + \cdots + t^{2S})^L,} \tag{32}$$

with the convention $b_S(L, -1) = 0$. This is the representation-theoretic target dimension for Bethe ansatz completeness [8, 9, Section 2], reducing to $d_{1/2}(L, N) = \binom{L}{N} - \binom{L}{N-1} = d(L, N)$ at $S = 1/2$.

B.3 Auxiliary ideals with fixed singular strings

Put $q = 2S + 1$ and define the singular string values

$$c_a = 2S - 2a, \qquad 0 \leq a \leq 2S.$$

In the original spectral parameter λ, a singular solution contains an exact Bethe string of length $q = 2S + 1$ centered at the origin,

$$\{\mathrm{i}S,\ \mathrm{i}(S - 1),\ \dots,\ -\mathrm{i}S\},$$

which in the rescaled x-coordinates becomes $\{c_0, c_1, \dots, c_{2S}\}$ [8, 9], generalizing the singular pair $(1, -1)$ for $S = 1/2$.

For $0 \le r \le \lfloor N/q \rfloor$, choose pairwise disjoint index subsets $A_0, \dots, A_{2S} \subseteq [N]$ with $|A_0| = \cdots = |A_{2S}| = r$, and pin $x_i = c_a$ for $i \in A_a$. Write

$$\mathbf{A} = (A_0, \dots, A_{2S}), \qquad R = [N] \setminus \bigcup_{a=0}^{2S} A_a, \qquad M = |R| = N - qr.$$

When $r \ge 1$, the Bethe equations $P_i^{(S)} = 0$ at the pinned indices $i \in \bigcup_{a=0}^{2S} A_a$ are satisfied identically: at each interior string value c_a $(1 \le a \le 2S-1)$, the adjacent string values $c_{a\mp1} = c_a \pm 2$ annihilate one interaction factor in each product of (31), while at the endpoints $c_0 = 2S$ and $c_{2S} = -2S$, one term vanishes via the vacuum factor $(x_i \mp 2S)^L$ and the other via the adjacent interaction factor. For each free index $i \in R$, we divide out the common factors arising from the r fixed strings to define the reduced Bethe polynomial

$$\begin{aligned} P_{i;\mathbf{A}}^{(S),\mathrm{red}}(x_R) &:= \frac{P_i^{(S)}(x)\big|_{x_j = c_a \ (j \in A_a)}}{\prod_{a=0}^{2S}(x_i - c_a)^r} \\ &= (x_i + 2S)^{L-r}(x_i - 2S - 2)^r \prod_{j \in R\setminus\{i\}} (x_i - x_j - 2) \\ &\quad - (x_i - 2S)^{L-r}(x_i + 2S + 2)^r \prod_{j \in R\setminus\{i\}} (x_i - x_j + 2). \end{aligned} \tag{33}$$

Here the second equality follows from the telescoping shifts $c_a + 2 = c_{a-1}$ $(1 \le a \le 2S)$ and $c_a - 2 = c_{a+1}$ $(0 \le a \le 2S-1)$, together with $r \le N/q \le SL/(2S+1) < L$, which ensures that $P_{i;\mathbf{A}}^{(S),\mathrm{red}}$ is a polynomial in the free variables x_R. For $M \ge 1$, its leading pure power x_i^{L+M-2} retains the nonzero coefficient $4(SL - N + 1)$, so $P_{i;\mathbf{A}}^{(S),\mathrm{red}}$ has degree

$$D = L + M - 2.$$

Given a partition $\pi = \{B_1, \dots, B_m\} \in \Pi(R)$ of the free indices with chosen block representatives $b_\alpha \in B_\alpha$, we define the auxiliary ideal

$$\begin{aligned} I_{\mathbf{A};\pi}^{(S)} = (&x_i - c_a \quad (0 \le a \le 2S,\ i \in A_a), \\ &P_{b_\alpha;\mathbf{A}}^{(S),\mathrm{red}} \quad (1 \le \alpha \le m), \\ &x_j - x_{b_\alpha} \quad (1 \le \alpha \le m,\ j \in B_\alpha \setminus \{b_\alpha\})) \subseteq \mathbb{C}[x_1, \dots, x_N], \end{aligned} \tag{34}$$

which is generated by $qr + m + (M - m) = N$ polynomials, is independent of the choice of block representatives, and has support contained in the full spin-S Bethe locus. Assuming $I_{\mathbf{A};\pi}^{(S)}$ is zero-dimensional, we denote its associated formal cycle by $Z_{\mathbf{A};\pi}^{(S)} = Z(I_{\mathbf{A};\pi}^{(S)})$.

B.4 Projective intersection numbers and the alternating identity

Homogenizing the m reduced equations to degree $D = L + M - 2$ in $\mathbb{P}^N$ before restricting to the linear subspace $\mathbb{P}^m \subseteq \mathbb{P}^N$ cut out by the qr fixed-coordinate hyperplanes and $M - m$ block equalities yields the projective intersection number

$$\deg_{\mathbb{P}^N} Z_{\mathbf{A};\pi}^{(S)} := \int_{\mathbb{P}^N} H^{N-m}(DH)^m = D^m = (L + M - 2)^m, \tag{35}$$

where H is the hyperplane class on $\mathbb{P}^N$ (for $M = 0$, $m = 0$ and $\deg_{\mathbb{P}^N} Z_{\mathbf{A};\varnothing}^{(S)} = 1$).

We define the depermuted fixed-string cycles and their S_N-averages by

$$\begin{aligned} Z_{\mathbf{A}}^{(S),\mathrm{dep}} &= \frac{1}{M!}\sum_{\pi\in\Pi(R)} \mu(\hat{0},\pi) Z_{\mathbf{A};\pi}^{(S)}, \\ Z_r^{(S),\mathrm{dep}} &= \frac{(r!)^q M!}{N!}\sum_{\mathbf{A}} Z_{\mathbf{A}}^{(S),\mathrm{dep}}, \end{aligned} \tag{36}$$

where the second sum ranges over all $N!/((r!)^q M!)$ tuples $\mathbf{A} = (A_0, \ldots, A_{2S})$ of pairwise disjoint subsets of $[N]$ of size r. By the same partition-lattice identity as in Proposition 2.7,

$$\deg_{\mathbb{P}^N} Z_{\mathbf{A}}^{(S),\mathrm{dep}} = \deg_{\mathbb{P}^N} Z_r^{(S),\mathrm{dep}} = \frac{1}{M!}\sum_{\pi\in\Pi(R)} \mu(\hat{0},\pi) D^{|\pi|} = \binom{D}{M} = \binom{L+M-2}{M}.$$

We then form the alternating candidate cycle

$$\tilde{Z}^{(S)} := \sum_{r=0}^{\lfloor N/q \rfloor} (-1)^r \binom{L}{r} Z_r^{(S),\mathrm{dep}}. \tag{37}$$

Proposition B.1 (Spin-S projective completeness count)**.** *With $q = 2S+1$, the projective intersection number of $\tilde{Z}^{(S)}$ satisfies*

$$\boxed{\deg_{\mathbb{P}^N} \tilde{Z}^{(S)} = \sum_{r=0}^{\lfloor N/(2S+1) \rfloor} (-1)^r \binom{L}{r} \binom{L+N-(2S+1)r-2}{N-(2S+1)r} = d_S(L,N).} \tag{38}$$

Proof. Substituting $\deg_{\mathbb{P}^N} Z_r^{(S),\mathrm{dep}} = \binom{L+M-2}{M}$ with $M = N - qr$ gives the first equality in (38). For the second equality, multiplying the generating series in $\mathbb{Q}[[t]]$ yields

$$\begin{aligned} \left(\sum_{r=0}^{L} (-1)^r \binom{L}{r} t^{qr}\right)\left(\sum_{M\geq 0} \binom{L+M-2}{M} t^M\right) &= \frac{(1-t^q)^L}{(1-t)^{L-1}} \\ &= (1-t)\left(1+t+\cdots+t^{2S}\right)^L. \end{aligned}$$

Extracting the coefficient of t^N and comparing with (32) completes the proof. □

Notice that the alternating sum in (38) matches the representation-theoretic formula obtained from $b_S(L,N) - b_S(L,N-1)$ in [8, Section 2] and [9, Section 2.5, eqs. (2.31)–(2.32)], reinterpreting each r-term as the depermuted projective Bézout degree of the reduced Bethe system with r pinned singular $(2S+1)$-strings. For $S = 1/2$ ($q = 2$), Proposition B.1 reduces to Proposition 2.7; for $S = 1$ ($q = 3$), each step pins a singular triplet $(2, 0, -2)$.

B.5 Local normalization and the comparison problem

A key new feature of higher-spin chains ($S \geq 1$) is the existence of *strange* physical solutions possessing repeated Bethe roots [8, 9]. Consequently, depermutation over $\Pi(R)$ should not annihilate every root configuration on the collision locus; rather, it should assign the appropriate orbit-normalized weight to physical repeated-root configurations while canceling nonphysical ones.

Specifically, let $p \in \mathbb{C}^N$ have m_ξ coordinates equal to ξ for each $\xi \in \mathbb{C}$, and set

$$h(p) = \prod_{\xi\in\mathbb{C}} m_\xi!, \qquad Q_p(t) = \prod_{i=1}^{N} (t - p_i),$$

so that $h(p)$ is the order of the stabilizer of p in S_N and the S_N-orbit of p has size $N!/h(p)$. Denoting by $\sigma : x \mapsto Q_x$ the quotient map to monic polynomials of degree N, we define the unordered local index $\iota_S(Q_p)$ by

$$\iota_S(Q_p) = w_{Q_p}(\sigma_* \tilde{Z}^{(S)}), \qquad \text{so that} \qquad w_p(\tilde{Z}^{(S)}) = \frac{h(p)}{N!}\, \iota_S(Q_p) \tag{39}$$

by S_N-invariance. The desired local completeness property is that $\iota_S(Q_p) = 1$ if Q_p corresponds to a physical solution (including physical singular or strange solutions) and $\iota_S(Q_p) = 0$ otherwise; in ordered coordinates, the target weight of p is then $h(p)/N!$.

To express $\iota_S(Q_p)$ in terms of the generalized TQ remainder ideals of Section 2.2, define the singular string polynomial

$$C_S(t) = \prod_{a=0}^{2S} (t - c_a)$$

and, for $0 \le r \le \min_{0\le a\le 2S} m_{c_a}$, the reduced polynomial

$$Q_{p,r}(t) = \frac{Q_p(t)}{C_S(t)^r}.$$

For a general monic polynomial $H(t)$ of degree $M = N - (2S+1)r$, set

$$\begin{aligned}\Phi_r^{(S)}(t; H) = {} & (t+2S)^{L-r}(t-2S-2)^r H(t-2) \\ & + (t-2S)^{L-r}(t+2S+2)^r H(t+2),\end{aligned}$$

let $J_r^{(S)}$ be the ideal in the coefficient space $\mathrm{Sym}^M(\mathbb{A}^1) \simeq \mathbb{A}^M$ generated by the M coefficients of the remainder $\mathrm{rem}_H \Phi_r^{(S)}$, and put

$$\nu_r(p) = \ell_{Q_{p,r}}(J_r^{(S)}),$$

with $\nu_r(p) = 0$ if $Q_{p,r} \notin V(J_r^{(S)})$ and $\nu_r(p) = 1$ when $M = 0$. Since there are $\prod_{a=0}^{2S} \binom{m_{c_a}}{r}$ choices of $\mathbf{A}$ supported at p, and each choice has residual stabilizer of order $\prod_{a=0}^{2S}(m_{c_a} - r)! \prod_{\xi \notin \{c_a\}} m_\xi! = h(p)/\big((r!)^{2S+1} \prod_{a=0}^{2S} \binom{m_{c_a}}{r}\big)$, multiplying by the prefactor $(r!)^{2S+1}/N!$ in (36) produces the exact orbit factor $h(p)/N!$ for every r. Under the pushforward relation $\sigma_* Z_{\mathbf{A}}^{(S),\mathrm{dep}} = Z(J_r^{(S)})$, we therefore obtain

$$\boxed{\iota_S(Q_p) = \sum_{r=0}^{\min_a m_{c_a}} (-1)^r \binom{L}{r} \nu_r(p).} \tag{40}$$

In particular, if p does not contain a complete singular string $\{c_0, \dots, c_{2S}\}$, then only $r = 0$ contributes and $\iota_S(Q_p) = \nu_0(p)$.

In the concrete examples below, the identity linking the depermuted local multiplicity in ordered coordinates to $\nu_r(p)$ in coefficient coordinates can be verified directly without invoking a general deformation argument:

- At a configuration with pairwise distinct free coordinates, the quotient map to coefficient space is locally étale and evaluating the remainder $\mathrm{rem}_H \Phi_r^{(S)}$ at the distinct roots is an invertible Vandermonde transformation of the remainder coefficients.
- At a configuration p with a single double root $p_1 = p_2$ and all other coordinates distinct, put

$$\delta = x_1 - x_2, \qquad K = \frac{F_1 - F_2}{\delta}, \qquad B = \widehat{\mathcal{O}}_{\mathbb{A}^N,p}/(F_1, F_3, \dots, F_N),$$

assuming the local intersections are zero-dimensional. Then the full and single-diagonal local algebras at p are $B/(\delta K)$ and $B/(\delta)$, respectively. Additivity of intersection multiplicities along the complete-intersection curve $\operatorname{Spec} B$ gives

$$\dim_{\mathbb{C}} B/(\delta K) - \dim_{\mathbb{C}} B/(\delta) = \dim_{\mathbb{C}} B/(K) = 2\nu_0(p),$$

where the factor of 2 comes from the degree-2 finite flat base change between ordered coordinates (x_1, x_2) and symmetric coordinates $(x_1 + x_2, x_1x_2)$, while replacing (F_1, F_2) by (F_1, K) corresponds via Newton interpolation to the linear remainder modulo $(t-x_1)(t-x_2)$. Since only the singleton partition and the partition pairing $\{1, 2\}$ contribute at p, their Möbius combination $\frac{1}{N!}(\dim_{\mathbb{C}} B/(\delta K) - \dim_{\mathbb{C}} B/(\delta)) = \frac{2}{N!}\nu_0(p)$ yields $\iota_S(Q_p) = \nu_0(p)$ prior to fixed-string corrections.

B.6 Explicit local comparisons

1. A physical nonsingular double root ($S = 3/2$, $L = 3$, $N = 2$). Writing the two ordered coordinates as (x, y), the polynomial Bethe equations (31) for $S = 3/2$, $L = 3$, $N = 2$ simplify to

$$P_1^{(3/2)}(x, y) = 14x^3 - 18x^2y - 54x - 54y = 0,$$
$$P_2^{(3/2)}(x, y) = 14y^3 - 18y^2x - 54y - 54x = 0.$$

Restricting to the collision diagonal $x = y$ gives $-4x(x^2 + 27) = 0$, so there are three diagonal roots: $(0, 0)$ and $(\pm 3\sqrt{3}\mathrm{i}, \pm 3\sqrt{3}\mathrm{i})$. At $p = (0, 0)$, taking the sum of the two equations yields

$$P_1^{(3/2)} + P_2^{(3/2)} = (x + y)(14(x + y)^2 - 60xy - 108).$$

Since the second factor is a unit in $\mathbb{C}[[x, y]]$, the full local ideal at $(0, 0)$ enforces $y = -x$, reducing $P_1^{(3/2)}$ to $32x^3 = 0$. Hence the full system has local multiplicity $\ell_{(0,0)}(P_1^{(3/2)}, P_2^{(3/2)}) = 3$, whereas the diagonal ideal $(P_1^{(3/2)}, x - y) = (-4x(x^2 + 27), x - y)$ has local multiplicity 1 at $(0, 0)$. Since $N = 2 < 2S + 1 = 4$, no complete singular string can occur, and we obtain

$$w_{(0,0)}(\tilde{Z}^{(3/2)}) = \frac{3 - 1}{2!} = 1, \qquad \iota_{3/2}(t^2) = 1.$$

By contrast, at each of the other two diagonal roots $(\pm 3\sqrt{3}\mathrm{i}, \pm 3\sqrt{3}\mathrm{i})$, both the full system and the diagonal system have simple zeros (multiplicity 1), so their Möbius difference vanishes: $\iota_{3/2}((t \mp 3\sqrt{3}\mathrm{i})^2) = 1 - 1 = 0$.

This matches the physicality criterion for repeated roots in [8, Section 3.2, eq. (3.2), and Appendix B.1]: for a double root $\{\mu, \mu\}$ in the original λ-coordinates, physicality requires

$$R(\mu) = -1, \qquad R'(\mu) = -4\mathrm{i}, \qquad \text{where} \qquad R(\mu) = \left(\frac{\mu - \mathrm{i}S}{\mu + \mathrm{i}S}\right)^L.$$

For $S = 3/2$ and $L = 3$, $R(\mu) = -1$ has the three roots $\mu = 0$ and $\mu = \pm\frac{3\sqrt{3}}{2}$, of which only $\mu = 0$ satisfies $R'(0) = -4\mathrm{i}$ (giving the simplest strange solution $\{0, 0\}$ in [8, Table 5]). Thus depermutation retains the unique physical double root with index 1 and cancels the two nonphysical double roots.

2. A single complete singular string ($S = 1$, $L = 3$, $N = 3$). Consider the singular string configuration $p = (2, 0, -2)$ for $S = 1$, $L = 3$, $N = 3$, corresponding to $\{\mathrm{i}, 0, -\mathrm{i}\}$ in λ-coordinates. Parametrize monic cubic polynomials near $Q_p(t) = t(t^2 - 4)$ by

$$Q(t) = t^3 + at^2 + (-4 + b)t + c.$$

Up to nonzero scalar multiples, the remainder coefficients of $\mathrm{rem}_Q\,\Phi_0^{(1)}$ generate the ideal

$$J_0^{(1)} = (ac + 4b,\ -ab + 8a + 6c,\ -a^2 + 3b) \subseteq \mathbb{C}[[a, b, c]].$$

Eliminating $b = a^2/3$ and $c = a^3/18 - 4a/3$ via the last two generators reduces the first generator to $a^4/18 = 0$, so $\nu_0(p) = \dim_{\mathbb{C}} \mathbb{C}[[a]]/(a^4) = 4$. Fixing the complete singular string ($r = 1$) leaves $M = 0$ free coordinates, so $\nu_1(p) = 1$ and (40) gives

$$\iota_1(t(t^2 - 4)) = \nu_0(p) - \binom{3}{1}\nu_1(p) = 4 - 3 \cdot 1 = 1, \qquad w_p(\tilde{Z}^{(1)}) = \frac{1}{3!}.$$

This agrees with the physicality condition $1 = (-1)^{2SL}$ for a single complete singular string with no additional roots [9, Section 4.1, eq. (4.14)], which holds identically for $S = 1$, $L = 3$ (see [9, Table 1]).

3. A repeated singular root ($S = 1$, $L = 4$, $N = 4$). Finally, consider $p = (2, 0, 0, -2)$ for $S = 1$, $N = 4$, corresponding to the strange singular solution $\{\mathrm{i}, 0, 0, -\mathrm{i}\}$ in [8, Section B.2.1 and Table 2] and [9, Section 4.1, Example 3]. Here $m_2 = m_{-2} = 1$ and $m_0 = 2$, so p contains only $\min_a m_{c_a} = 1$ complete singular string $(2, 0, -2)$ together with a repeated central root 0. Parametrize monic quartic polynomials near $Q_p(t) = t^2(t^2 - 4)$ by

$$Q(t) = t^4 + at^3 + (-4 + b)t^2 + ct + d.$$

For $L = 4$, dividing the four coefficients of $\mathrm{rem}_Q\,\Phi_0^{(1)}$ by 8 yields the generators

$$\begin{aligned}
r_0 &= a^2 d - 3bd + 16b - 8d,\\
r_1 &= a^2 c - ad - 16a - 3bc - 24c,\\
r_2 &= a^2 b - 4a^2 - ac - 3b^2 - 4b + 10d,\\
r_3 &= a^3 - 4ab + 4a + 6c.
\end{aligned}$$

Because the Jacobian matrix of (r_0, r_2, r_3) with respect to (b, c, d) is invertible at $(a, b, c, d) = (0, 0, 0, 0)$, the formal implicit function theorem expresses b, c, d as power series in a:

$$\begin{aligned}
b &= \frac{5}{24}a^2 - \frac{347}{18432}a^4 + O(a^6),\\
d &= \frac{5}{12}a^2 - \frac{167}{9216}a^4 + O(a^6),\\
c &= -\frac{1}{6}a^3 + \frac{2}{3}ab - \frac{2}{3}a.
\end{aligned}$$

Substituting these expansions into r_1 gives

$$r_1 = \frac{625}{2304}a^5 + O(a^7),$$

so $\mathbb{C}[[a, b, c, d]]/J_0^{(1)} \simeq \mathbb{C}[[a]]/(a^5)$ and $\nu_0(p) = 5$. After fixing one complete string $(2, 0, -2)$ ($r = 1$), the single remaining free root t satisfies the reduced equation

$$(t + 2)^3(t - 4) - (t - 2)^3(t + 4) = 4t(t^2 - 20) = 0,$$

for which $t = 0$ is a simple root, giving $\nu_1(p) = 1$. Equation (40) then yields

$$\iota_1(t^2(t^2 - 4)) = \nu_0(p) - \binom{4}{1}\nu_1(p) = 5 - 4 \cdot 1 = 1, \qquad w_p(\tilde{Z}^{(1)}) = \frac{2!}{4!} = \frac{1}{12}.$$

Summing $w_p(\tilde{Z}^{(1)}) = 1/12$ over the $4!/2! = 12$ distinct permutations of $p = (2, 0, 0, -2)$ gives total weight 1, retaining this physical repeated singular root.

For general chain length $L \geq 4$ at $N = 4$ (where there are no additional regular roots), the two physicality constraints for $\{\mathrm{i}, 0, 0, -\mathrm{i}\}$ in [8, Section B.2.1, eq. (B.16)] and [9, Section 4.1, eq. (4.21)] both reduce to $1 = (-1)^L$, so $p = (2, 0, 0, -2)$ is physical if and only if L is even. When $L \geq 5$ is odd, polynomiality of T already excludes $Q_p(t) = t^2(t^2 - 4)$ from $V(J_0^{(1)})$: writing $\Phi_0^{(1)}(t; Q_p) = t(t^2 - 4)B_L(t)$ with

$$B_L(t) = (t + 2)^{L-1}(t - 2)(t - 4) + (t - 2)^{L-1}(t + 2)(t + 4),$$

we have $B_L(0) = 2^{L+2}(1 + (-1)^{L-1}) = 2^{L+3} \neq 0$ for odd L, so $Q_p \nmid \Phi_0^{(1)}(t; Q_p)$, and similarly the reduced $r = 1$ polynomial does not vanish at $t = 0$; hence $\nu_0(p) = \nu_1(p) = 0$ and $\iota_1(Q_p) = 0$. For even $L = 6, 8$, repeating the power-series elimination above yields $\nu_0(p) = L + 1$ and $\nu_1(p) = 1$, as summarized below:

L	4	5	6	7	8
$\nu_0(p)$	5	0	7	0	9
$\nu_1(p)$	1	0	1	0	1
$\iota_1(Q_p) = \nu_0(p) - L\nu_1(p)$	1	0	1	0	1

In every tested case, the local index $\iota_1(Q_p)$ matches the physicality criterion $1 = (-1)^L$. Proving $\nu_0(p) = L + 1$ for arbitrary even L and extending the local multiplicity analysis to general higher-spin string configurations remain open problems for future work.

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