hexagon:2611.01354v1

BRESINSKY'S PROBLEM (1975) HAS A NEGATIVE ANSWER

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First version: November 28, 2026. This version (1.00): November 29, 2026.

ABSTRACT. Fix $e \geq 2$ positive integers $n_1 < \cdots < n_e$ with $\gcd(n_1, \ldots, n_e) = 1$, none a nonnegative integer sum of the others; the additive monoid $S = \langle n_1, \ldots, n_e \rangle \subseteq \mathbb{N}$ of all nonnegative integer combinations is a *numerical semigroup* of embedding dimension $\operatorname{edim}(S) = e$. Its minimal presentation size $\rho(S)$ is the minimum number of basic sum-relations from which all others follow—equivalently, the minimum number of binomial generators of the defining toric ideal of the monomial curve ring $\Bbbk[S] = \Bbbk[t^{n_1}, \ldots, t^{n_e}]$. The semigroup S is *symmetric* if every integer $z \in \mathbb{Z}$ satisfies $z \in S \iff F(S) - z \notin S$, where $F(S) = \max(\mathbb{Z} \setminus S)$ is its largest gap (equivalently, $\Bbbk[S]$ is Gorenstein). Setting $S(e) := \sup\{\rho(H) : H \text{ symmetric, } \operatorname{edim}(H) = e\}$, Bresinsky proved in 1975 that $S(4) = 5$ even though ρ is unbounded for general 4-generated numerical semigroups, and asked whether $S(e) < \infty$ for every e. Every case $e \geq 5$ has remained open for fifty-one years.

We settle Bresinsky's problem in opposite directions across embedding dimensions by self-contained, computer-free proofs. For $e = 8$, we construct an explicit family of symmetric semigroups $H_8(m)$ $(m \geq 2)$, with generators forming an affine image of the Boolean cube $\{0,1\}^3$, satisfying $\rho(H_8(m)) \geq m - 1$; inductive gluing yields symmetric e-generated semigroups $H_e(m)$ with $\rho(H_e(m)) \geq m + e - 9 \to \infty$, proving $S(e) = \infty$ for all $e \geq 8$. Along $H_e(m)$, all interior graded Betti numbers $b_i(\Bbbk[H_e(m)])$ $(1 \leq i \leq e-2$, the interior syzygy ranks) diverge together over every field $\Bbbk$ while the Cohen–Macaulay type b_{e-1} (the number of pseudo-Frobenius numbers, i.e., gaps $x \notin S$ with $x + n_j \in S$ for each generator n_j) stays 1, refuting both Stamate's (2018) problem and the Moscariello–Sammartano uniform boundedness problem in every interior homological degree for all $e \geq 8$. In contrast, for $e = 5$ an order-theoretic method on minimal relations grouped by generator support yields the quadratic bound $\rho(S) \leq 5 + 75\,t(S)^2$ in terms of Cohen–Macaulay type $t(S) \geq 1$. Combined with published type bounds of Moscariello ($t(S) \leq 473$ for almost-symmetric semigroups) and Moscariello–Strazzanti ($t(S) \leq 40$ for non-almost-symmetric nearly Gorenstein semigroups), this gives $\rho(S) \leq 16{,}779{,}680$ across both classes ($\leq 120{,}005$ in the latter). Refining the same method in the symmetric case ($t(S) = 1$, where $\rho \leq 80$ already follows) gives $S(5) \leq 60$ unconditionally, removing Bresinsky's 1979 four-sum hypothesis $n_i + n_j = n_h + n_k$ for four distinct indices and leaving only the dimensions $e \in \{6, 7\}$ open.[1] These results are verified in Lean.[2]

1. INTRODUCTION

Fix an integer $e \geq 2$ and let $S = \langle n_1, \ldots, n_e \rangle \subseteq \mathbb{N}$ be a numerical semigroup of embedding dimension $\operatorname{edim}(S) = e$, meaning that $n_1 < \cdots < n_e$ are positive integers with $\gcd(n_1, \ldots, n_e) = 1$ and no generator is a nonnegative integer combination of the others. Two distinct nonnegative integer combinations of $n_1, \ldots, n_e$ with the same sum define a relation of S, and the cardinality $\rho(S)$ of a minimal presentation of S is the minimum number of relations from which all others follow. Algebraically, over any field $\Bbbk$, $\rho(S)$ equals both the minimum number of binomial generators of the defining toric ideal $I_S \subset R = \Bbbk[x_1, \ldots, x_e]$ of the monomial curve ring $\Bbbk[S] = \Bbbk[t^{n_1}, \ldots, t^{n_e}] \cong R/I_S$ and the first Betti number $b_1(\Bbbk[S]) = \dim_{\Bbbk} \operatorname{Tor}_1^R(\Bbbk[S], \Bbbk)$ (equivalently $b_1(\Bbbk[[S]])$ for the completed local ring $\Bbbk[[t^{n_1}, \ldots, t^{n_e}]]$). More generally, the Betti numbers $b_i(\Bbbk[S])$ count minimal i-th syzygies, and $b_{e-1}(\Bbbk[S]) = t(S)$ is the Cohen–Macaulay type, i.e., the number of pseudo-Frobenius numbers (gaps $x \in \mathbb{Z} \setminus S$ with $x + n_j \in S$ for each generator n_j).

How large can $\rho(S)$ be when the embedding dimension e is fixed? For $e = 3$, Herzog [13] proved that $\rho(S) \in \{2, 3\}$, with $\rho(S) = 2$ precisely when S is symmetric. At $e = 4$, Henrik Bresinsky [1, 2] discovered a striking dichotomy in 1975. For arbitrary 4-generated numerical semigroups, $\rho(S)$ can be arbitrarily large [2]. Yet if S is *symmetric*—meaning that its largest gap (Frobenius number) $F(S) = \max(\mathbb{Z} \setminus S)$ satisfies $z \in$

[1] Expanded version, with further results [12]: https://doi.org/10.5281/zenodo.24633631

[2] The semigroup and graded Betti number results in Theorems 1.1–1.3, Corollary 1.4 (proving $t(S) \leq 473$ independently), and Theorem 1.5 are proved in Lean 4 from Lean's three standard axioms alone, and checked with lake comparator using nanoda 0.4.32: https://github.com/david44hello/paper_bresinsky-1975-problem/tree/v1.00/lean

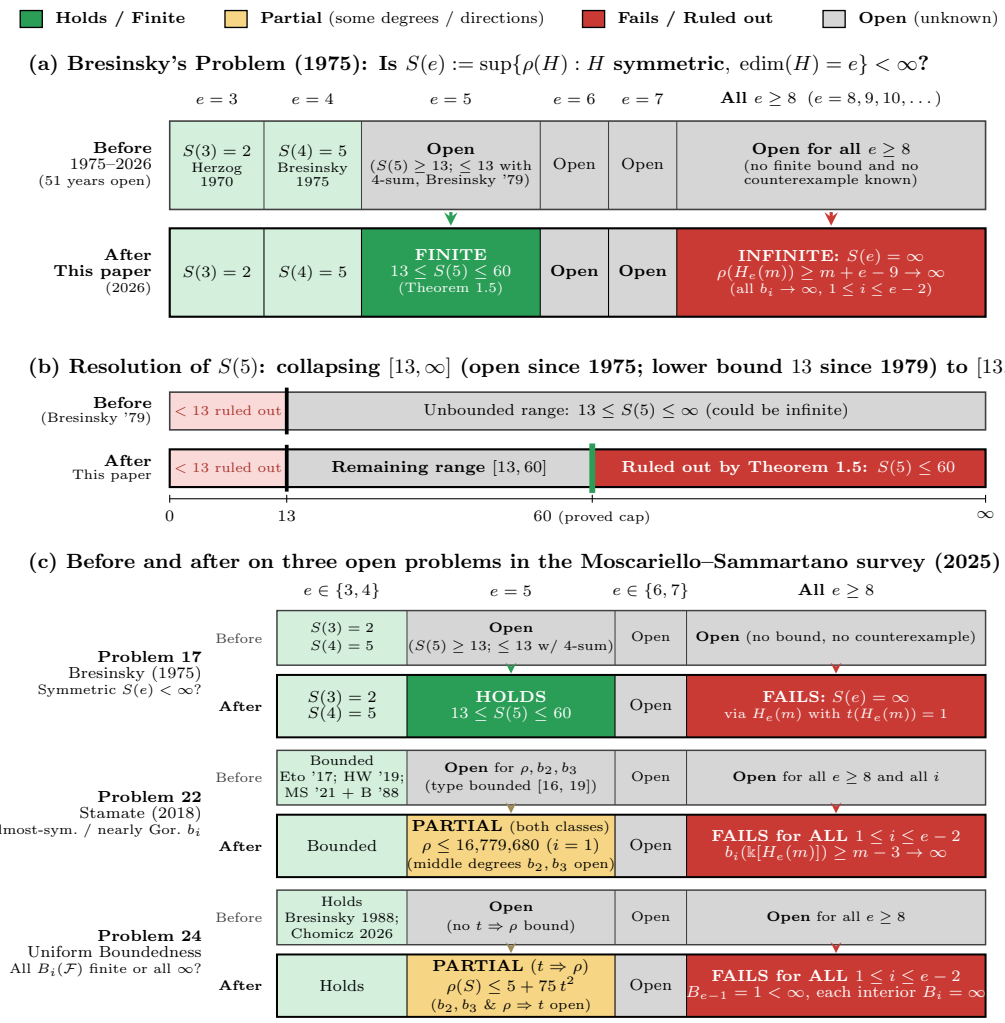

FIGURE 1. Landscape of Bresinsky's problem (1975) and its companion questions before and after this paper. Dark green marks new full positive finiteness results, light green marks previously known positive results, amber marks partial progress (specific degrees or one-way implications), red marks disproofs or ruled-out regions, and grey marks open cases. **(a)** Before this paper, $S(e)$ was known only for $e \leq 4$, leaving all $e \geq 5$ open for 51 years. We prove $S(5) \leq 60$ (Theorem 1.5) and $S(e) = \infty$ for every $e \geq 8$ (Theorem 1.1), leaving only $e \in \{6, 7\}$ open. **(b)** The unbounded range $[13, \infty]$ for $S(5)$ (open since 1975, with lower bound 13 since 1979) collapses to $[13, 60]$: values below 13 are ruled out by Bresinsky's 1979 examples (light red) and values above 60 are ruled out by Theorem 1.5 (dark red, green boundary bar at 60). **(c)** Before-and-after impact on Problems 17, 22 (Betti bounds for almost-symmetric and nearly Gorenstein semigroups), and 24 (whether the Betti suprema $B_i(\mathcal{F})$ of a fixed-e family $\mathcal{F}$ are all finite or all infinite) of the Moscariello–Sammartano survey [17]: all three were open across $e \geq 5$ (except the type bounds $t \leq 473$ [16] and $t \leq 40$ [19] for Problem 22 at $e = 5$); this paper proves full or partial positive bounds at $e = 5$ (bounding ρ across both Problem 22 classes, leaving only b_2, b_3 open), disproves Problem 17 for all $e \geq 8$, and disproves Problems 22 and 24 in every interior degree $1 \leq i \leq e - 2$ for all $e \geq 8$.

$S \iff F(S) - z \notin S$ for all $z \in \mathbb{Z}$, which by Kunz's theorem [15] holds precisely when $\Bbbk[S]$ is Gorenstein—then $\rho(S) \in \{3, 5\}$ [1]. Defining

$$S(e) := \sup\{\rho(H) : H \text{ is a symmetric numerical semigroup with } \operatorname{edim}(H) = e\},$$

we thus have $S(3) = 2$ and $S(4) = 5$.

Bresinsky's problem [1] asks whether $S(e) < \infty$ for every integer $e \geq 2$. Despite fifty-one years of research, every dimension $e \geq 5$ remained open; in their standard open-problems survey [17, Section 4, Problem 17], Moscariello and Sammartano describe it as "arguably the oldest open problem in the theory of numerical semigroups." Part of its difficulty is that Gorenstein symmetry alone cannot force a bound: by the Buchsbaum–Eisenbud structure theorem [6], general codimension-3 Gorenstein rings admit Betti vectors $(1, 2k+1, 2k+1, 1)$ for arbitrarily large k, so any bound on $S(e)$ must exploit the additive arithmetic of numerical semigroups rather than homological algebra alone.

Over the past half-century, partial progress appeared only under extra algebraic hypotheses or in $e = 4$:

- For a symmetric $e = 5$ semigroup H satisfying the special four-sum condition $n_i + n_j = n_h + n_k$ for four distinct indices, Bresinsky [3] proved $\rho(H) \leq 13$ and exhibited examples achieving equality, giving $S(5) \geq 13$.
- For arbitrary $e = 4$ semigroups of Cohen–Macaulay type $t(S)$, Bresinsky [4] proved $\rho(S) \leq 4 + 9\,t(S)$; very recently, Chomicz [8] sharpened this to $t(S) - 11 \leq \rho(S) \leq 4\,t(S) + 5$, settling Moscariello and Sammartano's Problem 25 and establishing uniform boundedness of all Betti numbers at $e = 4$.
- For almost-symmetric numerical semigroups (defined in Section 2) at $e = 4$, Eto [10] and Herzog–Watanabe [14] proved $\rho(S) \leq 7$ (bounding all Betti numbers alongside $t(S) \leq 3$); for nearly Gorenstein semigroups at $e = 4$, Moscariello–Strazzanti [18] proved $t(S) \leq 3$, which combined with Bresinsky's bound [4] yields bounded Betti numbers at $e = 4$. At $e = 5$, Moscariello [16] proved $t(S) \leq 473$ for almost-symmetric semigroups and Moscariello–Strazzanti [19, Theorem 2.10] proved $t(S) \leq 40$ for non-almost-symmetric nearly Gorenstein semigroups (bounding $t(S) \leq 473$ across both classes). This left boundedness of $\rho(S)$, b_2, and b_3 open for both classes at $e = 5$ and all Betti numbers open for both classes at $e \geq 6$ (Stamate's problem [23, Question 3.7], listed as Problem 22 in [17]).

More broadly, Moscariello and Sammartano [17, Section 6, Problem 24] asked whether for every family $\mathcal{F}$ of numerical semigroups of fixed embedding dimension e, the suprema $B_i(\mathcal{F}) := \sup\{b_i(\Bbbk[S]) : S \in \mathcal{F}\}$ $(1 \leq i \leq e-1)$ are either all finite or all infinite.

In this paper we resolve Bresinsky's problem in opposite directions across embedding dimensions—positively at $e = 5$ and negatively for all $e \geq 8$—and simultaneously refute Problems 22 and 24 in every interior homological degree for all $e \geq 8$. Figure 1 and Table 1 summarize the landscape before and after our results.

Problem / Regime	**Best Known Before**	**This Paper**	**Status**
Bresinsky $S(e)$, $e \in \{3,4\}$	$S(3) = 2$ [13], $S(4) = 5$ [1]	—	Known
Bresinsky $S(5)$ (Prob. 17)	Open ($S(5) \geq 13$; ≤ 13 with 4-sum only [3])	$S(5) \leq 60$ (so $13 \leq S(5) \leq 60$)	**HOLDS**
Bresinsky $S(e)$, $e \geq 8$ (Prob. 17)	Open for all $e \geq 8$	$S(e) = \infty$ ($\rho(H_e(m)) \geq m + e - 9$)	**FAILS**
Uniform bdd. $e = 5$ (Prob. 24, $t \Rightarrow \rho$)	No bound known	$\rho(S) \leq 5 + 75\,t^2$ (all $t \geq 1$)	**PARTIAL** ($t \Rightarrow \rho$)
Almost-sym. / n. Gor. $e = 5$ (Prob. 22)	$t \leq 473$ [16] (≤ 40 non-a.-s. [19]); ρ open	$\rho(S) \leq 16{,}779{,}680$ ($\leq 120{,}005$ non-a.-s., $i = 1$, Thm. 1.3 + [16, 19])	**PARTIAL** ($i = 1$)
Almost-sym. / n. Gor. $e \geq 8$ (Prob. 22)	Open for all $e \geq 8$, all i	$b_i(\Bbbk[H_e(m)]) \geq m - 3 \to \infty$ ($\forall\, 1 \leq i \leq e-2$)	**FAILS** ($1 \leq i \leq e-2$)
Uniform bdd. $e \geq 8$ (Prob. 24)	Open for all $e \geq 8$	$B_{e-1} = 1 < \infty$, every $B_i = \infty$ ($1 \leq i \leq e-2$)	**FAILS**
Bresinsky $S(e)$, $e \in \{6,7\}$	Open	Remaining open dimensions	Open

TABLE 1. Comparison of prior results and the theorems proved in this paper across embedding dimensions e and Problems 17, 22, and 24 of [17]. Theorems 1.1, 1.2, 1.3, and 1.5 are self-contained and computer-free; Row 5 combines Theorem 1.3 with Moscariello's $t(S) \leq 473$ [16, Remark 17] and Moscariello–Strazzanti's $t(S) \leq 40$ [19, Theorem 2.10]. In Row 6, type boundedness ($i = e - 1$) remains open.

Main results. Our first theorem refutes Bresinsky's problem for all embedding dimensions $e \geq 8$ via an explicit 8-generated Boolean-cube construction lifted by gluing.

Theorem 1.1 (Negative resolution of Bresinsky's problem for all $e \geq 8$)**.** *For every integer $m \geq 2$, let $a_m := m(m+1)(2m+3)$ and $D_m := 2(m+1)^2$. Define the 8-generated submonoid $H_8(m) \subseteq \mathbb{N}$ whose generators form an affine image of the three-dimensional Boolean cube $\{0,1\}^3$:*

$$H_8(m) := \Big\langle a_m + \epsilon_1 m + \epsilon_2(m+1) + \epsilon_3 D_m \;\Big|\; (\epsilon_1,\epsilon_2,\epsilon_3) \in \{0,1\}^3 \Big\rangle.$$

Then for every $m \geq 2$:

(1) $H_8(m)$ is a numerical semigroup of exact embedding dimension $\operatorname{edim}(H_8(m)) = 8$;

(2) $H_8(m)$ is symmetric ($t(H_8(m)) = 1$) with Frobenius number

$$F(H_8(m)) = 4m^4 + 12m^3 + 11m^2 + m - 1;$$

(3) its minimal presentation cardinality satisfies $\rho(H_8(m)) \geq m - 1$.

Moreover, inductively defining $H_{e+1}(m) := 2H_e(m) + \mathbb{N}d_e(m)$ for $e \geq 8$ with $d_8(m) := 2a_m + 2m + 1$ and $d_{e+1}(m) := 2\min(H_e(m)\setminus\{0\}) + d_e(m)$ produces for every $e \geq 8$ a family $\{H_e(m)\}_{m\geq 2}$ of symmetric numerical semigroups with $\operatorname{edim}(H_e(m)) = e$ and $\rho(H_e(m)) = \rho(H_8(m)) + (e-8) \geq m + e - 9$. In particular, $S(e) = \infty$ for every $e \geq 8$.

The same family $H_e(m)$ exhibits simultaneous divergence of *every* interior Betti number over every field $\Bbbk$, settling both Stamate's Problem 22 and the Moscariello–Sammartano Uniform Boundedness Problem 24 in every homological degree $1 \leq i \leq e-2$ for all $e \geq 8$.

Theorem 1.2 (Simultaneous divergence of all interior Betti numbers for $e \geq 8$)**.** *Let $\Bbbk$ be an arbitrary field, and let $H_e(m)$ ($e \geq 8$, $m \geq 2$) be the symmetric family of Theorem 1.1. Then:*

(1) For $H_8(m)$, the Betti numbers satisfy $b_0 = b_7 = 1$, $b_i = b_{7-i}$ for $0 \leq i \leq 7$, with $b_1(\Bbbk[H_8(m)]) \geq m-1$, $b_2(\Bbbk[H_8(m)]) \geq m-2$, and $b_3(\Bbbk[H_8(m)]) \geq m-3$.

(2) For every $e \geq 8$ and $0 \leq i \leq e-1$,

$$b_i(\Bbbk[H_e(m)]) = \sum_{j=0}^{e-8} \binom{e-8}{j} b_{i-j}(\Bbbk[H_8(m)])$$

(with $b_k := 0$ outside $[0,7]$). Consequently, for every $e \geq 8$ and every interior homological degree $1 \leq i \leq e-2$,

$$b_i(\Bbbk[H_e(m)]) \geq m - 3 \xrightarrow{m\to\infty} \infty,$$

while $b_0(\Bbbk[H_e(m)]) = b_{e-1}(\Bbbk[H_e(m)]) = 1$.

In particular, for every $e \geq 8$, both Stamate's Problem 22 and the Moscariello–Sammartano Uniform Boundedness Problem 24 have negative answers in every interior homological degree $1 \leq i \leq e-2$.

In sharp contrast to $e \geq 8$, our next two theorems establish affirmative finiteness in the first open dimension $e = 5$, not merely for symmetric semigroups but for *all* 5-generated numerical semigroups of bounded Cohen–Macaulay type.

Theorem 1.3 (Quadratic type-to-presentation bound in embedding dimension 5)**.** *For every numerical semigroup $S = \langle n_1, n_2, n_3, n_4, n_5\rangle$ of embedding dimension 5 and Cohen–Macaulay type $t = t(S) \geq 1$,*

$$\rho(S) \leq 5 + 75\,t^2.$$

Theorem 1.3 immediately yields two consequences in embedding dimension 5:

Corollary 1.4. *Let S be a numerical semigroup of embedding dimension 5.*

(1) (Almost-symmetric $e = 5$, Problem 22 for $i = 1$.) *If S is almost symmetric, then $\rho(S) \leq 5 + 75 \cdot 473^2 =$ 16,779,680.*

(2) (Uniform boundedness $t \Rightarrow \rho$ at $e = 5$, Problem 24.) *For any family $\mathcal{F}$ of 5-generated numerical semigroups, $\sup_{S\in\mathcal{F}} t(S) < \infty$ implies $\sup_{S\in\mathcal{F}} \rho(S) < \infty$.*

Combining Theorem 1.3 with the published bound $t(S) \leq 40$ of Moscariello and Strazzanti [19, Theorem 2.10] for non-almost-symmetric nearly Gorenstein 5-generated semigroups likewise gives $\rho(S) \leq 5 + 75 \cdot 40^2 = 120{,}005$ in that subclass (hence $\rho(S) \leq 16{,}779{,}680$ across all nearly Gorenstein 5-generated semigroups, settling Problem 22 for $i = 1$ in both classes at $e = 5$). At $t = 1$, Theorem 1.3 already gives $S(5) \leq 80$ unconditionally. Refining the same order-theoretic framework yields an improved symmetric bound.

Theorem 1.5 (Computer-free symmetric bound $S(5) \leq 60$)**.** *Every symmetric numerical semigroup H of embedding dimension 5 satisfies*

$$\rho(H) \leq 60.$$

Consequently, over any field $\Bbbk$, every Gorenstein monomial curve ring $\Bbbk[H]$ of embedding dimension 5 has Betti numbers $(1, \rho(H), 2\rho(H) - 2, \rho(H), 1)$ satisfying $b_1 = b_3 \leq 60$ and $b_2 \leq 118$.

Remark 1.6. In the expanded version [12], we determine the complete minimal presentation of $H_8(m)$ and $H_e(m)$, proving $\rho(H_8(m)) = 10m + 9$ and $\rho(H_e(m)) = 10m + e + 1$ for all $e \geq 8$ and $m \geq 2$ ([12, Section 7]), and we combine the local partition bounds of Section 6 with two computer-certified global partition-count exclusions to sharpen the symmetric $e = 5$ upper bound further to $S(5) \leq 41$ ([12, Section 8]). Every proof of Theorems 1.1, 1.2, 1.3, and 1.5 in the present focused version is self-contained and computer-free; Corollary 1.4(1) imports Moscariello's bound $t(S) \leq 473$ from [16, Remark 17].

Organization and proof ideas. Section 2 recalls the standard facts on Betti graphs, divisor complexes, symmetry, and gluing. Section 3 proves Theorem 1.1: symmetry of $H_8(m)$ is established in one shot by pairing every residue class modulo the smallest generator a_m to $F + a_m$ via two explicit representation regimes, while $\rho(H_8(m)) \geq m - 1$ follows directly from Euclidean division modulo D_m. Section 4 proves Theorem 1.2 by computing the exact fibers of two explicit families of degrees in $H_8(m)$ whose divisor complexes decompose into two contractible cones intersecting in S^0 and the 4-cycle C_4, respectively. Section 5 develops a mixed-union absorption method combining $(2, 2)$ and $(2, 3)$ Betti elements and proves Theorem 1.3 and Corollary 1.4 via an explicit 10-element cyclic matching of support partitions. Section 6 sharpens the same mixed-union and both-sides order inequalities to prove Theorem 1.5. Section 7 discusses the resulting picture and the remaining open dimensions $e \in \{6, 7\}$.

2. Preliminaries

We collect standard definitions and facts from [22, 17]. Let $S = \langle n_1, \dots, n_e \rangle \subseteq \mathbb{N}$ be a numerical semigroup with minimal generating set $\operatorname{msg}(S) = \{n_1, \dots, n_e\}$ ($n_1 < \cdots < n_e$, $\gcd(n_1, \dots, n_e) = 1$, and $e = \operatorname{edim}(S) \geq 2$). Since $S \neq \mathbb{N}$, its Frobenius number $F(S) := \max(\mathbb{Z} \setminus S)$ is positive. The set of *pseudo-Frobenius numbers* of S is

$$\operatorname{PF}(S) := \{x \in \mathbb{Z} \setminus S : x + s \in S \text{ for all } s \in S \setminus \{0\}\},$$

and the *Cohen–Macaulay type* of S is $t(S) := |\operatorname{PF}(S)| \geq 1$. For every $x \in \mathbb{Z} \setminus S$, taking $s \in S$ maximal with $x + s \notin S$ produces $f := x + s \in \operatorname{PF}(S)$ with $f - x = s \in S$. A numerical semigroup S is *symmetric* if $z \in S \iff F(S) - z \notin S$ for every $z \in \mathbb{Z}$, which is equivalent to $\operatorname{PF}(S) = \{F(S)\}$ (i.e., $t(S) = 1$) and, by Kunz's theorem [15], to $\Bbbk[S]$ (equivalently $\Bbbk[[S]]$) being Gorenstein. Two standard generalizations appear in Problem 22: S is *almost symmetric* if every $z \in \mathbb{Z} \setminus S$ satisfies either $F(S) - z \in S$ or both z and $F(S) - z$ lie in $\operatorname{PF}(S)$, and S is *nearly Gorenstein* if the trace ideal of the canonical module of $\Bbbk[S]$ contains the homogeneous maximal ideal of $\Bbbk[S]$ [18, 17]; every symmetric numerical semigroup is both almost symmetric and nearly Gorenstein.

For $h \in S \setminus \{0\}$, let $Z(h) := \{x \in \mathbb{N}^e : \sum_{i=1}^{e} x_i n_i = h\}$ be the set of factorizations of h, and for $x \in Z(h)$ write $\operatorname{supp}(x) := \{i \in \{1, \dots, e\} : x_i > 0\}$. We use three combinatorial objects attached to $h \in S \setminus \{0\}$ (with $\Delta_0 := \{\emptyset\}$):

- The *factorization graph* ∇_h has vertex set $Z(h)$, with an edge between $x, y \in Z(h)$ whenever $\operatorname{supp}(x) \cap \operatorname{supp}(y) \neq \emptyset$.
- The *Rosales* (or *generator*) *graph* G_h has vertex set $V_h := \{i \in \{1, \dots, e\} : h - n_i \in S\}$, with an edge $i \sim j$ whenever $h - n_i - n_j \in S$.
- The *squarefree divisor complex* is the abstract simplicial complex

$$\Delta_h := \Big\{ I \subseteq \{1, \dots, e\} : h - \sum_{i \in I} n_i \in S \Big\}$$

(which contains $\emptyset$ since $h \in S$).

For every $h \in S \setminus \{0\}$, every $x \in Z(h)$ has nonempty support forming a clique in G_h, and every vertex or edge of G_h extends to a factorization in $Z(h)$. Thus ∇_h, G_h, and the 1-skeleton of Δ_h have canonically bijective connected components; write $c(h)$ for their common number of connected components. By Rosales's theorem [20] (see [22, Chapter 8]), $h \in S \setminus \{0\}$ is a *Betti element* ($h \in \mathrm{Betti}(S)$) if and only if $c(h) \geq 2$, and the cardinality of every minimal presentation of S is

$$\rho(S) = \sum_{h \in \mathrm{Betti}(S)} \big(c(h) - 1\big). \tag{1}$$

More generally, over any field $\Bbbk$, let $R = \Bbbk[x_1, \ldots, x_e]$ with S-grading $\deg(x_i) = n_i$, and let $\beta_{i,h}(\Bbbk[S]) := \dim_\Bbbk \mathrm{Tor}_i^R(\Bbbk[S], \Bbbk)_h$ and $b_i(\Bbbk[S]) := \sum_{h \in S} \beta_{i,h}(\Bbbk[S])$ $(0 \leq i \leq e-1)$. By the Campillo–Marijuán and Bruns–Herzog formula [7, 5], for every $h \in S$ and every $i \geq 1$,

$$\beta_{i,h}(\Bbbk[S]) = \dim_\Bbbk \widetilde{H}_{i-1}(\Delta_h; \Bbbk), \tag{2}$$

where $\widetilde{H}_{i-1}$ denotes reduced simplicial homology over $\Bbbk$. In particular, $b_1(\Bbbk[S]) = \rho(S)$ and $b_{e-1}(\Bbbk[S]) = t(S)$; by flatness of completion, $b_i(\Bbbk[S]) = b_i(\Bbbk[[S]])$. When H is symmetric, $\Bbbk[H]$ is a 1-dimensional Gorenstein domain of codimension $e-1$, so its minimal graded free R-resolution is self-dual, giving $b_i(\Bbbk[H]) = b_{e-1-i}(\Bbbk[H])$ for $0 \leq i \leq e-1$.

Finally, if H is a numerical semigroup and $d \in H \setminus (\{0\} \cup \mathrm{msg}(H))$ is odd, then $H' := 2H + \mathbb{N}d$ is a *gluing* of H and $\mathbb{N}$ [9, 21]. By [22, Chapters 8–9] and [9, 21], H' is symmetric whenever H is symmetric, $\rho(H') = \rho(H) + 1$, and the defining ideal of $\Bbbk[H']$ in $R' = R[z]$ is $I_{H'} = I_H R' + (z^2 - M)$, where $M \in R$ is any monomial of degree d in H.

3. Negative Resolution of Bresinsky's Problem for All $e \geq 8$

Fix an integer $m \geq 2$, and set

$$A_m := (m+1)(2m+3) = 2m^2 + 5m + 3, \qquad a_m := mA_m, \qquad D_m := 2(m+1)^2 = A_m - (m+1).$$

Notice the fundamental identity $a_m = mD_m + m(m+1)$. Let $W_m := \{0, m, m+1, 2m+1\}$. Indexing the eight vertices of $\{0,1\}^3$ by $i = \epsilon_1 + 2\epsilon_2 + 4\epsilon_3 \in \{0, \ldots, 7\}$, the generators of $H_8(m)$ are

$$\begin{aligned} &g_0 = a_m, && g_1 = a_m + m, && g_2 = a_m + m + 1, && g_3 = a_m + 2m + 1, \\ &g_4 = a_m + D_m, && g_5 = a_m + D_m + m, && g_6 = a_m + D_m + m + 1, && g_7 = a_m + D_m + 2m + 1. \end{aligned}$$

Since $D_m = 2m^2 + 4m + 2 > 2m + 1$, we have $g_0 < g_1 < \cdots < g_7$. Figure 2 illustrates this affine Boolean cube of generators and the disjoint supports of the Betti witnesses $b_k(m)$ constructed in Subsection 3.4.

3.1. **Coprimality and exact embedding dimension.** Since $g_2 - g_1 = 1$, we have $\gcd(H_8(m)) = 1$. Moreover, for all $m \geq 2$,

$$2g_0 - g_7 = a_m - D_m - (2m+1) = 2m^3 + 3m(m-1) - 3 \geq 16 + 6 - 3 > 0.$$

Hence $g_7 < 2g_0$, so no generator can be expressed as a sum of two or more generators. Thus $\mathrm{msg}(H_8(m)) = \{g_0, \ldots, g_7\}$ and $\mathrm{edim}(H_8(m)) = 8$, proving part (1) of Theorem 1.1.

3.2. **Sumset lemmas for $W_m = \{0, m, m+1, 2m+1\}$.** For each integer $L \geq 0$, let $L \cdot W_m := \{\mu_1 m + \mu_2(m+1) + \mu_3(2m+1) : \mu_j \in \mathbb{N},\ \mu_0 + \mu_1 + \mu_2 + \mu_3 = L\}$. Clearly $L \cdot W_m \subseteq (L+1) \cdot W_m$ (as $0 \in W_m$), and swapping $(\mu_0, \mu_1) \leftrightarrow (\mu_3, \mu_2)$ gives the symmetry $L \cdot W_m = L(2m+1) - L \cdot W_m$. Define the *gap triangle*

$$\mathcal{G}_m := \big\{\alpha m + \beta : 0 \leq \alpha \leq m-2,\ \ \alpha + 1 \leq \beta \leq m-1\big\} \subseteq [1,\ m^2 - m - 1].$$

Lemma 3.1. *Let $m \geq 2$ and $L \geq 0$.*

1. *An integer u lies in $L \cdot W_m$ if and only if $u = qm + s$ for integers q, s with $0 \leq q \leq 2L$ and $\max(0, q - L) \leq s \leq \min(q, L)$.*
2. *If $x = Qm + R$ for integers $0 \leq R \leq Q$, then $x \notin \mathcal{G}_m$. In particular, $L \cdot W_m \cap \mathcal{G}_m = \emptyset$ and $L \cdot W_m \cap (L(2m+1) - \mathcal{G}_m) = \emptyset$.*
3. *For every $L \geq m$, $L \cdot W_m = [0, L(2m+1)] \setminus \big(\mathcal{G}_m \cup (L(2m+1) - \mathcal{G}_m)\big)$.*
4. *For every $v \in [0, m^2 - m - 1]$, $v \in \mathcal{G}_m \iff (m^2 - m - 1) - v \notin \mathcal{G}_m$.*
5. *If $u \in L \cdot W_m$ with $0 \leq L \leq m-1$, then $(2m^2 - 1) - u \notin \mathcal{G}_m$.*

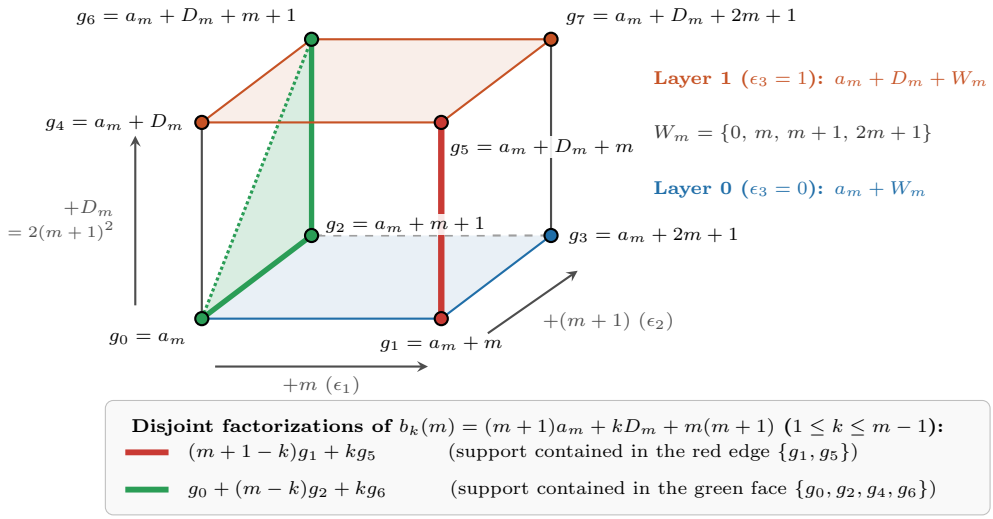

FIGURE 2. The three-dimensional affine Boolean cube $\{0,1\}^3$ of minimal generators of $H_8(m)$ ($a_m = m(m+1)(2m+3)$, $D_m = 2(m+1)^2$). Because $(m+1)(2m+1) < D_m$, base-D_m Euclidean division locks the Layer-1 multiplicity $L_1 = k$ for every factorization of $b_k(m)$ ($1 \le k \le m-1$), separating its Rosales graph into at least two disjoint faces (red vertical edge $\{g_1, g_5\}$ versus green face $\{g_0, g_2, g_4, g_6\}$).

Proof. (1) Given $\mu_0 + \mu_1 + \mu_2 + \mu_3 = L$, set $q = \mu_1 + \mu_2 + 2\mu_3$ and $s = \mu_2 + \mu_3$. Conversely, given (q, s) in the stated range, set $\mu_3 = \max(0, q - L)$, $\mu_2 = s - \mu_3$, $\mu_1 = q - s - \mu_3$, and $\mu_0 = L - \mu_1 - \mu_2 - \mu_3$; all four are nonnegative integers summing to L.

(2) Suppose $Qm + R = \alpha m + \beta \in \mathcal{G}_m$ with $0 \le R \le Q$. Since $R \equiv \beta \pmod m$ and $R \ge 0$ while $0 \le \beta \le m-1$, we have $R \ge \beta$, whence $Qm = \alpha m + \beta - R \le \alpha m$, so $Q \le \alpha < \beta \le R$, contradicting $R \le Q$. The second assertion follows from (1) and the reflection $u \mapsto L(2m+1) - u$.

(3) One inclusion holds by (2). For the converse, take $u \in [0, L(2m+1)] \setminus (\mathcal{G}_m \cup (L(2m+1) - \mathcal{G}_m))$ with $L \ge m$. By reflection $u \mapsto L(2m+1) - u$, assume $u \le Lm + L/2$, and write $u = qm + s$ with $0 \le s \le m-1$. If $q \le L$, then $s \le q$ holds automatically when $q \ge m-1$ and follows from $u \notin \mathcal{G}_m$ when $q \le m-2$; since $q - L \le 0 \le s$, (1) holds. If $q > L$, let $r \ge 0$ be minimal such that $s + r(m+1) \ge q - L$. If $r = 0$, then $0 \le s \le m-1 < L < q < 2L$ (as $u \le Lm + L/2$), so (q, s) satisfies (1). If $r \ge 1$, minimality gives $r(m+1) \le q - L - s + m$. Setting $q' = q - r$ and $s' = s + rm$, we have $u = q'm + s'$, $s' \ge q' - L$, $q' - s' = q - s - r(m+1) \ge L - m \ge 0$, and

$$(m+1)s' \le (m+1)s + m(q - L - s + m) = u - mL + m^2 \le \frac{L}{2} + m^2 < (m+1)L$$

since $(m + \frac{1}{2})L \ge m^2 + \frac{1}{2}m > m^2$ for $L \ge m$. Thus $s' \le L$, and (1) gives $u \in L \cdot W_m$.

(4) Writing $v = \alpha m + \beta$ with $0 \le \alpha \le m-2$ and $0 \le \beta \le m-1$, we have $(m^2 - m - 1) - v = (m - 2 - \alpha)m + (m - 1 - \beta)$, which lies in $\mathcal{G}_m$ iff $m - 1 - \beta \ge m - 1 - \alpha$, i.e., $\beta \le \alpha$, i.e., $v \notin \mathcal{G}_m$.

(5) If $(2m^2 - 1) - u = \alpha m + \beta \in \mathcal{G}_m$, then $u = (2m - 1 - \alpha)m + (m - 1 - \beta)$. Writing $u = q'm + s'$ via (1) gives $0 \le s' \le L \le m - 1 < m$, so uniqueness of Euclidean division by m forces $q' = 2m - 1 - \alpha$ and $s' = m - 1 - \beta$. Hence $L \ge q' - s' = m + (\beta - \alpha) \ge m + 1$, contradicting $L \le m - 1$. □

Lemma 3.2 (Exact $(2m+1)$-residue decomposition)**.** *Let $\mathcal{D}_m := \{c(m+1) : 0 \le c \le m\} \cup \{qm : 1 \le q \le m\} \subseteq [0, m(m+1)]$. Then $\mathcal{D}_m$ is a complete residue system modulo $2m+1$ invariant under $d_1 \mapsto m(m+1) - d_1$. Moreover, for every integer $u \ge 0$ with $u \notin \mathcal{G}_m$, the unique integer L with $d_1 := L(2m+1) - u \in \mathcal{D}_m$ satisfies $L \ge 0$ and $u \in L \cdot W_m$.*

Proof. Since $\gcd(2, 2m+1) = 1$ and multiplying $\mathcal{D}_m$ by 2 modulo $2m+1$ yields $\{0, 1, \dots, m\} \cup \{-1, -2, \dots, -m\}$, $\mathcal{D}_m$ is a complete residue system modulo $2m+1$; invariance under $d_1 \mapsto m(m+1) - d_1$ is immediate since $m(m+1) - c(m+1) = (m-c)(m+1)$ for $0 \le c \le m$ and $m(m+1) - qm = (m+1-q)m$ for $1 \le q \le m$. Clearly $L(2m+1) = u + d_1 \ge 0$ forces $L \ge 0$.

- If $d_1 = c(m+1)$ $(0 \le c \le m)$: when $c \le L$, $u = (L-c)(2m+1) + cm \in L \cdot W_m$. If $c > L$, then $0 \le L < c \le m$, so $\beta := m + L - c \in [0, m-1]$ and $u = \alpha m + \beta \ge 0$ with $\alpha := 2L - c - 1 \ge 0$ (as $\beta < m$). Then $\alpha \le L - 2 \le m - 3$ and $\alpha + 1 = 2L - c \le \beta$, giving $u \in \mathcal{G}_m$, contradiction.
- If $d_1 = qm$ $(1 \le q \le m)$: when $q \le L$, $u = (L-q)(2m+1) + q(m+1) \in L \cdot W_m$. If $q > L$, then $0 \le L < q \le m$, so $u = \alpha m + L \ge 0$ with $L \in [0, m-1]$ forces $\alpha := 2L - q \ge 0$, $\alpha \le L - 1 \le m - 2$, and $\alpha + 1 \le L$, giving $u \in \mathcal{G}_m$, contradiction. □

3.3. **Symmetry and exact Frobenius number of $H_8(m)$.** We use the following elementary criterion.

Lemma 3.3 (One-Shot Symmetry Criterion)**.** *Let $H \subseteq \mathbb{N}$ be a numerical semigroup containing a positive element g_0, and suppose $w_{\max} \in H$ satisfies:*

(i) $F := w_{\max} - g_0 \notin H$;

(ii) for every residue class modulo g_0, there exist $w, w' \in H$ with w in that residue class and $w + w' = w_{\max}$.

Then H is symmetric with exact Frobenius number $F(H) = F$.

Proof. Let $z \in \mathbb{Z} \setminus H$. By (ii), pick $w, w' \in H$ with $w \equiv z \pmod{g_0}$ and $w + w' = w_{\max}$. Since $w \in H$ and $z \notin H$, we cannot have $z \ge w$ (else $z = w + jg_0 \in H$), so $w - g_0 - z = kg_0$ for some integer $k \ge 0$. Hence $F - z = (w_{\max} - g_0) - z = w' + kg_0 \in H$. In particular, no $z > F$ can lie outside H (else $F - z < 0$ would lie in $H \subseteq \mathbb{N}$). Together with (i), this gives $F(H) = F$, and for every $z \in \mathbb{Z}$ at most one of $z, F - z$ lies in H (since $F \notin H$), proving symmetry. □

Set $F := (2m^2+m)A_m - (2m+1) = 4m^4+12m^3+11m^2+m-1$ and $w_{\max} := F + g_0 = 2m(m+1)A_m - (2m+1)$. Every element of $H_8(m)$ has the form

$$x(L_0, L_1, u) := \big(mL_0 + (m+1)L_1\big)A_m + \big(u - L_1(m+1)\big), \qquad L_0, L_1 \in \mathbb{N}, \quad u \in (L_0 + L_1) \cdot W_m,$$

since $g_i = mA_m + w$ on Layer 0 and $g_{i+4} = (m+1)A_m + (w - m - 1)$ on Layer 1 for $w \in W_m$.

Lemma 3.4. *$F \notin H_8(m)$.*

Proof. Suppose $F = x(L_0, L_1, u)$ with $L_0, L_1 \in \mathbb{N}$, $L := L_0 + L_1$, and $u \in L \cdot W_m$. Define $k := (2m^2 + m) - (mL + L_1) \in \mathbb{Z}$, so

$$mL + L_1 = 2m^2 + m - k, \qquad u = kA_m - (2m+1) + L_1(m+1).$$

We exclude every integer k:

- $k \le -1$: Since $(m+1)L_1 \le mL + L_1 = 2m^2 + m - k$, we have $u \le k(2m^2+5m+3) + (2m^2 - m - 1 - k) = 2(k+1)m^2 + (5k-1)m + (2k-1) < 0$, impossible.
- $k \ge 2$: From $mL \le 2m^2 + m - k < 2m^2 + m$, we have $L \le 2m$, hence $L(2m+1) \le 4m^2 + 2m < 4m^2 + 8m + 5 \le 2A_m - (2m+1) \le u$, impossible.
- $k = 1$: Here $mL + L_1 = 2m^2 + m - 1$, so $L_1 = qm + (m-1)$, $L = 2m - q$, and $L_0 = (1-q)(m+1) \ge 0$, forcing $q \in \{0, 1\}$. For $q = 1$, $u = 4m^2 + 4m + 1 > 4m^2 - 1 = L(2m+1)$, impossible. For $q = 0$, $L = 2m$ and $L(2m+1) - u = m^2 - m - 1 \in \mathcal{G}_m$, contradicting Lemma 3.1(2).
- $k = 0$: Here $mL + L_1 = 2m^2 + m$, so $L_1 = qm$, $L = 2m + 1 - q$, and $L_0 = (2-q)m + 1 - q \ge 0$, forcing $q \in \{0, 1\}$. For $q = 0$, $u = -(2m+1) < 0$, impossible. For $q = 1$, $u = m^2 - m - 1 \in \mathcal{G}_m$, contradicting Lemma 3.1(2). □

Next we verify condition (ii) of Lemma 3.3 modulo $g_0 = mA_m$. Because $\mathcal{I}_m := [-2m,\ 2m^2 + 3m + 2]$ is a contiguous integer interval of length $2m^2 + 5m + 3 = A_m$, the integers $jA_m + s$ for $(j, s) \in \{0, 1, \ldots, m-1\} \times \mathcal{I}_m$ form the contiguous interval $[-2m,\ mA_m - 2m - 1]$, and hence a complete residue system modulo mA_m. Set $U_1 := 3m^2 + 3m + 1$ and $U_2 := 2m^2 - 1$, and note two exact summation identities to $w_{\max}$:

- **Regime I:** For $L_0 \in [0, m+2]$, $j \in [0, m-1]$, $u \in (L_0 + j) \cdot W_m$, and $U_1 - u \in (2m + 1 - L_0 - j) \cdot W_m$,

$$w = x(L_0, j, u), \qquad w' = x(m + 2 - L_0,\ m - 1 - j,\ U_1 - u) \implies w + w' = w_{\max}.$$

- **Regime II:** For $L_1 \in [0, 2m]$, $u \in L_1 \cdot W_m$, and $U_2 - u \in (2m - L_1) \cdot W_m$,

$$w = x(0, L_1, u), \qquad w' = x(0,\ 2m - L_1,\ U_2 - u) \implies w + w' = w_{\max}.$$

Figure 3 displays the partition of $\{0, \ldots, m-1\} \times \mathcal{I}_m$ into four exhaustive cases.

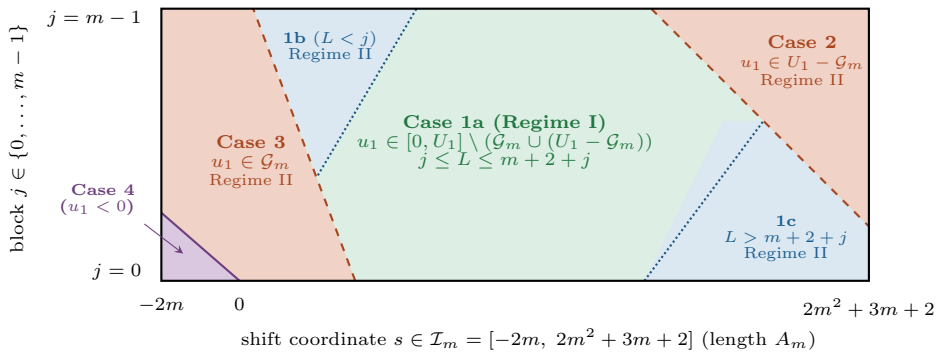

FIGURE 3. Schematic partition of the complete residue rectangle $\{0,\dots,m-1\}\times\mathcal{I}_m$ modulo mA_m (representing all mA_m residue classes via jA_m+s, with $u_1 = s+j(m+1)$) in the One-Shot Symmetry proof for $H_8(m)$. The central bulk is covered by Regime I pairs $x(L_0,j,u_1)+x(m+2-L_0,m-1-j,U_1-u_1) = w_{\max}$, while the upper-right gap triangle (Case 2, anchored at the top-right corner $u_1 = U_1$ and disjoint from $j=0$), left gap band (Case 3), negative corner (Case 4, only $j \in \{0,1\}$), and extreme-L wedges (Subcases 1b and 1c) are absorbed by pure Layer-1 Regime II pairs $x(0,L_1,u)+x(0,2m-L_1,U_2-u) = w_{\max}$.

Lemma 3.5. *For every $(j,s) \in \{0,\dots,m-1\}\times\mathcal{I}_m$, there exist $w,w' \in H_8(m)$ with $w \equiv jA_m+s \pmod{mA_m}$ and $w+w' = w_{\max}$.*

Proof. Set $u_1 := s+j(m+1)$. Since $s \le 2m^2+3m+2$ and $j \le m-1$, we have $u_1 \le U_1$.

Case 1: $u_1 \in [0,U_1]\setminus(\mathcal{G}_m \cup (U_1-\mathcal{G}_m))$. Let $L \in \mathbb{Z}$ be the unique integer with $d_1 := L(2m+1)-u_1 \in \mathcal{D}_m$. Because $(2m+1)^2 - U_1 = m(m+1)$, we have $(2m+1-L)(2m+1)-(U_1-u_1) = m(m+1)-d_1 \in \mathcal{D}_m$. By Lemma 3.2, $L \ge 0$, $2m+1-L \ge 0$, $u_1 \in L\cdot W_m$, and $U_1-u_1 \in (2m+1-L)\cdot W_m$.

- *Subcase 1a ($j \le L \le m+2+j$):* Regime I with $L_0 = L-j$ and $u = u_1$ gives $w = x(L_0,j,u_1) \equiv jA_m+s \pmod{mA_m}$.
- *Subcase 1b ($L < j$):* Take Regime II with $L_1 = j$, $L_1' = 2m-j \ge m+1$, and $u = u_1$. Since $L < j$, we have $u_1 \in L\cdot W_m \subseteq j\cdot W_m$ and $u_1 \le (m-2)(2m+1) < U_2$, so $u' := U_2-u_1 > 0$; by Lemma 3.1(5), $u' \notin \mathcal{G}_m$. Writing $u_1 = q_1m+s_1$ $(0 \le s_1 \le q_1)$ by Lemma 3.1(1) and $k = m-1-j \ge 0$, we have $L_1'(2m+1)-u' = (3+2k+q_1)m+(2+k+s_1) \notin \mathcal{G}_m$ by Lemma 3.1(2). Thus $u' \in L_1'\cdot W_m$ by Lemma 3.1(3), and $w \equiv jA_m+s \pmod{mA_m}$.
- *Subcase 1c ($L > m+2+j$):* Take Regime II with $L_1 = m+j+1 \ge m+1$, $L_1' = m-1-j$, and $u = (s-A_m)+L_1(m+1)$. Direct expansion gives $u' := U_2-u = U_1-u_1$. Since $L' := 2m+1-L < m-1-j = L_1'$, we have $u' \in L'\cdot W_m \subseteq L_1'\cdot W_m$ and $u' \le (m-2)(2m+1) < U_2$, so $u > 0$, $u \notin \mathcal{G}_m$ by Lemma 3.1(5), and writing $u' = q_2m+s_2$ $(0 \le s_2 \le q_2)$, $L_1(2m+1)-u = (3+2j+q_2)m+(2+j+s_2) \notin \mathcal{G}_m$ by Lemma 3.1(2). Thus $u \in L_1\cdot W_m$ by Lemma 3.1(3), and $w \equiv (j+1)A_m+(s-A_m) = jA_m+s \pmod{mA_m}$.

Case 2: $u_1 \in U_1-\mathcal{G}_m$. Write $x := U_1-u_1 = \alpha m+\beta \in \mathcal{G}_m$ $(0 \le \alpha \le m-2,\ \alpha+1 \le \beta \le m-1)$. Take Regime II with $L_1 = j+1$, $L_1' = 2m-j-1 \ge m$, and $u = (s-A_m)+L_1(m+1) = (m^2-m-1)-x$. Since $s \le 2m^2+3m+2$, we have $x \ge (m-1-j)(m+1)$, which forces $\alpha \ge m-2-j$ (else $\alpha m+\beta \le (m-2-j)m-1 < (m-1-j)(m+1)$). Thus $u = (m-2-\alpha)m+(m-1-\beta) =: q_um+s_u$ satisfies $0 \le s_u \le q_u \le j < L_1$ (since $q_u-s_u = \beta-\alpha-1 \ge 0$), so $u \in L_1\cdot W_m$ by Lemma 3.1(1). Meanwhile $u' = U_2-u = m^2+m+x \notin \mathcal{G}_m$, and with $k = m-1-j \ge 0$, $L_1'(2m+1)-u' = (1+2k+q_u)m+(1+k+s_u) \notin \mathcal{G}_m$ by Lemma 3.1(2). Hence $u' \in L_1'\cdot W_m$ by Lemma 3.1(3), and $w \equiv (j+1)A_m+(s-A_m) = jA_m+s \pmod{mA_m}$.

Case 3: $u_1 \in \mathcal{G}_m$. Write $u_1 = \alpha m+\beta \in \mathcal{G}_m$. Take Regime II with $L_1 = m+j \ge m$, $L_1' = m-j$, $u = s+L_1(m+1) = m^2+m+u_1$, and $u' = U_2-u = (m^2-m-1)-u_1$. Since $s \ge -2m$, we have $u_1 \ge (j-2)m+j$, forcing $\alpha \ge j-2$. Thus $u' = (m-2-\alpha)m+(m-1-\beta) =: q_um+s_u$ satisfies $0 \le s_u \le q_u \le m-j = L_1'$, so $u' \in L_1'\cdot W_m$ by Lemma 3.1(1). Meanwhile $u \ge m^2+m \implies u \notin \mathcal{G}_m$, and $L_1(2m+1)-u = (1+2j+q_u)m+(1+j+s_u) \notin \mathcal{G}_m$ by Lemma 3.1(2). Hence $u \in L_1\cdot W_m$ by Lemma 3.1(3), and $w \equiv jA_m+s \pmod{mA_m}$.

Case 4: $u_1 < 0$. Since $s \geq -2m$, $u_1 = s + j(m+1) < 0$ occurs only for $j \in \{0, 1\}$. Take Regime II with $L_1 = m + j$, $L_1' = m - j$, and $u = s + (m + j)(m + 1) = m^2 + m + u_1$.

- If $j = 0$ ($s \in [-2m, -1]$): both $u = m^2+m+s$ and $u' = m^2-m-1-s$ lie in $[m^2-m,\, m^2+m-1] \subseteq m \cdot W_m$ by Lemma 3.1(3), since $m(2m + 1) - (m^2 + m - 1) = m^2 + 1 > m^2 - m - 1$.
- If $j = 1$ ($s \in [-2m, -m-2]$): $u = (m+1)^2+s \in [m^2+1,\, m^2+m-1] \subseteq (m+1) \cdot W_m$ by Lemma 3.1(3), while $u' = m^2 - 2m - 2 - s \in [m^2 - m,\, m^2 - 2]$ has the explicit form $(m-1)m+s'$ with $0 \leq s' \leq m-2$, so $u' \in (m - 1) \cdot W_m$ by Lemma 3.1(1).

In both subcases $w \equiv jA_m + s \pmod{mA_m}$. □

Combining Lemmas 3.3, 3.4, and 3.5 (noting $w_{\max} = w + w' \in H_8(m)$) proves part (2) of Theorem 1.1.

3.4. **Lower bound $\rho(H_8(m)) \geq m - 1$ and inductive gluing.** For each $k \in \{1, 2, \ldots, m - 1\}$, consider the element

$$b_k(m) := (m + 1 - k)\, g_1 + k\, g_5 = (m + 1)a_m + kD_m + m(m + 1).$$

Since $(m + 1)m = 0 + m(m + 1)$, we also have the disjoint factorization

$$b_k(m) = g_0 + (m - k)\, g_2 + k\, g_6.$$

Let $b_k(m) = \sum_{i=0}^{3} c_i(a_m + w_i) + \sum_{i=0}^{3} e_i(a_m + D_m + w_i)$ be any factorization in $H_8(m)$, with total length $L := \sum(c_i + e_i)$ and Layer-1 count $L_1 := \sum e_i$ (where $w_0 = 0, w_1 = m, w_2 = m + 1, w_3 = 2m + 1$).

(1) *Exact length $L = m+1$:* Direct expansion yields $(m+2)g_0 - b_k(m) = (m-k)D_m > 0$ and $b_k(m) - mg_7 = 2m^2 + 5m + 2 + (k - 1)D_m > 0$, forcing $L = m + 1$.

(2) *Exact $L_1 = k$ and weight sum $u = m(m + 1)$:* Subtracting $(m + 1)a_m$ gives $L_1D_m + \sum_{i=0}^{3} \nu_i w_i = kD_m + m(m + 1)$, where $\nu_i := c_i + e_i$. Since $0 \leq \sum \nu_i w_i \leq (m + 1)(2m + 1) = 2m^2 + 3m + 1 < 2(m+1)^2 = D_m$ and $0 < m(m+1) < D_m$, uniqueness of Euclidean division by D_m forces $L_1 = k$ and $\sum_{i=0}^{3} \nu_i w_i = m(m + 1)$.

(3) *Disjointness:* Reducing $\nu_1 m + \nu_2(m + 1) + \nu_3(2m + 1) = m(m + 1)$ modulo m gives $\nu_2 + \nu_3 \equiv 0 \pmod{m}$. Since $0 \leq \nu_2 + \nu_3 \leq m + 1$ and $m \geq 2$ (so $2m > m + 1$), either $\nu_2 + \nu_3 = 0$, giving $(\nu_0, \nu_1, \nu_2, \nu_3) = (0, m+1, 0, 0)$ (support $\subseteq \{g_1, g_5\}$), or $\nu_2 + \nu_3 = m$, giving $(\nu_0, \nu_1, \nu_2, \nu_3) = (1, 0, m, 0)$ (support $\subseteq \{g_0, g_2, g_4, g_6\}$).

Thus each of the $m - 1$ strictly increasing elements $b_1(m) < \cdots < b_{m-1}(m)$ has $c(b_k(m)) \geq 2$. By (1), $\rho(H_8(m)) \geq m - 1$, proving part (3) of Theorem 1.1.

Finally, for the inductive gluing $H_{e+1}(m) = 2H_e(m) + \mathbb{N}d_e(m)$ ($e \geq 8$): at $e = 8$, $d_8(m) = g_1 + g_2 = 2a_m + 2m + 1$ is odd and a sum of two minimal generators of $H_8(m)$. Inductively, if $\mathrm{msg}(H_e(m))$ has size e and $d_e(m)$ is odd and a sum of two minimal generators of $H_e(m)$, then $d_e(m) \notin 2H_e(m)$ by parity; and if $2x = 2s + kd_e(m)$ for some $x \in \mathrm{msg}(H_e(m))$, $s \in H_e(m)$, and $k \geq 1$, then parity forces $k = 2j \geq 2$, giving $x = s + jd_e(m)$, impossible because $d_e(m)$ is a sum of two minimal generators of $H_e(m)$. Hence $\mathrm{msg}(H_{e+1}(m)) = (2 \cdot \mathrm{msg}(H_e(m))) \cup \{d_e(m)\}$ has exact cardinality $e + 1$, and $d_{e+1}(m) = 2\min(H_e(m) \setminus \{0\}) + d_e(m)$ is again odd and a sum of two minimal generators of $H_{e+1}(m)$. By the gluing theorem [9, 21, 22], each step preserves symmetry and satisfies $\rho(H_{e+1}(m)) = \rho(H_e(m)) + 1$. This completes the proof of Theorem 1.1. □

4. Simultaneous Divergence of All Interior Betti Numbers for $e \geq 8$

Fix an arbitrary field $\Bbbk$ and an integer $m \geq 2$. Write $\mathcal{C} := \{(L, L_1, c_1, c_2) \in \mathbb{N}^4 : 0 \leq L_1, c_1, c_2 \leq L\}$, define the coordinate-sum map $\sigma : \mathbb{N}^8 \to \mathcal{C}$ by

$$\sigma(z_0, \ldots, z_7) := \Big(\sum_{i=0}^{7} z_i,\ \sum_{i=0}^{7} \epsilon_3(i)z_i,\ \sum_{i=0}^{7} \epsilon_1(i)z_i,\ \sum_{i=0}^{7} \epsilon_2(i)z_i\Big),$$

and set $\Psi(L, L_1, c_1, c_2) := La_m + L_1D_m + c_1m + c_2(m + 1)$, so $\sum_{i=0}^{7} z_i g_i = \Psi(\sigma(z))$.

Lemma 4.1 (Cone-fiber criterion for Δ_h)**.** *The map $\sigma : \mathbb{N}^8 \to \mathcal{C}$ is surjective. Consequently, indexing the divisor complex Δ_h on $\{0, \ldots, 7\}$, for every $h \in H_8(m)$ and every subset $I \subseteq \{0, \ldots, 7\}$, writing $\chi_I \in \{0, 1\}^8$ for the indicator vector of I and $z_I := \sigma(\chi_I) = (|I|, L_1(I), c_1(I), c_2(I))$, we have*

$$I \in \Delta_h \iff \exists\, p \in \Psi^{-1}(h) \cap \mathcal{C} \text{ such that } p - z_I \in \mathcal{C}.$$

Proof. Given $p = (L, L_1, c_1, c_2) \in \mathcal{C}$, define the L Boolean-cube vertices $v^{(j)} := (\mathbf{1}[j > L - c_1], \mathbf{1}[j > L - c_2], \mathbf{1}[j > L - L_1]) \in \{0,1\}^3$ $(1 \le j \le L)$, where $\mathbf{1}[\cdot] \in \{0,1\}$ is the predicate indicator, and let $e_v \in \mathbb{N}^8$ denote the standard basis vector indexed by $v \in \{0,1\}^3$. Their coordinate sums equal (c_1, c_2, L_1), so $\sigma(\sum_{j=1}^{L} e_{v^{(j)}}) = p$, proving surjectivity. For the equivalence: if $h - \sum_{j\in I} g_j = \sum_{i=0}^{7} w_i g_i$ with $w \in \mathbb{N}^8$, then $p := \sigma(w + \chi_I) \in \Psi^{-1}(h) \cap \mathcal{C}$ satisfies $p - z_I = \sigma(w) \in \mathcal{C}$. Conversely, if $p \in \Psi^{-1}(h) \cap \mathcal{C}$ and $p - z_I \in \mathcal{C}$, surjectivity gives $w \in \mathbb{N}^8$ with $\sigma(w) = p - z_I$, so $\sum w_i g_i = \Psi(p - z_I) = h - \sum_{j\in I} g_j \in H_8(m)$. □

We now construct two explicit families of degrees in $H_8(m)$ whose divisor complexes have nonvanishing $\widetilde{H}_1$ and $\widetilde{H}_2$ (Figure 4).

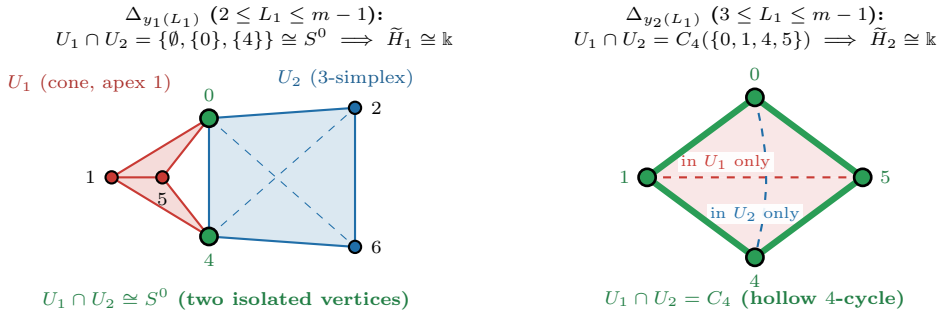

FIGURE 4. Mayer–Vietoris mechanism proving simultaneous divergence of $b_2(\Bbbk[H_8(m)])$ and $b_3(\Bbbk[H_8(m)])$. Both subcomplexes U_1 (cone with apex 1) and U_2 (cone with apex 0) are contractible. **Left:** For each $y_1(L_1)$ $(2 \le L_1 \le m-1)$, the intersection $U_1 \cap U_2$ consists of the two isolated vertices $\{0\}$ and $\{4\}$ (S^0), creating a 1-dimensional loop and yielding $\widetilde{H}_1(\Delta_{y_1(L_1)}; \Bbbk) \cong \Bbbk$. **Right:** For each $y_2(L_1)$ $(3 \le L_1 \le m-1)$, diagonal $\{1,5\}$ belongs only to U_1 (since $|I \cap \{1,3,5,7\}| \le 1$ in U_2) and diagonal $\{0,4\}$ belongs only to U_2 (since $|I \cap \{0,4\}| \le 1$ in U_1), leaving the hollow 4-cycle C_4 as $U_1 \cap U_2$, which creates a 2-dimensional cavity and yields $\widetilde{H}_2(\Delta_{y_2(L_1)}; \Bbbk) \cong \Bbbk$.

Lemma 4.2 (Lower bound $b_2(\Bbbk[H_8(m)]) \ge m-2$)**.** *For every $m \ge 3$ and every integer $L_1 \in \{2, 3, \ldots, m-1\}$, the element*

$$y_1(L_1) := (m+2)a_m + L_1 D_m + m(m+1) \in H_8(m)$$

satisfies $\beta_{2,y_1(L_1)}(\Bbbk[H_8(m)]) = 1$. Hence $b_2(\Bbbk[H_8(m)]) \ge m-2$ for all $m \ge 2$.

Proof. For $m = 2$ the bound $b_2 \ge 0$ is trivial; assume $m \ge 3$ and fix $2 \le L_1 \le m-1$. Let $p = (L', L_1', c_1', c_2') \in \Psi^{-1}(y_1(L_1)) \cap \mathcal{C}$ and $u' := c_1' m + c_2'(m+1) \in [0, L'(2m+1)]$. Since $L_1 D_m + m(m+1) \le (m-1)D_m + m(m+1) < mD_m + m(m+1) = a_m$, we have $y_1(L_1) < (m+3)a_m$, forcing $L' \le m+2$. Conversely,

$$y_1(L_1) - (m+1)g_7 = (L_1 - 1)D_m - (m+1) \ge D_m - (m+1) = 2(m+1)^2 - (m+1) > 0,$$

forcing $L' \ge m+2$, so $L' = m+2$. Then $L_1' D_m + u' = L_1 D_m + m(m+1)$. If $L_1' \ge L_1 + 1$, then $u' \le m(m+1) - D_m < 0$; if $L_1' \le L_1 - 1$, then $u' \ge D_m + m(m+1) = 3m^2 + 5m + 2 > 2m^2 + 5m + 2 = (m+2)(2m+1)$, impossible. Hence $L_1' = L_1$ and $c_1' m + c_2'(m+1) = m(m+1)$. Reducing modulo $m+1$ gives $(m+1) \mid c_1' \in [0, m+2]$, so $c_1' \in \{0, m+1\}$, yielding exactly two points in $\Psi^{-1}(y_1(L_1)) \cap \mathcal{C}$:

$$p_0 = (m+2,\ L_1,\ 0,\ m), \qquad p_1 = (m+2,\ L_1,\ m+1,\ 0).$$

By Lemma 4.1, $\Delta_{y_1(L_1)} = U_1 \cup U_2$, where:

- $p_0 - z_I \in \mathcal{C}$ requires $c_1(I) = 0$, i.e., $I \subseteq \{0,2,4,6\}$. Conversely, every such I has $|I| \le 4 < m+2$, $L_1(I) \le 2 \le L_1$, $|I| - L_1(I) \le 2 < 3 \le (m+2) - L_1$, $c_2(I) \le 2 < m$, and $|I| - c_2(I) \le 2 = (m+2) - m$, so $p_0 - z_I \in \mathcal{C}$. Thus $U_2 = 2^{\{0,2,4,6\}}$ (a 3-simplex).
- $p_1 - z_I \in \mathcal{C}$ requires $c_2(I) = 0$ $(I \subseteq \{0,1,4,5\})$ and $(m+1) - c_1(I) \le (m+2) - |I|$, i.e., $|I \cap \{0,4\}| \le 1$, so $I \subseteq \{0,1,5\}$ or $I \subseteq \{1,4,5\}$. Every such I has $|I| \le 3$, $L_1(I) \le 2 \le L_1$, $|I| - L_1(I) \le 2 < (m+2) - L_1$, and $c_1(I) \le 2 < m+1$, so $U_1 = 2^{\{0,1,5\}} \cup 2^{\{1,4,5\}}$.

Both U_1 (a cone with apex 1) and U_2 (a cone with apex 0) are contractible, while $U_1 \cap U_2 = \{\emptyset, \{0\}, \{4\}\} \cong S^0$. The reduced Mayer–Vietoris exact sequence for $\Delta_{y_1(L_1)} = U_1 \cup U_2$ gives

$$\widetilde{H}_1(\Delta_{y_1(L_1)}; \Bbbk) \cong \widetilde{H}_0(U_1 \cap U_2; \Bbbk) \cong \Bbbk.$$

By (2), $\beta_{2,y_1(L_1)}(\Bbbk[H_8(m)]) = 1$. Summing over the $m-2$ strictly increasing elements $y_1(2) < \cdots < y_1(m-1)$ gives $b_2(\Bbbk[H_8(m)]) \geq m-2$. □

Lemma 4.3 (Lower bound $b_3(\Bbbk[H_8(m)]) \geq m-3$)**.** *For every $m \geq 4$ and every integer $L_1 \in \{3, 4, \ldots, m-1\}$, the element*

$$y_2(L_1) := y_1(L_1) + g_1 = (m+3)a_m + L_1 D_m + m(m+2) \in H_8(m)$$

satisfies $\beta_{3,y_2(L_1)}(\Bbbk[H_8(m)]) = 1$. Hence $b_3(\Bbbk[H_8(m)]) \geq m-3$ for all $m \geq 2$.

Proof. For $m \in \{2, 3\}$ the bound $b_3 \geq m-3$ is trivial; assume $m \geq 4$ and fix $3 \leq L_1 \leq m-1$. Let $p = (L', L_1', c_1', c_2') \in \Psi^{-1}(y_2(L_1)) \cap \mathcal{C}$ and $u' := c_1' m + c_2'(m+1)$. Since $L_1 D_m + m(m+2) \leq (m-1)D_m + m(m+2) = a_m - (2m^2+3m+2) < a_m$, we have $L' \leq m+3$. Conversely,

$$y_2(L_1) - (m+2)g_7 = (L_1 - 2)D_m - 2(m+1) \geq D_m - 2(m+1) = 2m(m+1) > 0,$$

forcing $L' = m+3$. Then $L_1' D_m + u' = L_1 D_m + m(m+2)$. If $L_1' \geq L_1 + 1$, then $u' \leq m(m+2) - D_m < 0$; if $L_1' \leq L_1 - 1$, then $u' \geq D_m + m(m+2) = 3m^2 + 6m + 2 > 2m^2 + 7m + 3 = (m+3)(2m+1)$ (difference $m^2 - m - 1 \geq 11 > 0$ for $m \geq 4$), impossible. Hence $L_1' = L_1$ and $c_1' m + c_2'(m+1) = m + m(m+1)$. Reducing modulo $m+1$ gives $c_1' \equiv 1 \pmod{m+1}$ with $0 \leq c_1' \leq m+3$, yielding exactly two points in $\Psi^{-1}(y_2(L_1)) \cap \mathcal{C}$:

$$q_0 = (m+3,\ L_1,\ 1,\ m), \qquad q_1 = (m+3,\ L_1,\ m+2,\ 0).$$

By Lemma 4.1, $\Delta_{y_2(L_1)} = U_1 \cup U_2$, where:

- $q_0 - z_I \in \mathcal{C}$ requires $c_1(I) \leq 1$, i.e., $|I \cap \{1,3,5,7\}| \leq 1$, so $I \subseteq \{0,2,4,6\} \cup \{j\}$ for some $j \in \{1,3,5,7\}$. Every such I satisfies $|I| \leq 5 < m+3$, $L_1(I) \leq 3 \leq L_1$, $|I| - L_1(I) \leq 3 < 4 \leq (m+3) - L_1$, $c_1(I) \leq 1$, $|I| - c_1(I) \leq 4 < m+2$, $c_2(I) \leq 3 < m$ (as $m \geq 4$), and $|I| - c_2(I) \leq 3 = (m+3) - m$. Thus $U_2 = \bigcup_{j \in \{1,3,5,7\}} 2^{\{0,2,4,6,j\}}$.
- $q_1 - z_I \in \mathcal{C}$ requires $c_2(I) = 0$ ($I \subseteq \{0,1,4,5\}$) and $|I \cap \{0,4\}| \leq 1$, with all remaining bounds automatic for $m \geq 4$ and $3 \leq L_1 \leq m-1$. Thus $U_1 = 2^{\{0,1,5\}} \cup 2^{\{1,4,5\}}$.

Again U_1 (cone with apex 1) and U_2 (cone with apex 0) are contractible. A subset I lies in $U_1 \cap U_2$ iff $I \subseteq \{0,1,4,5\}$, $|I \cap \{0,4\}| \leq 1$, and $|I \cap \{1,5\}| \leq 1$—precisely the 4-cycle C_4 with vertices $\{0,1,4,5\}$ and edges $\{0,1\}, \{1,4\}, \{4,5\}, \{5,0\}$. By reduced Mayer–Vietoris,

$$\widetilde{H}_2(\Delta_{y_2(L_1)}; \Bbbk) \cong \widetilde{H}_1(C_4; \Bbbk) \cong \Bbbk,$$

so $\beta_{3,y_2(L_1)}(\Bbbk[H_8(m)]) = 1$ by (2). Summing over $y_2(3) < \cdots < y_2(m-1)$ gives $b_3(\Bbbk[H_8(m)]) \geq m-3$. □

Proof of Theorem 1.2. By Theorem 1.1 and Kunz's theorem [15], $\Bbbk[H_8(m)]$ is a 1-dimensional Gorenstein domain of codimension 7 over $\Bbbk$, so $b_0 = b_7 = 1$ and $b_i = b_{7-i}$ for $0 \leq i \leq 7$. Together with Theorem 1.1(3) ($b_1 = \rho \geq m-1$) and Lemmas 4.2 and 4.3, this establishes (1).

For (2), at each gluing step $H_{e+1}(m) = 2H_e(m) + \mathbb{N}d_e(m)$, let $M_e \in R = \Bbbk[x_0, \ldots, x_{e-1}]$ be a monomial of degree $d_e(m)$ in $H_e(m)$ (so $\deg(M_e) \geq 2$ in the standard polynomial grading, as $d_e(m) \notin \{0\} \cup \mathrm{msg}(H_e(m))$). By Section 2, the defining ideal of $\Bbbk[H_{e+1}(m)]$ in $R' = R[z]$ is $I_{H_{e+1}(m)} = I_{H_e(m)}R' + (z^2 - M_e)$. Since $\Bbbk[H_e(m)][z] \cong R'/I_{H_e(m)}R'$ is a domain and $z^2 - M_e$ is monic in z, $z^2 - M_e$ is a nonzerodivisor on $\Bbbk[H_e(m)][z]$.

Therefore, if $\mathbb{F}_\bullet$ is a minimal graded free R-resolution of $\Bbbk[H_e(m)]$, the tensor product $\mathbb{F}_\bullet \otimes_R (0 \to R' \xrightarrow{z^2 - M_e} R' \to 0)$ is a graded free R'-resolution of $\Bbbk[H_{e+1}(m)]$ (see also [11]). It is minimal because all differential entries of $\mathbb{F}_\bullet$ lie in $\mathfrak{m}_R \subset \mathfrak{m}_{R'}$ and $z^2 - M_e \in \mathfrak{m}_{R'}$. Hence $b_i(\Bbbk[H_{e+1}(m)]) = b_i(\Bbbk[H_e(m)]) + b_{i-1}(\Bbbk[H_e(m)])$, and induction on $e - 8 \geq 0$ gives the binomial convolution formula in (2).

Finally, given $e \geq 8$ and $1 \leq i \leq e-2$, the integer interval $[i-(e-8), i]$ has left endpoint $i - e + 8 \leq 6$ and right endpoint $i \geq 1$, so it contains some $k = i - j_0 \in \{1,2,3,4,5,6\}$ with $0 \leq j_0 \leq e-8$. By part (1) and Gorenstein symmetry, $b_k(\Bbbk[H_8(m)]) \geq m-3$, so $b_i(\Bbbk[H_e(m)]) \geq \binom{e-8}{j_0} b_k(\Bbbk[H_8(m)]) \geq m-3 \to \infty$, while $b_0 = b_{e-1} = 1$. Since every symmetric numerical semigroup is almost symmetric and nearly Gorenstein, this refutes both Stamate's Problem 22 and the Moscariello–Sammartano Uniform Boundedness Problem 24 in every interior degree $1 \leq i \leq e-2$ for all $e \geq 8$. □

5. Quadratic Uniform Boundedness in Embedding Dimension 5

Throughout Sections 5 and 6, let $S = \langle n_1, n_2, n_3, n_4, n_5\rangle$ be any numerical semigroup of embedding dimension 5 and Cohen–Macaulay type $t = t(S) = |\operatorname{PF}(S)| \geq 1$.

5.1. **Pure-5 and non-redundant support-partition accounting.** For each $i \in \{1,\dots,5\}$, let $\alpha_i := \min\{k \in \mathbb{Z}_{\geq 1} : kn_i \in \langle n_j : j \neq i\rangle\}$ and put $P := \{\alpha_1 n_1, \dots, \alpha_5 n_5\} \subseteq \operatorname{Betti}(S)$. Let $\mathcal{P}_{2,3}$ be the set of 10 unordered $(2,3)$ partitions $\sigma = (A, A^c)$ of $\{1,\dots,5\}$ ($|A| = 2$), and let $\mathcal{P}_{2,2}$ be the set of 15 unordered $(2,2)$ partitions $\pi = (A, B)$ of 4-element subsets of $\{1,\dots,5\}$ ($|A| = |B| = 2$, $A \cap B = \emptyset$).

For each $\sigma = (A, A^c) \in \mathcal{P}_{2,3}$, let $\mathcal{N}_\sigma \subseteq \operatorname{Betti}(S)$ be the set of Betti elements admitting factorizations of full support A and full support A^c in distinct connected components of ∇_h. For each $\pi = (A, B) \in \mathcal{P}_{2,2}$ omitting $m_\pi \in \{1,\dots,5\} \setminus (A \cup B)$, let $\mathcal{M}_\pi \subseteq \operatorname{Betti}(S)$ be the set of Betti elements admitting factorizations of full support A and full support B in distinct components of ∇_h, and define the *non-redundant* $(2,2)$ Betti set

$$\mathcal{M}^\circ_{(A,B)} := \mathcal{M}_{(A,B)} \setminus \big(\mathcal{N}_{(A,\, B\cup\{m_\pi\})} \cup \mathcal{N}_{(B,\, A\cup\{m_\pi\})}\big).$$

Lemma 5.1 (Pure-5 Betti accounting)**.** *For every 5-generated numerical semigroup S,*

$$\rho(S) \leq 5 + \sum_{\pi\in\mathcal{P}_{2,2}} |\mathcal{M}^\circ_\pi| + \sum_{\sigma\in\mathcal{P}_{2,3}} |\mathcal{N}_\sigma|.$$

Proof. Consider any $h \in \operatorname{Betti}(S)$.

- If $h \in P$, let $s(h) \geq 1$ be the number of components of ∇_h whose union of supports is a singleton $\{i\}$ (each $i \in \{1,\dots,5\}$ forms a singleton component of ∇_h iff $h = \alpha_i n_i$ by minimality of α_i, so $\sum_{h\in P} s(h) = 5$), and let $p(h) \geq 0$ be the number of components of ∇_h whose support union has size at least 2. Since distinct components have disjoint support unions inside $\{1,\dots,5\}$, we have $s(h) + 2p(h) \leq 5$. If $p(h) \leq 1$, then $c(h) - 1 = s(h) + p(h) - 1 \leq s(h)$. If $p(h) = 2$, then $s(h) = 1$, $c(h) - 1 = 2 = s(h) + 1$, and up to relabeling the three components have support unions $\{5\}$, $\{1,2\}$, and $\{3,4\}$; hence h admits no full-support $\{1,2,5\}$- or $\{3,4,5\}$-factorization (which would connect $\{5\}$ to $\{1,2\}$ or $\{3,4\}$), so $h \in \mathcal{M}^\circ_{(\{1,2\},\{3,4\})}$.
- If $h \notin P$, every component C of ∇_h contains a factorization of support size at least 2: if $ae_i \in C$, then $h \notin P$ and $c(h) \geq 2$ force $a > \alpha_i$, so replacing $\alpha_i n_i$ by a factorization outside $\{i\}$ yields a factorization in C of support size ≥ 2. Since $2c(h) \leq 5$, we have $c(h) = 2$, $c(h) - 1 = 1$, and the two components have support-union sizes either $(2,2)$ or $(2,3)$: if h admits full-support $(2,3)$ factorizations in its two components then $h \in \mathcal{N}_\sigma$ for some σ; otherwise each component contains a support-size-2 factorization (if a component has support union $\{3,4,5\}$ with no full-support $\{3,4,5\}$-factorization, it still contains a support-size-2 factorization), placing $h \in \mathcal{M}^\circ_\pi$ for some π.

Summing $c(h) - 1$ over all $h \in \operatorname{Betti}(S)$ via (1) yields the stated inequality. □

5.2. **Shared mixed-union order machinery.** Without loss of generality, fix an ordered $(2,2)$ partition $\pi = (\{1,2\},\{3,4\})$ (whose second block combines with the omitted generator 5 to form the adjacent $(2,3)$ partition $\sigma = (\{1,2\},\{3,4,5\})$), and define the *mixed disjoint union*

$$\mathcal{U}_{(\{1,2\},\{3,4\})} := \mathcal{M}^\circ_{(\{1,2\},\{3,4\})} \sqcup \mathcal{N}_{(\{1,2\},\{3,4,5\})}$$

(disjoint by definition of $\mathcal{M}^\circ$). For each $h \in \mathcal{U}_{(\{1,2\},\{3,4\})}$, let $h = a_h n_1 + b_h n_2$ ($a_h, b_h \geq 1$) be the unique factorization of h with $a_h \geq 1$ *minimal among all support-*$\{1,2\}$ *factorizations of* h, and choose any factorization $h = c_h n_3 + d_h n_4 + e_h n_5$ in a distinct component of ∇_h with $c_h, d_h \geq 1$ and $e_h = 0$ if $h \in \mathcal{M}^\circ_\pi$, $e_h \geq 1$ if $h \in \mathcal{N}_\sigma$.

Notice that $\{(a_h, b_h) : h \in \mathcal{U}_\pi\}$ is a strict antichain in $\mathbb{Z}^2_{\geq 1}$: if $a_h \leq a_{h'}$ and $b_h \leq b_{h'}$ for distinct $h, h' \in \mathcal{U}_\pi$, then $h' = (a_{h'} - a_h)n_1 + (b_{h'} - b_h)n_2 + c_h n_3 + d_h n_4 + e_h n_5$ connects the $\{1,2\}$-component of $\nabla_{h'}$ to $\{3,4\}$, contradiction (regardless of whether h, h' lie in $\mathcal{M}^\circ_\pi$ or $\mathcal{N}_\sigma$).

Furthermore, for every $h \in \mathcal{U}_\pi$ and every cross-pair $(\alpha,\beta) \in \{1,2\} \times \{3,4\}$, we have $h - n_\alpha - n_\beta \notin S$ (else adding $n_\alpha + n_\beta$ bridges the two components of ∇_h). Hence there exist four pseudo-Frobenius numbers $f_u(h), f_p(h), f_q(h), f_v(h) \in \operatorname{PF}(S)$ (not necessarily distinct) such that

$$f_u(h) - h + n_1 + n_3 \in S, \quad f_p(h) - h + n_1 + n_4 \in S, \quad f_q(h) - h + n_2 + n_3 \in S, \quad f_v(h) - h + n_2 + n_4 \in S.$$

Suppose $h_1, \dots, h_N \in \mathcal{U}_\pi$ are N distinct elements ordered strictly by decreasing a:

$$a_1 > a_2 > \cdots > a_N \geq 1, \qquad 1 \leq b_1 < b_2 < \cdots < b_N$$

(with *no* ordering assumption on c_k, d_k, e_k). Whenever $h_1, \dots, h_N$ share a common $f_u \in \mathrm{PF}(S)$, pick any factorization $u^{(k)} \in Z(f_u - h_k + n_1 + n_3)$ for each k; since $h_k - n_1, h_k - n_3 \in S$ and $f_u \notin S$, we have $u_1^{(k)} = u_3^{(k)} = 0$. Write $U_k := u_5^{(k)} \geq 0$, giving two exact representations of f_u:

$$f_u^{\mathrm{A}}(k) = (a_k - 1)n_1 + (b_k + u_2^{(k)})n_2 - n_3 + u_4^{(k)}n_4 + U_k n_5,$$
$$f_u^{\mathrm{B}}(k) = -n_1 + u_2^{(k)}n_2 + (c_k - 1)n_3 + (d_k + u_4^{(k)})n_4 + (e_k + U_k)n_5.$$

Similarly, whenever $f_p, f_q, f_v \in \mathrm{PF}(S)$ are shared on $h_1, \dots, h_N$, pick factorizations $p^{(k)} \in Z(f_p - h_k + n_1 + n_4)$ ($p_1^{(k)} = p_4^{(k)} = 0$, $P_k := p_5^{(k)}$), $q^{(k)} \in Z(f_q - h_k + n_2 + n_3)$ ($q_2^{(k)} = q_3^{(k)} = 0$, $Q_k := q_5^{(k)}$), and $v^{(k)} \in Z(f_v - h_k + n_2 + n_4)$ ($v_2^{(k)} = v_4^{(k)} = 0$, $V_k := v_5^{(k)}$), with analogous A- and B-forms.

Lemma 5.2 (Ordering-free step bounds)**.** *For every $1 \leq i < j \leq N$, whenever the corresponding pseudo-Frobenius element is shared on $h_1, \dots, h_N$, we have*

$$u_2^{(i)},\ p_2^{(i)} < b_j - b_i \qquad \text{and} \qquad q_1^{(j)},\ v_1^{(j)} < a_i - a_j.$$

Proof. Set $\Delta_{ij} := (a_i - a_j)n_1 - (b_j - b_i)n_2 - (c_i - c_j)n_3 - (d_i - d_j)n_4 - (e_i - e_j)n_5 = (h_i - h_j) - (h_i - h_j) = 0$. Adding Δ_{ij} to $f_u^{\mathrm{B}}(i)$ cancels c_i, d_i, e_i identically (regardless of whether e_i, e_j are 0 or ≥ 1):

$$f_u = (a_i - a_j - 1)n_1 + \big(u_2^{(i)} - (b_j - b_i)\big)n_2 + (c_j - 1)n_3 + (d_j + u_4^{(i)})n_4 + (e_j + U_i)n_5 \notin S.$$

Since $a_i > a_j$, $c_j, d_j \geq 1$, and $e_j \geq 0$, every coefficient except that of n_2 is nonnegative, forcing $u_2^{(i)} < b_j - b_i$. The bound $p_2^{(i)} < b_j - b_i$ follows identically from $f_p^{\mathrm{B}}(i) + \Delta_{ij} \notin S$, and $q_1^{(j)}, v_1^{(j)} < a_i - a_j$ follow symmetrically by subtracting Δ_{ij} from $f_q^{\mathrm{B}}(j)$ and $f_v^{\mathrm{B}}(j)$. □

Lemma 5.3 (Strict antitonicity on $\mathcal{U}_\pi$)**.**

(1) For every $1 \leq i < j \leq N - 1$, if f_u is shared then $U_i \neq U_j$ and $U_i < U_j \iff u_4^{(i)} > u_4^{(j)}$; if f_p is shared then $P_i \neq P_j$ and $P_i < P_j \iff p_3^{(i)} > p_3^{(j)}$.

(2) For every $2 \leq j < k \leq N$, if f_q is shared then $Q_j \neq Q_k$ and $Q_j < Q_k \iff q_4^{(j)} > q_4^{(k)}$; if f_v is shared then $V_j \neq V_k$ and $V_j < V_k \iff v_3^{(j)} > v_3^{(k)}$.

Proof. For (1), fix $1 \leq i < j \leq N - 1$ (so $j + 1 \leq N$ exists). Equating $f_u^{\mathrm{A}}(i) = f_u^{\mathrm{A}}(j)$ cancels f_u and gives

$$(a_i - a_j)n_1 + (u_4^{(i)} - u_4^{(j)})n_4 + (U_i - U_j)n_5 = Mn_2, \qquad M := (b_j + u_2^{(j)}) - (b_i + u_2^{(i)}).$$

By Lemma 5.2 on (i, j) and $(j, j + 1)$, we have $1 \leq M < b_{j+1}$, while $1 \leq a_i - a_j < a_i$.

- If $u_4^{(i)} \geq u_4^{(j)}$ and $U_i \geq U_j$ with at least one strict, substituting Mn_2 into $h_{j+1} = a_{j+1}n_1 + b_{j+1}n_2$ produces $h_{j+1} = (a_{j+1} + a_i - a_j)n_1 + (b_{j+1} - M)n_2 + (u_4^{(i)} - u_4^{(j)})n_4 + (U_i - U_j)n_5$. If $h_{j+1} \in \mathcal{N}_\sigma$, this bridges $\{1, 2\}$ to $\{3, 4, 5\}$; if $h_{j+1} \in \mathcal{M}_\pi^\circ$, it either bridges $\{1, 2\}$ to n_4 (when $u_4^{(i)} > u_4^{(j)}$) or yields a full-support $\{1, 2, 5\}$-factorization of h_{j+1} (contradicting $h_{j+1} \in \mathcal{M}_\pi^\circ$, including the three-component case $h_{j+1} = \alpha_5 n_5 \in P$).
- Symmetrically, if $u_4^{(i)} \leq u_4^{(j)}$ and $U_i \leq U_j$ with at least one strict, substituting $(a_i - a_j)n_1$ into h_i gives the same contradiction at h_i.
- If $u_4^{(i)} = u_4^{(j)}$ and $U_i = U_j$, then $(a_i - a_j)n_1 = Mn_2$ gives $h_i = a_j n_1 + (b_i + M)n_2$, a support-$\{1, 2\}$ factorization of h_i with $1 \leq a_j < a_i$, contradicting the minimality of a_i.

The proof for p is identical ($n_3 \leftrightarrow n_4$). For part (2), equating $f_q^{\mathrm{A}}(j) = f_q^{\mathrm{A}}(k)$ ($2 \leq j < k \leq N$) yields

$$An_1 + (q_4^{(j)} - q_4^{(k)})n_4 + (Q_j - Q_k)n_5 = (b_k - b_j)n_2, \qquad A := (a_j + q_1^{(j)}) - (a_k + q_1^{(k)}),$$

with $1 \leq A < a_{j-1}$ (using $j - 1 \geq 1$ and Lemma 5.2 on (j, k) and $(j - 1, j)$) and $1 \leq b_k - b_j < b_k$. The two same-sign strict cases substitute $(b_k - b_j)n_2$ into h_k or An_1 into h_{j-1} just as above; in the equality case ($q_4^{(j)} = q_4^{(k)}$ and $Q_j = Q_k$), substituting $An_1 = (b_k - b_j)n_2$ into h_{j-1} gives $h_{j-1} = (a_{j-1} - A)n_1 + (b_{j-1} + b_k - b_j)n_2$ with $1 \leq a_{j-1} - A < a_{j-1}$, contradicting the minimality of a_{j-1} among support-$\{1, 2\}$ factorizations of h_{j-1}. The proof for v is identical ($n_3 \leftrightarrow n_4$). □

Lemma 5.4 (Four pure A-form order rules)**.** *Whenever the relevant pseudo-Frobenius elements are shared on $h_1, \dots, h_N \in \mathcal{U}_\pi$, the following strict inequalities hold:*

(A-Bwd) *For all* $1 \le r \le i < j \le N-1$*:* $U_i > U_j \implies P_r, V_r < U_i$*, and* $P_i > P_j \implies U_r, Q_r < P_i$.
(A-Fwd) *For all* $1 \le i < j < k \le N$ *(with* $j \le N-1$*):* $U_i < U_j \implies P_k, V_k < U_j$*, and* $P_i < P_j \implies U_k, Q_k < P_j$.
(A-Rev-Fwd) *For all* $2 \le j < k \le r \le N$*:* $Q_j < Q_k \implies V_r, P_r < Q_k$*, and* $V_j < V_k \implies Q_r, U_r < V_k$.
(A-Rev-Bwd) *For all* $1 \le r < j < k \le N$ *(with* $j \ge 2$*):* $Q_j > Q_k \implies V_r, P_r < Q_j$*, and* $V_j > V_k \implies Q_r, U_r < V_j$.

Proof. Each implication follows from a single three-term A-form identity of the shape $f_X^{\mathrm{A}}(r)+f_Y^{\mathrm{A}}(u)-f_Y^{\mathrm{A}}(v) = f_X \notin S$ (with $u \neq v$), in which f_Y cancels identically (so no cross-f equality is needed). We write out two representative identities; all others are obtained by $n_3 \leftrightarrow n_4$ or $(n_1, a) \leftrightarrow (n_2, b)$ symmetry.

- For (A-Bwd) ($P_i > P_j \implies U_r < P_i$, $1 \le r \le i < j \le N-1$):

$$f_u^{\mathrm{A}}(r) + f_p^{\mathrm{A}}(j) - f_p^{\mathrm{A}}(i) = \big((a_r - a_i) + (a_j - 1)\big)n_1 + \big(b_r + u_2^{(r)} + p_2^{(j)} + (b_j - b_i - p_2^{(i)})\big)n_2 + (p_3^{(j)} - p_3^{(i)} - 1)n_3 + u_4^{(r)}n_4 + (U_r + P_j - P_i)n_5 = f_u \notin S.$$

  By $r \le i$, Lemma 5.2 ($b_j - b_i > p_2^{(i)}$), and Lemma 5.3(1) ($P_i > P_j \implies p_3^{(j)} > p_3^{(i)}$), the first four coordinates are nonnegative, forcing $U_r + P_j < P_i$ and hence $U_r < P_i$. Replacing $f_u^{\mathrm{A}}(r)$ by $f_q^{\mathrm{A}}(r)$ gives $(a_r - a_i + a_j + q_1^{(r)})n_1 + (b_r - 1 + p_2^{(j)} + b_j - b_i - p_2^{(i)})n_2 + (p_3^{(j)} - p_3^{(i)} - 1)n_3 + q_4^{(r)}n_4 + (Q_r + P_j - P_i)n_5 = f_q \notin S$, forcing $Q_r < P_i$.
- For (A-Fwd) ($P_i < P_j \implies U_k < P_j$, $1 \le i < j < k \le N$):

$$f_u^{\mathrm{A}}(k) + f_p^{\mathrm{A}}(i) - f_p^{\mathrm{A}}(j) = \big((a_i - a_j) + (a_k - 1)\big)n_1 + \big(b_i + u_2^{(k)} + p_2^{(i)} + (b_k - b_j - p_2^{(j)})\big)n_2 + (p_3^{(i)} - p_3^{(j)} - 1)n_3 + u_4^{(k)}n_4 + (U_k + P_i - P_j)n_5 = f_u \notin S.$$

  By $i < j$, Lemma 5.2 ($b_k - b_j > p_2^{(j)}$), and Lemma 5.3(1) ($P_i < P_j \implies p_3^{(i)} > p_3^{(j)}$), the first four coordinates are nonnegative, forcing $U_k + P_i < P_j$ and hence $U_k < P_j$; replacing $f_u^{\mathrm{A}}(k)$ by $f_q^{\mathrm{A}}(k)$ gives $Q_k < P_j$.

For (A-Rev-Fwd) and (A-Rev-Bwd), evaluate $f_X^{\mathrm{A}}(r) + f_q^{\mathrm{A}}(j) - f_q^{\mathrm{A}}(k)$ (using $q_1^{(k)}, v_1^{(k)} < a_j - a_k$) and $f_X^{\mathrm{A}}(r) + f_q^{\mathrm{A}}(k) - f_q^{\mathrm{A}}(j)$ (using $q_1^{(j)}, v_1^{(j)} < a_r - a_j$) for $X \in \{v, p\}$, together with their $n_3 \leftrightarrow n_4$ mirrors, by Lemma 5.2 and Lemma 5.3(2). □

5.3. **Proof of Theorem 1.3 and Corollary 1.4.**

Proposition 5.5 (Mixed-union quadratic bound)**.** *For every* 5*-generated numerical semigroup* S *of type* $t = t(S) \ge 1$ *and every mixed disjoint union* $\mathcal{U}_\pi$*, we have* $|\mathcal{U}_\pi| \le 5\,t^2$.

Proof. Assume for contradiction that $|\mathcal{U}_\pi| \ge 5\,t^2 + 1$. By the pigeonhole principle over the t^2 pairs $(f_u, f_p) \in \mathrm{PF}(S)^2$, there exist $N = 6$ distinct elements $h_1, \dots, h_6 \in \mathcal{U}_\pi$ sharing one fixed pair $(f_u, f_p) \in \mathrm{PF}(S)^2$. Order them by $a_1 > \cdots > a_6 \ge 1$. Lemma 5.4 (A-Bwd) and (A-Fwd) apply on $\{1, 2, 3, 4, 5\}$ and yield two immediate consequences:

- *No co-drop (*$1 \le i < j \le 5$*):* $U_i > U_j$ and $P_i > P_j$ cannot both hold, else (A-Bwd) at $r = i$ gives $P_i < U_i$ and $U_i < P_i$.
- *Rise forces future rise (*$1 \le i < j < k < \ell \le 5$*):* If $U_i < U_j$, then (A-Fwd) on (i, j, k) gives $P_k < U_j$, forcing $P_k < P_\ell$ (else $P_k > P_\ell$ gives $U_j < P_k$ by (A-Bwd) on (j, k, ℓ)). Symmetrically, $P_i < P_j \implies U_k < U_\ell$.

We now derive a contradiction in six lines:

(1) By no co-drop on $(1, 2)$, swapping $n_3 \leftrightarrow n_4$ (which swaps $(f_u, u, U) \leftrightarrow (f_p, p, P)$) if needed, assume without loss of generality that $U_1 < U_2$.
(2) By rise-forces-future-rise on $(1, 2, 3, 4)$ and $(1, 2, 4, 5)$, we have $P_3 < P_4 < P_5$.
(3) By (A-Fwd) on $(1, 2, 4)$ ($U_1 < U_2$) and $(3, 4, 5)$ ($P_3 < P_4$), we obtain $U_5 < P_4 < U_2$, so $U_2 > U_5$.
(4) By (A-Bwd) on $(2, 2, 5)$ ($U_2 > U_5$), we have $P_2 < U_2$, hence $P_2 < P_3$ (else $P_2 > P_3$ gives $U_2 < P_2$ by (A-Bwd) on $(2, 2, 3)$).
(5) By (A-Fwd) on $(2, 3, 4)$ ($P_2 < P_3$), we obtain $U_4 < P_3 < P_4 < P_5$.

(6) Hence $U_1 > U_4$ (else $U_1 < U_4$ gives $P_5 < U_4$ by (A-Fwd) on $(1,4,5)$). By (A-Bwd) on $(1,1,4)$, $P_1 < U_1$, so $P_1 < P_2$ (else (A-Bwd) on $(1,1,2)$ gives $U_1 < P_1$). Finally, rise-forces-future-rise on $(1,2,3,4)$ ($P_1 < P_2$) yields $U_3 < U_4$, whence (A-Fwd) on $(3,4,5)$ gives $P_5 < U_4$, contradicting line (5). □

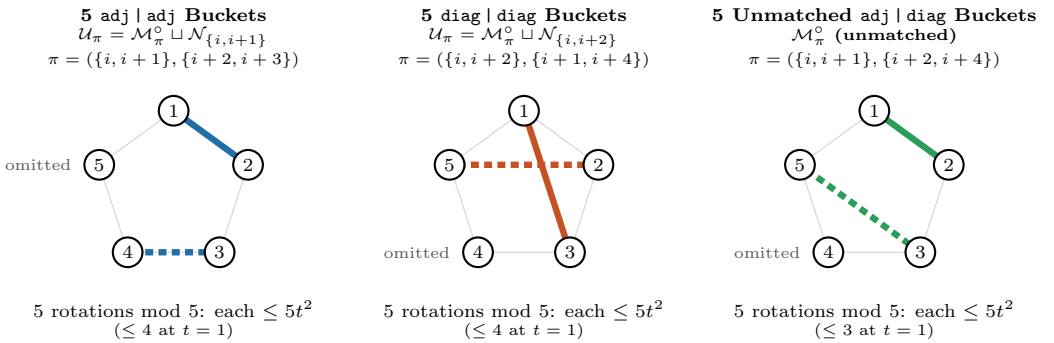

FIGURE 5. The explicit 10-element cyclic matching on $\{1,2,3,4,5\}$ modulo 5 (illustrated at $i = 1$). The ten $(2,3)$ support partitions (indexed by their 2-element component, shown solid) match injectively to ten distinct $(2,2)$ partitions—five `adj | adj` and five `diag | diag`—forming mixed disjoint unions $\mathcal{U} = \mathcal{M}^\circ \sqcup \mathcal{N}$, while the five remaining `adj | diag` partitions stay unmatched as $\mathcal{M}^\circ$.

Proof of Theorem 1.3. Index $\{1,2,3,4,5\}$ cyclically modulo 5 (Figure 5). Match all 10 partitions in $\mathcal{P}_{2,3}$ (indexed by their 2-element block A) injectively to 10 distinct $(2,2)$ partitions $(A,B) \in \mathcal{P}_{2,2}$ with $B \subset A^c$:

- the 5 adjacent pairs: $\{i, i+1\} \mapsto (\{i, i+1\}, \{i+2, i+3\})$ (omitting $i+4$, all 5 distinct);
- the 5 diagonal pairs: $\{i, i+2\} \mapsto (\{i, i+2\}, \{i+1, i+4\})$ (omitting $i+3$, all 5 distinct and disjoint from the adjacent pairs).

The remaining 5 partitions in $\mathcal{P}_{2,2}$ are $(\{i, i+1\}, \{i+2, i+4\})$ (each unmatched, contributing $|\mathcal{M}^\circ_\pi| \leq |\mathcal{U}_\pi|$). Applying Lemma 5.1 and Proposition 5.5 to all 15 buckets gives

$$\rho(S) \leq 5 + \sum_{k=1}^{10} |\mathcal{U}_{\pi_k}| + \sum_{k=11}^{15} |\mathcal{M}^\circ_{\pi_k}| \leq 5 + 15 \cdot 5\, t^2 = 5 + 75\, t^2.$$ □

Proof of Corollary 1.4. (1) By Moscariello [16, Theorem 2 and Remark 17], every almost-symmetric numerical semigroup S with $\operatorname{edim}(S) = 5$ satisfies $t(S) \leq 473$. Theorem 1.3 then yields $\rho(S) \leq 5 + 75 \cdot 473^2 = 16{,}779{,}680$. (2) Immediate from Theorem 1.3. □

6. Computer-Free Symmetric Bound $S(5) \leq 60$

When H is symmetric ($t(H) = 1$), all four complements (u, p, q, v) share the unique Frobenius number $F = F(H)$ (more generally, pigeonholing over $\mathrm{PF}(S)^4$ gives factors of t^4 for arbitrary $t \geq 1$). We sharpen Proposition 5.5's bound 5 to 4 on the 10 matched unions $\mathcal{U}_\pi$ and to 3 on the 5 unmatched sets $\mathcal{M}^\circ_\pi$.

Proposition 6.1 (Mixed-union bound $|\mathcal{U}_\pi| \leq 4\,t^4$)**.** *For every 5-generated numerical semigroup S of type $t \geq 1$ and every mixed disjoint union $\mathcal{U}_\pi$, $|\mathcal{U}_\pi| \leq 4\,t^4$. In particular, $|\mathcal{U}_\pi| \leq 4$ when S is symmetric.*

Proof. Assume $|\mathcal{U}_\pi| \geq 4\,t^4 + 1$. By pigeonhole over $\mathrm{PF}(S)^4$, pick $N = 5$ distinct elements $h_1, \ldots, h_5 \in \mathcal{U}_\pi$ sharing $(f_u, f_p, f_q, f_v) \in \mathrm{PF}(S)^4$, ordered by $a_1 > \cdots > a_5 \geq 1$. Lemma 5.4 holds on $\{1, \ldots, 5\}$ and gives four two-line facts:

- *(F1)* For $1 \leq i < j \leq 4$, $\neg(U_i > U_j \wedge P_i > P_j)$, by (A-Bwd) at $r = i$.
- *(F2)* For $2 \leq j < k \leq 5$, $\neg(Q_j < Q_k \wedge V_j < V_k)$, by (A-Rev-Fwd) at $r = k$.

- *(F3)* For $1 \le i < j < k < \ell \le 5$, $U_i < U_j \implies V_k < V_\ell$: if $U_i < U_j$ and $V_k > V_\ell$, then (A-Fwd) on (i,j,k) gives $V_k < U_j$, while (A-Rev-Bwd) on (j,k,ℓ) gives $U_j < V_k$, contradiction.
- *(F4)* Symmetrically ($n_3 \leftrightarrow n_4$), for $1 \le i < j < k < \ell \le 5$, $P_i < P_j \implies Q_k < Q_\ell$.

By *(F1)* on $(1,2)$, swapping $n_3 \leftrightarrow n_4$ if needed, assume without loss of generality that $U_1 < U_2$.

(1) By *(F3)* on $(1,2,3,4)$ and $(1,2,4,5)$, $V_3 < V_4 < V_5$ (so $v_3^{(3)} > v_3^{(4)} > v_3^{(5)}$ by Lemma 5.3(2)).
(2) By *(F2)* on $(3,4)$ and $(4,5)$, $Q_3 > Q_4 > Q_5$ (so $q_4^{(3)} < q_4^{(4)} < q_4^{(5)}$).
(3) By the contrapositive of *(F4)* on $(1,2,4,5)$ and $(2,3,4,5)$ (since $Q_4 > Q_5$), $P_1 > P_2 > P_3$ (so $p_3^{(1)} < p_3^{(2)} < p_3^{(3)}$).
(4) By (A-Bwd) on $(2,2,3)$ ($P_2 > P_3$), $Q_2 < P_2 < P_1$.
(5) Now evaluate $f_p^{\mathrm{A}}(1) + f_q^{\mathrm{A}}(5) - f_q^{\mathrm{A}}(2) = f_p \notin S$:

$$\big(a_1 - a_2 + a_5 + q_1^{(5)} - q_1^{(2)} - 1\big)n_1 + \big(b_1 + p_2^{(1)} + b_5 - b_2\big)n_2 + p_3^{(1)} n_3 + \big(q_4^{(5)} - q_4^{(2)} - 1\big)n_4 + \big(P_1 + Q_5 - Q_2\big)n_5 \notin S.$$

Here $a_1 - a_2 > q_1^{(2)}$ (Lemma 5.2), $b_5 > b_2$, and $P_1 + Q_5 \ge P_1 > Q_2$ by (4), forcing $q_4^{(5)} \le q_4^{(2)}$; hence $q_4^{(4)} < q_4^{(5)} \le q_4^{(2)}$, so $Q_2 < Q_4$ by Lemma 5.3(2).
(6) By (A-Rev-Fwd) on $(2,4,5)$ ($Q_2 < Q_4$), $V_5 < Q_4$.
(7) By (A-Rev-Fwd) on $(3,4,4)$ ($V_3 < V_4$), we obtain $Q_4 < V_4 < V_5$, contradicting (6). □

Proposition 6.2 (Both-sides bound $|\mathcal{M}_\pi^\circ| \le 3\,t^4$)**.** *For every 5-generated numerical semigroup S of type $t \ge 1$ and every $(2,2)$ partition π, $|\mathcal{M}_\pi^\circ| \le 3\,t^4$. In particular, $|\mathcal{M}_\pi^\circ| \le 3$ when S is symmetric.*

Proof. Assume $|\mathcal{M}_\pi^\circ| \ge 3\,t^4 + 1$ for $\pi = (\{1,2\},\{3,4\})$. By pigeonhole over $\mathrm{PF}(S)^4$, pick $N = 4$ distinct elements $h_1, h_2, h_3, h_4 \in \mathcal{M}_\pi^\circ$ sharing $(f_u, f_p, f_q, f_v) \in \mathrm{PF}(S)^4$. Because the $\{1,2\}$- and $\{3,4\}$-components of ∇_{h_k} are distinct, choose simultaneously for each $k \in \{1,2,3,4\}$:

- the unique support-$\{1,2\}$ factorization $h_k = a_k n_1 + b_k n_2$ with $a_k \ge 1$ minimal among support-$\{1,2\}$ factorizations of h_k, and
- the unique support-$\{3,4\}$ factorization $h_k = c_k n_3 + d_k n_4$ with $c_k \ge 1$ minimal among support-$\{3,4\}$ factorizations of h_k.

Order the four elements as:

- *A-order* $(1,2,3,4)$ by strictly decreasing a: $a_1 > a_2 > a_3 > a_4 \ge 1$ ($1 \le b_1 < b_2 < b_3 < b_4$);
- *B-order* $(\tau_1,\tau_2,\tau_3,\tau_4)$ (some permutation $\tau \in S_4$) by strictly decreasing c: $c_{\tau_1} > c_{\tau_2} > c_{\tau_3} > c_{\tau_4} \ge 1$ ($1 \le d_{\tau_1} < d_{\tau_2} < d_{\tau_3} < d_{\tau_4}$).

By Lemma 5.4 at $N = 4$, the four rules (A-Bwd), (A-Fwd), (A-Rev-Fwd), and (A-Rev-Bwd) hold on A-order $(1,2,3,4)$ for $(U,P;Q,V)$ (with antitonicity of U,P on $\{1,2,3\}$ and Q,V on $\{2,3,4\}$). Because $\mathcal{M}^\circ_{(\{1,2\},\{3,4\})}$ is invariant under swapping $\{1,2\} \leftrightarrow \{3,4\}$, which sends $(n_1,n_2,n_3,n_4) \mapsto (n_3,n_4,n_1,n_2)$ and swaps $(u,p,q,v) \mapsto (u,q,p,v)$, the exact same four rules (B-Bwd), (B-Fwd), (B-Rev-Fwd), and (B-Rev-Bwd) hold on B-order $(\tau_1,\tau_2,\tau_3,\tau_4)$ for $(U,Q;P,V)$ (with antitonicity of U,Q on $\{\tau_1,\tau_2,\tau_3\}$ and P,V on $\{\tau_2,\tau_3,\tau_4\}$).

Step 1 (Global phase lock and involutive normalization). On A-order $(1,2,3,4)$, (A-Bwd) on $(1,1,2)$ forbids $U_1 > U_2 \wedge P_1 > P_2$, and (A-Rev-Fwd) on $(3,4,4)$ forbids $Q_3 < Q_4 \wedge V_3 < V_4$. Moreover, if $U_1 < U_2$ then (A-Fwd) on $(1,2,3)$ gives $V_3 < U_2$, forcing $V_3 < V_4$ (else (A-Rev-Bwd) on $(2,3,4)$ gives $U_2 < V_3$); identically, $P_1 < P_2 \implies Q_3 < Q_4$. In particular, $U_1 < U_2$ and $P_1 < P_2$ cannot both hold (else $V_3 < V_4$ and $Q_3 < Q_4$, already forbidden), so $(U_1 - U_2, P_1 - P_2, Q_3 - Q_4, V_3 - V_4)$ is locked into exactly two phases:

- **Phase A+:** $U_1 < U_2,\ P_1 > P_2,\ Q_3 > Q_4,\ V_3 < V_4$; or
- **Phase A−:** $U_1 > U_2,\ P_1 < P_2,\ Q_3 < Q_4,\ V_3 > V_4$.

Identically, B-order is locked into **Phase B+** ($U_{\tau_1} < U_{\tau_2},\ Q_{\tau_1} > Q_{\tau_2},\ P_{\tau_3} > P_{\tau_4},\ V_{\tau_3} < V_{\tau_4}$) or **Phase B−** (all four reversed).

This proved inequality system on the 16 numbers $\{U_k, P_k, Q_k, V_k\}$ is invariant under three involutive relabelings:

- ι_1: reverse A-order $(1,2,3,4) \mapsto (4,3,2,1)$ and swap $(U,P) \leftrightarrow (Q,V)$ (pairing A-Bwd $\leftrightarrow$ A-Rev-Fwd and A-Fwd $\leftrightarrow$ A-Rev-Bwd; fixes Phase A+, flips Phase B+ $\leftrightarrow$ Phase B−);
- ι_2: reverse B-order $(\tau_1,\tau_2,\tau_3,\tau_4) \mapsto (\tau_4,\tau_3,\tau_2,\tau_1)$ and swap $(U,Q) \leftrightarrow (P,V)$ (fixes Phase B+, flips Phase A+ $\leftrightarrow$ Phase A−);

- ι_3: swap A-order $\leftrightarrow$ B-order and swap $P \leftrightarrow Q$ (fixes both Phase A+ and Phase B+, swaps $U_2 \leftrightarrow U_{\tau_2}$).

Applying ι_2, ι_1, and ι_3 as needed, assume without loss of generality **Phase A+**, **Phase B+**, and $U_{\tau_2} \le U_2$.

Step 2 (Elimination of all 24 permutations $\tau \in S_4$). Set $M_P := \max_k P_k$, $M_Q := \max_k Q_k$, and $M_V := \max_k V_k$.

(1) *From Phase A+:* (A-Fwd) on $(1,2,3)$ and $(1,2,4)$ gives $P_3, V_3, P_4, V_4 < U_2$; (A-Rev-Bwd) on $(1,3,4)$ and $(2,3,4)$ gives $P_1, V_1, P_2, V_2 < Q_3$; (A-Bwd) on $(1,1,2)$ gives $U_1, Q_1 < P_1 < Q_3$; and (A-Rev-Fwd) on $(3,4,4)$ gives $U_4, Q_4 < V_4 < U_2$. In particular, $\max(M_P, M_V) < \max(U_2, Q_3)$.

(2) *From Phase B+ (ι_3-mirror):* $Q_{\tau_3}, V_{\tau_3}, Q_{\tau_4}, V_{\tau_4} < U_{\tau_2}$; $Q_{\tau_1}, V_{\tau_1}, Q_{\tau_2}, V_{\tau_2} < P_{\tau_3}$; $U_{\tau_1}, P_{\tau_1} < Q_{\tau_1} < P_{\tau_3}$; and $U_{\tau_4}, P_{\tau_4} < V_{\tau_4} < U_{\tau_2}$. In particular, $\max(M_Q, M_V) < \max(U_{\tau_2}, P_{\tau_3}) \le \max(U_2, P_{\tau_3})$. Furthermore, $P_{\tau_2} > P_{\tau_4}$ (else $P_{\tau_2} < P_{\tau_4}$ gives $V_{\tau_4} < P_{\tau_4}$ by (B-Rev-Fwd) on (τ_2, τ_4, τ_4), contradicting $P_{\tau_4} < V_{\tau_4}$).

(3) *Global dominance:* Since $P_{\tau_3} \le M_P < \max(U_2, Q_3)$ and $Q_3 \le M_Q < \max(U_2, P_{\tau_3})$, we have $\max(P_{\tau_3}, Q_3) < U_2$, so (1) and (2) immediately yield $\max(M_P, M_Q, M_V) < U_2$.

(4) *Forced minimum at P_2:* We have $P_2 < P_3$ (else (A-Bwd) on $(2,2,3)$ gives $U_2 < P_2 \le M_P < U_2$). Together with $P_1 > P_2$ (Phase A+), this yields $P_2 < \min(P_1, P_3)$.

(5) *Locating 2 and forcing $\tau_4 = 4$:* Since $U_{\tau_1} < Q_{\tau_1} \le M_Q < U_2$ and $U_{\tau_4} < V_{\tau_4} \le M_V < U_2$, element 2 cannot be τ_1 or τ_4; hence $2 \in \{\tau_2, \tau_3\}$. In both cases ($2 = \tau_2$ via $P_{\tau_2} > P_{\tau_4}$ in (2), or $2 = \tau_3$ via Phase B+'s $P_{\tau_3} > P_{\tau_4}$), we have $P_2 > P_{\tau_4}$. Since $P_2 < \min(P_1, P_3)$ and $\tau_4 \neq 2$, this forces $\tau_4 = 4$.

(6) *Final three-line contradiction:*

- If $2 = \tau_3$: since $\tau_4 = 4$, we have $3 \in \{\tau_1, \tau_2\}$, so (2) gives $Q_3 < P_{\tau_3} = P_2$, contradicting $P_2 < Q_3$ from (1).
- If $2 = \tau_2$ and $(\tau_1, \tau_3) = (3,1)$: then (2) gives $Q_3 = Q_{\tau_1} < P_{\tau_3} = P_1$, contradicting $P_1 < Q_3$ from (1).
- If $2 = \tau_2$ and $(\tau_1, \tau_3) = (1,3)$: then (2) gives $P_1 = P_{\tau_1} < Q_{\tau_1} = Q_1$, contradicting $Q_1 < P_1$ from (1). □

Proof of Theorem 1.5. In the 10-element cyclic matching of Theorem 1.3 (Figure 5), apply Proposition 6.1 ($|\mathcal{U}_\pi| \le 4$) to the 10 matched buckets and Proposition 6.2 ($|\mathcal{M}_\pi^\circ| \le 3$) to the 5 unmatched buckets:

$$\rho(H) \le 5 + 10 \cdot 4 + 5 \cdot 3 = 60.$$

Finally, since $\Bbbk[H]$ is a 1-dimensional Gorenstein domain of codimension 4, its minimal free resolution over $R = \Bbbk[x_1, \dots, x_5]$ is self-dual ($b_0 = b_4 = 1$, $b_1 = b_3 = \rho(H)$) and has vanishing Euler characteristic $\sum_{i=0}^{4} (-1)^i b_i = 0$ (as $\dim \Bbbk[H] = 1 < 5$, so $\Bbbk[H]$ is a torsion R-module), giving $b_2 = 2\rho(H) - 2 \le 118$. □

7. Discussion

Bresinsky's 1975 question—whether Gorenstein symmetry uniformly bounds the minimal presentation size of a numerical semigroup of fixed embedding dimension—turns out to have an unexpectedly nuanced answer. Rather than holding in every dimension or failing immediately after $e = 4$, finiteness persists through $e = 5$ (in fact for arbitrary Cohen–Macaulay type via the quadratic bound $\rho(S) \le 5 + 75\, t(S)^2$) and fails by $e = 8$, where every interior Betti number $b_1, \dots, b_{e-2}$ of the symmetric Boolean-cube family $H_e(m)$ diverges simultaneously over every field.

The sharpest remaining question is to locate the exact threshold between $e = 5$ and $e = 8$: does $S(e)$ remain finite at $e = 6$ or $e = 7$, or does an unbounded symmetric family already exist in one of those two dimensions? Even at $e = 5$, narrowing $S(5)$ inside the finite range $[13, 60]$ (or $[13, 41]$ via [12, Section 8])—and determining whether Bresinsky's 1979 four-sum maximum 13 is already optimal—remains open. Equally inviting at $e = 5$ are bounding $b_2(\Bbbk[S])$ and $b_3(\Bbbk[S])$ for non-symmetric semigroups of bounded type and settling the reverse implication $\rho(S) \Rightarrow t(S)$ (the $e = 5$ analogue of Problem 25, matching Chomicz's recent resolution at $e = 4$ [8]), which together with bounding b_2 and b_3 would resolve Problem 24 at $e = 5$. Finally, it would be interesting to determine whether the full Betti vector of $H_8(m)$ satisfies the linear law $(1, 10m + 9, 50m + 16, 100m + 10, 100m + 10, 50m + 16, 10m + 9, 1)$ for all $m \ge 3$ (observed for small m in [12, Section 7], where $b_1 = 10m + 9$ is proved for all $m \ge 2$; exact formulas for b_2 and b_3 remain open).

Acknowledgments

This work was done overwhelmingly by AI models, steered by the author in a continuous critical dialogue, variously with Anthropic's Claude, OpenAI's ChatGPT and Google's Gemini. The author takes full responsibility for the paper.

References

[1] H. Bresinsky, *Symmetric semigroups of integers generated by* 4 *elements*, Manuscripta Math. **17** (1975), no. 3, 205–219.

[2] H. Bresinsky, *On prime ideals with generic zero* $x_i = t^{n_i}$, Proc. Amer. Math. Soc. **47** (1975), 329–332.

[3] H. Bresinsky, *Monomial Gorenstein ideals*, Manuscripta Math. **29** (1979), 159–181.

[4] H. Bresinsky, *Binomial generating sets for monomial curves, with applications in* $\mathbb{A}^4$, Rend. Sem. Mat. Univ. Politec. Torino **46** (1988), no. 3, 353–370.

[5] W. Bruns and J. Herzog, *Semigroup rings and simplicial complexes*, J. Pure Appl. Algebra **122** (1997), 185–208.

[6] D. A. Buchsbaum and D. Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension* 3, Amer. J. Math. **99** (1977), 447–485.

[7] A. Campillo and C. Marijuán, *Higher order relations for a numerical semigroup*, Sém. Théor. Nombres Bordeaux **3** (1991), 249–260.

[8] K. Chomicz, *The type and cardinality of minimal presentations of numerical semigroups with embedding dimension four*, preprint (2026), https://arxiv.org/abs/2609.04000

[9] C. Delorme, *Sous-monoïdes d'intersection complète de* $\mathbb{N}$, Ann. Sci. Éc. Norm. Supér. **9** (1976), 145–154.

[10] K. Eto, *Almost Gorenstein monomial curves in affine four space*, J. Algebra **488** (2017), 362–387.

[11] P. Gimenez and H. Srinivasan, *The structure of the minimal free resolution of semigroup rings obtained by gluing*, J. Pure Appl. Algebra **223** (2019), no. 4, 1411–1426.

[12] D. Goel, *Bresinsky's problem (1975) has a negative answer, and other miscellanea*, expanded version (2026), https://doi.org/10.5281/zenodo.24633631

[13] J. Herzog, *Generators and relations of abelian semigroups and semigroup rings*, Manuscripta Math. **3** (1970), 175–193.

[14] J. Herzog and K. Watanabe, *Almost symmetric numerical semigroups*, Semigroup Forum **98** (2019), 589–630.

[15] E. Kunz, *The value-semigroup of a one-dimensional Gorenstein ring*, Proc. Amer. Math. Soc. **25** (1970), 748–751.

[16] A. Moscariello, *On the boundedness of the type of an almost Gorenstein monomial curve in* $\mathbb{A}^5$, Comm. Algebra **51** (2023), no. 3, 1179–1185.

[17] A. Moscariello and A. Sammartano, *Open problems on relations of numerical semigroups*, in *Recent Progress in Ring and Factorization Theory*, Springer Proc. Math. Stat., Vol. 477, Springer, 2025, pp. 365–380; updated as https://arxiv.org/abs/2406.00790

[18] A. Moscariello and F. Strazzanti, *Nearly Gorenstein vs almost Gorenstein affine monomial curves*, Mediterr. J. Math. **18** (2021), 127.

[19] A. Moscariello and F. Strazzanti, *Nearly Gorenstein numerical semigroups with five generators have bounded type*, Comm. Algebra **53** (2025), no. 10, 4041–4052.

[20] J. C. Rosales, *An algorithmic method to compute a minimal relation for any numerical semigroup*, Int. J. Algebra Comput. **6** (1996), no. 4, 441–455.

[21] J. C. Rosales, *On presentations of subsemigroups of* $\mathbb{N}^n$, Semigroup Forum **55** (1997), no. 2, 152–159.

[22] J. C. Rosales and P. A. García-Sánchez, *Numerical Semigroups*, Developments in Mathematics, Vol. 20, Springer, New York, 2009.

[23] D. I. Stamate, *Betti numbers for numerical semigroup rings*, in *Multigraded Algebra and Applications*, Springer Proc. Math. Stat., Vol. 238, Springer, 2018, pp. 133–157.