Exact results for self-avoiding billiards

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Abstract

We extend the spatial-memory billiard of Albers, Delnoij, Schramma, and Jalaal (2024) by deriving exact results for finite collisions and infinite capture. For polygonal tables, a fixed collision word gives affine collision positions and linear first-contact conditions. We bound the number of collision words using the planar boundary, distinguish endpoint returns from normal impacts in regular polygons, and give explicit endpoint families. We derive the complete third-incidence law in the equilateral triangle and the joint first-memory-impact law in the disk. The latter has a cubic survival tail for the collision count and infinitely many radial density singularities. Clean triangular regions admit a finite classification for rational angles and an explicit nonperiodic capture family. Exact capture maps and path-length distributions follow for families in the equilateral triangle and regular pentagon.

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1 Introduction

In *Billiards with Spatial Memory* (2024), Thijs Albers, Stijn Delnoij, Nico Schramma, and Maziyar Jalaal introduced the self-avoiding billiard in which a particle reflects from its own trajectory as well as from a fixed boundary [1]. This article extends that study with exact results for polygonal and circular tables. The questions are finite collision geometry, the laws of early collisions, the distinction between endpoint and normal impacts, and the construction of trajectories with infinitely many reflections and finite total length.

The first results describe a collision history by its directions and supporting lines. This gives linear conditions on the starting point at a fixed direction and permits a combinatorial bound on the number of histories. Two probability laws use the same contact rule in different geometries. For the equilateral triangle the third incidence angle has an atom at $\pi/3$. For the disk the first memory impact has a collision-count tail of order n^{-3} and a radial law with infinitely many singularities. The later sections give arithmetic criteria for two kinds of finite terminal contact and explicit finite-length capture families. Polygonal unfolding follows the standard construction surveyed in [2].

1.1 AI contribution and human review

Yi-Hsien Lai directed the project and wrote the original model definitions. OpenAI Codex carried out all the mathematical derivations presented in this manuscript, drafted the English text, and prepared the Hexagon deposit. Codex handles the mechanical submission on Lai's behalf. Lai is the submitting contributor.

Lai has reviewed and understands the reasoning in the following parts of the work. In Section 2, *Collision geometry and symbolic histories*, his review covers the model definitions. In Section 2.2, *Affine branches and endpoint boundaries*, it covers the integer representation of flight directions and the parallelism of corresponding flights that share an initial direction and collision history. His review of Section 8, *Triangular confinement and finite-length capture*, and Section 8.1, *Rational angles*, covers the obtuse triangular cage with rational angles, the finite angle state space and its stopping or cycling alternatives, and why a repeating shape implies contraction, convergence to a

point, and finite total path length. Lai has yet to finish reading the manuscript and commits to reading in full and proofreading again the portions within his ability to assess.

2 Collision geometry and symbolic histories

Let P_0 be inside a convex polygon or disk and let u_0 be a unit direction. At each step take the first positive-distance contact with the table or any completed segment $L_j = [P_{j-1}, P_j]$. A regular contact is a unique transverse hit in an open wall segment, or on the smooth circle, with nonnormal incidence. Reflect in the local tangent. A vertex, a historical endpoint, a multiple contact, or the onset of an overlap ends the regular history. We record a normal impact separately and stop there because specular reflection would retrace the incoming segment. Incidence angles are measured from the tangent and folded into $[0, \pi/2]$. In probabilistic statements the starting point is uniform in area and the initial direction is independent and uniform, unless specified otherwise.

2.1 The first-contact test

Write $[v, w] = v_x w_y - v_y w_x$. If $[u, v] \neq 0$, the intersection of $P + tu$ with a segment $Q + [0, l]v$ is given by

$$t = \frac{[Q - P, v]}{[u, v]}, \qquad s = \frac{[Q - P, u]}{[u, v]}.$$

Keep $t > 0$ and $0 \leq s \leq l$ and minimize t over all walls; a regular hit also requires $0 < s < l$. A parallel noncollinear wall gives no candidate. In the collinear case, intersect the closed parameter interval with the positive ray, distinguishing a first endpoint from overlap issuing from the ray origin. The reflection is $u' = 2(u \cdot v)v/(v \cdot v) - u$. In particular, equality of the intersection times of two *supporting lines* does not suffice for a multiple contact. With both finite-segment conditions satisfied, equal times give the same spatial point. On a regular prefix two existing walls cannot cross in the interiors of both segments. Initially this holds for the table walls. Each new segment meets the old walls only at its own endpoints, since it is stopped at the first contact. Thus a positive-distance tie entails an endpoint of at least one wall.

2.2 Affine branches and endpoint boundaries

A collision history is the word of wall labels, using the original wall labels and the creation indices of the L_j. The direction at every step is determined by the word and the initial direction. A reflection in a wall with tangent angle β takes θ to $2\beta - \theta$. For an old segment its tangent is a previous flight direction, modulo π. In a regular h-gon with tangent angles $\beta_j = \beta_0 + 2\pi j/h$, put $\psi = \theta_0 - \beta_0$. The direction of each flight has the form

$$\theta = \beta_0 + a\psi + 4\pi m/h \pmod{2\pi}, \qquad a \in 2\mathbb{Z} + 1, \quad m \in \mathbb{Z}.$$

The initial label is $(1, 0)$. A fixed wall j sends (a, m) to $(-a, j - m)$. A memory wall with flight label (b, k) sends it to $(2b - a, 2k - m)$. These updates prove the claim by induction, including the oddness of a.

Fix the direction and write $P_0 = su_0 + \eta u_0^{\perp}$, where $u_0^{\perp}$ is the counterclockwise quarter-turn of u_0. For each feasible word, the post-impact positions depend affinely on η alone. The first flight line depends on η, so its intersection with the chosen table side does too. Successive flight lines intersect fixed-direction supports of the specified walls. Using the determinant formula with unit tangents, the denominators are constant and the numerators are affine. Each determined intersection therefore

remains affine in η. Only the free endpoint P_0 of L_1 depends on s, and this affects membership of an intersection in L_1 without changing its location.

For a more direct recurrence, define a flight support by $[u_i, X] = d_i$. If the wall has unit tangent v and equation $[v, X] = e$, substituting any point of intersection in the reflected support yields

$$d_{i+1} = -d_i + 2(u_i \cdot v)e.$$

Consequently $d_i = c_i\eta + b_i$ with $c_0 = 1$. Fixed walls send c_i to $-c_i$. A hit on an earlier support with coefficient c_j sends c_i to $-c_i + 2\gamma c_j$, where $\gamma = u_i \cdot u_j$. The quadratic form $c_i^2 + c_j^2 - 2\gamma c_i c_j$ is invariant under this single update and is positive definite for a transverse impact. Numbering flights from 1, their first three coefficients have absolute value 1 because the first two regular hits are on the table. The flight leaving a point cannot transversely meet the segment just completed at a later point. Thus the memory label of a possible target at flight i is at most $i-2$. The bounds $F_1 = F_2 = F_3 = 1$ and $F_{i+1} = F_i + 2F_{i-2}$ bound both $|c_i|$ and the absolute direction coefficient $|a_i|$, by induction.

For fixed θ_0, legality and the ordering of transverse contacts are Boolean combinations of linear inequalities. In a regular history all old endpoints and lengths are affine, and a potential hit is tested against the finite interval above. Parallel supports can coincide only on affine equalities and the collinear interval test uses linear comparisons there. The resulting parameter sets are finite unions of relatively open polygonal cells. For varying θ_0, the direction updates make the trigonometric coefficients rational in $\tan(\theta_0/2)$ on the charts excluding its pole. Clearing denominators with their signs gives a semialgebraic description. A history count and a count of distinct analytic state maps are different quantities. Multiple words can define the same map on different domains.

For a candidate endpoint E, the equality $[u_i, E - P_i] = 0$ is necessary. For all endpoints other than P_0, a nontrivial equality fixes η. For P_0 it can involve both s and η. In complex notation a support equation is $\mathrm{Im}(ze^{-i\alpha}) = p\eta + q$. A fixed vertex $e^{2\pi im/h}$ of the circumradius-one regular h-gon furnishes the candidate $\eta = (\sin(2\pi m/h - \alpha) - q)/p$ when $p \neq 0$. For a moving endpoint shared by two nonparallel supports, set the determinant of the three support equations to zero. The result is affine in η, since only the column of offsets depends on it. For the free endpoint the condition is

$$\sin(\theta_0 - \alpha)s + (\cos(\theta_0 - \alpha) - p)\eta - q = 0.$$

An identically vanishing equation requires the forward and first-contact tests. At a unique transverse regular first contact with strictly interior wall parameter, the label is locally constant under perturbations that keep the prefix regular and exclude extra contact at distance zero. A switch otherwise requires the loss of a finite-segment intersection or a tie, which entails an endpoint when the regularity conditions hold. For all walls other than L_1, the forward contact distances are affine functions of η on their intervals of validity. Their lower envelope gives the first such contact; the intersecting support of L_1 contributes only where the free endpoint lies behind that intersection and the distance beats the envelope. These observations justify cutting a convex parameter cell by candidate lines and checking each resulting regular cell.

2.3 Bounds by recording the accessible face

Write $B_n^{(h)}$ for the number of length-n regular words at a fixed direction in a regular h-gon. At the first hit at most $c_h = \lceil h/2 \rceil$ sides have outward-pointing flux. The next hit cannot be on the departure wall or the segment just completed, so counting all other labels gives the elementary bound $B_n^{(h)} \leq c_h(h+n-2)!/(h-1)!$ for $n \geq 1$.

A stronger bound follows from a planar graph with edges subdivided at every contact. Attach the first segment to the table boundary. Its free end is the unique leaf. Each later segment joins two existing graph points and cuts the accessible face. The only bridge is the surviving piece of L_1 incident to the free end: adding an edge between connected vertices puts it in a cycle, and hitting the bridge puts its part between the attachment and the hit into a cycle. The accessible face may lose the bridge. Root its oriented boundary walk at the sector occupied by the outgoing ray. Occurrences of a bridge in that walk are distinct. Label all subedges with the original object label.

For a face without a bridge write P_l for a walk of length l. With a bridge write $Q_{a,b}$, with a occurrences before and b after the two adjacent bridge occurrences. Their adjacency follows from the degree-one bridge tip. A transverse ray cannot strike either occurrence incident to its departure point. A candidate edge and the retained side specify the new walk and new root: split the struck edge, inherit its label, and add the flight with its creation label. For a bridge, both occurrences must be split. The record thus recovers the next object label and iterates, even if different starting points produce different walks for a given word. Allowing both retained sides only increases the number of words.

For P_l, cutting edge i with $1 \le i \le l-2$ gives P_{i+2} and P_{l-i+1}. Thus each P_j, $3 \le j \le l$, occurs twice as a child. For $Q_{a,b}$, striking one of the $a-1$ eligible edges before the bridge gives pairs $(Q_{r,b+1}, P_{r+2})$ with $1 \le r < a$, and striking after it gives $(Q_{a+1,r}, P_{r+2})$ with $1 \le r < b$. The first pair follows by counting the r edges on the retained bridge side after replacing the struck edge by its remaining piece; the other side is closed by the new flight. If $a, b \ge 1$, the additional children are $Q_{0,b+2}$, $Q_{a+2,0}$, P_{a+2}, and P_{b+2}. For example, on cutting the first bridge occurrence, the retained tip side has the new opposing bridge pieces adjacent to its root and two extra nonbridge occurrences on the b side. The opposite side has $a+2$ occurrences. The mirror cut gives the other two. A bridge incident to the root is excluded on both occurrences. The initial state is $Q_{0,h+1}$ or its mirror. One may include the formal state $Q_{1,1}$ to close the relaxed system.

For $2 < x < 3$ set $\lambda = 2x/(x-1)$, $A = 4/(x(3-x))$, and $D = A/2$. Assign weights

$$p_l = x^{l-3}, \quad q_{a,b} = Ax^{a+b} - x^a - x^b \ (a, b \ge 1), \quad q_{0,b} = Dx^b - 1 \ (b \ge 3).$$

Use the mirrored weight when $b = 0$. Each is at least one. The child sum subtracted from λ times the parent weight is $2/(x-1)$ for P_l, $x((b-1)x^a + (a-1)x^b)$ for $Q_{a,b}$ with $a, b \ge 1$, and

$$x^{b-1} + \frac{Ax^2 - 3x + 1}{x-1} + x(b-1)$$

for $Q_{0,b}$. These identities follow by summing the finite geometric progressions in the child lists and using $A = 2D$. All differences are nonnegative. Iteration bounds the number of leaves by the initial weight times λ^{n-1}, so

$$B_n^{(h)} \le c_h \left(\frac{2x^h}{3-x} - 1 \right) \left(\frac{2x}{x-1} \right)^{n-1}.$$

Choosing $x = 3 - 1/n$ for $n \ge 2$ gives $B_n^{(h)} \le 2e^{1/6} c_h 3^{h-1} n 3^n$. Indeed, $\lambda/3 = 1 + 1/(3(2n-1))$, and the initial weight is at most $2n3^h$.

For a tighter finite-depth comparison, replace the sum over the two sides of each nonbridge target by their maximum. The outgoing ray lies on a side of the new chord determined by the sign of $[u, u']$, which is fixed by the directions throughout a labeled prefix. The bridge is a single transverse target, approached from the one side prescribed by the direction. Replace all four bridge alternatives by their maximum. With initial values one on valid states, these recurrences bound the number of suffixes by induction. A bridge-free triangle has at most one suffix at each depth, because it has

a single admissible target and each retained face is again triangular. A bridge-free quadrilateral has relaxed bound 2^m at depth m, and for fixed $l \geq 4$ the bound is $O_l((m+1)^{l-4}2^m)$. There are exactly two opportunities to retain length l, and the others have smaller length, giving the estimate by induction on l.

2.4 An angle-only bound for an interior launch

An independent bound keeps the oriented edge directions of the rooted boundary walk and the current direction α. A candidate target edge with direction β_i can be approached only if $\sin(\alpha - \beta_i) < 0$, since the interior lies to its left. Omit the two root-adjacent occurrences. At the root, and again at the new root after reflection, require strict inward orientation with respect to both adjacent edges. Write $\alpha' = 2\beta_i - \alpha$ and $\delta = \sin(\alpha' - \alpha)$. If $\delta < 0$, the retained word of edge directions is $(\alpha + \pi, \beta_0, \ldots, \beta_i)$. If $\delta > 0$, it is $(\beta_i, \ldots, \beta_{l-1}, \alpha)$. This follows by closing the retained boundary arc with the chord in the direction that leaves the retained face on the left. The zero case is singular. Every regular history generates an allowed walk in this graph. The converse may fail because the graph tracks directions without specifying finite lengths.

In the equilateral triangle with circumradius one, vertices at angles $0, 2\pi/3, 4\pi/3$, and initial direction $\pi/6$, there is one possible initial wall. Normalize each direction word by subtracting its first edge direction. With angles in multiples of $\pi/6$, the outgoing angle is 2 in each of the seven reachable states, and the direction words and transitions are

0	$(0, 6, 10, 2, 6, 10)$	$1, 2$
1	$(0, 4, 10)$	3
2	$(0, 4, 10, 2, 6, 10)$	$0, 4$
3	$(0, 6, 10)$	1
4	$(0, 4, 8, 2, 6, 10)$	$2, 5$
5	$(0, 4, 8, 0, 6, 10)$	$4, 6$
6	$(0, 4, 8, 0, 4, 10)$	$5, 3$

The first row follows from the two copies of the initial flight. Each subsequent row follows from the angle tests. If A is the matrix of this table, the count at impact n is bounded by $C_n = e_0^{\mathsf{T}} A^{n-1}\mathbf{1}$. Solving the seven linear generating-function equations gives $\sum_{n\geq 1} C_n z^{n-1} = (1+z-2z^2-z^3)/((1-z)(1-3z^2))$. In particular $C_1 = 1$, $C_{2k} = (3^k+1)/2$, and $C_{2k+1} = (5\cdot 3^k + 3)/6$ for $k \geq 1$. This is an $O(3^{n/2})$ upper bound at this direction.

3 The third-collision law in an equilateral triangle

Use the unit triangle with vertices $(0,0), (1,0), (1/2, \sqrt{3}/2)$ and area $A = \sqrt{3}/4$. The first two regular impacts are on distinct table sides. On leaving the second side, the next regular target is the first side, the remaining side, or L_1. The second flight cannot be struck from one of its own endpoints. The dihedral group acts transitively on the six ordered pairs of distinct sides. It suffices to take the bottom side followed by the right side.

Write $\theta_0 = -\theta$, $h = \sqrt{3}$, $s = \sin\theta$, and $c = \cos\theta$ in this section. The first hit is $Q = (a, 0)$, and the starting point is $(a - zc, zs)$ with Jacobian s. Backward from Q, the exit distance ℓ is the least positive value among $ha/(s + hc)$ and $h(1-a)/(s - hc)$. The first reflected ray hits the right side exactly when $a > \max(0, (s - hc)/(2s))$ and $0 < \theta < 2\pi/3$. These tests follow by intersecting the backward and forward rays with the two lines of slope $\pm h$.

The first two reflections give directions θ and $4\pi/3 - \theta$. The third ray intersects the first support at backward distance

$$r = (1-a)(hs-c)$$

from Q. To reach a point behind Q and ahead of the second impact requires $\pi/6 < \theta < \pi/2$. Within that range the finite-memory condition is $r < z < \ell$. The two supporting directions differ by $4\pi/3$, so the incidence at every such hit is $\pi/3$. Outside the memory class the angle varies with θ with slope of magnitude one.

Let M, B, L denote starting areas at a fixed θ for third targets L_1, the bottom side, and the left side, respectively. Write

$$D = s + hc, \quad H = hs - c, \quad T = \frac{DH}{2s(hs+c)}, \quad b = \frac{3s - hc}{2s}, \quad a_* = \frac{D}{2s}.$$

For $\pi/6 < \theta < \pi/3$, the area above $z = r$ starts at $a = T$ and is a triangle in (a, z) coordinates. Its area times s is $s(h + DH)(1-T)^2/(2D)$. For $\pi/3 < \theta < \pi/2$ the upper boundary ℓ changes at a_* and the two pieces are triangles with a common height $\ell(a_*) - r(a_*)$. Hence

$$M(\theta) = \begin{cases} 3/[4(h + 2\sin 2\theta)], & \pi/6 < \theta < \pi/3, \\ hc/[2(hs+c)], & \pi/3 < \theta < \pi/2, \\ 0, & \text{otherwise.} \end{cases}$$

Without L_1, the third ray reaches the bottom side throughout $0 < \theta < \pi/6$, and when $\pi/6 < \theta < \pi/3$ it reaches the bottom side precisely at $a > b$. This follows by testing its passage through the bottom-left vertex. In the latter interval $0 < T < b < 1$; a ray missing L_1 requires $z < r$ when $a > T$. For $\pi/3 < \theta < 2\pi/3$ the outside candidate is always the left side. Integrating the appropriate lengths in z times s gives the table below. Put $J = h(s^2 + 4hsc - 3c^2)/(8s(s+hc))$.

θ	$B(\theta)$	$L(\theta)$
$(0, \pi/6)$	$hs/(2D)$	0
$(\pi/6, \pi/3)$	$sH(1-b)^2/2$	$s\{hT^2/(2D) + H(b - T - (b^2 - T^2)/2)\}$
$(\pi/3, \pi/2)$	0	$J - M(\theta)$
$(\pi/2, 2\pi/3)$	0	$hD^2/(8s(s-hc))$.

The functions vanish outside the displayed intervals. For example J is the full area of the first two specified impacts in its interval, obtained by integrating $s\ell$ above $a = (s - hc)/(2s)$ with a change of upper boundary at a_*. For the last interval this same lower cutoff already lies beyond a_*, giving the single triangular area in the last row.

With uniform direction and starting area, the three total probabilities W, R, O for memory, return to the bottom, and the remaining side are the integrals of M, B, L times $6/(2\pi A)$. With $q = \tan\theta$, the first integral splits into rational functions $3/[4(q+h)(hq+1)]$ and $h/[2(hq+1)(1+q^2)]$ on $(1/h, h)$ and (h, ∞). Their integrals are $3\log(4/3)/8$ and $h\pi/48 - 3\log(4/3)/16$. The return integrands are $hq/[2(q+h)(1+q^2)]$ on $(0, 1/h)$ and $(hq-1)(h-q)^2/[8q(1+q^2)^2]$ on $(1/h, h)$. For the second of these an antiderivative is

$$-\tfrac{3}{8}\log q + \tfrac{3}{16}\log(1+q^2) + \tfrac{3h}{8}\arctan q + \frac{hq+1}{4(1+q^2)}.$$

Evaluating, and using $O = 1 - W - R$, yields

$$\begin{aligned} W &= \tfrac{1}{4} + \tfrac{3h}{4\pi}\log(4/3), \\ R &= 1 - \tfrac{h}{2\pi}(1 + 3\log 2), \\ O &= -\tfrac{1}{4} + \tfrac{h}{4\pi}(2 + 3\log 3). \end{aligned}$$

Each class has positive area for an open direction interval, so six labels occur in each class and the first three history counts are $3, 6, 18$. Each individual third history has one-sixth of the corresponding total probability. The boundary equalities in the geometric tests and the exceptional angles have measure zero.

At any fixed depth, as the initial direction varies through a full turn, the admissible history count is divisible by three. Rotation by $2\pi/3$ changes the first wall label and preserves creation indices, so every orbit has size three. Distinct words need not give distinct output formulas. For example, in the equilateral triangle the reflections R_i, R_j in any two fixed sides satisfy $R_iR_jR_i = R_jR_iR_j$. Both compositions are reflections with the same axis through the intersection of the sides, by the 60° line angle. Therefore the words (E_i, E_j, E_i, E_k) and (E_j, E_i, E_j, E_k) have the same fourth-impact state formula on their respective domains: they reflect the initial oriented support to the same line after the third impact, then meet and reflect in E_k.

The third-angle distribution is $W\delta_{\pi/3} + f(\alpha)\,d\alpha$, with

$$f(\alpha) = \frac{4h}{\pi}\big[B(\pi/3-\alpha) + L(\alpha) + L(\pi-\alpha)\big], \qquad 0 < \alpha < \pi/2.$$

This is the push-forward by $\alpha = \pi/3 - \theta$ on the bottom class and $\alpha = \min(\theta, \pi-\theta)$ on the left class, with the zero extensions understood. At $\alpha = \pi/6$ the left-side terms tend to zero. The bottom term changes from zero to $h/8$, so $f(\pi/6+) - f(\pi/6-) = 3/(2\pi)$. The atom and the jump concern the third impact specifically.

4 The first memory impact in the unit disk

Let $\alpha \in (0, \pi/2)$ be the angle from the tangent at the first circle impact A_0. Choose the counterclockwise orientation by reflection symmetry. In ordinary circle billiards the wall points A_j advance by 2α and the complete chords have length $d = 2\sin\alpha$. The starting point lies on the chord from A_{-1} to A_0 at distance zd from A_0, with $0 < z < 1$. For area-uniform starts and independent uniform directions, α has density $p(\alpha) = 4\sin^2\alpha/\pi$ and z is independent and uniform. Indeed, at fixed direction the unsigned perpendicular distance b of the chord from the center has density $4\sqrt{1-b^2}/\pi$. Substitute $b = \cos\alpha$ and use the uniform coordinate along a chord conditional on b.

Set $m = \lfloor \pi/\alpha \rfloor$, $\beta = \pi - m\alpha$, and exclude equalities at the endpoints of these angle and length intervals. Define

$$u = \tfrac{1}{2} + \frac{\tan\beta}{2\tan\alpha}, \qquad v = \tfrac{1}{2} - \frac{\tan(\alpha-\beta)}{2\tan\alpha}, \qquad 0 < v < \tfrac{1}{2} < u < 1.$$

The competing flights in a full circuit first reach the initial chord on the flight from A_{m-1} and then on the flight from A_m. Until then the chord endpoints do not alternate, so no chord interior can intersect an earlier one. The flight from A_{m-1} can meet only the initial partial chord. If this misses the partial chord, the next flight meets the initial support and then the first full reflected chord. To see the intersection parameters, put the initial support at $X = \cos\alpha$ with A_0 at height $\sin\alpha$. Its two crossing chords have support normals rotated by -2β and $2(\alpha-\beta)$. Solving for Y on the initial line gives $-\cos\alpha\tan\beta$ and $\cos\alpha\tan(\alpha-\beta)$, so their backward fractions are u, v. Reflection in the bisector between two chord normals swaps their supports and sends the backward parameter on one chord to the forward parameter on the other. Consequently the two forward fractions on the flight from A_m are v, u in that order. All later chords are irrelevant once an intersection in a completed segment has been reached.

Count the first table hit as impact 1. The full joint law of the first memory impact number N, radius r, incidence γ, and distance S is specified by

z	N	x	S/d
$(u,1)$	$m+1$	β	$z+m-1+u$
(v,u)	$m+2$	$\alpha-\beta$	$z+m+v$
$(0,v)$	$m+2$	β	$z+m+u.$

In each row $r=\cos\alpha/\cos x$ and $\gamma=\min(2x,\pi-2x)$. This follows from the intersection height and the difference of the chord normals. To compute the probability of any Borel condition on the four observables, integrate its indicator against $p(\alpha)\,d\alpha\,dz$ on the three rows for each $m\geq 2$, with $\pi/(m+1)<\alpha<\pi/m$. The position has an additional independent uniform azimuth by rotational symmetry. For a partial sum through $m=M$, the omitted mass is exactly $(2a-\sin 2a)/\pi$ with $a=\pi/(M+1)$.

For $n\geq 3$ and $F(a)=(2a-\sin 2a)/\pi$, the tail is

$$\Pr(N>n)=F(\pi/n)+\int_{\pi/n}^{\pi/(n-1)}p(\alpha)u_{n-1}(\alpha)\,d\alpha.$$

Here u_{n-1} uses $m=n-1$. The tail lies between $F(\pi/n)$ and $F(\pi/(n-1))$, giving $\Pr(N>n)\sim 4\pi^2/(3n^3)$. For the point mass at n, only the bands for $m=n-1,n-2$ contribute. In each band substitute $\alpha=\pi/(m+t)$ with $0<t<1$. The density times the Jacobian is uniformly asymptotic to $4\pi^2/m^4$, while the applicable weights are $1-u$ and u, tending to $(1-t)/2$ and $(1+t)/2$. They integrate to a combined factor one. Thus $\Pr(N=n)\sim 4\pi^2/n^4$, and the positive moment of order p is finite exactly for $p<3$.

The distance has infimum 4. For $m\geq 3$ it is at least $2m\sin(\pi/(m+1))\geq 3\sqrt{2}$. The last inequality follows since $\sin y/y$ decreases on $(0,\pi/2)$ and $m/(m+1)$ increases. For $m=2$ the second and third rows give $S>4$ using $u>1/2$ and $v>1/3$. The latter follows from $\tan(3\alpha-\pi)/\tan\alpha=(\tan^2\alpha-3)/(3\tan^2\alpha-1)<1/3$. For the first row the lower endpoint is $4(1-x)^{3/2}/(1-2x)$ with $x=\cos^2\alpha\in(0,1/4)$. Squaring its ratio to 4 gives an excess with numerator $x(1-x-x^2)>0$. Taking $z\downarrow u$ as $\alpha\uparrow\pi/2$ approaches 4.

The exact distance supremum is the supremum over the angle bands of $H_m(\alpha)=2\sin\alpha(m+u+v)$, obtained by taking a row's upper endpoint in z. Convexity of the tangent bounds this by $2\pi+2\pi/m$, so the distance is globally bounded. For an exact finite maximization, it suffices to take $2\leq m\leq 27$ and the one-sided endpoints of those bands. Indeed the bound for $m\geq 28$ is $29\pi/14$, whereas the value with $m=7$ and $\alpha=5\pi/36$ is larger. This last comparison can be verified directly using the sine and tangent Taylor bounds at α, $\pi/36$, and $\pi/9$. Thus the maximum of those finitely many continuous upper envelopes gives a sharp bound. As $\alpha\to 0$ the first-impact distance tends uniformly in z to 2π, since $m\alpha\to\pi$ and u,v,z are bounded.

4.1 Radial and angular laws

For the first and third rows, $r=\cos\alpha/\cos(\pi-m\alpha)$ is strictly decreasing in each angle band. For the middle row put $y=(m+1)\alpha-\pi$. Its radius is $r_1=\cos\alpha/\cos y$. For $m=2$, $r_1=1/(3-4\cos^2\alpha)$ tends to $1/3$ with quadratic contact as $\alpha\uparrow\pi/2$. The radial density there has singular term $1/(2\pi\sqrt{r-1/3})$. Indeed, with $e=\pi/2-\alpha$, one has $r-1/3\sim 4e^2/9$, $p(\alpha)\to 4/\pi$, and the middle-row weight $u-v\to 1/6$.

For $m\geq 3$, the logarithmic derivative of r_1 is $(m+1)\tan y-\tan\alpha$. It is negative at the lower endpoint, positive at the upper endpoint, and has derivative at least $(m+1)^2-4>0$. It vanishes at exactly one nondegenerate minimum. Its radius r_m lies strictly between $\cos(\pi/m)$ and $\cos(\pi/(m+1))$, by the lower bound $r_1\geq\cos\alpha$ and the value at the lower endpoint. Hence the radii increase to one. For example $r_4=4/5$, since the reciprocal radius is $-16x^2+20x-5$ with $x=\cos^2\alpha$, and

its interior maximum is $5/4$. At each interior minimum, putting $w = p(\alpha)(u - v)$ and writing α_m for the minimizing angle, the two inverse branches contribute

$$\frac{\sqrt{2}\, w(\alpha_m)}{\sqrt{r_1''(\alpha_m)}}(r - r_m)^{-1/2}$$

to leading order. The contributions are locally integrable. The planar density equals the radial density divided by $2\pi r$. These radii describe first memory impacts.

To obtain the angle density, put $a_m = (\pi - x)/m$, $b_m = (\pi + x)/(m + 1)$, and $w_0(a, x) = \cos a/(2\cos x \cos(a - x))$. The first and third rows have combined weight $1 - u + v = 1 - w_0(\alpha, \beta)$. The second row has weight $u - v = w_0(\alpha, \alpha - \beta)$, by the tangent addition identity. The change of variables gives

$$\begin{aligned} g(x) = &\sum_{\substack{m\geq 2\\ x<\pi/(m+1)}} \frac{p(a_m)}{m}[1 - w_0(a_m, x)] \\ &+ \sum_{\substack{m\geq 2\\ x<\pi/m}} \frac{p(b_m)}{m+1} w_0(b_m, x), \qquad 0 < x < \pi/2. \end{aligned}$$

The incidence density is $[g(t/2) + g((\pi - t)/2)]/2$ for $0 < t < \pi/2$. There are no atoms, since each angle comes from at most countably many α values. At $x = \pi/k$, $k \geq 3$, one term from each sum vanishes across the threshold, and the downward jump is $D_k = 4k\sin^2(\pi/k)/(\pi(k^2 - 1))$. This follows by substituting $w_0 = 1/2$ into the two disappearing terms with denominators $k - 1, k + 1$. The folded density has downward jumps $D_k/2$ at $2\pi/k$ for $k \geq 5$, $k \neq 6$. At $\pi/3$ the net jump is upward and equals $(D_3 - D_6)/2 = 267/(560\pi)$.

5 Normal impact and arithmetic of regular polygons

A normal impact on a fixed wall specifies finitely many initial angles on any fixed word, since the incident direction coefficient a is odd and nonzero. On a memory wall the direction difference is $(a-b)\psi+4\pi(m-k)/h$. If $a \neq b$, a prescribed angle again restricts ψ to finitely many values per word. The union over finite words is countable. If $a = b$, normal incidence requires $8(m - k) = h(2j + 1)$ for an integer j. Thus $8 \mid h$ is necessary for a finite normal impact of positive probability. More generally, putting $q = h/\gcd(h, 4)$, the only possible nonzero atomic angles on finite memory impacts are $j\pi/q$ for $1 \leq j \leq \lfloor q/2 \rfloor$. This lattice identifies candidates.

For sufficiency let $8 \mid h$ and normalize the inradius to one. Set $\gamma = 3\pi/8$, $c = \cos\gamma$, $s = \sin\gamma$, and select the two walls with unit outward normals $n_- = (c, -s)$ and $n_+ = (c, s)$. At their midpoints $P_1 = n_-$ and $P_2 = n_+$, an incident direction $(1, -1)/\sqrt{2}$ reflects to $(0, 1)$ and then to $(-1, -1)/\sqrt{2}$. The third flight meets the first at $(c - s, 0)$, strictly within the incircle. Start a small positive distance farther back on the first line. The first two hits are open-side impacts, and the third is a transverse hit on the interior of L_1. Convexity and the transverse first-contact tests leave strict margins under a perturbation of position and direction. The two fixed reflections always compose to the same rotation, so the third incidence stays normal in this open set. The first two hits are on fixed walls and cannot give a positive-probability normal impact. This proves both the $8 \mid h$ criterion and sharpness of the third-impact threshold.

Here is the exact third-impact probability and joint law. For normal impact on L_1, the three collision points form a right triangle. At its first two vertices the inward normals bisect the interior angles by reflection. The angle at the meeting of these bisectors facing those vertices is $135°$, so the

selected wall normals must be separated by $3\pi/4$. There are $2h$ such ordered pairs when $8 \mid h$, all related by symmetries. Let $t = \tan(\pi/h)$ and vary the first two hits on the pair already specified:

$$P_1 = (c + su, -s + cu), \qquad P_2 = (c - sv, s + cv), \qquad |u|, |v| < t.$$

Put $\ell = |P_2 - P_1|$, $d_1 = (P_2 - P_1)/\ell$, and $d_0 = d_1 - 2(n_- \cdot d_1)n_-$, $d_2 = d_1 - 2(n_+ \cdot d_1)n_+$. The latter two directions are perpendicular. For $D = P_2 - P_1$ the intersection is $X = P_1 - r_1 d_0 = P_2 + r_2 d_2$, with $r_1 = -D \cdot d_0$ and $r_2 = -D \cdot d_2$. Take those (u, v) for which $r_1, r_2 > 0$ and X is interior. Denote this set by $\mathcal{D}_h$. The entire starting set for these boundary points is $P_0 = X - z d_0$, $0 < z < w$, with

$$w = \min_{j : n_j \cdot (-d_0) > 0} \frac{1 - n_j \cdot X}{n_j \cdot (-d_0)},$$

where n_j are all the outward normals. The strict inequalities put X on the existing segment and rule out premature boundary hits by convexity.

At fixed first-hit coordinate u, the angle changes with v at rate $|n_+ \cdot d_1|/\ell$. At fixed angle the area element is $|n_- \cdot d_0|\, du\, dz$. The possible tangential shift between $X - z d_0$ and $P_1 - z d_0$ adds a multiple of the z-derivative and does not affect this determinant. Thus $J = |n_- \cdot d_1|\, |n_+ \cdot d_1|/\ell$ is the full Jacobian. The table area is ht, which gives

$$P_3(h) = \frac{1}{\pi t} \iint_{\mathcal{D}_h} J w \, du \, dv \quad (8 \mid h).$$

The probability is zero for the other h. Conditional on this event the standardized triple (u, v, z) has density $J/(\pi t P_3(h))$ on its allowed set. The ordered pair is uniform over its symmetry orbit. The terminal point and total distance are X and $r_1 + \ell + r_2 + z$, respectively, giving their joint law.

Along multiples of eight, rescale $u = tU, v = tV$. On the whole fixed square $|U|, |V| \le 1$, the hits converge uniformly to the two midpoints, and X stays in a compact subset of the incircle. The inradius-one tables are between disks of radii 1 and $\sec(\pi/h)$, giving uniform convergence of the backward exit distances to the disk exit distance $s - c$. Also $J \to s/2$, so $Jw \to s(s - c)/2 = 1/4$. The entire open square is eventually admissible. The integral is asymptotic to t^2, and $hP_3(h) \to 1$ since $ht/\pi \to 1$.

6 Endpoint returns before a regular memory reflection

6.1 Return to the free starting point

Consider first memory contact exactly at the initial interior point P, after m regular wall reflections $Q_1, \ldots, Q_m$. The resulting closed polygonal path is simple. Its wall points must follow the cyclic order on the boundary. To see this, an edge $Q_i Q_{i+1}$ that separates nonempty sets of the other wall points would force the complementary arc through those points to change sides, crossing the chord: the arc cannot go around a chord endpoint within the convex table. Thus every consecutive pair of wall points in the path are neighbors in the circular order of the set. The omitted pair Q_m, Q_1 is joined through P. The same wall can contain two consecutive points in boundary order only if they are Q_1, Q_m, since movement along a wall is excluded. Up to orientation and rotation, the possible histories are strictly increasing side numbers beginning with 0, with at most one additional 0 at the end. Hence $m \le h + 1$ and the endpoint has event number at most $h + 2$.

For a wall of unit outward normal n_j at distance one, define $R_j(X) = X + 2(1 - n_j \cdot X)n_j$. The composition $T = R_{j_1} \circ \cdots \circ R_{j_m}$ sends the final copy of P to the unfolded endpoint. Thus the candidate direction and length are $(T(P) - P)/|T(P) - P|$ and $|T(P) - P|$ when the difference

is nonzero. A candidate is valid exactly when the segment from P to $T(P)$ crosses the unfolded finite walls transversely in their open interiors in the specified order, the folded path has no normal impacts, and its nonconsecutive segments have no contacts except the intended return. These tests imply the collision order by convexity of the table and the absence of earlier memory contacts. They also prove necessity by unfolding. For each fixed P there are finitely many candidate angles and therefore the probability under an absolutely continuous joint starting law is zero.

The bound $h+2$ is attained. Use the word $0, 1, \dots, h-1, 0$ and start at $P = n_0 - \epsilon n_0$ for small $\epsilon > 0$. At $\epsilon = 0$ the unfolded segment follows the midpoint loop, with its first and last side-0 crossings at the respective segment endpoints, and with nonzero total unfolded length. The first crossing moves to a positive flight distance because P moves inside the initial table and its launch direction still points toward side 0. The last crossing likewise precedes $T(P)$, since the terminal copy of P moves strictly into the last reflected table. The other crossings remain strictly ordered and open by continuity. In the folded table the two side-0 points move to opposite sides of the midpoint: reflection in its normal axis reverses the word and fixes P, so the crossings are symmetric, and the first ray has a nonzero tangential component. The first and last small segments form an interior notch between these crossings. They meet only at P, are separated from the nonadjacent midpoint chords, and meet the two adjoining chords solely at the intended reflections. The strict-order tests hold for small ϵ.

At a fixed direction u, a positive-area family for one word requires $[(T - I)P, u] = 0$ identically as an affine equation. Write $T(P) = AP + b$. If A is a proper rotation other than the identity, $A - I$ is invertible. If A is a reflection, the identity requires u perpendicular to the reflection axis and b parallel to u. Then the final direction $A^{\mathsf{T}}u$ is $-u$, causing earlier contact with the initial segment. Hence a positive-area family requires a nonzero translation $T(P) = P + b$ and $u = b/|b|$. A word that returns to the first side cannot give a translation family, since Q_m, P, Q_1 would be collinear with the two side points on the same supporting line, while P is interior.

For the remaining words with $m = 2r$, the orientation condition on T is $4\pi S/h \equiv 0 \pmod{2\pi}$, where $S = \sum_{i=1}^{r}(j_{2i} - j_{2i-1})$. Since $0 < S < h$, this requires $S = h/2$. It is impossible for odd h. In the even case each eligible word gives a candidate translation and direction. At that direction the ordered open-side intersection conditions are linear inequalities in P. When these give a polygon of positive area, the folded cycle is simple because it connects the wall points in order with P on the closing edge. The nonnormal incidence conditions exclude the two-side backtrack. These tests characterize all positive-area families. For the eligible cyclic gap patterns with fixed starting side and orientation there are $\binom{h-2}{h/2-1} - 1$ candidates. Indeed the alternating groups of gaps are positive compositions of $h/2$ into r parts, so the sum over $2 \le r \le h/2$ follows by Vandermonde's identity.

For an explicit family when h is even, let $a = \pi/h$, $n_j = (\cos 2ja, \sin 2ja)$ and $e_j = (-\sin 2ja, \cos 2ja)$. Put $Q_j = n_j + (-1)^j s e_j$, $|s| < \tan a$. The chord $Q_{j+1} - Q_j$ equals $2(\sin a - (-1)^j s \cos a)e_{j+1/2}$. These coefficients are positive, the directions satisfy reflection, and the path closes after h walls. Start on the open chord from Q_{h-1} to Q_0 with direction $(\sin a, \cos a)$; the first memory contact is at P on impact $h + 1$. As s varies, these parallel chords fill the triangle of three consecutive table vertices: at one end they collapse to the central vertex and at the other they span the opposite side of that triangle. Its area is $4\sin^3 a/\cos a$, using the area of the two equal table sides meeting at interior angle $\pi - 2a$. The opposite cap is obtained with the same launch direction and reversed traversal. These two caps have total area fraction $8\sin^2 a/h$, and summing the chord lengths gives the constant total distance $2h \sin a$. This proves that a positive-area fixed-direction family exists exactly for even h. For $h = 4$ only the full four-wall translations are possible, in the four diagonal directions, with the two caps filling the square up to their boundaries. A uniform choice among the 360 integer-degree directions then gives probability $1/90$ for this endpoint event.

6.2 Return to a previous wall impact

Suppose the first memory contact is a return to a previous wall point Q. The closed loop from Q has all its vertices on the boundary. The same chord separation argument shows its corners follow the boundary order, one per wall. With Q itself the first wall impact, there are at most h wall visits before the return, and the return event is at most $h+1$. For a later Q the bound applies to the loop, including its starting wall point.

On a specified cyclic word, take Q along its open wall E_0 and unfold the reflections at the remaining loop vertices. The displacement $d(Q) = T(Q) - Q$ is affine in its one side coordinate, and the prospective outgoing direction at Q is $d(Q)/|d(Q)|$. The unfolded finite-wall and strict order tests determine the loop. The incoming direction is the reflection of the outgoing direction in E_0. Extend this incoming ray backward from Q and take the starting point on its open initial portion before the first intersection with the loop or another table wall. Exclude collinearity with an outgoing or returning loop edge whenever it overlaps the initial segment. This procedure gives exactly the allowed initial segments for a first-wall return. The loop tests can be reduced to sign comparisons of polynomials of degree at most two: use the unnormalized d in each line intersection, multiply by denominators with known sign, and compare affine numerators and denominators pairwise. The wall segment constraints can also be obtained from the side orientation determinants, with numerator degree at most two. Splitting the side coordinate at their real roots gives the intervals of validity.

For fixed direction and word, the determinant equation $[T(Q)-Q, w] = 0$ is affine along E_0, where w is the fixed unit outgoing direction. It cannot hold identically for a regular, unobstructed loop. Indeed, if $T(Q) - Q = \ell(Q)w$ throughout the side, differentiation along the unit tangent e gives $Ae = e + \ell' w$ for the orthogonal linear part A of T. Hence $\ell' = 0$ or $\ell' = -2e \cdot w$. The final direction $A^{\mathsf{T}} w$ has the opposite normal sign from w, and its tangential component is $e \cdot w + \ell'$. In the first case the loop approaches Q along the original incident direction, and in the second case it retraces the first loop segment. Both produce earlier memory contact. Thus there is at most one valid Q for each fixed direction and word. A later anchor Q admits the same loop reduction by starting on the preceding flight close to Q. For fixed initial direction, ordinary wall reflections have only finitely many possible direction values in a regular polygon. This gives finitely many possible anchors, and the admissible earlier starting points lie along their backward ordinary billiard paths. Allowing all finite prefixes covers the event by countably many line segments, so its area is zero at each fixed direction.

For odd h the upper bound for a first-wall return is attained by a full-wall loop. Let $a = \pi/h$, $t = \tan a$, $-1 < s < 1$, $s \neq 0$, and $\kappa = \arctan(st/h)$. The loop vertices are $Q_j = n_j + y_j e_j$, where

$$y_j = \begin{cases} st(1 - 2j/(h+s)), & j \text{ even}, \\ st(2j - h - s)/(h - s), & j \text{ odd}. \end{cases}$$

The chord from Q_j has angle $\pi/2 + (2j+1)a + (-1)^j \kappa$. To verify it, the intersection with the next side sends y_j to $[2 \sin a \sin \varepsilon - y_j \sin(a+\varepsilon)]/\sin(a-\varepsilon)$, with $\varepsilon = (-1)^j \kappa$. Applying this twice subtracts $4st/(h+s)$ for even j, and odd h gives $y_h = y_0 = st$. Each $|y_j| < t$ so the loop is simple, and reflection at each intermediate wall follows from the angles. The incident angle coordinate at launch is $\theta(s) = \pi/2 - a - \kappa$. The incoming and returning lines at Q_0 differ by 2κ, so a sufficiently short initial segment is eligible on either side of the missing symmetric case $s = 0$. The loop length is $2 \sin a \sqrt{h^2 + s^2 t^2}$: its unfolded displacement has orthogonal components $2h \sin a$ and $2st \sin a$. The full distance adds the initial segment length. The required angle band has width $2 \arctan(t/h)$ and h^2 times this width tends to 2π as odd h tends to infinity.

7 Endpoint impact after a memory reflection

On a convex table consider a terminal contact at an old wall-impact point, after a regular reflection in memory. The earliest memory reflection is the third impact, at X on the first flight. If the fourth impact were a return to the first wall point it would be tangent to the memory wall just struck. If it were a return to the second wall point it would retrace the incoming flight at X. Both are excluded by regularity. Thus the earliest terminal impact of this type is the fifth.

For the square $[-1,1]^2$, fix the first wall to be the top and the initial direction proportional to $(1,r)$ by symmetry. All fifth-impact examples in this orientation have the word top, right, bottom, L_1, then the old right impact. Their data are

$$D=3r^2-1,\quad K=\frac{2(r^2+1)}{rD},\quad T=\frac{4(r^2-1)}{rD},\quad C=\frac{r^2-3}{D},\qquad r>1.$$

$$Q_1=(1-K,1),\qquad Q_2=(1,C),\qquad Q_3=(1-T,-1),$$
$$X=(1-2/r,-C),\qquad P_0=Q_1-\tau(1,r),\quad T<\tau<\min(2-K,2/r).$$

To verify completeness, the first two square reflections make the third flight parallel to the first, so a transverse hit on memory first occurs at the fourth. Hitting L_2 at that step leaves its two endpoints on the tangent and the most recent impact on the backtrack. Hitting L_1 can leave only the second wall point as the intended target. Write $Q_{1x}=1-k$ and intersect the fourth ray with $Q_1-u(1,r)$, using direction components of magnitudes $1,r$ at each impact. For the words top-right-bottom, top-right-left, top-bottom-right, and top-bottom-top, respectively, the fourth forward parameter and the backward parameter u are $(k,2/r-k)$, $(-k,2)$, $(2/r,2/r-k)$, and $(-2/r,-2/r)$. In the third word, order of the second and third wall hits requires $k>2/r$, so $u<0$. Only the first pair can have both parameters positive. There $r>1$ and the four points form a parallelogram spanned by $A=(1,r)$ and $B=(1,-r)$, with $Q_2=X+uA+kB$. The reflection at X has outgoing direction $\alpha A+B$ with $\alpha=2(r^2-1)/(r^2+1)$. Reaching Q_2 requires $u=\alpha k$, which gives $k=K$ and $u=T$. The interval in τ puts P_0 inside the square and X inside L_1; it is nonempty since $K+T=2/r<2$. The final open segment is a diagonal inside the parallelogram, giving no earlier contact.

Changing Q_{1x} from $1-K$ to $1-K+\epsilon$ changes k to $K-\epsilon$ and u to $T+\epsilon$. The same regular first four hits persist for sufficiently small ϵ. The ray from X intersects the line $x=1$ at a height differing from Q_{2y} by $-2r\epsilon$. Equivalently, its two parallelogram exit parameters are u/α and k, and these exchange order at $\epsilon=0$. A negative ϵ gives the wall Q_1Q_2 and a positive ϵ gives Q_2Q_3. The contact points approach Q_2 from these opposite sides. The limiting reflected directions make angle $2\pi-4\arctan r$, twice the acute angle between the two target lines.

8 Triangular confinement and finite-length capture

Call a triangular region clean if its interior has no earlier wall, its open sides have no historical endpoint, and the current impact is regular. Use ordered vertices (P,A,B), with the particle at P, the wall just struck along PA, and the previous flight BP. Denote the angles at A,B by a,b. To enter the triangle after reflection one needs $a,b>0$ and $a+b<\pi/2$. Indeed the angle at P is $\pi-a-b$, and the reflected ray makes angle $a+b$ with PA inside the triangle. The next hit Q is strictly inside AB. Its two angles in the cut triangles PAQ, PBQ are $\pi-2a-b$ and $2a+b$, respectively. Specular reflection enters the component with the obtuse angle at Q because the two old side rays at Q are opposite and the reflected ray must lie on the inward side of AB. If the angles are both right, the ray retraces QP. Thus the angle update is

$$(a,b) \longmapsto \begin{cases} (a, a+b), & 2a+b < \pi/2, \\ (b, \pi - 2a - 2b), & 2a+b > \pi/2, \end{cases}$$

with equality a normal impact. The new ordered vertices are (Q, A, P) in the first case and (Q, B, P) in the second. The new triangle is clean: the new side ran inside an empty interior, P, Q are vertices, and inherited open sides contain no older endpoint. This proves by induction that no finite historical-endpoint impact occurs in such a triangle.

If $s = |PA|$, the sine rule gives

$$Q = A + \lambda(B - A), \qquad \lambda = \frac{\sin b}{\sin(2a+b)}, \qquad |PQ| = \frac{s \sin a}{\sin(2a+b)}.$$

The identity $\sin(2a+b) - \sin b = 2\cos(a+b)\sin a > 0$ gives $0 < \lambda < 1$. The retained area is λ times the parent area in the first branch and $1 - \lambda$ times in the second. The two child diameters relative to $|AB|$ are $\sin b / \sin(a+b)$ and $\sin a / \sin(a+b)$, since the child diameter is the side opposite its new obtuse angle.

8.1 Rational angles

For $a = i\pi/q$, $b = j\pi/q$, the same denominator q works at every step. The map on (i, j) is $(i, i+j)$ when $2(2i+j) < q$ and $(j, q-2i-2j)$ when $2(2i+j) > q$. The legal state space has size $M(M-1)/2$ where $M = \lfloor (q-1)/2 \rfloor$. Therefore the orbit reaches either equality and stops, or a repeated state and a deterministic cycle of shapes. If q is odd the equality is impossible. For example, $q = 5, (i, j) = (1, 1)$ is fixed and $q = 7$ gives $(1,1) \to (1,2) \to (2,1) \to (1,1)$. A trajectory in a regular polygon that begins at a rational multiple of π relative to a side direction has rational angles in any clean triangle it reaches, by the integer direction recurrence.

For a repeating shape, let ρ be the square root of the product of the retained area fractions over one cycle. The corresponding marked triangles are similar and $0 < \rho < 1$. The flights at matching phases are corresponding sides of the child triangles, so their lengths have the same factor ρ. This argument distinguishes single-step area factors from the full-cycle similarity. The product is independent of the phase chosen as start of the same cycle. Write the corresponding full-cycle similarity as $x \mapsto Mx + d$, allowing orthogonal reflection in its linear part. The nested triangles contract to the single point $X_* = (I - M)^{-1} d$. If G is the distance in the first full cycle and S_{pre} is the distance before it, the total distance is $S_{\mathrm{pre}} + G/(1-\rho)$. The remaining distance after k full cycles is exactly $G\rho^k/(1-\rho)$. At unit speed this is finite-time accumulation of infinitely many regular reflections. When $a = b = \pi/5$ and $|PA| = |PB| = 1$, the one-step ratio is $1/(2\cos(\pi/5))$ and the infinite distance from P is $(1+\sqrt{5})/2$.

8.2 Real angles with bounded gaps between returns

Let D_n be the diameter of the nth triangle and ℓ_n the next flight length. The diameter ratios above and the identity $\sin(a+b)\cos a - \sin b = \sin a \cos(a+b) > 0$ give

$$D_{n+1} \le \cos(\min(a_n, b_n)) D_n, \qquad \ell_n \le D_n.$$

Suppose no normal impact occurs and there are times $0 = n_0 < n_1 < \cdots$ with gaps at most M and $\min(a_{n_k}, b_{n_k}) \ge \varepsilon > 0$. Put $r = \cos\varepsilon < 1$. Then $D_{n_{k+1}} \le r D_{n_k}$ and, by a geometric series, the remaining distance at n_k is at most $M D_{n_k}/(1-r)$. The nested compact triangles have diameters tending to zero, so their one-point intersection is the capture point. A finite prefix gives a coordinate rectangle enclosing it via the minimum and maximum vertex coordinates, and the same tail bound

encloses the total distance. More generally, summability of the products $\prod_{j<n} \cos(\min(a_j, b_j))$ suffices for finite total distance.

For an explicit family normalize angles by π, with $x = a/\pi, y = b/\pi$, and put $\Omega = \{x, y > 0 : x + y < 1/2\}$ and $E = \Omega \cap \{y > x\}$. Denote the first and second forward branches by A, B. Their inverses are $C(x, y) = (x, y - x)$ on E, and $D(x, y) = (1/2 - x - y/2, x)$ on Ω. Substitution verifies the strict forward inequalities on these inverse images. The map $F = D \circ D \circ C$ preserves E and is $(y/2, (1 - x - y)/2)$. Its cube and the eighth iterate of D are

$$F^3(x, y) = (1/8, 1/4) + (x, y)/8, \qquad D^8(x, y) = (3/16, 3/16) + (x, y)/16.$$

Take $H_0 = F^3$ and $H_1 = F^3 \circ D^8$. Their forward blocks are $(BBA)^3$ and $(BBA)^3B^8$. Let K be the triangle with vertices $(1/8, 1/4), (1/8, 5/16), (5/32, 9/32)$. It equals $F^3(\overline{E})$ and is contained in E. Both H_0, H_1 send K into K, since $D^8(E) \subset E$. Their images are disjoint: the largest first coordinate of $H_0(K)$ is $148/1024$ and the smallest of $H_1(K)$ is $153/1024$. Thus an infinite binary word determines a unique angle pair by nesting its images of K, with different words giving different pairs. At each block boundary the orbit is in K; all trajectories meet the bounded-gap condition with $M = 17$ and $\varepsilon = \pi/8$.

For a specific nonperiodic example, let the bits be the parity of the binary digit sum of n. If they were eventually periodic with period $p > 0$, the bits at $2^k - p$ would equal those at 2^k for all large k. The first has parity $k - s_2(p - 1)$ modulo two, alternating with k, and the second is always one. This gives a contradiction. The forward blocks contain three A's each, with the gap of B's after the last A either two or ten. Eventual periodicity of the forward word would imply eventual periodicity of this decoded binary word. The associated angle pair is therefore outside the double-rational lattice. In fact both angles are irrational multiples of π: the maps preserve $3x + 2y = 1$ and contract distances to that line, so a rational coordinate would force the other rational as well.

8.3 Entry from an equilateral table

Use the unit equilateral triangle and the coordinates of its third-impact calculation. Fix $0 < \theta < \pi/6$. For $0 < a < 1$ and $0 < z < Ca$, where $C = \sqrt{3}/(\sin\theta + \sqrt{3}\cos\theta)$, the launch $(a - z\cos\theta, z\sin\theta)$ hits the base at $Q_1 = (a, 0)$, then the right side at Q_2. The second length is $C(1 - a)$. Write $V = (1, 0)$. The region Q_2VQ_1 is clean with ordered angles $(\pi/3, \theta)$ at V, Q_1 and with Q_2 the current obtuse vertex. The first memory segment is outside it because it points left from Q_1 while V, Q_2 lie to the right. Its size in the side Q_2V is $F(1 - a)$ with $F = 2\sin\theta/(\sin\theta + \sqrt{3}\cos\theta)$. The starting region occupies fraction F of the table by the Jacobian $\sin\theta$.

For example at $\theta = \pi/21$ the initial integer state $(7, 1)$ enters a ten-step cycle after one step. For any capturing state in this family, take the normalized marked triangle with current vertex at $(0, 0)$ and next vertex at $(1, 0)$, and let its capture coordinates and distance be (x_*, y_*) and L_*. The initial normalization is chosen so its third vertex has positive second coordinate. Let R be rotation by $-\pi/3$ and put $\mathbf{W} = FR(x_* - 1, -y_*)$ and $K = C + FL_*$. Similarity gives the full capture map and length,

$$X_\infty = V + (1 - a)\mathbf{W}, \qquad S_\infty = z + K(1 - a).$$

The map holds because the vector from Q_2 to V has angle $-\pi/3$ and the oriented triangle in the table is the reflected normalized triangle. Uniform starting area in this family corresponds to uniform (a, z) on its triangle. With $r = 1 - a$, the joint density of (r, S_∞) is $2/C$ on $0 < r < 1$, $Kr < S_\infty < Kr + C(1 - r)$. In particular r has density $2(1 - r)$, and the length density is

$$f_S(\ell) = \begin{cases} 2\ell/(CK), & 0 < \ell < C, \\ 2(K-\ell)/(K(K-C)), & C < \ell < K, \\ 0, & \text{otherwise.} \end{cases}$$

The mean length is $(C+K)/3$, and for $0 < \epsilon < C$ the probability of length at most ϵ is $\epsilon^2/(CK)$. This gives arbitrarily short examples with infinitely many reflections. Also $a = x + y\cot\theta$ in table coordinates, so the capture map has Lipschitz constant $|\mathbf{W}|/\sin\theta$ on this family.

8.4 An exact golden-ratio family in a regular pentagon

Use inradius one, outward normals $n_j = (\cos(2\pi j/5), \sin(2\pi j/5))$, and initial direction $\pi/10$. Put $\varphi = (1+\sqrt{5})/2$, $\rho = 1/\varphi$, $t = \tan(\pi/10)$, $H = \tan(\pi/5)$. The following open starting triangle gives the word E_0, E_2, L_1:

$$V_0 = (1-\sqrt{5}, 0), \quad V_1 = (\varphi - 2, -2\sin(\pi/5)), \quad V_2 = -\rho^2 V_1.$$

Indeed for $P = (x, y)$, set $u = y + t(1-x)$, $Q_1 = (1, u)$ and $d_1 = (-1, t)$. The next hit is $Q_2 = Q_1 + sd_1$ with $s = 1 + \cos(\pi/5) - u\sin(\pi/5)$ since $n_2 \cdot d_1 = 1$. Put $d_0 = (1, t)$, $d_2 = d_1 - 2(d_1 \cdot n_2)n_2$, and

$$f = \frac{[Q_1 - Q_2, d_0]}{[d_2, d_0]}, \qquad X = Q_2 + fd_2.$$

The complete conditions are P interior, Q_1, Q_2 on their open sides, $s, f > 0$, and $0 < 1 - X_x < 1 - x$. They express positive flight and membership in the finite L_1. The three open chords have no earlier blocking segment by convexity and their transverse line intersections. Equivalently the conditions are $-H < u < H$, with x strictly between the lower boundary of the table at that u and X_x. At $u = -H$ the bounds coincide at V_1. At $u = H$ the ends are V_0, V_2. This reduction follows by substituting the side half-planes $n_j \cdot (x, y) < 1$ into $y = u - t(1-x)$ and using the signed coordinate along E_2 for Q_2. The resulting triangular area divided by the table area $5H$ is $(5-\sqrt{5})/10$, using the determinant of $V_1 - V_0, V_2 - V_0$.

In the marked triangle $(A, B, C) = (X, Q_2, Q_1)$ the current angle is $3\pi/5$ and the two other angles are $\pi/5$. The segment of L_1 toward P is outside the triangle; all three open sides are free of old endpoints, and reflection at X points upward into the triangle. The one-step update of vertices is $(A, B, C) \mapsto (\rho B + (1-\rho)C, A, B)$, from the sine rule. If $\ell = |XQ_2|$, the remaining distance is $\varphi\ell$. Invariance of the barycentric weights $1 : 1 : (1-\rho)$ gives the exact capture point F, and the total distance is S:

$$F = \frac{X + Q_2 + (1-\rho)Q_1}{3-\rho}, \qquad S = |PQ_1| + |Q_1Q_2| + (1+\varphi)|Q_2X|.$$

The capture point stays strictly inside every subsequent triangle by positivity of the weights. The start-to-capture map depends affinely on u alone, and the distance is affine in P because each chord has a fixed direction and positive length. In the starting triangle, set $Z = (u+H)/(2H)$ and $W = (S - 2\cot(\pi/10))/(2H)$. At the three boundary vertices V_0, V_1, V_2 the affine limits are $(1, 0), (0, 1), (1, -1)$. Thus uniform starting area in the family has joint density 2 on $0 < z < 1$, $1 - 2z < w < 1 - z$. The marginal densities are $2z$ and $1 - |w|$ on their respective intervals, by straight cross-sections. Integrating gives means $2/3, 0$, variances $1/18, 1/6$, and covariance $-1/12$, hence correlation $-\sqrt{3}/2$. The length ranges between the infimum $2\cot(\pi/10) - 2H$ and the supremum $2\cot(\pi/10) + 2H$, with mean $2\cot(\pi/10)$ and variance $2H^2/3$.

9 Boundary launch without a free initial endpoint

In this section the particle starts on the open boundary of a convex h-gon and points inward. Each new flight is a chord of the current convex region, so the reflected ray is confined to one of its two convex children. After the first chord there are at most $h+1$ corners. After each later one the current launch point is already a vertex, and the number of vertices cannot increase. Count geometric vertices only, and terminate a word if an impact lands on a historical endpoint along an open edge. For a specified initial side and first target, the reflected sector fixes one retained side. At any regular prefix the next wall and continuation sector specify each new labeled boundary, and all supporting directions are fixed across that prefix.

Let a retained polygon have m corners and launch vertex V_0. The only way to retain $m > 3$ corners is to strike V_1V_2 or $V_{m-2}V_{m-1}$, denoted L and R. Indeed the children of a cut to side V_iV_{i+1} have $i+2$ and $m-i+1$ vertices, for $1 \le i \le m-2$. For the following integer angle calculations express angles in degrees. Write the directions of counterclockwise edges as $0 = a_0 < a_1 < \cdots < a_{m-1} < 360$ and the launch direction as θ in $(0, a_{m-1} - 180)$. The necessary conditions and updates are

$$\begin{aligned} L:\quad & 0 < a_1 - \theta < 90, \qquad \theta' = a_1 - \theta, \\ & a' = (0, a_2 - a_1, \ldots, a_{m-1} - a_1, 360 + \theta - a_1), \\ R:\quad & 90 < a_{m-2} - \theta < 180, \qquad \theta' = 2a_{m-2} - 2\theta - 180, \\ & a' = (0, 180 - \theta, 180 - \theta + a_1, \ldots, 180 - \theta + a_{m-2}). \end{aligned}$$

These formulas are obtained by replacing the discarded edge by the chord, starting at the new collision point on the retained counterclockwise boundary, and subtracting the first edge angle. The inequalities ensure reflection points into the side retaining the corner count. They are necessary for all shapes with the given directions.

For $m \ge 5$, the three-step group LLL, LLR, RLL cannot all be realized from the same direction data. Their conditions imply respectively $a_3 > \theta + 2(a_2 - a_1)$, $a_2 - a_1 > 90$, and $2a_{m-2} < \theta + 360$. These contradict $a_{m-2} \ge a_3$ and $\theta > 0$. The disjoint group LRL, RRL, RRR gives $a_1 > \theta$, $a_{m-2} - a_{m-3} > 90$, and $a_{m-4} < \theta + 180 - 2(a_{m-2} - a_{m-3})$, contradicting $a_{m-4} \ge a_1$. Thus at most six of the eight three-step words can retain m.

For $m = 4$ use the following exact contradiction certificates. For a word w and step j write w_j^- for the lower-bound difference and w_j^+ for the upper-bound difference of that step's inequality above. For example, $LLLLL_2^+ = -\theta + 2a_1 - a_2 + 90$. Every term in each row is strictly positive if all the words of that row are realized. Successive use of the direction updates yields the stated sums.

$2LLLLL_2^+ + RRLRL_5^-$	0
$LLLRR_5^- + LRRRR_5^-$	0
$LLRLR_5^+ + 2RRRRL_2^-$	0
$RLLLL_5^+ + RRRLL_5^+$	0
$2LLLLR_4^+ + 4LRLLL_5^- + RRLRR_2^- + 7RRLRR_4^+$	-90
$4LLLRL_5^- + 2LRLLR_4^+ + 5LRLLR_5^+ + 2LRLRL_5^-$	0
$LLRLL_2^+ + 7LLRLL_4^- + 4RLRRR_5^+ + 2RRRRR_4^-$	-90
$LLRRL_3^+ + 8LLRRL_4^- + 2LRLRR_5^- + 5RLLRL_3^+ + 5RLLRL_5^-$	-270
$5LLRRR_4^- + LRRRL_2^+ + 2LRRRL_3^- + 4RLLRR_3^+ + 4RLLRR_5^-$	-90
$8LRRLL_3^- + 8LRRLL_5^+ + 5RLLLR_3^+ + RRLLL_2^- + 10RRLLL_4^+$	-180
$5LRRLR_3^- + 5LRRLR_5^+ + 2RLRLR_5^+ + 2RRLLR_3^- + 8RRLLR_4^+$	-270
$RLRLL_4^- + RLRRL_5^- + RRRLR_5^+$	0

The word sets in the twelve rows are disjoint and cover all thirty-two words, giving at most twenty five-step words. Put $\rho = 20^{1/5}$ and $\sigma = 6^{1/3} < \rho$. The numbers $A_m(t)$ of words retaining the corner

count throughout t steps satisfy $A_4(t) \le 1.5\rho^t$ and $A_m(t) \le 1.25\sigma^t$ for $m \ge 5$, by grouping into five-step or three-step blocks. For the unrestricted continuation bound $F_m(t)$, classify by the time of the first strict decrease of corner count, or its absence. There are at most $m-2$ choices at that step, giving

$$F_m(t) \le A_m(t) + (m-2)\sum_{j=0}^{t-1} A_m(j) \max_{3\le l<m} F_l(t-j-1).$$

A triangle has at most one continuation, so the recurrence gives $F_4(t) \le 6\rho^t$. For $m \ge 5$, the sum of $(\sigma/\rho)^j$ is bounded by $\rho/(\rho-\sigma)$, and constants $C_m = 400mC_{m-1}$, $C_4 = 6$, suffice for $F_m(t) \le C_m\rho^t$. The projection perpendicular to the launch direction splits the initial chords into at most $h-1$ edge-pair classes, since the pair changes only at a vertex projection. It follows that

$$B^{\partial}_{h,n} \le 6(h-1)[400(h+1)]^{h-3}20^{(n-1)/5}.$$

The distinction between boundary and interior launch is essential because the first chord of an interior launch has a free endpoint and does not separate the table. There is also a connectedness property at a fixed direction in the boundary model. On a fixed initial side and fixed regular word, all hit positions and target endpoints are affine in the one initial side coordinate. Membership in the target's open segment and forward travel are strict linear inequalities there. Convexity makes an interior target of a positive ray from the root its first hit. The retained face at each step is determined by the fixed directions and is convex. Thus the realized interval for a fixed word is a single open interval or empty.

9.1 Improved bound by exact affine infeasibility

One can sharpen the previous estimate to $O_h(26^{n/6})$ by considering longer words. Here are the finite inequalities and the counting rule for this refinement. Measure angles in quarter-turns. For m edges use variables $a_1, \dots, a_{m-1}, \theta$ with $a_0 = 0$. Impose $0 < \theta < a_{m-1} - 2$, $a_{m-1} < 4$, and $0 < a_{j+1} - a_j < 2$ for $0 \le j < m-1$. For each binary word append its two strict inequalities per letter, propagating the displayed angle updates in the same units. A pair or triple of words is forbidden when this common system is infeasible. Each affine certificate consists of positive integer multiples of selected strict inequalities whose sum is a nonpositive constant.

The exact finite tests give the following bounds. For $m = 4$ and length six, 440 forbidden pairs or triples suffice to exclude any compatible subset of size 27. For length five, 53 common forbidden pairs or triples suffice to exclude any compatible subset of size 15 for $5 \le m \le 11$ and the relaxed $m = 12$ system. In the relaxed system omit the upper bound on $a_6 - a_5$. To reproduce the finite tests without choosing certificates, one may take every pair and triple of the words. Eliminate one variable at a time from each system: retain inequalities with zero coefficient of that variable and, for each opposite-sign pair, add positive multiples cancelling its coefficient. At the end the system is infeasible exactly when a nonpositive constant is required to be positive. For the length-five test retain the pair or triple if it is infeasible for every listed m. To check a threshold k in either list, recursively include or discard one available word. Reject an inclusion completing a forbidden group, and reject a branch when the count already included plus the count available is less than k. The stated finite outcomes follow using the 440 and 53 certified groups, respectively. Adding further forbidden groups cannot change an exclusion to a compatible subset.

For $m \ge 12$, only the first five original nonzero angles and the last six are used in the five-letter inequalities. This follows by following the two angle updates from the ends of the ordered list. Each step deletes one angle next to an end and inserts a value determined by the current direction and end data. Relabel the retained eleven angles in order as an abstract twelve-edge state. The

gap between the first five and last six can exceed two quarter-turns, explaining the omitted central upper bound. All other initial inequalities and all five-step tests remain unchanged. Thus the two block bounds are 26 in six steps for quadrilaterals and 14 in five steps for larger polygons.

Set $r = 26^{1/6}$ and $s = 14^{1/5} < r$. The preceding first-decrease recurrence now uses $A_4(t) \le 3r^t$ and $A_m(t) \le 2s^t$ for $m \ge 5$. The triangle case and geometric sum give $F_4(t) < 12r^t$. For the induction at $m \ge 5$, the coefficient is bounded by $2 + 2(m-2)C_{m-1}/(r-s)$, which is at most $80mC_{m-1}$ when $C_{m-1} \ge 12$. The same initial projection argument yields

$$B^{\partial}_{h,n} \le 12(h-1)[80(h+1)]^{h-3}26^{(n-1)/6}.$$

9.2 Exponential realization for a boundary launch

For one boundary direction in the unit equilateral triangle, both of two return words can be concatenated arbitrarily. Here are explicit length inequalities for this construction. Describe a rooted quadrilateral by absolute directions $(0, b, c, d)$ in order and launch direction θ. Its first and last side lengths x, y determine the middle two via closure. If $S(q) = \sin(\pi q/180)$, the length vector is $N(x, y)^{\mathsf{T}}$ with

$$N = \begin{pmatrix} 1 & 0 \\ -S(c)/S(c-b) & -S(c-d)/S(c-b) \\ S(b)/S(c-b) & -S(d-b)/S(c-b) \\ 0 & 1 \end{pmatrix}.$$

Intersecting the ray with the chosen side and taking the length of the retained portion gives two linear updates for $(x, y)^{\mathsf{T}}$. For L their rows are $(N_{21} - S(\theta)/S(b-\theta), N_{22})$ and $(S(b)/S(b-\theta), 0)$. For R the rows are $(0, S(d-c)/S(c-\theta))$ and $(N_{31}, N_{32} + S(d-\theta)/S(c-\theta))$. These formulas come from resolving the two adjacent sides and the chord into the two known directions by the sine rule. A state satisfying the strict angle inequalities whose updated lengths are all positive has a forward cut on the interior of the designated side: the old length is the sum of the piece retained in the child and the positive piece adjacent to the removed corner, the latter obtained by the same sine rule.

Start at $(b, c, d, \theta) = (100, 220, 300, 60)$. The words $LLRLLLR$ and $LLRRLLLLR$ return to that angle state. Their length matrices are invertible. For each word pull back target vectors $(1-r, r)$ at $r = 2/5, 7/10$ by the inverse product. The following rational enclosures for the input ratios $y/(x+y)$ are obtained by substitution into the two-by-two matrices. All intermediate lengths, including at the input, are strictly positive.

	$r = 2/5$	$r = 7/10$
$LLRLLLR$	$(507/1000, 508/1000)$	$(482/1000, 483/1000)$
$LLRRLLLLR$	$(644/1000, 645/1000)$	$(637/1000, 638/1000)$

These are finite trigonometric inequalities. One way to check them is to reduce each sine to an argument in $[0, \pi/2]$, use the first twenty terms of the alternating sine series and the next term as error, and bound π by Machin's identity and the alternating arctangent series. The linear equations for all intermediate lengths preserve positivity on the convex hull of the two pulled-back vectors. The corresponding projective maps therefore take the target interval back into two disjoint subintervals of its interior. Each pulled-back state realizes the whole word.

For the entry, launch at $(s, 0)$ at 80° in the unit equilateral triangle, with $2/5 \le s \le 7/10$. The first target is the right side, the second is the left side and the third is the first chord. To check this, apply the determinant intersection test on the retained quadrilateral after each cut. The launch angles along the three flights are $80^\circ, 160^\circ, 320^\circ$. At both endpoints of the s interval the target

parameters are strictly between zero and one and the flight parameters positive. Since the vertices are affine in s with constant supporting directions, the distances to both ends of each target edge and the flight parameters are affine in s. Thus all three cuts are valid throughout the interval. The retained quadrilateral after the third impact has directions $(0, 100, 220, 300)$ after subtracting its first edge angle $140°$, and launch direction $60°$ in these coordinates. Its endpoint ratios lie in $(906/1000, 907/1000)$ and $(286/1000, 287/1000)$, respectively, by the same sine bounds. Continuity therefore provides every ratio in $[2/5, 7/10]$. Pulling back any successive choice of the two words inside this interval yields a realizable prefix. The words share their first three letters and differ in their fourth, so the choice of each block distinguishes histories. Their lengths are seven and nine. For every integer $k \geq 1$, extending each of the 2^k choices by more blocks gives at least 2^k distinct histories of length $3 + 9k$ at this fixed boundary direction.

10 An algebraic certificate for the disk distance supremum

The finite disk maximum can be expressed as an algebraic number. Let $t = \cos\alpha$ and $x = t^2$. Take the polynomials T_j, U_j defined by $T_j(\cos\alpha) = \cos j\alpha$ and $U_j(\cos\alpha) = \sin((j+1)\alpha)/\sin\alpha$. Both satisfy the recurrence $F_{j+1}(t) = 2tF_j(t) - F_{j-1}(t)$, with $T_0 = U_0 = 1$, $T_1 = t$, $U_1 = 2t$. The quotient $H_m(\alpha)/\sqrt{1-x} = p_m(x)/q_m(x)$ follows directly by cancellation from

$$2(m+1) - t\left(\frac{U_{m-1}(t)}{T_m(t)} + \frac{U_m(t)}{T_{m+1}(t)}\right).$$

Equivalently put the expression over $T_m(t)T_{m+1}(t)$ and divide numerator and denominator by t before writing both in x. The endpoints in x are $\cos^2(\pi/m)$ and $\cos^2(\pi/(m+1))$. For $2 \leq m \leq 27$, $m \neq 7$, set $B_m = (6511/1000)^2 q_m^2 - (1-x)p_m^2$. The finite sign certificate is that q_m and B_m have zero roots on the sector, with B_m positive at its midpoint. For $m = 7$, the stationary polynomial $2(1-x)(p_7'q_7 - p_7q_7') - p_7q_7$ has two interior roots, and the one near $82106954/10^8$ gives the larger value. The evaluation rules for checking these entries are to take the signed remainder sequence of the polynomial and its derivative, count sign variations at the endpoints, and subtract the counts. The sector endpoints may be enclosed by rationals of denominator 10^{60} using the alternating cosine series and Machin's identity for π. The variations at the two rational bounds for each endpoint coincide. For $m = 7$ the two stationary values lie in $(6511317/10^6, 6511318/10^6)$ and $(6504282/10^6, 6504283/10^6)$, while both endpoint values are less than $6511/1000$. These inequalities follow by isolating the stationary roots with the same signed remainders and substituting the isolated intervals in the quotient.

Consequently the value at the first stationary root for $m = 7$ is the global supremum. Its upper length fraction in the last row of the disk law is approached as z increases to v, an excluded endpoint event. Thus regular trajectories have $4 < S < S_{\mathrm{sup}}$, where S_{sup} is that algebraic value.

References

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