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# No ambidexterity for $\mathbb{P}$-invariant sheaves and for the birational motivic category

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**Abstract**

Let $\mathrm{Sh}_{\mathbb{P}}(T)$ be the $\infty$-category of Nisnevich sheaves of spectra on smooth $T$-schemes which are invariant under the projections $X \times \mathbb{P}^n \to X$, and let $\mathcal{SH}^b(T) \subseteq \mathrm{Sh}_{\mathbb{P}}(T)$ be the stable version of the birational motivic homotopy category of D. Maity. R. Iwasa asked whether a smooth proper morphism $f$ has ambidexterity in these theories, that is, whether $f_* \simeq f_\sharp(- \otimes L)$ for an invertible object $L$, as in the motivic spectra of Annala–Hoyois–Iwasa. We show that it does not. For a smooth morphism $f\colon T \to \operatorname{Spec} k$, such an $L$ exists only if $T$ is a finite disjoint union of copies of $\operatorname{Spec} k$; and if $T$ is geometrically connected of positive dimension, for example $T = \mathbb{P}^1$, then $f_*$ is not of the form $f_\sharp(- \otimes \omega)$ for any object $\omega$ whatsoever. The same holds unstably, over every integral base, and for the theories in between, such as Nisnevich sheaves and $\mathbb{A}^1$-invariant sheaves of spectra. The reason is that the unit is a constant sheaf; the only dualizable objects are the constant ones. For $\mathbb{P}$-invariant modules over the structure sheaf $\mathcal{O}$, we show that the Serre duality form of ambidexterity fails in characteristic $p$, already for $\mathbb{P}^1$, and that no twist at all works for a supersingular elliptic curve; in characteristic 0 we reduce the case of $\mathbb{P}^1$ to a divisibility question, which we leave open. We also show that naive purity fails for $\mathbb{P}$-invariant sheaves.

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## 1 Introduction

### 1.1 The questions

Let $f\colon T \to S$ be a smooth morphism of schemes. In a six-functor formalism of motivic type there are adjunctions $f_\sharp \dashv f^* \dashv f_*$, and when $f$ is also proper one expects $f_*$ to be a twist of $f_\sharp$. In $\mathbb{A}^1$-homotopy theory this is relative Atiyah duality. Annala, Hoyois and Iwasa showed that it does not depend on $\mathbb{A}^1$-invariance: in their category MS of motivic spectra there is a canonical equivalence $f_\sharp \Sigma^{-\Omega_f} \simeq f_*$ for $f$ smooth and locally projective [AHI, Thm. 1.1]. Let us say that $f$ *has ambidexterity* in a given theory if there are a $\otimes$-invertible object $L$ over $T$ and an equivalence of functors $f_\sharp(- \otimes L) \simeq f_*$.

In MS the twist is a Thom spectrum, and the pointed projective line is invertible. This note is about theories in which the pointed projective line is zero. For a scheme $T$ let $\mathrm{Sh}_{\mathrm{Nis}}(T)$ be the $\infty$-category of Nisnevich sheaves of spectra on the category $\mathrm{Sm}_T$ of smooth $T$-schemes, and let

$$\mathrm{Sh}_{\mathbb{P}}(T) \subseteq \mathrm{Sh}_{\mathrm{Nis}}(T)$$

be the full subcategory of $\mathbb{P}$-*invariant* sheaves, those $G$ with $G(X) \xrightarrow{\sim} G(X \times \mathbb{P}^n)$ for all $X \in \mathrm{Sm}_T$ and $n \geq 1$. The structure sheaf $\mathcal{O}\colon X \mapsto R\Gamma(X, \mathcal{O}_X)$ is a commutative algebra in $\mathrm{Sh}_{\mathbb{P}}(T)$. R. Iwasa asked the following questions.

(Q1) Does a smooth proper morphism have ambidexterity in $\mathrm{Sh}_{\mathbb{P}}$? Such an equivalence would encode Serre duality.

(Q2) The weaker question for modules over $\mathcal{O}$ in $\mathrm{Sh}_{\mathbb{P}}$: is $f_\sharp(- \otimes_{\mathcal{O}} \mathcal{L}) \simeq f_*$ for an invertible $\mathcal{O}$-module $\mathcal{L}$? Serre duality dictates $\mathcal{L} = \omega_f^\vee[-d]$, where $d$ is the relative dimension.

(Q3) D. Maity has introduced the birational motivic homotopy category $\mathcal{H}^0(S)$, the localization of the Morel–Voevodsky category at the dense open immersions between smooth $S$-schemes [Mai26a, Mai26b]. Does it have ambidexterity?

The answer to (Q1) and (Q3) is no, in a strong sense, and the two are closely related: birational sheaves of spectra are $\mathbb{P}$-invariant Nisnevich sheaves, and the birational category is the part of $\mathrm{Sh}_{\mathbb{P}}$ which already carries the obstruction. The answer to (Q2) is no in positive characteristic: the Serre twist fails for $\mathbb{P}^1$, and for a supersingular elliptic curve no twist works. In characteristic 0 we leave it open.

### 1.2 Results in the birational category

Let $\mathcal{V}$ be one of Spc (spaces), $\mathrm{Spc}_\bullet$ (pointed spaces), Sp (spectra). For a scheme $T$ we write $\mathcal{B}_{\mathcal{V}}(T)$ for the $\infty$-category of $\mathcal{V}$-valued presheaves on smooth $T$-schemes that send finite coproducts to products and dense open immersions to equivalences (Definition 2.1), and

$$\mathcal{H}^b(T) := \mathcal{B}_{\mathrm{Spc}}(T), \qquad \mathcal{H}^b_\bullet(T) := \mathcal{B}_{\mathrm{Spc}_\bullet}(T), \qquad \mathcal{SH}^b(T) := \mathcal{B}_{\mathrm{Sp}}(T).$$

The first is Maity's category, and the other two are its pointed objects and its stabilization (Remark 2.2). All three are symmetric monoidal, and a smooth morphism $f$ induces adjunctions $f_\sharp \dashv f^* \dashv f_*$ (§2). We write $\mathrm{Sm}_k$ for the category of smooth separated $k$-schemes of finite type.

**Theorem 1.1.** *Let $k$ be a field, $T \in \mathrm{Sm}_k$ non-empty, and $f\colon T \to \operatorname{Spec} k$ the structure map.*

(i) *In $\mathcal{SH}^b$, there are an invertible object $L$ over $T$ and an equivalence $f_\sharp(- \otimes L) \simeq f_*$ if and only if $T$ is a finite disjoint union of copies of $\operatorname{Spec} k$.*

(ii) *In $\mathcal{H}^b$ and in $\mathcal{H}^b_\bullet$, there are an invertible object $L$ over $T$ and an equivalence $f_\sharp(- \otimes L) \simeq f_*$ if and only if $T \cong \operatorname{Spec} k$.*

In particular $\mathbb{P}^1_k \to \operatorname{Spec} k$ has no ambidexterity. Theorem 1.1 only excludes invertible twists, and there are very few of those (Proposition 1.4). The next statement removes this restriction.

**Theorem 1.2.** *Let $k$ be a field, $T \in \mathrm{Sm}_k$ geometrically connected of dimension $\geq 1$, and $f\colon T \to \operatorname{Spec} k$. In each of $\mathcal{H}^b$, $\mathcal{H}^b_\bullet$ and $\mathcal{SH}^b$ there is no object $\omega$ over $T$, invertible or not, with an equivalence of functors $f_\sharp(- \otimes \omega) \simeq f_*$.*

**Corollary 1.3.** *Let $S$ be an integral quasi-compact quasi-separated scheme with generic point $\eta$, and let $f\colon T \to S$ be smooth, separated and of finite presentation, with $T \neq \emptyset$. Then $f$ has ambidexterity in $\mathcal{SH}^b$ if and only if the generic fibre $T_\eta$ is a finite disjoint union of copies of $\eta$, and in $\mathcal{H}^b$ or $\mathcal{H}^b_\bullet$ if and only if $T_\eta \cong \eta$. If $T_\eta$ is geometrically connected of positive dimension over $k(\eta)$, for instance if $T = \mathbb{P}^n_S$ with $n \geq 1$, then $f_*$ is not of the form $f_\sharp(- \otimes \omega)$ for any object $\omega$.*

So a smooth proper morphism of positive relative dimension with non-empty source never has ambidexterity in the birational category.

**Proposition 1.4.** *Let $T$ be a connected normal noetherian scheme, for instance a connected smooth scheme over a field. An object of $\mathcal{B}_{\mathcal{V}}(T)$ is dualizable if and only if it is constant with dualizable value in $\mathcal{V}$. In particular the invertible objects are: only the unit $\mathbf{1}_T$, if $\mathcal{V} = \mathrm{Spc}$ or $\mathrm{Spc}_\bullet$; the shifts $\mathbf{1}_T[n]$, $n \in \mathbb{Z}$, if $\mathcal{V} = \mathrm{Sp}$.*

### 1.3 Results for $\mathbb{P}$-invariant sheaves

**Theorem 1.5.** *Let $k$ be a field and $T \in \mathrm{Sm}_k$ non-empty, with structure map $f\colon T \to \operatorname{Spec} k$.*

(i) *The unit of $\mathrm{Sh}_{\mathbb{P}}(T)$ is the constant sheaf: $\mathbf{1}_T(X) = \mathbb{S}^{\pi_0(X)}$ for $X \in \mathrm{Sm}_T$. If $T$ is connected, an object of $\mathrm{Sh}_{\mathbb{P}}(T)$ is dualizable if and only if it is a constant sheaf with value a finite spectrum. In particular every invertible object of $\mathrm{Sh}_{\mathbb{P}}(T)$ is, on each connected component, a shift of the unit.*

(ii) *There are an invertible $L \in \mathrm{Sh}_{\mathbb{P}}(T)$ and an equivalence $f_\sharp(- \otimes L) \simeq f_*$ if and only if $T$ is a finite disjoint union of copies of $\operatorname{Spec} k$. If $T$ is geometrically connected of dimension $\geq 1$, there is no object $\omega \in \mathrm{Sh}_{\mathbb{P}}(T)$ at all with $f_\sharp(- \otimes \omega) \simeq f_*$.*

(iii) *Let $T$ be geometrically connected and let $L \in \mathrm{Sh}_{\mathbb{P}}(T)$ be any object. If $f_\sharp(f^*M \otimes L) \simeq f_* f^* M$ for the two objects $M = \mathbf{1}$ and $M = \mathcal{O}$, then $R\Gamma(T, \mathcal{O}_T) \simeq k$ as spectra; in particular $H^i(T, \mathcal{O}_T) = 0$ for all $i > 0$.*

So ambidexterity fails in $\mathrm{Sh}_{\mathbb{P}}$ for $\mathbb{P}^1$, for every curve, and already for $\operatorname{Spec} k' \to \operatorname{Spec} k$ with $k'/k$ a non-trivial finite separable extension. By (i), $L \otimes \mathcal{O} \simeq \mathcal{O}[n]$ for every invertible $L$: the Serre twist $\omega_f[d]$ has no invertible lift to $\mathrm{Sh}_{\mathbb{P}}(T)$ unless $\omega_f \cong \mathcal{O}_T$ (Corollary 8.7). By (iii), if $H^i(T, \mathcal{O}_T) \neq 0$ for some $i > 0$, no functor $f_\sharp(- \otimes L)$ agrees with $f_*$ even on the two objects $f^*\mathbf{1}$ and $f^*\mathcal{O}$, and the projection formula $f_* f^* M \simeq f_* \mathbf{1}_T \otimes M$ fails for $M = \mathcal{O}$. Parts (i) and (ii) hold, with the same proofs, for Nisnevich sheaves of spectra, for $\mathbb{P}^1$-invariant sheaves, and for $\mathbb{A}^1$-invariant sheaves (Remark 8.9). When $f$ is proper, the failure in (ii) can also be seen on open subschemes of $T$ (Proposition 8.11).

For modules over $\mathcal{O}$ we write $\mathrm{Mod}_{\mathcal{O}}(T) := \mathrm{Mod}_{\mathcal{O}}(\mathrm{Sh}_{\mathbb{P}}(T))$. A quasi-coherent sheaf on $T$ defines an object of $\mathrm{Mod}_{\mathcal{O}}(T)$, and line bundles are invertible (§9).

**Theorem 1.6.** *Let $k$ be a field of characteristic $p > 0$.*

(i) *For $f\colon \mathbb{P}^1_k \to \operatorname{Spec} k$ one has $f_\sharp(\mathcal{O}(1)) \neq 0$ in $\mathrm{Mod}_{\mathcal{O}}(k)$, but $f_*(\mathcal{O}(1) \otimes \omega_f[1]) = 0$. So $f_\sharp \not\simeq f_*(- \otimes \omega_f[1])$: the Serre duality form of ambidexterity fails for $\mathbb{P}^1$.*

(ii) *For a supersingular elliptic curve $f\colon E \to \operatorname{Spec} k$, the object $\mathcal{O} \otimes E_+$ is not dualizable in $\mathrm{Mod}_{\mathcal{O}}(k)$; equivalently, the Künneth map $f_*\mathcal{O}_E \otimes_{\mathcal{O}} M \to f_* f^* M$ fails to be an equivalence for some $\mathcal{O}$-module $M$. Consequently there is no object $\mathcal{L} \in \mathrm{Mod}_{\mathcal{O}}(E)$ with an equivalence of functors $f_\sharp(- \otimes_{\mathcal{O}} \mathcal{L}) \simeq f_*$ on $\mathcal{O}$-modules.*

In characteristic 0 the $\mathcal{O}$-linear question is not decided here. Proposition 9.4 reduces its first test case, $f_\sharp(\mathcal{O}_{\mathbb{P}^1}(1)) = 0$, to a concrete divisibility property of sections over $\mathbb{A}^1$ (Question 9.5). Finally, Proposition 10.1 shows that the analogue of the homotopy purity theorem fails in $\mathrm{Sh}_{\mathbb{P}}$, even for a closed subscheme with trivial normal bundle.

### 1.4 Why

*The unit is constant.* On connected smooth schemes, limits, colimits and tensor products in $\mathcal{B}_{\mathcal{V}}(T)$ are computed objectwise (Lemma 2.6), and the unit is the constant presheaf. Objectwise tensor products leave no room for duality: a dualizable object must send every morphism to an equivalence (Lemma 3.1). This gives Proposition 1.4.

*$f_*$ annihilates geometric covers.* Let $C \to U$ be a connected finite étale cover of a dense open $U \subseteq T$ which stays connected over $\bar{k}$ and has degree $> 1$. Then $(f_* C_{\mathcal{V}})(Y) = C_{\mathcal{V}}(Y \times T)$ is built from sections of $Y \times C \to Y \times U$, and there are none: $f_*(C_{\mathcal{V}})$ is the initial object, while $f_\sharp(C_{\mathcal{V}})$ is the class of the variety $C$ and is not. Such covers exist on every geometrically connected $T$ of positive dimension (Lemma 4.1); on $\mathbb{P}^1$ one takes $s \mapsto s^\ell$ on $\mathbb{G}_m$. Together with a test on the unit, this proves Theorem 1.1.

*Constant sheaves do not see fibres.* If $\mathcal{Y} \to T$ is a family with geometrically connected fibres over a dense open, then $\underline{\mathrm{Hom}}(\mathcal{Y}_{\mathcal{V}}, G) \simeq G$ for every constant $G$. If $f_*$ were $f_\sharp(- \otimes \omega)$, this would force $f_*(\mathcal{Y}_{\mathcal{V}})$ to admit as many maps to constant objects as $f_*(\mathbf{1}_T)$ does, whatever $\omega$ is. But $f_*(\mathcal{Y}_{\mathcal{V}})$ is initial as soon as $\mathcal{Y}$ has no rational section over any $Z \times T$, and an isotrivial family of genus one curves twisted by a geometric cover has none. This proves Theorem 1.2.

*From birational to $\mathbb{P}$-invariant sheaves.* A birational presheaf of spectra is a $\mathbb{P}$-invariant Nisnevich sheaf. So $\mathcal{SH}^b(T)$ is a reflective subcategory of $\mathrm{Sh}_{\mathbb{P}}(T)$; the reflection is symmetric monoidal and commutes with $f_\sharp$ and $f^*$, and $f_*$ preserves $\mathcal{SH}^b$. Since the unit is birational, the dual of any object of $\mathrm{Sh}_{\mathbb{P}}(T)$ is birational, so dualizable objects are birational, hence constant; and an equivalence $f_* \simeq f_\sharp(- \otimes \omega)$ in $\mathrm{Sh}_{\mathbb{P}}$ restricts to one in $\mathcal{SH}^b$. This proves Theorem 1.5(i),(ii). Part (iii) is a direct computation: the sphere has no cohomology in $\mathrm{Sh}_{\mathbb{P}}$, so $f_*\mathbf{1}_T$ only remembers $\pi_0(T)$, and an ambidexterity statement would give $M(X \times T) \simeq M(X)$ for every $\mathbb{P}$-invariant $M$; but $\mathcal{O}(k) \to \mathcal{O}(T)$ is not an equivalence in general. In MS the dual of $T_+$ is a Thom spectrum, a desuspension by a power of $\mathbb{P}^1$; in $\mathrm{Sh}_{\mathbb{P}}$ the pointed projective line is zero, and duality is exactly what is lost.

## 1.5 Comments

(a) In dimension zero, the theorems say that a non-split finite étale morphism has no ambidexterity. This is the familiar absence of transfers before $\mathbb{P}^1$-stabilization; compare naive and genuine equivariant spectra. The point is that in positive dimension the failure persists over algebraically closed fields, for $\mathbb{P}^1$, and for arbitrary twists.

(b) The proof of Theorem 1.2 shows that an equivalence $f_* \simeq f_\sharp(- \otimes \omega)$ in the birational category would force $T$ to be birationally contractible, i.e. $T_{\mathcal{V}} \simeq \mathbf{1}$. Rational varieties have this property [Mai26a, Ex. 4.2.14], and for them $f^*$ is fully faithful [Mai26b, Cor. 4.4.5], so that $f_*f^* \simeq \mathrm{id} \simeq f_\sharp f^*$. Thus for $\mathbb{P}^1$ the two functors agree on objects pulled back from the base, and differ elsewhere.

(c) Birational localizations of categories of varieties and of motives were studied by Kahn and Sujatha [KS15, KS16, KS17]. Maity's papers do not develop the stable theory of the birational category, and note the absence of an exceptional pushforward for closed immersions [Mai26b, §1.3]. The unstable $\mathbb{P}^1$-homotopy category and its stabilization are studied in [Mai26c], where the failure of gluing and the lack of a purity theorem are pointed out; $\mathrm{Sh}_{\mathbb{P}}(T)$ is the full subcategory of that stabilization consisting of the sheaves invariant under all $\mathbb{P}^n$, and it is an open question whether the two coincide [Mai26c, Rem. 2.2.20]. We have not found the statements of this note in the literature, but we have not searched it systematically. We only treat the 0-birational category; Maity's $n$-birational categories for $n \geq 1$ are not considered.

## 1.6 Conventions

We use the language of $\infty$-categories. For a scheme $T$, $\mathrm{Sm}_T$ is the category of smooth, separated, finitely presented $T$-schemes, as in [Mai26b]. A variety over a field is *geometrically connected* if it stays connected over an algebraic closure. Mapping spaces and mapping spectra are denoted Map, internal mapping objects $\underline{\mathrm{Hom}}$. All statements labelled Theorem, Proposition, Lemma or Corollary are proved in full; remarks and questions are not claimed.

**Provenance.** The text and the proofs of this note were generated by an AI system (Claude, developed by Anthropic) in response to questions of Ryomei Iwasa, relayed by Markus Lien, who prompted the system. The system checked the arguments by hand; no formal verification was carried out. The results quoted from [AHI, Mai26a, Mai26b, Mai26c] were checked against the arXiv versions listed in the references, and are used only for context and attribution: the proofs

below are self-contained apart from standard facts.

# 2 Birational presheaves

## 2.1 Coefficients

Throughout, $\mathcal{V}$ is one of the presentably symmetric monoidal $\infty$-categories $(\mathrm{Spc}, \times)$, $(\mathrm{Spc}_\bullet, \wedge)$, $(\mathrm{Sp}, \otimes)$. We write $\otimes$ for its tensor product, $\mathbf{1}_\mathcal{V}$ for its unit ($*$, $S^0$, $\mathbb{S}$), and $F_\mathcal{V}\colon \mathrm{Set} \to \mathcal{V}$ for the free functor $A \mapsto A$, $A_+$, $\Sigma^\infty_+ A$. This functor sends products to tensor products and $\emptyset$ to the initial object, and a map of sets $\varphi$ is bijective if $F_\mathcal{V}(\varphi)$ is an equivalence. The *underlying space* of $V \in \mathcal{V}$ is $\mathrm{Map}(\mathbf{1}_\mathcal{V}, V)$. We fix

$$V_2 := F_\mathcal{V}(\{a, b\}),$$

whose underlying space is not contractible. For $\mathcal{V} = \mathrm{Sp}$ we write $[n]$ for the $n$-fold suspension.

## 2.2 Definition and formal properties

For a scheme $T$ let $\mathrm{PSh}_\mathcal{V}(T) := \mathrm{Fun}(\mathrm{Sm}_T^{\mathrm{op}}, \mathcal{V})$, with the objectwise symmetric monoidal structure, and let $h_X$ be the presheaf of sets represented by $X \in \mathrm{Sm}_T$.

**Definition 2.1.** A presheaf $G \in \mathrm{PSh}_\mathcal{V}(T)$ is *birational* if

(Σ) $G(\emptyset)$ is terminal and $G(X \sqcup Y) \to G(X) \times G(Y)$ is an equivalence for all $X, Y \in \mathrm{Sm}_T$;

(D) $G(X) \to G(U)$ is an equivalence for every dense open immersion $U \to X$ in $\mathrm{Sm}_T$.

We write $\mathcal{B}_\mathcal{V}(T) \subseteq \mathrm{PSh}_\mathcal{V}(T)$ for the full subcategory of birational presheaves.

*Remark* 2.2 (relation to Maity's categories). $\mathcal{B}_{\mathrm{Spc}}(T)$ is the category $\mathcal{H}^b(T)$ of birational local presheaves of [Mai26a, Def. 2.0.2], [Mai26b, §2.3]. For quasi-compact quasi-separated $T$, birational local presheaves are Nisnevich sheaves [Mai26a, Thm. 2.1.3], [Mai26b, Thm. 2.3.5] and are $\mathbb{A}^1$-invariant, and $\mathcal{H}^b(T)$ coincides with the localization $\mathcal{H}^{b\mathbb{A}^1}(T) = \mathcal{H}^0(T)$ of the Morel–Voevodsky category at dense open immersions [Mai26a, Cor. 2.2.12], [Mai26b, Cor. 2.3.7]. (Theorem numbers refer to the arXiv versions cited in the references.) Since (Σ) and (D) can be tested on the spaces $\Omega^{\infty-n}G$, the category $\mathcal{B}_{\mathrm{Sp}}(T)$ is the stabilization of $\mathcal{H}^b(T)$, and $\mathcal{B}_{\mathrm{Spc}_\bullet}(T)$ is the category of pointed objects of $\mathcal{H}^b(T)$. We do not use any of these comparisons.

**Lemma 2.3** (formal structure)**.** (i) *The inclusion $\mathcal{B}_\mathcal{V}(T) \subseteq \mathrm{PSh}_\mathcal{V}(T)$ has a left adjoint $L$, and $L$ is compatible with the symmetric monoidal structure. Hence $\mathcal{B}_\mathcal{V}(T)$ is presentably symmetric monoidal, $L$ is symmetric monoidal, and $\mathcal{B}_\mathcal{V}(T)$ has internal mapping objects.*

(ii) *For $X \in \mathrm{Sm}_T$ put $X_\mathcal{V} := L(F_\mathcal{V} h_X)$. Then $\mathrm{Map}(X_\mathcal{V}, G)$ is the underlying space of $G(X)$ for $G \in \mathcal{B}_\mathcal{V}(T)$; moreover $X_\mathcal{V} \otimes Y_\mathcal{V} \simeq (X \times_T Y)_\mathcal{V}$, the unit is $\mathbf{1}_T = T_\mathcal{V}$, and $\underline{\mathrm{Hom}}(X_\mathcal{V}, G)(Y) \simeq G(X \times_T Y)$.*

*Proof.* (i) A presheaf is birational if and only if it is local for the set $W$ of maps $F_\mathcal{V} h_U \to F_\mathcal{V} h_X$ ($U \to X$ a dense open immersion), $F_\mathcal{V} h_X \sqcup F_\mathcal{V} h_Y \to F_\mathcal{V} h_{X \sqcup Y}$ and $\emptyset \to F_\mathcal{V} h_\emptyset$, together with their desuspensions if $\mathcal{V} = \mathrm{Sp}$. So $L$ exists [HTT, §5.5.4]. Dense open immersions and coproduct decompositions are stable under $- \times_T Z$ for $Z \in \mathrm{Sm}_T$, so $W$ is stable under tensoring with the generators $F_\mathcal{V} h_Z$; hence the class of $L$-equivalences is stable under tensoring with arbitrary objects, and the localization is symmetric monoidal [HA, §2.2.1]. (ii) The first assertion is the Yoneda lemma. The second holds because $L$ and $F_\mathcal{V}$ are symmetric monoidal and $h_X \times h_Y = h_{X \times_T Y}$. The last follows: $\mathrm{Map}(Y_\mathcal{V}, \underline{\mathrm{Hom}}(X_\mathcal{V}, G)) = \mathrm{Map}((X \times_T Y)_\mathcal{V}, G)$. □

**Lemma 2.4** (smooth morphisms)**.** *Let $g\colon T' \to T$ be smooth, separated and of finite presentation.*

(i) *Restriction along* $\mathrm{Sm}_{T'} \to \mathrm{Sm}_T$, $(X \to T') \mapsto (X \to T' \to T)$, *defines a symmetric monoidal functor* $g^*\colon \mathcal{B}_{\mathcal{V}}(T) \to \mathcal{B}_{\mathcal{V}}(T')$ *with* $g^*(Y_{\mathcal{V}}) = (Y \times_T T')_{\mathcal{V}}$.

(ii) $g^*$ *has a left adjoint* $g_\sharp$, *with* $g_\sharp(X_{\mathcal{V}}) = X_{\mathcal{V}}$, *and a right adjoint* $g_*$, *with* $(g_*G)(Y) = G(Y \times_T T')$.

(iii) (Projection formula.) $g_\sharp(N \otimes g^*M) \simeq g_\sharp N \otimes M$, *naturally in* $N$ *and* $M$. *Hence* $g_\sharp g^* M \simeq T'_{\mathcal{V}} \otimes M$ *and* $g_* g^* M \simeq \underline{\mathrm{Hom}}(T'_{\mathcal{V}}, M)$.

For $\mathcal{V} = \mathrm{Spc}$ the adjunctions are [Mai26b, Prop. 3.1.9].

*Proof.* Let $u\colon \mathrm{Sm}_{T'} \to \mathrm{Sm}_T$ be the forgetful functor and $v = -\times_T T'$ its right adjoint. Restriction $u^*$ on presheaves is strictly symmetric monoidal for the objectwise structures, preserves birational presheaves, and is left Kan extension along $v$; so $u^*(F_{\mathcal{V}} h_Y) = F_{\mathcal{V}} h_{vY}$, and $u^*$ preserves $L$-equivalences because $v$ preserves dense open immersions and coproducts. This gives (i). The left adjoint is $g_\sharp = L \circ u_!$, and $u_!(F_{\mathcal{V}} h_X) = F_{\mathcal{V}} h_{uX}$. The right adjoint of $u^* = v_!$ is $v^*$, which preserves birational presheaves. This gives (ii). For (iii) it suffices to treat $N = X_{\mathcal{V}}$, $M = Y_{\mathcal{V}}$, as both sides preserve colimits in each variable; there both sides are $(X \times_T Y)_{\mathcal{V}}$. The last two formulas follow by taking $N = \mathbf{1}_{T'}$ and passing to right adjoints. □

**Lemma 2.5** (dense opens)**.** *Let* $j\colon U \to T$ *be a quasi-compact dense open immersion. Then* $j^*\colon \mathcal{B}_{\mathcal{V}}(T) \to \mathcal{B}_{\mathcal{V}}(U)$ *is an equivalence with inverse* $j_* \simeq j_\sharp$. *Explicitly, for* $X \in \mathrm{Sm}_U$ *and* $Y \in \mathrm{Sm}_T$,

$$X_{\mathcal{V}}(Y) \simeq X_{\mathcal{V}}(Y \times_T U),$$

*where* $X_{\mathcal{V}}$ *is formed in* $\mathcal{B}_{\mathcal{V}}(T)$ *on the left and in* $\mathcal{B}_{\mathcal{V}}(U)$ *on the right.*

*Proof.* For $Y \in \mathrm{Sm}_T$ the open immersion $Y \times_T U \to Y$ is dense, because $Y \to T$ is open. So the unit $G \to j_* j^* G$, which on $Y$ is $G(Y) \to G(Y \times_T U)$, is an equivalence by (D). The counit $j^* j_* H \to H$ is the identity on every $Y \in \mathrm{Sm}_U$. Hence $j^*$ is an equivalence with inverse $j_*$, and its left adjoint $j_\sharp$ agrees with $j_*$. The formula is $j_\sharp(X_{\mathcal{V}}) = j_*(X_{\mathcal{V}})$. □

## 2.3 Objectwise structure on connected schemes

Let $T$ be a normal noetherian scheme, for example the spectrum of a field or a smooth scheme over a field. Then every $X \in \mathrm{Sm}_T$ is a finite disjoint union of normal integral schemes, and a dense open subscheme of a connected $X$ is connected. Let $\mathrm{Sm}_T^{\mathrm{c}} \subseteq \mathrm{Sm}_T$ be the full subcategory of non-empty connected schemes.

**Lemma 2.6.** *Let* $T$ *be a normal noetherian scheme.*

(i) *Restriction to* $\mathrm{Sm}_T^{\mathrm{c}}$ *identifies* $\mathcal{B}_{\mathcal{V}}(T)$ *with the full subcategory of* $\mathrm{Fun}((\mathrm{Sm}_T^{\mathrm{c}})^{\mathrm{op}}, \mathcal{V})$ *of functors that invert dense open immersions.*

(ii) *If* $P \in \mathcal{B}_{\mathcal{V}}(T)$ *and* $Q \to P$ *is a map of presheaves which is an equivalence on connected schemes, then* $L(Q) \simeq P$.

(iii) *Limits, colimits and tensor products in* $\mathcal{B}_{\mathcal{V}}(T)$ *are computed objectwise on* $\mathrm{Sm}_T^{\mathrm{c}}$. *In particular, for* $V \in \mathcal{V}$ *the constant functor* $V_T$ *with value* $V$ *on* $\mathrm{Sm}_T^{\mathrm{c}}$ *lies in* $\mathcal{B}_{\mathcal{V}}(T)$, *the unit is* $\mathbf{1}_T = (\mathbf{1}_{\mathcal{V}})_T$, *the initial object is constant, and* $V \mapsto V_T$ *is symmetric monoidal.*

(iv) *Suppose that* $X \in \mathrm{Sm}_T$ *is* birationally rigid, *i.e.* $h_X$ *satisfies* (D). *Then* $X_{\mathcal{V}}(Y) = F_{\mathcal{V}} \mathrm{Hom}_T(Y, X)$ *for* $Y \in \mathrm{Sm}_T^{\mathrm{c}}$.

*Proof.* (i) Since $\mathrm{Hom}(X, \bigsqcup_j Y_j) = \bigsqcup_j \mathrm{Hom}(X, Y_j)$ for connected $X$, the category $\mathrm{Sm}_T$ is obtained from $\mathrm{Sm}_T^{\mathrm{c}}$ by freely adjoining finite coproducts. So restriction $i^*$ along $i\colon \mathrm{Sm}_T^{\mathrm{c}} \to \mathrm{Sm}_T$ identifies

presheaves satisfying ($\Sigma$) with all presheaves on $\mathrm{Sm}_T^{\mathrm{c}}$, the inverse being right Kan extension $i_*$ [HTT, §5.5.8]. A dense open immersion in $\mathrm{Sm}_T$ is a disjoint union of dense open immersions in $\mathrm{Sm}_T^{\mathrm{c}}$. (ii) For $G \in \mathcal{B}_{\mathcal{V}}(T)$ we have $G = i_* i^* G$, so $\mathrm{Map}(Q, G) = \mathrm{Map}(i^*Q, i^*G) = \mathrm{Map}(i^*P, i^*G) = \mathrm{Map}(P, G)$. (iii) Functors inverting dense open immersions are stable under objectwise limits and colimits. Let $G, G' \in \mathcal{B}_{\mathcal{V}}(T)$ and let $Q$ be their objectwise tensor product on $\mathrm{Sm}_T$, so that $G \otimes G' = L(Q)$. The functor $X \mapsto G(X) \otimes G'(X)$ on $\mathrm{Sm}_T^{\mathrm{c}}$ inverts dense open immersions, so it defines $P \in \mathcal{B}_{\mathcal{V}}(T)$ with a map $Q \to P$ as in (ii). Hence $G \otimes G' = P$. The remaining assertions are clear. (iv) Apply (ii) to $Q = F_{\mathcal{V}} h_X$. □

The term "birationally rigid" is from [Mai26a, §4.1], where the next examples (a), (b) also appear.

**Lemma 2.7** (rigid objects)**.** (a) *Let $T$ be normal noetherian and $E \to T$ finite étale. Then $E$ is birationally rigid in* $\mathrm{Sm}_T$.

(b) *Let $A$ be an abelian variety over a field $k$. Then $A$ is birationally rigid in* $\mathrm{Sm}_k$.

(c) *Let $B \in \mathrm{Sm}_k$ be affine, let $\ell \neq \mathrm{char}\, k$ be a prime, $P \to B$ a $\mu_\ell$-torsor, and $A$ an abelian variety over $k$ with a closed subgroup scheme $\mu_\ell \subseteq A$. Let $\mathcal{Y} := (P \times_k A)/\mu_\ell \to B$ be the quotient by the diagonal action, $\mu_\ell$ acting on $A$ by translations. Then $\mathcal{Y}$ is smooth and proper over $B$ with geometrically connected fibres, for $X \in \mathrm{Sm}_B$ the $B$-morphisms $X \to \mathcal{Y}$ correspond to the $\mu_\ell$-equivariant morphisms $X \times_B P \to A$, and $\mathcal{Y}$ is birationally rigid in* $\mathrm{Sm}_B$.

*Proof.* Let $X$ be a connected object of the relevant category and $U \subseteq X$ a dense open. Restriction of morphisms from $X$ to $U$ is injective since the targets are separated and $X$ is reduced; we show surjectivity.

(a) $E_X := E \times_T X$ is finite étale over the normal integral scheme $X$, hence a finite disjoint union of normal integral schemes, each finite étale over $X$. A section over $U$ is an open and closed immersion $U \to E_U$; its closure in $E_X$ is a connected component of degree 1 over $X$, hence maps isomorphically to $X$.

(b) This is Weil's theorem: a rational map from a smooth variety to an abelian variety is defined everywhere [Mil, §3].

(c) The quotient exists as a scheme because $P \times_k A$ is quasi-projective and the action is free. $P \times_k A \to \mathcal{Y}$ is a $\mu_\ell$-torsor, and $\mathcal{Y} \to B$ is a torsor under the abelian scheme $A \times_k B$ which becomes trivial over the finite étale cover $P \to B$; this gives the first assertion. The second is descent along the $\mu_\ell$-torsor $X \times_B P \to X$. Now $X \times_B P$ is finite étale over $X$, hence smooth over $k$, and $U \times_B P$ is dense in it. By (b) an equivariant morphism $U \times_B P \to A$ extends to $X \times_B P$, and the extension is equivariant by uniqueness. □

**Lemma 2.8** ($\mathbb{A}^1$-invariance)**.** *Let $T$ be any scheme and $X \in \mathrm{Sm}_T$. The projections induce equivalences $(X \times \mathbb{A}^1)_{\mathcal{V}} \simeq X_{\mathcal{V}}$ and $(X \times \mathbb{P}^n)_{\mathcal{V}} \simeq X_{\mathcal{V}}$ in $\mathcal{B}_{\mathcal{V}}(T)$. Equivalently, birational presheaves are invariant under $\mathbb{A}^1$ and under $\mathbb{P}^n$. In particular $X_{\mathcal{V}} \simeq \mathbf{1}$ for every rational variety $X$ over a field.*

For the 1-categorical localization this goes back to Kahn–Sujatha [KS15], and in the present setting it is [Mai26a, Lemma 2.2.6, Cor. 2.2.12]. We include a short argument.

*Proof.* The map $\rho(x, a, b) = (x, a, ab)$ of $X \times \mathbb{A}^2$ restricts to an open immersion on the dense open $\{a \neq 0\}$; as dense open immersions become equivalences, so does $\rho$. The maps $i(x, b) = (x, 0, b)$ and $i'(x, b) = (x, 0, 0)$ from $X \times \mathbb{A}^1$ satisfy $\rho i = \rho i'$, so $i$ and $i'$ become equivalent. Composing with $(x, a, b) \mapsto (x, b)$ gives $\mathrm{id} \simeq s_0 q$ on $(X \times \mathbb{A}^1)_{\mathcal{V}}$, where $q$ is the projection to $X$ and $s_0$ the zero section. As $q s_0 = \mathrm{id}$, $q$ becomes an equivalence. By induction $(X \times \mathbb{A}^n)_{\mathcal{V}} \simeq X_{\mathcal{V}}$, and $X \times \mathbb{A}^n \to X \times \mathbb{P}^n$ is a dense open immersion. □

## 3 Dualizable and invertible objects

**Lemma 3.1.** *Let $\mathcal{C}$ be an $\infty$-category and $\mathcal{D} \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{V})$ a full subcategory containing the constant functor $\mathbf{1}_{\mathcal{V}}$ and closed under objectwise tensor products. If $A \in \mathcal{D}$ is dualizable in $\mathcal{D}$, then $A(u)$ is an equivalence for every morphism $u$ of $\mathcal{C}$.*

*Proof.* Let $A^\vee \in \mathcal{D}$ be the dual, with evaluation $e\colon A \otimes A^\vee \to \mathbf{1}$ and coevaluation $\eta\colon \mathbf{1} \to A^\vee \otimes A$; these are natural transformations satisfying the triangle identities objectwise. Let $u\colon c \to c'$, and put $\varphi = A(u)$, $\gamma = A^\vee(u)$. Naturality gives $e_{c'}(\varphi \otimes \gamma) = e_c$ and $(\gamma \otimes \varphi)\eta_c = \eta_{c'}$. Define

$$\gamma^t := (e_{c'} \otimes 1)(1 \otimes \gamma \otimes 1)(1 \otimes \eta_c)\colon A(c') \longrightarrow A(c).$$

Then $\gamma^t\varphi = (e_{c'}(\varphi \otimes \gamma) \otimes 1)(1 \otimes \eta_c) = (e_c \otimes 1)(1 \otimes \eta_c) = \mathrm{id}$, and $\varphi\gamma^t = (e_{c'} \otimes 1)(1 \otimes (\gamma \otimes \varphi)\eta_c) = (e_{c'} \otimes 1)(1 \otimes \eta_{c'}) = \mathrm{id}$. These identities hold in the homotopy category of $\mathcal{V}$, which suffices. □

*Proof of Proposition 1.4.* By Lemma 2.6, $\mathcal{B}_{\mathcal{V}}(T)$ is a full subcategory of $\mathrm{Fun}((\mathrm{Sm}^{\mathrm{c}}_T)^{\mathrm{op}}, \mathcal{V})$ as in Lemma 3.1. So a dualizable $A$ inverts all morphisms of $\mathrm{Sm}^{\mathrm{c}}_T$. As $T$ is connected, it is a terminal object of $\mathrm{Sm}^{\mathrm{c}}_T$; a functor on a category with a terminal object which inverts all morphisms is constant. Thus $A = V_T$ with $V = A(T)$, and $V$ is dualizable since evaluation at $T$ is symmetric monoidal. Conversely $V \mapsto V_T$ is symmetric monoidal, so it preserves dualizable objects. The invertible objects of $\mathcal{V}$ are $*$, $S^0$, and the spheres $\mathbb{S}[n]$, respectively. □

*Remark* 3.2. Let $T$ be as in Proposition 1.4 and $X \in \mathrm{Sm}^{\mathrm{c}}_T$. Then $X_+ \in \mathcal{SH}^b(T)$ is dualizable if and only if the canonical map $X_+ \to \mathbf{1}_T$ is an equivalence. Indeed, if $X_+ \simeq V_T$ then $\mathrm{Map}(V, W) = \mathrm{Map}(V_T, W_T) = W_T(X) = W$ for all $W \in \mathrm{Sp}$, so $V \simeq \mathbb{S}$ and $X_+ \simeq \mathbf{1}_T$; the canonical map is then an equivalence by the argument of Step 1 in §6. For example, over a field $k$ the object $(\mathrm{Spec}\, k')_+$ is not dualizable for a non-trivial finite separable extension $k'/k$, and $X_+$ is not dualizable if $X$ has a morphism to an abelian variety which does not factor through $\mathrm{Spec}\, k$ (Lemma 2.7(b)).

## 4 Geometric covers

**Lemma 4.1** (geometric covers exist)**.** *Let $k$ be a field, $T \in \mathrm{Sm}_k$ geometrically connected of dimension $\geq 1$, $K = k(T)$, and $\ell \neq \mathrm{char}\, k$ a prime. There are a dense affine open $U^\circ \subseteq T$ and a unit $g \in \mathcal{O}(U^\circ)^\times$ such that the $\mu_\ell$-torsor*

$$C^\circ := \mathrm{Spec}\, \mathcal{O}_{U^\circ}[x]/(x^\ell - g) \longrightarrow U^\circ$$

*is geometrically connected over $k$. Thus $K' := k(C^\circ) = K[x]/(x^\ell - g)$ is a separable field extension of $K$ of degree $\ell$ such that $K' \otimes_k \bar{k}$ is a field. For $T = \mathbb{P}^1$ with coordinate $t$ one can take $U^\circ = \mathbb{G}_m$ and $g = t$.*

*Proof.* Some non-empty affine open $U \subseteq T$ admits an étale map to $\mathbb{A}^n_k$, $n = \dim T$; let $u\colon U \to \mathbb{A}^1_k$ be its first coordinate. As $u$ is smooth, its image is a non-empty open subset, and contains a closed point $a$ with $k(a)/k$ separable. Let $h \in k[z]$ be the minimal polynomial of $a$, and $g := h(u) \in \mathcal{O}(U)$. The zero scheme $D = u^{-1}(a)$ of $g$ is non-empty and smooth over $k(a)$, hence over $k$. Put $U^\circ := U \smallsetminus D$, a principal open of $U$.

Let $\Omega := K \otimes_k \bar{k}$, the function field of the integral scheme $T_{\bar{k}}$, and let $v$ be the valuation of $\Omega$ given by an irreducible component of $D_{\bar{k}}$. Since $D_{\bar{k}}$ is smooth, hence reduced, $v(g) = 1$. So $g$ is not an $\ell$-th power in $\Omega$, and $x^\ell - g$ is irreducible over $\Omega$ [Lang, VI, Thm. 9.1], a fortiori over $K$. Hence $K'$ is a field, separable of degree $\ell$ over $K$, and $K' \otimes_k \bar{k} = \Omega[x]/(x^\ell - g)$ is a field. Finally $C^\circ_{\bar{k}}$ is finite étale over the integral scheme $U^\circ_{\bar{k}}$ with connected generic fibre $\mathrm{Spec}(K' \otimes_k \bar{k})$, so it is connected. □

**Lemma 4.2** ($f_*$ annihilates geometric covers)**.** *Let $T$, $U^\circ$, $C^\circ$ be as in Lemma 4.1, and regard $C^\circ$ as an object of $\mathrm{Sm}_T$. Then $C^\circ_{\mathcal{V}}(Y \times_k T)$ is the initial object of $\mathcal{V}$ for every non-empty connected $Y \in \mathrm{Sm}_k$. In other words, $f_*(C^\circ_{\mathcal{V}})$ is the initial object of $\mathcal{B}_{\mathcal{V}}(k)$, where $f\colon T \to \operatorname{Spec} k$.*

*Proof.* $Y \times_k T$ is connected, hence integral, and its restriction to $U^\circ$ is $Y \times_k U^\circ$. By Lemma 2.5, Lemma 2.7(a) and Lemma 2.6(iv),

$$C^\circ_{\mathcal{V}}(Y \times_k T) = C^\circ_{\mathcal{V}}(Y \times_k U^\circ) = F_{\mathcal{V}} \operatorname{Hom}_{U^\circ}(Y \times_k U^\circ, C^\circ),$$

so we must show that $Y \times C^\circ \to Y \times U^\circ$ has no section. A section induces, at the generic points, a homomorphism of $K$-algebras $K' \to F$, where $F$ is the function field of $Y \times_k T$.

Let $k_Y$ be the algebraic closure of $k$ in $\mathcal{O}(Y)$, a finite separable extension of $k$ over which $Y$ is geometrically integral, and choose an embedding $k_Y \subseteq \bar{k}$. Then $\bar{Y} := Y \times_{k_Y} \operatorname{Spec} \bar{k}$ is integral, $\bar{Y} \times_{\bar{k}} T_{\bar{k}}$ is integral and dominates $Y \times_k T$, and its function field $\bar{F}$ contains $F$ and $\Omega = K \otimes_k \bar{k}$. Moreover $\Omega$ is algebraically closed in $\bar{F}$, because $\bar{F}$ is the function field of the $\Omega$-variety $\bar{Y} \times_{\bar{k}} \operatorname{Spec} \Omega$, which is geometrically integral. The image of $K' \to F \subseteq \bar{F}$ is algebraic over $K$, hence over $\Omega$, and therefore lies in $\Omega$, since $\Omega$ is algebraically closed in $\bar{F}$. Extending scalars we get a surjection of $\Omega$-algebras $K' \otimes_K \Omega \to \Omega$. But $K' \otimes_K \Omega = K' \otimes_k \bar{k}$ is a field of degree $\ell > 1$ over $\Omega$. Contradiction. □

*Example* 4.3 ($T = \mathbb{P}^1$). Here $C^\circ = \mathbb{G}_m \to \mathbb{G}_m = U^\circ$, $s \mapsto s^\ell = t$, and $N := C^\circ_+ \in \mathcal{SH}^b(\mathbb{P}^1)$ is the "sheaf of $\ell$-th roots of the coordinate": $N(X) = \Sigma^\infty_+\{r \in k(X) : r^\ell = t\}$ for connected $X \in \mathrm{Sm}_{\mathbb{P}^1}$. We have $f_\sharp N = (\mathbb{G}_m)_+ \simeq \mathbf{1}$ by Lemma 2.8, whereas $f_* N = 0$, since $t$ has no $\ell$-th root in $k(Y)(t)$.

## 5 Invertible twists: proof of Theorem 1.1

Let $T_1, \dots, T_r$ be the connected components of $T$, and let $k_i$ be the algebraic closure of $k$ in $\mathcal{O}(T_i)$; it is a finite separable extension of $k$, and $T_i$ is geometrically connected as a $k_i$-scheme. We have $\mathcal{B}_{\mathcal{V}}(T) = \prod_i \mathcal{B}_{\mathcal{V}}(T_i)$.

*"If".* For $T = \bigsqcup_{i=1}^r \operatorname{Spec} k$ we have $\mathcal{B}_{\mathcal{V}}(T) = \mathcal{B}_{\mathcal{V}}(k)^r$, and $f_\sharp$ is the coproduct and $f_*$ the product. They agree if $\mathcal{V} = \mathrm{Sp}$, and if $r = 1$.

*"Only if".* Suppose $L$ is invertible and $f_\sharp(- \otimes L) \simeq f_*$.

*Step 1: $L$ is a shift of the unit.* By Proposition 1.4, $L|_{T_i} \simeq \mathbf{1}_{T_i}[n_i]$ with $n_i \in \mathbb{Z}$ if $\mathcal{V} = \mathrm{Sp}$, and $L \simeq \mathbf{1}_T$ otherwise. In the latter case put $n_i = 0$.

*Step 2:* $\dim T = 0$. Suppose $\dim T_1 \geq 1$. Apply Lemma 4.1 to the geometrically connected $k_1$-scheme $T_1$ to get $C^\circ \to U^\circ \subseteq T_1$. For connected $Y \in \mathrm{Sm}_k$,

$$(f_* C^\circ_{\mathcal{V}})(Y) = C^\circ_{\mathcal{V}}(Y \times_k T) = \prod_i C^\circ_{\mathcal{V}}(Y \times_k T_i).$$

The factors with $i \neq 1$ are initial, because $C^\circ_{\mathcal{V}}$ lies in the factor $\mathcal{B}_{\mathcal{V}}(T_1)$ of $\mathcal{B}_{\mathcal{V}}(T)$. The factor $i = 1$ is $C^\circ_{\mathcal{V}}(Y' \times_{k_1} T_1)$ with $Y' := Y \times_k \operatorname{Spec} k_1 \in \mathrm{Sm}_{k_1}$, which is initial by Lemma 4.2 over the field $k_1$, applied to the components of $Y'$. So $f_*(C^\circ_{\mathcal{V}})$ is initial. On the other hand, by Lemma 2.6(iii), $C^\circ_{\mathcal{V}} \otimes L = C^\circ_{\mathcal{V}}[n_1]$, and $f_\sharp(C^\circ_{\mathcal{V}}[n_1]) = C^\circ_{\mathcal{V}}[n_1] \in \mathcal{B}_{\mathcal{V}}(k)$. This is not initial, because $C^\circ_{\mathcal{V}} \in \mathcal{B}_{\mathcal{V}}(k)$ is not: $\mathrm{Map}(C^\circ_{\mathcal{V}}, G)$ is the underlying space of $G(C^\circ)$, which is not contractible for $G = (V_2)_k$. This contradicts the hypothesis. Hence $T = \bigsqcup_i \operatorname{Spec} k_i$.

*Step 3: the unit.* Apply the hypothesis to $\mathbf{1}_T$ and evaluate at $\operatorname{Spec} k$. On one side, $f_\sharp(L) = \bigsqcup_i (\operatorname{Spec} k_i)_{\mathcal{V}}[n_i]$, the coproduct being computed objectwise, and by Lemma 2.7(a) and Lemma 2.6(iv)

$$(\operatorname{Spec} k_i)_{\mathcal{V}}(\operatorname{Spec} k) = F_{\mathcal{V}} \operatorname{Hom}_k(\operatorname{Spec} k, \operatorname{Spec} k_i) = \begin{cases} \mathbf{1}_{\mathcal{V}} & \text{if } k_i = k, \\ \text{initial} & \text{otherwise.} \end{cases}$$

On the other side, $(f_*\mathbf{1}_T)(\operatorname{Spec} k) = \mathbf{1}_T(T) = \prod_{i=1}^r \mathbf{1}_{\mathcal{V}}$.

- $\mathcal{V} = \mathrm{Sp}$: $\bigoplus_{i:\, k_i=k} \mathbb{S}[n_i] \simeq \mathbb{S}^{\oplus r}$. Comparing homology, $k_i = k$ and $n_i = 0$ for all $i$.
- $\mathcal{V} = \mathrm{Spc}_\bullet$: $\bigvee_{i:\, k_i=k} S^0$ has $1 + \#\{i : k_i = k\} \leq 1 + r$ points and $(S^0)^{\times r}$ has $2^r$. So $r = 1$ and $k_1 = k$.
- $\mathcal{V} = \mathrm{Spc}$: $f_\sharp(*) = T_{\mathcal{V}} = h_T$ is equivalent to $f_*(*) = *$, so $\mathrm{Hom}_k(Z, T)$ is a point for every $Z \in \mathrm{Sm}_k^{\mathrm{c}}$. For $Z = \operatorname{Spec} k$ this gives an index $i_0$ with $k_{i_0} = k$. For $Z = T_j$, the inclusion $T_j \subseteq T$ and the composite $T_j \to \operatorname{Spec} k = T_{i_0} \subseteq T$ must coincide, so $j = i_0$. Hence $T = \operatorname{Spec} k$.

This proves Theorem 1.1. □

## 6 Arbitrary twists: proof of Theorem 1.2

Let $T$ be geometrically connected of dimension $\geq 1$, and suppose that $\omega \in \mathcal{B}_{\mathcal{V}}(T)$ satisfies $f_\sharp(- \otimes \omega) \simeq f_*$. For connected $Z \in \mathrm{Sm}_k$ the scheme $Z \times_k T$ is connected, so $f_*\mathbf{1}_T = \mathbf{1}_k$ and, more generally, $f^*(V_k) = V_T$ and $f_*(V_T) = V_k$ for $V \in \mathcal{V}$.

*Step 1: $T$ is birationally contractible.* For $M \in \mathcal{B}_{\mathcal{V}}(k)$, the hypothesis and the projection formula give $f_*f^*M \simeq f_\sharp(f^*M \otimes \omega) \simeq M \otimes f_\sharp\omega$, naturally in $M$. For $M = \mathbf{1}$ this reads $f_\sharp\omega \simeq f_*\mathbf{1}_T = \mathbf{1}$. Hence $\underline{\mathrm{Hom}}(T_{\mathcal{V}}, M) = f_*f^*M \simeq M$ naturally in $M$, so $\mathrm{Map}(T_{\mathcal{V}}, M) \simeq \mathrm{Map}(\mathbf{1}, M)$ naturally in $M$, and $T_{\mathcal{V}} \simeq \mathbf{1}$ by the Yoneda lemma. Let $p\colon T_{\mathcal{V}} \to \mathbf{1}$ be the canonical map. Since $\mathbf{1}_k$ is constant and $T$ is connected, composition with $p$ is an equivalence $\mathrm{Map}(\mathbf{1}, \mathbf{1}) \to \mathrm{Map}(T_{\mathcal{V}}, \mathbf{1})$, so every map $T_{\mathcal{V}} \to \mathbf{1}$ is of the form $ap$ with $a$ an endomorphism of $\mathbf{1}$. If $ap$ is an equivalence with inverse $\psi$, then $a(p\psi) = \mathrm{id}$, so $a$ is invertible in the commutative monoid $\pi_0 \mathrm{Map}(\mathbf{1}, \mathbf{1})$, and $p$ is an equivalence. Tensoring with $Z_{\mathcal{V}}$ we get, for all $G \in \mathcal{B}_{\mathcal{V}}(k)$ and $Z \in \mathrm{Sm}_k$,

$$G(Z) \xrightarrow{\sim} G(Z \times_k T).$$

Let $A$ be an abelian variety over $k$. By Lemma 2.7(b) and Lemma 2.6(iv), $A_{\mathcal{V}}(X) = F_{\mathcal{V}}(A(X))$ for connected $X$. We conclude that

$$A(Z) = A(Z \times_k T) \qquad \text{for every connected } Z \in \mathrm{Sm}_k. \tag{1}$$

*Step 2: constant sheaves do not see fibres.* Let $\mathcal{Y} \in \mathrm{Sm}_T$ be such that $\mathcal{Y} \times_T X$ is connected and non-empty for every $X \in \mathrm{Sm}_T^{\mathrm{c}}$. Then for every $V \in \mathcal{V}$ the map $V_T = \underline{\mathrm{Hom}}(\mathbf{1}_T, V_T) \to \underline{\mathrm{Hom}}(\mathcal{Y}_{\mathcal{V}}, V_T)$ induced by $\mathcal{Y}_{\mathcal{V}} \to \mathbf{1}_T$ is an equivalence: on $X \in \mathrm{Sm}_T^{\mathrm{c}}$ it is the map $V_T(X) \to V_T(\mathcal{Y} \times_T X)$, the identity of $V$. Therefore, using the hypothesis twice,

$$\begin{aligned}\mathrm{Map}(f_*\mathcal{Y}_{\mathcal{V}}, V_k) &\simeq \mathrm{Map}(f_\sharp(\mathcal{Y}_{\mathcal{V}} \otimes \omega), V_k) \simeq \mathrm{Map}(\mathcal{Y}_{\mathcal{V}} \otimes \omega, V_T) \simeq \mathrm{Map}(\omega, \underline{\mathrm{Hom}}(\mathcal{Y}_{\mathcal{V}}, V_T)) \\ &\simeq \mathrm{Map}(\omega, V_T) \simeq \mathrm{Map}(f_\sharp(\mathbf{1}_T \otimes \omega), V_k) \simeq \mathrm{Map}(f_*\mathbf{1}_T, V_k) = \mathrm{Map}(\mathbf{1}_k, V_k),\end{aligned}$$

which is the underlying space of $V$. For $V = V_2$ it is not contractible. Hence $f_*(\mathcal{Y}_{\mathcal{V}})$ is not the initial object.

*Step 3: a family without rational sections.* We now construct $\mathcal{Y}$ as in Step 2 for which $f_*(\mathcal{Y}_{\mathcal{V}})$ is initial; this contradiction proves the theorem. Let $\ell \in \{2, 3\}$ with $\ell \neq \operatorname{char} k$, and let $C^\circ \to U^\circ$ be the $\mu_\ell$-torsor of Lemma 4.1. Let $E$ be an elliptic curve over $k$ with a $k$-point $R$ of order $\ell$; for instance $y^2 = x^3 - x$ if $\ell = 2$ and $y^2 + y = x^3$ if $\ell = 3$, with $R = (0, 0)$. Let $E_0 := E/\langle R \rangle$ and let $\psi\colon E_0 \to E$ be the isogeny dual to $E \to E_0$. Its kernel is the Cartier dual of $\mathbb{Z}/\ell$, that is $\mu_\ell$ [Mil, §11]. So $\psi$ is a $\mu_\ell$-torsor, with $\mu_\ell$ acting on $E_0$ by translations. Put

$$\mathcal{Y} := (C^\circ \times_k E_0)/\mu_\ell \longrightarrow U^\circ \subseteq T$$

as in Lemma 2.7(c), with $B = U^\circ$, $P = C^\circ$ and $A = E_0$.

*$\mathcal{Y}$ satisfies the condition of Step 2.* Let $X \in \mathrm{Sm}_T^{\mathrm{c}}$. Then $X_{U^\circ} := X \times_T U^\circ$ is a non-empty connected open subscheme of $X$, and $\mathcal{Y} \times_T X \to X_{U^\circ}$ is smooth, proper and surjective with geometrically connected fibres. So $\mathcal{Y} \times_T X$ is connected and non-empty.

*$f_*(\mathcal{Y}_\mathcal{V})$ is initial.* Let $Z \in \mathrm{Sm}_k^{\mathrm{c}}$. By Lemma 2.5, Lemma 2.7(c) and Lemma 2.6(iv),

$$(f_*\mathcal{Y}_\mathcal{V})(Z) = \mathcal{Y}_\mathcal{V}(Z \times_k T) = \mathcal{Y}_\mathcal{V}(Z \times_k U^\circ) = F_\mathcal{V} \operatorname{Hom}_{U^\circ}(Z \times_k U^\circ, \mathcal{Y}),$$

and we claim that the last set is empty. Suppose not. A $U^\circ$-morphism $Z \times U^\circ \to \mathcal{Y}$ is a $\mu_\ell$-equivariant morphism $x\colon Z \times C^\circ \to E_0$. The composite $\psi x$ is $\mu_\ell$-invariant, so it descends to $y\colon Z \times U^\circ \to E$, and $x$ identifies the $\mu_\ell$-torsor $Z \times C^\circ \to Z \times U^\circ$ with the pullback of $\psi$ along $y$. By Lemma 2.7(b) and (1), $E(Z \times U^\circ) = E(Z \times T) = E(Z)$; so $y = y_0 \circ \mathrm{pr}_Z$ with $y_0\colon Z \to E$. Therefore $Z \times C^\circ \cong Z' \times U^\circ$ over $Z \times U^\circ$, where $Z' \to Z$ is the $\mu_\ell$-torsor $y_0^*(\psi)$. Restricting to a geometric point of $Z$ gives $C^\circ_{\bar{k}} \cong \mu_\ell \times U^\circ_{\bar{k}}$, which is disconnected. This contradicts Lemma 4.1. □

# 7 General bases: proof of Corollary 1.3

**Lemma 7.1** (generic decomposition)**.** *Let $S$ be an integral quasi-compact quasi-separated scheme with generic point $\eta$, let $T \in \mathrm{Sm}_S$, and let $j\colon T_\eta \to T$ be the generic fibre. There is a symmetric monoidal equivalence*

$$j^*\colon \mathcal{B}_\mathcal{V}(T) \xrightarrow{\sim} \mathcal{B}_\mathcal{V}(T_\eta), \qquad (j^*G)(X_\eta) = G(X) \quad \text{for } X \in \mathrm{Sm}_T,$$

*with $j^*(X_\mathcal{V}) = (X_\eta)_\mathcal{V}$. For a morphism $p\colon T' \to T$ in $\mathrm{Sm}_S$ with generic fibre $p_\eta$, these equivalences identify $p_\sharp$, $p^*$, $p_*$ with $(p_\eta)_\sharp$, $p_\eta^*$, $(p_\eta)_*$.*

For $\mathcal{V} = \mathrm{Spc}$ and $T = S$ this is a special case of Maity's generic decomposition theorem [Mai26b, Cor. 4.3.4], which applies to all quasi-compact quasi-separated schemes with finitely many generic points.

*Proof.* Let $w\colon \mathrm{Sm}_T \to \mathrm{Sm}_{T_\eta}$ be the base change functor, and $j^* := L \circ w_!$, where $w_!$ is left Kan extension; it is symmetric monoidal since $w$ preserves fibre products, and it has the right adjoint $j_* = w^*$, $(j_*H)(X) = H(X_\eta)$, which preserves birational presheaves. For a non-empty affine open $U \subseteq S$ and $X \in \mathrm{Sm}_T$, the open immersion $X_U \to X$ is quasi-compact, and dense because $X$ is flat over $S$, so that its generic points lie over $\eta$. Hence $G(X) \simeq G(X_U)$ for $G \in \mathcal{B}_\mathcal{V}(T)$. Since $T_\eta$ is the limit of the $T_U$, the standard limit arguments [EGA, 8.8.2, 8.10.5, 17.7.8] show that $\mathrm{Sm}_{T_\eta}$ is the filtered colimit of the categories $\mathrm{Sm}_{T_U} \subseteq \mathrm{Sm}_T$: every $Y \in \mathrm{Sm}_{T_\eta}$ is $X_\eta$ for some $X \in \mathrm{Sm}_{T_U}$, unique up to shrinking $U$, and likewise for morphisms; a dense open immersion in $\mathrm{Sm}_{T_\eta}$ comes from an open immersion of models, which is dense since the generic points of the models lie in the generic fibre. Therefore $(w_!G)(X_\eta) = \operatorname{colim}_U G(X_U) = G(X)$, and $w_!G$ is already birational, so $j^*G = w_!G$. With this formula the unit $G \to j_*j^*G$ and the counit $j^*j_*H \to H$ are equivalences. The compatibility with $p^*$ and $p_*$ follows from the formulas of Lemma 2.4, and that with $p_\sharp$ by adjunction. □

*Proof of Corollary 1.3.* By Lemma 7.1, applied to $S$ and to $T$, the functors $f_\sharp$, $f^*$, $f_*$ and the symmetric monoidal structures are identified with those of the generic fibre $f_\eta\colon T_\eta \to \eta$, and $T_\eta \in \mathrm{Sm}_{k(\eta)}$. Since $T$ is flat over $S$ and non-empty, $T_\eta \neq \emptyset$. Apply Theorems 1.1 and 1.2 to $f_\eta$. □

# 8 $\mathbb{P}$-invariant sheaves

In this section $T$ is a normal noetherian scheme of finite Krull dimension, for example a smooth scheme over a field. We take $\mathcal{V} = \mathrm{Sp}$ and compare the birational category $\mathcal{SH}^b(T) = \mathcal{B}_{\mathrm{Sp}}(T)$ with $\mathbb{P}$-invariant sheaves.

## 8.1 Definitions and formal structure

A *distinguished square* in $\mathrm{Sm}_T$ consists of an open immersion $U \to V$ and an étale morphism $p\colon W \to V$ which is an isomorphism over the reduced complement $V \smallsetminus U$, together with their fibre product $W \times_V U$. A presheaf $G \in \mathrm{PSh}_{\mathrm{Sp}}(T)$ is a *Nisnevich sheaf* if $G(\emptyset) = 0$ and $G(V) \simeq G(U) \times_{G(W\times_V U)} G(W)$ for every distinguished square; this is equivalent to descent for the Nisnevich topology [MV, §3.1]. It is $\mathbb{P}$-*invariant* if $G(X) \to G(X \times \mathbb{P}^n)$ is an equivalence for all $X \in \mathrm{Sm}_T$ and $n \geq 1$. We write

$$\mathrm{Sh}_{\mathbb{P}}(T) \subseteq \mathrm{Sh}_{\mathrm{Nis}}(T) \subseteq \mathrm{PSh}_{\mathrm{Sp}}(T)$$

for the full subcategories of $\mathbb{P}$-invariant Nisnevich sheaves and of Nisnevich sheaves. If invariance is required only for $n = 1$, one obtains the stable $\mathbb{P}^1$-homotopy category of [Mai26c, §5.1]; whether the two notions agree is open [Mai26c, Rem. 2.2.20]. Everything below holds for both, with the same proofs.

**Lemma 8.1** (formal structure)**.** *Lemmas 2.3 and 2.4 hold for* $\mathrm{Sh}_{\mathbb{P}}(T)$ *in place of* $\mathcal{B}_{\mathrm{Sp}}(T)$*. That is: the inclusion* $\mathrm{Sh}_{\mathbb{P}}(T) \subseteq \mathrm{PSh}_{\mathrm{Sp}}(T)$ *has a symmetric monoidal left adjoint* $L_{\mathbb{P}}$*; with* $X_+ := L_{\mathbb{P}}(\Sigma^\infty_+ h_X)$ *we have* $\mathrm{Map}(X_+, G) = G(X)$*,* $X_+ \otimes Y_+ = (X \times_T Y)_+$ *and* $\underline{\mathrm{Hom}}(X_+, G)(Y) = G(X \times_T Y)$*; and a smooth, separated, finitely presented morphism* $g\colon T' \to T$ *induces adjunctions* $g_\sharp \dashv g^* \dashv g_*$ *given by the same formulas and satisfying the projection formula. Moreover the structure sheaf* $\mathcal{O}\colon X \mapsto R\Gamma(X, \mathcal{O}_X)$ *is a commutative algebra in* $\mathrm{Sh}_{\mathbb{P}}(T)$*, with* $\mathrm{Map}(X_+, \mathcal{O}) = R\Gamma(X, \mathcal{O}_X)$ *and* $g^*\mathcal{O} = \mathcal{O}$*.*

*Proof.* $\mathrm{Sh}_{\mathbb{P}}(T)$ consists of the presheaves which are local for the following maps and their desuspensions: $0 \to \Sigma^\infty_+ h_\emptyset$; $\Sigma^\infty_+(h_U \sqcup_{h_{W\times_V U}} h_W) \to \Sigma^\infty_+ h_V$ for distinguished squares; and $\Sigma^\infty_+ h_{X\times\mathbb{P}^n} \to \Sigma^\infty_+ h_X$. The proofs of Lemmas 2.3 and 2.4 only used that the generating set of maps is stable under $- \times_T Z$ for $Z \in \mathrm{Sm}_T$, under base change along $g$, and under the forgetful functor $\mathrm{Sm}_{T'} \to \mathrm{Sm}_T$. This holds here. Finally, coherent cohomology satisfies Nisnevich descent, and $R\Gamma(X \times \mathbb{P}^n, \mathcal{O}) = R\Gamma(X, \mathcal{O})$. □

**Lemma 8.2.** *Every birational presheaf of spectra on* $\mathrm{Sm}_T$ *is a* $\mathbb{P}$*-invariant (and* $\mathbb{A}^1$*-invariant) Nisnevich sheaf. Thus* $\mathcal{SH}^b(T) \subseteq \mathrm{Sh}_{\mathbb{P}}(T)$*.*

For presheaves of spaces the sheaf property is [Mai26a, Thm. 2.1.3].

*Proof.* Let $G$ be birational. Invariance under $\mathbb{A}^1$ and $\mathbb{P}^n$ is Lemma 2.8. Consider a distinguished square given by $U \subseteq V$ and $p\colon W \to V$. By $(\Sigma)$ we may assume that $V$ is connected, hence integral. If $U = \emptyset$ then $p$ is an isomorphism and there is nothing to prove. Otherwise $U$ is dense in $V$. Each connected component of $W$ is integral and étale over $V$, so its image is a non-empty open subset of $V$ and meets $U$; thus $W \times_V U$ is dense in $W$. Both $G(V) \to G(U)$ and $G(W) \to G(W \times_V U)$ are then equivalences, and the square of sections is cartesian. □

**Proposition 8.3** (comparison)**.** (i) *The inclusion of* $\mathcal{SH}^b(T)$ *into* $\mathrm{Sh}_{\mathbb{P}}(T)$ *has a symmetric monoidal left adjoint* $L_b$*. For* $g$ *as in Lemma 8.1, the functor* $L_b$ *commutes with* $g^*$ *and with* $g_\sharp$*, and* $g_*$ *preserves birational sheaves.*

(ii) *If* $X \in \mathrm{Sm}_T$ *is birationally rigid, then* $X_+ \in \mathrm{Sh}_{\mathbb{P}}(T)$ *is birational, and* $X_+(Y)$ *is* $\Sigma^\infty_+ \mathrm{Hom}_T(Y, X)$ *for* $Y \in \mathrm{Sm}^{\mathrm{c}}_T$. *In particular the unit of* $\mathrm{Sh}_{\mathbb{P}}(T)$ *is the constant sheaf, with* $\mathbf{1}_T(X) = \mathbb{S}^{\pi_0(X)}$. *For every spectrum* $V$, *the constant sheaf* $V_T$ *with* $V_T(X) = V^{\pi_0(X)}$ *is* $V \otimes \mathbf{1}_T$.

(iii) *For* $A \in \mathrm{Sh}_{\mathbb{P}}(T)$ *and* $G \in \mathcal{SH}^b(T)$, *the object* $\underline{\mathrm{Hom}}(A, G)$ *of* $\mathrm{Sh}_{\mathbb{P}}(T)$ *is birational. It is the internal mapping object from* $L_bA$ *to* $G$ *in* $\mathcal{SH}^b(T)$.

*Proof.* (i) The localization $\mathrm{PSh}_{\mathrm{Sp}}(T) \to \mathcal{SH}^b(T)$ of Lemma 2.3 inverts the maps of Lemma 8.1, because birational presheaves are local for them (Lemma 8.2). So it factors through a functor $L_b \colon \mathrm{Sh}_{\mathbb{P}}(T) \to \mathcal{SH}^b(T)$, left adjoint to the inclusion, and $L_b$ is symmetric monoidal because both localizations of $\mathrm{PSh}_{\mathrm{Sp}}(T)$ are. On both categories $g^*$ and $g_*$ are given by the same formulas, so the inclusions commute with them; passing to left adjoints, $L_b$ commutes with $g_\sharp$ and with $g^*$.

(ii) Let $P \in \mathcal{SH}^b(T)$ be the birational sheaf of Lemma 2.6(iv). A Nisnevich sheaf satisfies $(\Sigma)$, by the distinguished square for $X \sqcup Y$. So for $G \in \mathrm{Sh}_{\mathbb{P}}(T)$, the argument of Lemma 2.6(ii) gives $\mathrm{Map}(P, G) = \mathrm{Map}(\Sigma^\infty_+ h_X, G)$. Hence $P = L_{\mathbb{P}}(\Sigma^\infty_+ h_X) = X_+$. The case $X = T$ gives the unit, and $V \otimes \mathbf{1}_T$ is the localization of the presheaf $X \mapsto V \otimes \mathbb{S}^{\pi_0(X)} = V^{\pi_0(X)}$, which is already birational.

(iii) Let $U \to X$ be a dense open immersion in $\mathrm{Sm}_T$. Then $U_+ \to X_+$ becomes an equivalence under $L_b$, and so does $A \otimes U_+ \to A \otimes X_+$, since $L_b$ is symmetric monoidal. As $G$ is birational, $\underline{\mathrm{Hom}}(A, G)(X) = \mathrm{Map}(A \otimes X_+, G) \to \mathrm{Map}(A \otimes U_+, G) = \underline{\mathrm{Hom}}(A, G)(U)$ is an equivalence. So $\underline{\mathrm{Hom}}(A, G)$ is birational, and $\mathrm{Map}(B, \underline{\mathrm{Hom}}(A, G)) = \mathrm{Map}(L_bB \otimes L_bA, G)$ for all $B$. □

**Corollary 8.4** (the unit)**.** (i) $\pi_0\mathbf{1}_T(X) = \mathbb{Z}^{\pi_0(X)}$, *and* $\mathrm{GL}_1(\mathbf{1}_T)$ *is the constant sheaf* $\mathrm{GL}_1(\mathbb{S})$. *If* $T$ *is connected, the endomorphism spectrum of* $\mathbf{1}_T$ *is* $\mathbb{S}$.

(ii) *The unit map* $\mathbf{1}_T \to \mathcal{O}$ *factors through the constant sheaf* $(H\mathbb{Z})_T$. *Hence the image of* $\pi_0\mathbf{1}_T(X) \to H^0(X, \mathcal{O}_X)$ *consists of locally constant functions with values in the prime ring, and there is no map of sheaves of sets* $\mathbb{G}_m \to \Omega^\infty\mathbf{1}_T$ *whose composite with* $\mathbf{1}_T \to \mathcal{O}$ *is the tautological inclusion* $\mathbb{G}_m \subset \mathcal{O}$.

(iii) *Let* $W \subset T$ *be a nowhere dense closed subset and* $j \colon U = T \smallsetminus W \to T$. *Then* $\mathbf{1}_T \to j_*j^*\mathbf{1}_T$ *is an equivalence: the unit has no cohomology with supports in* $W$.

*Proof.* (i) is Proposition 8.3(ii). (ii) $R\Gamma(X, \mathcal{O}_X)$ is functorially an $H\mathbb{Z}$-algebra, so the unit of $\mathcal{O}$ factors through $H\mathbb{Z} \otimes \mathbf{1}_T = (H\mathbb{Z})_T$; apply $\pi_0$. The coordinate of $\mathbb{G}_m$ is not in the image. (iii) For $X \in \mathrm{Sm}_T$ the map is $\mathbb{S}^{\pi_0(X)} \to \mathbb{S}^{\pi_0(X \times_T U)}$, and $X \times_T U$ is dense in the normal scheme $X$. □

**Theorem 8.5** (dualizable objects)**.** *Let* $T$ *be connected. An object of* $\mathrm{Sh}_{\mathbb{P}}(T)$ *is dualizable if and only if it is equivalent to* $V_T$ *for a finite spectrum* $V$. *In particular the invertible objects of* $\mathrm{Sh}_{\mathbb{P}}(T)$ *are the shifts* $\mathbf{1}_T[n]$. *For* $X \in \mathrm{Sm}^{\mathrm{c}}_T$, *the object* $X_+$ *is dualizable if and only if* $X_+ \to \mathbf{1}_T$ *is an equivalence, i.e. if and only if* $G(T) \to G(X)$ *is an equivalence for every* $G \in \mathrm{Sh}_{\mathbb{P}}(T)$.

*Proof.* The functor $V \mapsto V \otimes \mathbf{1}_T = V_T$ is symmetric monoidal, so $V_T$ is dualizable for finite $V$. Conversely let $A$ be dualizable. By Proposition 8.3(ii),(iii) the dual $A^\vee = \underline{\mathrm{Hom}}(A, \mathbf{1}_T)$, formed in $\mathrm{Sh}_{\mathbb{P}}(T)$, is birational, and so is $A \simeq \underline{\mathrm{Hom}}(A^\vee, \mathbf{1}_T)$. Thus $A = L_bA$ is a dualizable object of $\mathcal{SH}^b(T)$, hence constant by Proposition 1.4. The last assertion follows as in Remark 3.2. □

*Example* 8.6. Let $T = \mathrm{Spec}\, k$ for a field $k$, and $X \in \mathrm{Sm}_k$ connected. If the canonical map $k \to R\Gamma(X, \mathcal{O}_X)$ is not an equivalence, then $X_+$ is not dualizable in $\mathrm{Sh}_{\mathbb{P}}(k)$: take $G = \mathcal{O}$ in Theorem 8.5. This applies to $\mathrm{Spec}\, k'$ for a non-trivial finite separable extension $k'/k$, to $\mathbb{A}^1$ and $\mathbb{G}_m$, and to smooth proper geometrically connected $X$ with $H^i(X, \mathcal{O}_X) \neq 0$ for some $i > 0$, such as curves of positive genus and abelian varieties. On the other hand $\mathbb{P}^n_+ \simeq \mathbf{1}$.

The *Tate object* $Q := \operatorname{cofib}(\mathbb{A}^1_+ \to \mathbf{1}) \in \mathrm{Sh}_{\mathbb{P}}(k)$ is non-zero, since $\operatorname{Map}(Q, \mathcal{O}) = \operatorname{fib}(k \to k[t]) \neq 0$. By the distinguished square covering $\mathbb{P}^1$ by two affine lines, and $\mathbb{P}^1_+ \simeq \mathbf{1}$, it is also the cofibre $\mathbb{A}^1/\mathbb{G}_m$ of $(\mathbb{G}_m)_+ \to \mathbb{A}^1_+$. We have $\underline{\mathrm{Hom}}(Q, \mathbf{1}) = 0$, because the unit is $\mathbb{A}^1$-invariant, and $L_b Q = 0$ by Lemma 2.8. If $Q$ were dualizable it would be birational (Theorem 8.5), hence equal to $L_b Q = 0$. So $Q$ is not dualizable, and in particular not invertible.

**Corollary 8.7** (no spherical Serre twist)**.** *Let $k$ be a field, $T \in \mathrm{Sm}_k$, and $L \in \mathrm{Sh}_{\mathbb{P}}(T)$ invertible. Then $L \otimes \mathcal{O} \simeq \mathcal{O}[n]$ for a locally constant function $n$ on $T$. If $\mathcal{L}$ is a line bundle on $T$ and $L \otimes \mathcal{O} \simeq \mathcal{L}[m]$ as $\mathcal{O}$-modules, where $\mathcal{L}$ denotes the module $(g\colon X \to T) \mapsto R\Gamma(X, g^*\mathcal{L})$, then $\mathcal{L} \cong \mathcal{O}_T$. In particular $\omega_f[d]$ and $\omega_f^\vee[-d]$ are of the form $L \otimes \mathcal{O}$ with $L$ invertible only if $\omega_f \cong \mathcal{O}_T$.*

*Proof.* By Theorem 8.5, $L$ is a shift of the unit on each component. For the second assertion restrict $\mathcal{L}[m] \simeq \mathcal{O}[n]$ to the small Zariski site of $T$. □

## 8.2 No ambidexterity

*Proof of Theorem 1.5.* (i) is Proposition 8.3(ii) and Theorem 8.5.

(ii) Let $\omega \in \mathrm{Sh}_{\mathbb{P}}(T)$ and suppose $f_\sharp(- \otimes \omega) \simeq f_*$ on $\mathrm{Sh}_{\mathbb{P}}(T)$. For $N \in \mathcal{SH}^b(T)$ the object $f_* N$ is birational. Applying $L_b$ and using Proposition 8.3(i) we get, naturally in $N \in \mathcal{SH}^b(T)$,

$$f^b_\sharp(N \otimes L_b\omega) = L_b f_\sharp(N \otimes \omega) \simeq L_b f_* N = f_* N,$$

where $f^b_\sharp$ and the first tensor product are those of $\mathcal{SH}^b$. So $f^b_\sharp(- \otimes L_b\omega) \simeq f_*$ on $\mathcal{SH}^b(T)$, and $L_b\omega$ is invertible if $\omega$ is. Now apply Theorem 1.1(i) and Theorem 1.2. Conversely, if $T = \bigsqcup_{i=1}^r \operatorname{Spec} k$ then $\mathrm{Sh}_{\mathbb{P}}(T) = \mathrm{Sh}_{\mathbb{P}}(k)^r$ and $f_\sharp = \bigoplus = \prod = f_*$.

(iii) Put $D := f_\sharp L$. By the projection formula, $f_\sharp(f^* M \otimes L) \simeq M \otimes D$, and $f_* f^* M \simeq \underline{\mathrm{Hom}}(T_+, M)$. For $M = \mathbf{1}$ the hypothesis gives $D \simeq \underline{\mathrm{Hom}}(T_+, \mathbf{1})$, which is $\mathbf{1}$: its sections on $X$ are $\mathbb{S}^{\pi_0(X \times T)} = \mathbb{S}^{\pi_0(X)}$, because $T$ is geometrically connected. For $M = \mathcal{O}$ it then gives $\mathcal{O} \simeq \mathcal{O} \otimes D \simeq \underline{\mathrm{Hom}}(T_+, \mathcal{O})$, and evaluating at $\operatorname{Spec} k$ yields $k \simeq R\Gamma(T, \mathcal{O}_T)$. □

**Corollary 8.8.** *Let $S$ be an integral normal noetherian scheme of finite Krull dimension with generic point $\eta$, and $f\colon T \to S$ smooth, separated and of finite type, with $T \neq \emptyset$. If $f$ has ambidexterity in $\mathrm{Sh}_{\mathbb{P}}$, then $T_\eta$ is a finite disjoint union of copies of $\eta$. If $T_\eta$ is geometrically connected of positive dimension, then $f_*$ is not of the form $f_\sharp(- \otimes \omega)$ for any $\omega \in \mathrm{Sh}_{\mathbb{P}}(T)$.*

*Proof.* The argument for Theorem 1.5(ii) reduces this to $\mathcal{SH}^b$, where it is Corollary 1.3. □

*Remark* 8.9 (other theories). The proofs of Theorem 8.5 and of Theorem 1.5(i),(ii) used only that $\mathrm{Sh}_{\mathbb{P}}(T)$ is a full subcategory of $\mathrm{PSh}_{\mathrm{Sp}}(T)$ for which Lemma 8.1 holds, and that it contains $\mathcal{SH}^b(T)$. So they apply equally to Nisnevich sheaves of spectra, to $\mathbb{P}^1$-invariant Nisnevich sheaves [Mai26c], and to $\mathbb{A}^1$-invariant Nisnevich sheaves, i.e. to $S^1$-spectra in $\mathbb{A}^1$-homotopy theory: in each of these the unit is the constant sheaf, the dualizable objects over a connected base are the constant ones, and $f_*$ is not a twist of $f_\sharp$. Part (iii) of Theorem 1.5 needs in addition that $\mathcal{O}$ belongs to the theory, which excludes the $\mathbb{A}^1$-invariant case. The unstable analogues follow from Theorems 1.1 and 1.2 in the same way.

## 8.3 Failure on open subschemes

In $\mathrm{Sh}_{\mathbb{P}}$ the failure of ambidexterity for a proper $f$ is visible on the simplest objects, the open subschemes of $T$, with $\mathcal{O}$ as detector.

**Lemma 8.10.** *Let $k$ be a field, $B \in \mathrm{Sm}_k$, $W \subset B$ a closed subscheme which is smooth over $k$, and $U = B \smallsetminus W$. Then in* $\mathrm{Sh}_{\mathbb{P}}(B)$

$$U_+(X) = \mathrm{fib}\big(\mathbb{S}^{\pi_0(X)} \to \mathbb{S}^{\pi_0(X_W)}\big), \qquad X_W := X \times_B W.$$

*Proof.* Let $G(X)$ be the right-hand side. The presheaf $X \mapsto \mathbf{1}_W(X_W) = \mathbb{S}^{\pi_0(X_W)}$ is the pushforward of $\mathbf{1}_W$ along the base change functor $\mathrm{Sm}_B \to \mathrm{Sm}_W$, which preserves distinguished squares and products with $\mathbb{P}^n$; so it is a $\mathbb{P}$-invariant sheaf, and so is $G$. The presheaf $h_U$ is a point on $X$ if $X_W = \emptyset$ and is empty otherwise. This gives a map $\Sigma^\infty_+ h_U \to G$ which is an equivalence on connected $X$ with $X_W = \emptyset$. At a Henselian local point $x$ of some $X$ with $x \in X_W$ both stalks vanish: that of $\Sigma^\infty_+ h_U$ trivially, and that of $G$ because $V_W$ is connected for a cofinal system of étale neighbourhoods $V$ of $x$, as $X_W$ is regular. A map of presheaves which is an equivalence on stalks induces an equivalence of Nisnevich sheafifications [MV]. Hence $G$ is the sheafification of $\Sigma^\infty_+ h_U$, and it is already $\mathbb{P}$-invariant. □

**Proposition 8.11.** *Let $k$ be a field and $f\colon T \to \operatorname{Spec} k$ smooth and proper. Suppose that a component $T_1$ of $T$ has dimension $\geq 1$, let $z \in T_1$ be a closed point with $k(z)/k$ separable, and $U := T \smallsetminus \{z\}$. Then $f_\sharp(U_+ \otimes L) \not\simeq f_*(U_+)$ in $\mathrm{Sh}_{\mathbb{P}}(k)$ for every invertible $L \in \mathrm{Sh}_{\mathbb{P}}(T)$.*

*Proof.* Suppose $f_\sharp(U_+ \otimes L) \simeq f_*(U_+)$. Let $k'/k$ be a finite Galois extension containing $k(z)$ and the constant fields of all components of $T$, let $g\colon \operatorname{Spec} k' \to \operatorname{Spec} k$, and denote base changes to $k'$ by a prime. By the formulas of Lemma 8.1, $g^* f_\sharp = f'_\sharp g'^*$ and $g^* f_* = f'_* g'^*$, and $g'^*(U_+) = U'_+$. So $f'_\sharp(U'_+ \otimes L') \simeq f'_*(U'_+)$ in $\mathrm{Sh}_{\mathbb{P}}(k')$, where $L' = g'^* L$ is invertible. Now the components $T'_j$ of $T'$ are geometrically connected, $U' = T' \smallsetminus z'$, and $z'$ is a finite non-empty set of $k'$-rational points; let $m_j$ be the number of those lying on $T'_j$, and let $n_j$ be the shift of $L'$ on $T'_j$ (Theorem 8.5).

By Lemma 8.10, for connected $Y \in \mathrm{Sm}_{k'}$,

$$(f'_* U'_+)(Y) = U'_+(Y \times T') = \mathrm{fib}\big(\mathbb{S}^{\pi_0(Y \times T')} \to \mathbb{S}^{\pi_0(Y \times z')}\big) = \bigoplus_j \mathrm{fib}\big(\mathbb{S} \xrightarrow{\mathrm{diag}} \mathbb{S}^{m_j}\big),$$

naturally in $Y$. So $f'_*(U'_+)$ is a finite sum of shifts of the unit, and $\mathrm{Map}(f'_*(U'_+), \mathcal{O})$ is a finite sum of shifts of $k'$. On the other hand $f'_\sharp(U'_+ \otimes L') = \bigoplus_j (U'_j)_+[n_j]$ with $U'_j = T'_j \smallsetminus z'$, so

$$\mathrm{Map}(f'_\sharp(U'_+ \otimes L'), \mathcal{O}) = \prod_j R\Gamma(U'_j, \mathcal{O})[-n_j].$$

Mapping spectra into $\mathcal{O}$ are modules over $\mathrm{Map}(\mathbf{1}, \mathcal{O}) = k'$, functorially in the source, so the two mapping spectra would be equivalent as $k'$-modules. But the second one has infinite total dimension over $k'$. Indeed choose $j$ with $m_j \geq 1$, and let $d = \dim T'_j \geq 1$. If $d = 1$, then $U'_j$ is an affine curve and $H^0(U'_j, \mathcal{O})$ is infinite-dimensional. If $d \geq 2$, the exact sequence $H^{d-1}(U'_j, \mathcal{O}) \to H^d_{z'}(T'_j, \mathcal{O}) \to H^d(T'_j, \mathcal{O})$, in which the middle term is infinite-dimensional and the last one is finite-dimensional since $T'_j$ is proper, shows that $H^{d-1}(U'_j, \mathcal{O})$ is infinite-dimensional. Contradiction. □

# 9 Modules over the structure sheaf

Let $k$ be a field and $T \in \mathrm{Sm}_k$. Recall that $\mathrm{Mod}_{\mathcal{O}}(T) = \mathrm{Mod}_{\mathcal{O}}(\mathrm{Sh}_{\mathbb{P}}(T))$. For smooth $f$ the functors $f_\sharp \dashv f^* \dashv f_*$ pass to $\mathcal{O}$-modules, by Lemma 8.1. A quasi-coherent sheaf $\mathcal{F}$ on $T$ defines the module $(g\colon X \to T) \mapsto R\Gamma(X, g^*\mathcal{F})$ on $\mathrm{Sm}_T$, again denoted $\mathcal{F}$; it is a $\mathbb{P}$-invariant Nisnevich sheaf, and line bundles are invertible objects of $\mathrm{Mod}_{\mathcal{O}}(T)$. If $\operatorname{char} k = p$, the Frobenius $F\colon \mathcal{O} \to \mathcal{O}$ is an endomorphism of the sheaf of rings $\mathcal{O}$, and $\Phi := F_*\mathcal{O}$, the sheaf $\mathcal{O}$ with module structure $a \cdot n = a^p n$, is an object of $\mathrm{Mod}_{\mathcal{O}}(T)$.

**Proposition 9.1** (linear ambidexterity forces Künneth)**.** *Let $f\colon T \to \operatorname{Spec} k$ be smooth and proper. Suppose there are an object $\mathcal{L} \in \operatorname{Mod}_{\mathcal{O}}(T)$ and an equivalence $f_\sharp(- \otimes_{\mathcal{O}} \mathcal{L}) \simeq f_*$ of functors $\operatorname{Mod}_{\mathcal{O}}(T) \to \operatorname{Mod}_{\mathcal{O}}(k)$. Then*

$$\mathcal{O} \otimes T_+ \simeq \mathcal{O} \otimes_k R\Gamma(T, \mathcal{O}_T)^\vee \quad \text{in } \operatorname{Mod}_{\mathcal{O}}(k).$$

*In particular $\mathcal{O} \otimes T_+$ is dualizable, and the Künneth map $\underline{\operatorname{Hom}}(T_+, \mathcal{O}) \otimes_{\mathcal{O}} M \to \underline{\operatorname{Hom}}(T_+, M)$ is an equivalence for every $M \in \operatorname{Mod}_{\mathcal{O}}(k)$: that is, $M(X) \otimes_k R\Gamma(T, \mathcal{O}_T) \simeq M(X \times T)$ for every $\mathbb{P}$-invariant $\mathcal{O}$-module $M$ on $\operatorname{Sm}_k$ and every $X \in \operatorname{Sm}_k$.*

*Proof.* Put $D := f_\sharp \mathcal{L}$. The hypothesis for $\mathcal{O}_T$ gives $D \simeq f_* \mathcal{O}_T = \underline{\operatorname{Hom}}(T_+, \mathcal{O})$, which is $\mathcal{O} \otimes_k R\Gamma(T, \mathcal{O}_T)$ by the Künneth formula for coherent cohomology. As $T$ is proper this is a finite sum of shifts of $\mathcal{O}$, so $D$ is dualizable, with dual $D^\vee = \mathcal{O} \otimes_k R\Gamma(T, \mathcal{O}_T)^\vee$. For $M \in \operatorname{Mod}_{\mathcal{O}}(k)$ the hypothesis and the projection formula give

$$\underline{\operatorname{Hom}}(T_+, M) = f_* f^* M \simeq f_\sharp(f^* M \otimes_{\mathcal{O}} \mathcal{L}) \simeq M \otimes_{\mathcal{O}} D,$$

naturally in $M$. Taking global sections over $\operatorname{Spec} k$, we get $\operatorname{Map}_{\mathcal{O}}(\mathcal{O} \otimes T_+, M) \simeq \operatorname{Map}_{\mathcal{O}}(\mathcal{O}, M \otimes_{\mathcal{O}} D) = \operatorname{Map}_{\mathcal{O}}(D^\vee, M)$, naturally in $M$. So $\mathcal{O} \otimes T_+ \simeq D^\vee$ by the Yoneda lemma. For a dualizable $\mathcal{O}$-module $P$ the canonical map $\underline{\operatorname{Hom}}_{\mathcal{O}}(P, \mathcal{O}) \otimes_{\mathcal{O}} M \to \underline{\operatorname{Hom}}_{\mathcal{O}}(P, M)$ is an equivalence; apply this to $P = \mathcal{O} \otimes T_+$. □

The Künneth property holds for quasi-coherent $M$, but $\operatorname{Mod}_{\mathcal{O}}(k)$ is larger. The analogous property over the sphere is what fails in Theorem 1.5.

*Proof of Theorem 1.6.* (i) Since $\mathcal{O}(1)$ is invertible with inverse $\mathcal{O}(-1)$,

$$\operatorname{Map}_{\mathcal{O}}(f_\sharp \mathcal{O}(1), \Phi) = \operatorname{Map}_{\mathcal{O}_T}(\mathcal{O}(1), f^*\Phi) = (\mathcal{O}(-1) \otimes_{\mathcal{O}_T} F_* \mathcal{O}_T)(\mathbb{P}^1) = R\Gamma(\mathbb{P}^1, \mathcal{O}(-p)),$$

because $\mathcal{O}(-1) \otimes_{\mathcal{O}} F_* \mathcal{O} = F_*(F^* \mathcal{O}(-1)) = F_* \mathcal{O}(-p)$: a transition function $u$ acts on $F_* \mathcal{O}$ as $u^p$. This is a $\mathbb{P}$-invariant sheaf, so no further localization intervenes. As $H^1(\mathbb{P}^1, \mathcal{O}(-p))$ has dimension $p - 1 \geq 1$, we get $f_\sharp \mathcal{O}(1) \neq 0$. On the other hand $\omega_f = \mathcal{O}(-2)$, and $f_*(\mathcal{O}(1) \otimes \omega_f[1])(Y) = R\Gamma(Y \times \mathbb{P}^1, \operatorname{pr}_2^* \mathcal{O}(-1))[1] = 0$.

(ii) $f_* \mathcal{O}_E = \underline{\operatorname{Hom}}(E_+, \mathcal{O}) = \mathcal{O} \otimes_k R\Gamma(E, \mathcal{O})$ by the Künneth formula for coherent cohomology, and $f_* f^* \Phi = \underline{\operatorname{Hom}}(E_+, \Phi)$. At $\operatorname{Spec} k$ the Künneth map for $M = \Phi$ is $R\Gamma(E, \mathcal{O}) \otimes_k \Phi(k) \to \Phi(E)$, $a \otimes c \mapsto c \cdot F(a)$, the linearization of Frobenius on $R\Gamma(E, \mathcal{O})$. On $H^1$ it is zero, since $F = 0$ on $H^1(E, \mathcal{O})$ for supersingular $E$, while $H^1(\Phi(E)) = H^1(E, \mathcal{O}) \neq 0$. So $\mathcal{O} \otimes E_+$ is not dualizable over $\mathcal{O}$, since dualizability would make this map an equivalence. The last assertion of (ii) now follows from Proposition 9.1. □

*Remark* 9.2 (normalization). The twist in Theorem 1.6(i) is the one dictated by Serre duality. For $f\colon T \to \operatorname{Spec} k$ smooth and proper of dimension $d$ one has $\operatorname{Map}_{\mathcal{O}}(f_\sharp \mathcal{O}_T, \mathcal{O}) = R\Gamma(T, \mathcal{O})$, and $f_*(\omega_f[d])$ is the free $\mathcal{O}$-module on $R\Gamma(T, \omega_f)[d]$, so that $\operatorname{Map}_{\mathcal{O}}(f_*(\omega_f[d]), \mathcal{O}) = R\Gamma(T, \omega_f)^\vee[-d]$, which is the same by Serre duality. So the natural candidate is $f_\sharp \simeq f_*(- \otimes \omega_f[d])$, i.e. $\mathcal{L} = \omega_f^\vee[-d]$ in (Q2).

*Remark* 9.3 (non-cartesian modules). $\operatorname{Mod}_{\mathcal{O}}(T)$ consists of modules on the big site $\operatorname{Sm}_T$, and it contains objects which are not determined by their restriction to $T$, such as $\Phi$: its restriction to the small site of each $X$ is $F_{X*} \mathcal{O}_X$, which is not the pullback of its value on $T$. Six-functor formalisms for quasi-coherent sheaves only see modules which are cartesian in this sense. This is why Serre duality for quasi-coherent sheaves does not help in Theorem 1.6.

## 9.1 Characteristic zero: the first test

**Proposition 9.4.** *Let $k$ be any field, $N \in \mathrm{Mod}_{\mathcal{O}}(k)$, $X \in \mathrm{Sm}_k$, $z$ the coordinate of $\mathbb{A}^1$, and*

$$\widetilde{N}(X \times \mathbb{A}^1) := \mathrm{fib}\big(N(X \times \mathbb{A}^1) \to N(X)\big),$$

*the fibre of the restriction to $z = 0$.*

(i) *$N(X \times \mathbb{G}_m) \simeq N(X \times \mathbb{A}^1_z) \oplus \widetilde{N}(X \times \mathbb{A}^1_w)$, where $w = z^{-1}$.*

(ii) *Multiplication by $z$ on $\pi_* N(X \times \mathbb{A}^1)$ is injective, and $z$ factors through $\widetilde{N}(X \times \mathbb{A}^1)$.*

(iii) *For $f\colon \mathbb{P}^1 \to \operatorname{Spec} k$,*

$$\mathrm{Map}_{\mathcal{O}}(f_\sharp \mathcal{O}(1) \otimes X_+, N) \simeq \mathrm{fib}\big(z\colon N(X \times \mathbb{A}^1) \to \widetilde{N}(X \times \mathbb{A}^1)\big).$$

(iv) *$f_\sharp(\mathcal{O}_{\mathbb{P}^1}(1)) = 0$ in $\mathrm{Mod}_{\mathcal{O}}(k)$ if and only if*

$$z\colon N(\mathbb{A}^1) \xrightarrow{\sim} \widetilde{N}(\mathbb{A}^1) \qquad \text{for every } N \in \mathrm{Mod}_{\mathcal{O}}(k), \tag{$\star$}$$

*i.e. every section over $\mathbb{A}^1$ of a $\mathbb{P}$-invariant $\mathcal{O}$-module that vanishes at the origin is uniquely divisible by the coordinate.*

*Proof.* (i) Zariski descent for $\mathbb{P}^1_X = (X \times \mathbb{A}^1_z) \cup (X \times \mathbb{A}^1_w)$ and $N(\mathbb{P}^1_X) = N(X)$ give a cofibre sequence $N(X) \to N(X \times \mathbb{A}^1_z) \oplus N(X \times \mathbb{A}^1_w) \to N(X \times \mathbb{G}_m)$; split $N(X \times \mathbb{A}^1_w) = N(X) \oplus \widetilde{N}(X \times \mathbb{A}^1_w)$. (ii) If $zn = 0$ then $n|_{X \times \mathbb{G}_m} = z^{-1}(zn)|_{X \times \mathbb{G}_m} = 0$, and restriction to $X \times \mathbb{G}_m$ is split injective by (i). The restriction to $z = 0$ is linear over $\mathcal{O}(X \times \mathbb{A}^1) \to \mathcal{O}(X)$, $z \mapsto 0$. (iii) By adjunction the left side is the spectrum of sections over $\mathbb{P}^1$ of $\mathcal{O}(-1) \otimes_{\mathcal{O}_T} f^* \underline{\mathrm{Hom}}(X_+, N)$. With local generators $e_0 = (1, z)$, $e_1 = (w, 1)$ of $\mathcal{O}(-1) \subset \mathcal{O}^2$, $e_0 = z e_1$, a section is a pair $(a, b) \in N(X \times \mathbb{A}^1_z) \times N(X \times \mathbb{A}^1_w)$ with $a = wb$ on $X \times \mathbb{G}_m$. By (i) and (ii), in the decomposition $N(X \times \mathbb{G}_m) = N(X \times \mathbb{A}^1_z) \oplus \widetilde{N}(X \times \mathbb{A}^1_w)$ the map $(a, b) \mapsto a - wb$ is $(a, b) \mapsto (a, -wb)$, with fibre $\mathrm{fib}(w\colon N(X \times \mathbb{A}^1_w) \to \widetilde{N}(X \times \mathbb{A}^1_w))$. (iv) follows from (iii), applied to all $N$ and to the modules $\underline{\mathrm{Hom}}(X_+, N)$. □

By Theorem 1.6(i), ($\star$) fails in characteristic $p$: for $N = \Phi$ the cokernel of $z$ is spanned by $z, \dots, z^{p-1}$. Since $f_*(\mathcal{O}(1) \otimes \omega_f[1]) = 0$ in every characteristic, ($\star$) is a *necessary* condition for the $\mathcal{O}$-linear Serre form of ambidexterity for $\mathbb{P}^1 \to \operatorname{Spec} k$. Here is what we know in its favour. It holds for $N = \underline{\mathrm{Hom}}(M, \mathcal{O})$ with $M \in \mathrm{Sh}_{\mathbb{P}}(k)$, since then $N(\mathbb{A}^1) = \mathrm{Map}(M, \bigoplus_{n \geq 0} \mathcal{O} z^n)$ and $z$ acts by the shift. The class of $N$ for which $z\colon N(X \times \mathbb{A}^1) \to \widetilde{N}(X \times \mathbb{A}^1)$ is an equivalence for all $X$ is closed under limits, colimits, retracts and $\underline{\mathrm{Hom}}(M, -)$. And for $k = \mathbb{C}$ it holds for the non-quasi-coherent module $N(X) = R\Gamma(X^{\mathrm{an}}, \mathcal{O}^{\mathrm{an}})$, by the exact sequence $0 \to \mathcal{O}^{\mathrm{an}} \xrightarrow{z} \mathcal{O}^{\mathrm{an}} \to \mathcal{O}^{\mathrm{an}}|_{z=0} \to 0$.

**Question 9.5.** Let $\operatorname{char} k = 0$. Does ($\star$) hold? Equivalently, does $f_\sharp(\mathcal{O}_{\mathbb{P}^1}(1))$ vanish in $\mathrm{Mod}_{\mathcal{O}}(\mathrm{Sh}_{\mathbb{P}}(k))$?

This is the first test that an $\mathcal{O}$-linear ambidexterity in characteristic 0 must pass. We could neither prove it nor find a counterexample. In characteristic 0 the sheaf of rings $\mathcal{O}$ on $\mathrm{Sm}_k$ has no endomorphism other than the identity, which is what removes the Frobenius obstruction.

*Remark* 9.6 (what $\mathbb{P}$-localization adds to $\mathcal{O} \otimes \mathbb{A}^1_+$). By Proposition 9.4(i) every $N \in \mathrm{Mod}_{\mathcal{O}}(k)$ has natural "Taylor coefficient" maps $\tau_j\colon N(\mathbb{A}^1) \to N(k)$, $\tau_j(n) = \mathrm{ev}_0 \mathrm{pr}_z(z^{-j} n)$, where $\mathrm{pr}_z$ is the projection of $N(\mathbb{G}_m)$ onto $N(\mathbb{A}^1_z)$; for $N = \mathcal{O}$, $\tau_j(z^n) = \delta_{jn}$. By the Yoneda lemma they are classes $D_j \in \pi_0(\mathcal{O} \otimes \mathbb{A}^1_+)(k)$, "derivatives of the delta function at 0". They do not exist before localization: $(\mathcal{O} \otimes \Sigma^\infty_+ \mathbb{A}^1)(k) = k[k]$ is the free vector space on the points $[a]$, on which the functionals $z^n$ take the values $\sum_i c_i a_i^n$. So $L_{\mathbb{P}}$ completes $k[k] \ni [a] \leftrightarrow 1/(1 - aw)$ towards a ring of power series in $w = z^{-1}$. A natural guess is the ring of algebraic power series, the Henselization of $k[w]$ at 0, rather than $k[[w]]$; we have not proved this.

## 10 Naive purity fails

In $\mathbb{A}^1$-homotopy theory, the homotopy purity theorem identifies $X/(X \smallsetminus Z)$, for a smooth closed pair $Z \subset X$, with the Thom space of the normal bundle; if the normal bundle is trivial of rank one, this is $Z_+ \otimes (\mathbb{A}^1/\mathbb{G}_m)$. The analogous statement in $\mathrm{Sh}_{\mathbb{P}}$, with the Tate object $Q = \mathbb{A}^1/\mathbb{G}_m$ of Example 8.6, is false.

**Proposition 10.1.** *Let* $\operatorname{char} k = 0$*, let* $E$ *be an elliptic curve over* $k$*,* $0 \to \mathcal{O}_E \to A \to \mathcal{O}_E \to 0$ *the non-split extension,* $X = \mathbb{P}(A) \to E$ *the projective bundle of lines in* $A$*, and* $Z \subset X$ *the section given by the line subbundle* $\mathcal{O}_E \subset A$*. Then* $N_{Z/X} \cong \mathcal{O}_Z$*, but in* $\mathrm{Sh}_{\mathbb{P}}(k)$

$$\operatorname{cofib}\big((X \smallsetminus Z)_+ \to X_+\big) \;\not\simeq\; Z_+ \otimes Q.$$

*Proof.* $W := X \smallsetminus Z \to E$ is the $\mathbb{G}_a$-torsor of splittings of $A$, with class the extension class $\neq 0$ in $H^1(E, \mathcal{O})$, and $N_{Z/X} = \mathcal{H}om(\mathcal{O}, A/\mathcal{O}) = \mathcal{O}$. The algebra of functions on $W$ over $E$ is $\operatorname{colim}_n F_n$, where $F_n = \operatorname{Sym}^{n-1} A$ is Atiyah's indecomposable bundle of rank $n$ and degree 0 [Ati], with $F_n \subset F_{n+1}$, $F_{n+1}/F_n \cong \mathcal{O}$ and $h^0(F_n) = h^1(F_n) = 1$. In the long exact sequence of $0 \to F_n \to F_{n+1} \to \mathcal{O} \to 0$ the map $H^0(F_n) \to H^0(F_{n+1})$ is therefore an isomorphism, so $H^0(\mathcal{O}) \to H^1(F_n)$ is an isomorphism and $H^1(F_n) \to H^1(F_{n+1})$ is zero. Hence $R\Gamma(W, \mathcal{O}) = k$. As $R\Gamma(X, \mathcal{O}) = R\Gamma(E, \mathcal{O}) = k \oplus k[-1]$,

$$\operatorname{Map}\big(\operatorname{cofib}(W_+ \to X_+), \mathcal{O}\big) = \operatorname{fib}\big(R\Gamma(X, \mathcal{O}) \to R\Gamma(W, \mathcal{O})\big) = k[-1],$$

whereas $\operatorname{Map}(Z_+ \otimes Q, \mathcal{O}) = R\Gamma(E, \mathcal{O}) \otimes_k \operatorname{fib}(k \to k[t])$ is infinite-dimensional. Since mapping spectra into $\mathcal{O}$ are $k$-modules functorially in the source, the two objects are not equivalent. □

The object $\operatorname{cofib}((X \smallsetminus Z)_+ \to X_+)$ depends only on a Nisnevich neighbourhood of $Z$ in $X$, i.e. on the Henselian germ of $X$ along $Z$, which here is not that of the zero section of the normal bundle. In $\mathbb{A}^1$-homotopy theory the deformation to the normal cone removes this dependence; in $\mathrm{Sh}_{\mathbb{P}}$ it does not.

## 11 Remarks

*Remark* 11.1 (an exceptional pullback exists, but it is not a twist). In $\mathcal{SH}^b$ and in $\mathrm{Sh}_{\mathbb{P}}$ the functor $f_*$ preserves colimits. So $f_*$ has a right adjoint $f^!$. The right adjoint of $f_\sharp(- \otimes \omega)$ is $\underline{\mathrm{Hom}}(\omega, f^*(-))$, so Theorems 1.2 and 1.5 say that $f^!$ is not of the form $\underline{\mathrm{Hom}}(\omega, f^*(-))$ for any $\omega$. Since $\mathcal{B}_{\mathcal{V}}(T)$ only depends on the generic points of $T$ (Lemma 7.1), in the birational category there are no localization sequences along closed immersions with which one could try to repair this.

*Remark* 11.2 (restriction of scalars). Under Lemma 7.1, the morphism $f\colon T \to \operatorname{Spec} k$ acts on the birational category through the field extension $K = k(T)$ of $k$: $(f_*H)(Y) = H(Y \times_k \operatorname{Spec} K)$ is restriction of scalars. Lemma 4.2 says that it annihilates $(\operatorname{Spec} K')_{\mathcal{V}}$ for the geometric extensions $K'/K$ considered there, and Step 3 of §6 that it annihilates the class of a genus one curve over $K$ which has no point over the function field of any $Y \times_k \operatorname{Spec} K$. The functor $f_\sharp$ sends these objects to the classes of the corresponding $k$-varieties.

*Remark* 11.3 (weights). In MS, ambidexterity reads $f_\sharp \Sigma^{-\Omega_f} \simeq f_*$ [AHI, Thm. 1.1]: the twist is a Thom spectrum. In Hodge-theoretic terms, Serre duality pairs $H^i(\Omega^j)$ with $H^{d-i}(\Omega^{d-j})$, i.e. weight $j$ with weight $d - j$: $\mathcal{O}$ has weight 0 and $\omega_f[d]$ has weight $d$. The category $\mathrm{Sh}_{\mathbb{P}}$ is by construction the weight 0 part of the story, since the pointed projective line is zero in it, so the two halves of the pairing cannot both live there. The situation resembles that of effective motives before the Tate object is inverted, where smooth projective varieties are in general not dualizable; compare [KS16, KS17], where killing the Lefschetz motive produces birational motives.

*Remark* 11.4 (what a positive statement would have to look like). By Proposition 9.1, ambidexterity for $\mathcal{O}$-modules requires a Künneth theorem for all $\mathbb{P}$-invariant $\mathcal{O}$-modules, and Step 1 of §6 is the analogous statement over the sphere. It fails over the sphere and, in characteristic $p$, over $\mathcal{O}$. The version left standing is the one over $\mathcal{O}$ in characteristic 0, and its first test is Question 9.5.

*Remark* 11.5 (what is not proved). In Theorem 1.2, and hence in the second part of Theorem 1.5(ii), we assume that $T$ is geometrically connected. We expect the statement for every $T$ with a component of positive dimension, but we have not proved it. Question (Q2) is open in characteristic 0. We have not considered Maity's $n$-birational categories for $n \geq 1$.

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