hexagon:2611.01351v1

RESOLUTION OF HOLLAND'S (1977, 1984) PROBLEMS ON BLOCH FUNCTIONS AND SINGULAR ZYGMUND MEASURES

DAVE GOEL

First version: November 29, 2026. This version (1.00): November 29, 2026.

Abstract. We resolve two open problems of Finbarr Holland from the *Bulletin of the London Mathematical Society* (Hayman and Lingham, *Research Problems in Function Theory*, 2019). Via Herglotz's formula $F(z) = 1 + 2\sum_{n=1}^{\infty} \hat{\mu}(n) z^n$, probability measures μ on $\mathbb{T}$ in Zygmund's class Λ_* correspond to Bloch functions $F \in \mathcal{B}$ on $\mathbb{D}$ with $\operatorname{Re} F > 0$ and $F(0) = 1$. Problem 8.13 (1977) asks whether a nonzero singular $\mu \in \Lambda_*$ can have $\hat{\mu}(n) \geq 0$; Problem 5.69 (1984) asks whether $F(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$ with $\operatorname{Re} F > 0$ satisfies (a) $(a_n) \in \ell^2$ when $a_n \geq 0$, and (b) $(a_n) \in \ell^4$ (or ℓ^p, $p > 4$) in general. Previously, no known singular Λ_* measure had $\hat{\mu} \geq 0$, singular Riesz products fail to lie in Λ_*, and no proof of 5.69(b) was recorded in Hayman's list (though $q > 3$ already follows from classical H^p theory). First, we disprove 5.69(a) and answer 8.13 affirmatively via an even, purely singular, atomless probability measure μ in the little Zygmund class $\lambda_* \subset \Lambda_*$ with $\hat{\mu} \geq 0$ and $\sum_{n=1}^{\infty} \hat{\mu}(n)^2 = \infty$. Second, a Green–Hardy–Stein identity applied directly to $\operatorname{Re} F$ settles 5.69(b) explicitly: $\sum_{n=1}^{\infty} |a_n|^4 \leq 48 \|F\|_{\mathcal{B}}^2 (\operatorname{Re} F(0))^2$, $(a_n) \in \ell^q$ for all $q > 3$, and a log-weighted ℓ^3 bound at $q = 3$; unweighted $(2, 3]$ stays open here. Third, for any probability measure μ on $\mathbb{T}$ with bounded dyadic block sums $s_k(\mu) := \sum_{2^k \leq n < 2^{k+1}} |\hat{\mu}(n)| \leq M$ (which characterizes $\mu \in \Lambda_*$ when $\hat{\mu} \geq 0$), finite Riesz-product tests yield a sharp weak-ℓ^2 bound on arbitrary block subsets. Consequently $\hat{\mu} \in \ell^q$ for every $q > 2$ ($q = 2$ sharp), $\sum_{n=1}^{N} |\hat{\mu}(n)|^2 = O_M(\ln \ln(4N))$, and a bounded non-increasing non-negative sequence $b(k)$ forces $\hat{\mu} \in \ell^2$ whenever $\hat{\mu} \geq 0$ and $s_k(\mu) \leq b(k)$ if and only if $\sum_{k=0}^{\infty} b(k)/(k+1) < \infty$. Theorems 1.1–1.3 settle both of Holland's problems.[1] These results are verified in Lean.[2]

1. Introduction

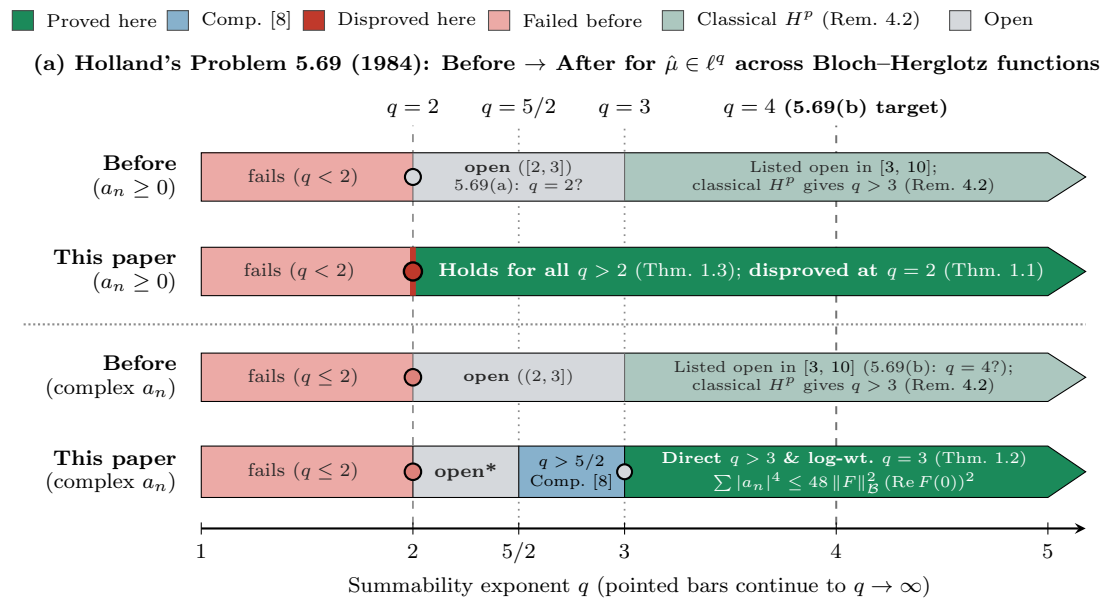
Can a purely singular Borel probability measure on the unit circle—one concentrated on a set of length zero—belong to Zygmund's smoothness class Λ_* while having all Fourier–Stieltjes coefficients non-negative? For more than four decades, this question and its companion problem on the square-summability of Taylor coefficients stood open in Walter Hayman's celebrated problem series (*Hayman's List*, initiated in [9] and updated across [1, 3, 4, 10]). Posed by Finbarr Holland in the *Bulletin of the London Mathematical Society* problem collections—as **Problem 8.13** in Anderson, Barth, and Brannan [1] (1977) and **Problem 5.69** in Barth, Brannan, and Hayman [3] (1984)—both were republished unchanged in the Fiftieth Anniversary Edition by Hayman and Lingham [10] (2019), where each still carried the status *"No progress on this problem has been reported to us."*

Holland's problems sit at a classic meeting point of geometric function theory and harmonic analysis. Identify the unit circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ with normalized Lebesgue measure $dm(\theta) = d\theta/(2\pi)$, and let $M^+(\mathbb{T})$ denote the cone of finite positive Borel measures on $\mathbb{T}$. A finite real Borel measure μ on $\mathbb{T}$ belongs to *Zygmund's class* Λ_* if its cumulative distribution function $G_\mu(\theta) = \mu([0, \theta])$ ($\theta \in [0, 2\pi)$, extended to $\mathbb{R}$ via $G_\mu(\theta + 2\pi) = G_\mu(\theta) + \mu(\mathbb{T})$) is continuous on $\mathbb{R}$ and satisfies $|G_\mu(\theta + h) + G_\mu(\theta - h) - 2G_\mu(\theta)| = O(h)$ uniformly in $\theta \in \mathbb{R}$ as $h \to 0^+$, and to the *little Zygmund class* λ_* if this symmetric difference is $o(h)$ uniformly in θ. On the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, a holomorphic function F lies in the *Bloch space* $\mathcal{B}$ if $\|F\|_{\mathcal{B}} := \sup_{z \in \mathbb{D}} (1 - |z|^2)|F'(z)| < \infty$, and in the *little Bloch space* $\mathcal{B}_0$ if $(1 - |z|^2)|F'(z)| \to 0$ as $|z| \to 1^-$. By Herglotz's theorem [11], holomorphic functions F on $\mathbb{D}$ with $\operatorname{Re} F > 0$ and $F(0) = 1$ correspond bijectively to Borel probability measures $\mu \in M^+(\mathbb{T})$ via $F(z) = \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\mu(\theta) = 1 + 2\sum_{n=1}^{\infty} \hat{\mu}(n) z^n$ (where $\hat{\mu}(n) := \int_{\mathbb{T}} e^{-in\theta} \, d\mu(\theta)$), and $F \in \mathcal{B}$ if and only if $\mu \in \Lambda_*$ (likewise $F \in \mathcal{B}_0 \iff \mu \in \lambda_*$; see Zygmund [19] and Duren–Shapiro–Shields [7], and Lemma 2.1 in Section 2 for the self-contained proof when $\hat{\mu} \geq 0$). We refer to a Bloch function $F \in \mathcal{B}$ with $\operatorname{Re} F > 0$ as a *Bloch–Herglotz function*.

For any conformal map ψ of $\mathbb{D}$ onto a rectifiable Jordan domain, $\log \psi' \in \mathcal{B}$ [17, 2], and the failure of the Smirnov property (that $\log |\psi'|$ is the Poisson integral of its boundary values) is governed by a singular boundary measure in Λ_* [19, 6]. In classic constructions, Duren, Shapiro, and Shields [7], Piranian [16], Shapiro [18], and Kahane [14] showed that positive singular probability measures can belong to Λ_*, proving that Bloch–Herglotz functions need not belong to the Hardy space $H^2(\mathbb{D})$ (i.e., need not have $\sum_{n=1}^{\infty} |a_n|^2 < \infty$). None of those classical singular Λ_* measures, however, has non-negative Fourier–Stieltjes coefficients (the coefficients of the even examples change sign infinitely often). Conversely, infinite Riesz products $\prod_k (1 + a_k \cos(n_k \theta))$ [20]—the classical mechanism for singular probability measures with non-negative Fourier–Stieltjes coefficients—can never remain inside Λ_* while singular, as

[1] Expanded version, with further results [8]: https://doi.org/10.5281/zenodo.24642945

[2] Theorems 1.1–1.3 are proved in Lean 4 from Lean's three standard axioms alone, and checked with lake comparator using nanoda 0.4.32: https://github.com/david44hello/paper_holland-1977-1984-bloch-zygmund/tree/v1.00/lean



*This paper settles $q = 4$ directly and proves log-weighted $q = 3$ (unweighted $(2, 3]$ open here); companion [8] reaches $q > 5/2$.

(b) Holland's Problem 8.13 (1977): Before → After for singular measures in Zygmund's class Λ_*

	General / oscillating coefficients ($\hat{\mu}$ not all ≥ 0)	**Non-negative Fourier coefficients ($\hat{\mu}(n) \geq 0$ for all n)**
Positive singular measure (Problem 8.13)	**Before (1966–1969): Prior constructions** Duren–Shapiro–Shields [7], Pirarian [16], Shapiro [18], Kahane [14]. None has $\hat{\mu} \geq 0$ (even ones oscillate).	**Before (1977–2026):** Open (Problem 8.13) Singular Riesz products fail Λ_* [5, 13]. **After (This paper): Problem 8.13 YES** **Theorem 1.1 & Cor. 3.1:** Purely singular $\mu \in \lambda_* \subset \Lambda_*$, topological support $[-\delta, \delta]$, $\hat{\mu} > 0$ strictly.
Companion [8]: Compact null support K	**Before: Kahane (1969** [14]**)** Positive measure on compact null set E_∞^{Kah}. **Companion** [8, Thm. 6.2]**:** $\hat{\mu} \geq 0$ is *impossible* on E_∞^{Kah} for any nonzero real distribution.	**Companion** [8, Thm. 6.1]**: Signed measure exists** Signed $\mu \in \Lambda_*$ on compact null K_∞, $\hat{\mu}(0) = 1$, total variation $\|\mu\|_{\mathrm{TV}} \leq 1 + \varepsilon$, $\hat{\mu} \geq 0$. **Follow-up open (beyond 8.13):** Positive compact-null support for $\hat{\mu} \geq 0$.

FIGURE 1. Before-and-after overview of Holland's Problems 5.69 (1984) and 8.13 (1977) [1, 3, 10]. **(a)** Summability exponents q for which $\hat{\mu} \in \ell^q(\mathbb{Z})$ holds for every Bloch–Herglotz function $F(z) = 1 + 2\sum_{n=1}^{\infty} \hat{\mu}(n) z^n \in \mathcal{B}$ ($\mathrm{Re}\, F > 0$, $a_n = 2\hat{\mu}(n)$ for $n \geq 1$). For $a_n \geq 0$ (top two bars), every $q < 2$ already failed for elementary $L^2(\mathbb{T})$ densities ($P = |\sum_{j=1}^{\infty} c_j e^{i16^j\theta}|^2$ with $P\, dm \in \lambda_*$, taking $c_j \in \ell^2 \setminus \bigcup_{q<2} \ell^q$), while $q > 3$ followed from classical H^p theory (Remark 4.2); Theorem 1.1 disproves $q = 2$, and Theorem 1.3 proves $\hat{\mu} \in \ell^q$ for every $q > 2$, closing the genuinely open interval $[2, 3]$ with explicit bounds across all $q > 2$. For general complex a_n (bottom two bars), failure at $q \leq 2$ followed from [7, 16, 18, 14], and Problem 5.69(b) ($q = 4$, or any $p > 4$) was listed open in [3, 10] (though $q > 3$ already follows from Smirnov + Hardy–Stein + Bloch interpolation; Remark 4.2); Theorem 1.2 gives a direct conjugate-free proof of $\sum |a_n|^4 \leq 48\|F\|_{\mathcal{B}}^2 (\mathrm{Re}\, F(0))^2$, explicit $q > 3$ bounds, and a log-weighted $q = 3$ bound (leaving unweighted $(2, 3]$ open in this paper, marked by the open dot at $q = 3$), while the external companion [8, Theorem 8.1] reaches $q > 5/2$. **(b)** Before-and-after landscape for singular measures in Zygmund's class Λ_*: singular classical Riesz products fail Λ_* [5, 13], while none of the classical singular Λ_* measures [7, 16, 18, 14] (including Kahane's measure on his compact null Cantor set E_∞^{Kah}) has $\hat{\mu} \geq 0$. Theorem 1.1 and Corollary 3.1 settle Problem 8.13 affirmatively via a positive measure whose full mass lives on a dense Borel set of Lebesgue measure zero inside $[-\delta, \delta]$ (so its closed topological support is $[-\delta, \delta]$; dashed bottom row summarizes compact-null extensions in companion [8] and the open follow-up of a *closed* null support); here $\|\mu\|_{\mathrm{TV}}$ denotes total variation.

proved by Duren [5] and Holland and Twomey [13] (building on Holland [12]). Distilling this clash, Holland [1, 3, 10] posed Problems 8.13 and 5.69, surveyed visually in Figure 1 and stated precisely below.

- **Problem 5.69 ([3]; [10, Problem 5.69]). (a)** Let $F(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ ($|z| < 1$) be a Bloch function with $\operatorname{Re} F > 0$ in $\mathbb{D}$ and $a_n \geq 0$ for all $n \geq 1$—equivalently, let $\mu \in M^+(\mathbb{T})$ be a probability measure in Λ_* with $\hat{\mu}(n) \geq 0$ for all $n \in \mathbb{Z}$. Must $\sum_{n=1}^{\infty} a_n^2 < \infty$ (equivalently, $\hat{\mu} \in \ell^2(\mathbb{Z})$)? **(b)** As Holland observed in the same entry, an affirmative answer would imply that every Bloch function $F(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ with $\operatorname{Re} F > 0$ (with complex, not necessarily non-negative, coefficients a_n) obeys $\sum_{n=1}^{\infty} |a_n|^4 < \infty$: letting ν be its Herglotz probability measure and $\tilde{\nu}(E) := \nu(-E)$, the convolution square $\mu := \nu * \tilde{\nu} \in \Lambda_*$ satisfies $\hat{\mu}(n) = |\hat{\nu}(n)|^2 = \frac{1}{4}|a_n|^2 \geq 0$. Holland asked whether $\sum_{n=1}^{\infty} |a_n|^4 < \infty$ holds, or at least whether $\sum_{n=1}^{\infty} |a_n|^p < \infty$ for some $p > 4$.
- **Problem 8.13 ([1]; [10, Problem 8.13]).** Does there exist a singular measure in Zygmund's class Λ_* (printed A^* in [10]) all of whose Fourier–Stieltjes coefficients are non-negative?

In this paper we resolve both of Holland's problems completely, revealing a sharp split across his program. First, Problem 5.69(a) has a **negative** answer and Problem 8.13 has an **affirmative** answer: positive purely singular probability measures with $\hat{\mu} \geq 0$ exist even inside the strictly smoother little Zygmund class λ_*. Second, Problem 5.69(b) has an **affirmative** answer without assuming $a_n \geq 0$, proved directly with explicit bounds and a log-weighted $q = 3$ endpoint. Third, for non-negative coefficients, $q = 2$ is the exact sharp threshold, with $\hat{\mu} \in \ell^q$ holding for every $q > 2$. Figure 1 surveys both problems before and after this paper, and Table 1 summarizes the main advances.

TABLE 1. Summary of Holland's Problems 5.69 (1984) and 8.13 (1977) [1, 3, 10]: state of the art before and after this paper. Here $\|F\|_{\mathcal{B}}$ is the Bloch seminorm, $s_k(\mu) := \sum_{2^k \leq n < 2^{k+1}} |\hat{\mu}(n)|$ are the dyadic ℓ^1 block sums, and $M := \sup_{k \geq 0} s_k(\mu)$.

Question	Best known before this paper	This paper
Problem 5.69(a) (1984): $\mu \in M^+(\mathbb{T}) \cap \Lambda_*$, $\hat{\mu} \geq 0 \implies \hat{\mu} \in \ell^2$?	Open since Holland [3, 10]; singular Riesz products ruled out [5, 13].	**FALSE** (Theorem 1.1, Prop. 3.2): fails in $\lambda_*/\mathcal{B}_0$ for both a purely singular $\mu \perp m$ and an $L^1 \setminus L^2$ density $\lvert G\rvert^2/\|G\|_2^2$.
Problem 8.13 (1977): Singular $\mu \in \Lambda_*$ with $\hat{\mu} \geq 0$?	Open since Holland [1, 10]; singular Riesz products fail Λ_* [5, 13]; prior singular Λ_* measures [7, 16, 18, 14] do not have $\hat{\mu} \geq 0$.	**TRUE** (Theorem 1.1, Cor. 3.1): positive purely singular atomless $\mu \in \lambda_*$, with topological support $[-\delta, \delta]$ and $\hat{\mu} > 0$ strictly (companion [8, Thm. 6.1] gives signed compact-null K_∞).
Problem 5.69(b) (1984): General Bloch F, $\operatorname{Re} F > 0$: $\sum \lvert a_n\rvert^4 < \infty$?	Listed open in [3, 10] (fails at $q = 2$ [7, 16, 18, 14]); $q > 3$ already follows from Smirnov [6] + Hardy–Stein + Bloch interpolation (Remark 4.2).	**TRUE** (Theorem 1.2): direct conjugate-free proof with $\sum \lvert a_n\rvert^4 \leq 48\|F\|_{\mathcal{B}}^2 (\operatorname{Re} F(0))^2$, explicit $q > 3$ bounds, and critical $q = 3$ log-weighted bound (companion [8, Thm. 8.1] reaches $q > 5/2$).
Sharp threshold & dyadic envelope (Thm. 1.3(i)–(iii): $\hat{\mu} \in \mathbb{C}$; (iv): $\hat{\mu} \geq 0$)	Since $\sup s_k < \infty \implies \mu \in \Lambda_*$, classical $q > 3$ and Hölder interpolation give $O_\varepsilon((\ln N)^{1/2+\varepsilon})$ partial ℓ^2 (vs. $O(\ln N)$ for general Bloch [17, 2]); only trivial $\sum b(k)^2 < \infty$ sufficiency was known.	**Exact $q = 2$ threshold & if-and-only-if criterion** (Theorem 1.3): weak-ℓ^2 and $\hat{\mu} \in \ell^q$ ($\forall q > 2$), $\sum_{n=1}^{N} \lvert\hat{\mu}(n)\rvert^2 = O_M(\ln\ln(4N))$, and ℓ^2 holds for monotone $s_k \leq b(k) \iff \sum \frac{b(k)}{k+1} < \infty$.

1.1. **Main results.** For a finite real Borel measure μ on $\mathbb{T}$, let $B_k := [2^k, 2^{k+1}) \cap \mathbb{Z}$ ($k \geq 0$) and write $s_k(\mu) := \sum_{n \in B_k} |\hat{\mu}(n)|$ for its k-th dyadic ℓ^1 block sum. Our first theorem refutes the first half of Problem 5.69 and resolves Problem 8.13 affirmatively, even in the little classes λ_* and $\mathcal{B}_0$.

Theorem 1.1 (Disproof of Problem 5.69(a) and positive resolution of Problem 8.13)**.** *There exists an even, positive, purely singular, atomless Borel probability measure μ on $\mathbb{T}$ ($\mu \perp m$) such that:*

(i) $\hat{\mu}(n) \geq 0$ *for every* $n \in \mathbb{Z}$, *with* $\hat{\mu}(0) = 1$;
(ii) $s_k(\mu) \leq 1$ *for every* $k \geq 0$ *and* $\lim_{k \to \infty} s_k(\mu) = 0$; *consequently* $\mu \in \lambda_* \subset \Lambda_*$, *and its Herglotz function* $F(z) = 1 + 2\sum_{n=1}^{\infty} \hat{\mu}(n) z^n$ *satisfies* $\operatorname{Re} F > 0$ *on* $\mathbb{D}$ *and lies in the little Bloch space* $\mathcal{B}_0 \subset \mathcal{B}$;
(iii) $\sum_{n=1}^{\infty} \hat{\mu}(n)^2 = +\infty$ *(so* $F \notin H^2(\mathbb{D})$*), whereas* $\sum_{n=1}^{\infty} \hat{\mu}(n)^q < \infty$ *for every* $q > 2$.

Theorem 1.1 is proved in Section 3 via an inductive dilated product of non-flat lacunary analytic squares $Q_R(\theta) = \left|\sum_{j=1}^{R} (jH_R)^{-1/2} e^{i16^j\theta}\right|^2$ (where $H_R := \sum_{j=1}^{R} j^{-1}$), bypassing Duren's Riesz-product barrier directly. In the same section we localize μ to any interval $[-\delta, \delta]$ while enforcing *strict* positivity $\widehat{\mu_\delta}(n) > 0$ for every $n \in \mathbb{Z}$ (Corollary 3.1), and construct an absolutely continuous probability density $P = |G|^2/\|G\|_2^2 \in L^1(\mathbb{T}) \setminus L^2(\mathbb{T})$ with $P\,dm \in \lambda_*$ and $\hat{G} \geq 0$ (Proposition 3.2).

Although Problem 5.69(a) fails, Holland's second question in Problem 5.69—whether every Bloch function with positive real part has ℓ^4-summable Taylor coefficients—holds without assuming $a_n \geq 0$, and in fact extends to every $q > 3$ via an elementary real-part Green–Hardy–Stein identity (Section 4).

Theorem 1.2 (Positive resolution of Holland's Bloch–Herglotz ℓ^4 question)**.** *Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be holomorphic on $\mathbb{D}$ with $\operatorname{Re} F > 0$ and finite Bloch seminorm $M_{\mathcal{B}} := \|F\|_{\mathcal{B}} < \infty$ (with arbitrary complex coefficients a_n). Then:*

(i) $\displaystyle\sum_{n=1}^{\infty} |a_n|^4 \leq 48\, M_{\mathcal{B}}^2 \left(\operatorname{Re} F(0)\right)^2$;

(ii) *for every $q > 3$,*

$$\sum_{n=1}^{\infty} |a_n|^q \leq \frac{2(q-1)^2}{q-3}\, (2\sqrt{e})^q\, M_{\mathcal{B}}^{q-2} \left(\operatorname{Re} F(0)\right)^2;$$

(iii) *at the critical exponent $q = 3$, for every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} \frac{|a_n|^3}{\left(1 + \ln\!\big(1 + \frac{M_{\mathcal{B}}}{2\operatorname{Re} F(0)} \ln(16n)\big)\right)^{1+\varepsilon}} \leq \frac{64\, e^{3/2}\,(1+\varepsilon)}{\varepsilon}\, M_{\mathcal{B}} \left(\operatorname{Re} F(0)\right)^2.$$

Returning to positive probability measures with bounded dyadic ℓ^1 sums (in particular, all probability measures $\mu \in \Lambda_*$ with $\hat{\mu} \geq 0$), our third theorem shows that $q = 2$ is the exact sharp threshold of Problem 5.69, caps partial ℓ^2 growth universally at $O_M(\ln\ln(4N))$, and characterizes completely which monotone dyadic block envelopes force $\hat{\mu} \in \ell^2$ (Section 5). Write $\ln_+(x) := \max(0, \ln x)$ for $x > 0$ and $\ln_+(0) := 0$.

Theorem 1.3 (Sharp weak-ℓ^2 bound, $q > 2$ summability, and exact monotone envelope criterion)**.** *Let $\mu \in M^+(\mathbb{T})$ be any Borel probability measure (not necessarily even, with arbitrary complex $\hat{\mu}$), write $s_k := s_k(\mu)$ ($s_{-1} := 0$) and $\tilde{s}_k := \max(s_{k-1}, s_k, s_{k+1})$, and suppose $M := \sup_{k \geq 0} s_k < \infty$.*

(i) Local-collar weak-ℓ^2: *For every index set $I \subset \mathbb{Z}_{\geq 0}$ with $\sup_{k \in I} \tilde{s}_k \leq w$ ($w > 0$) and every $\delta > 0$,*

$$\#\Big\{n \in \bigcup_{k \in I} B_k : |\hat{\mu}(n)| \geq \delta\Big\} \leq \frac{56\, w}{\delta^2},$$

with strict inequality when I is finite.

(ii) Global ℓ^q ($q > 2$) and log-weighted ℓ^2: *For every $q > 2$ and every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} |\hat{\mu}(n)|^q \leq \frac{56\, q}{q-2}\, M \left(\min(1, M)\right)^{q-2}, \qquad \sum_{\substack{n \geq 1 \\ \hat{\mu}(n) \neq 0}} \frac{|\hat{\mu}(n)|^2}{\left(1 + \ln(1/|\hat{\mu}(n)|)\right)^{1+\varepsilon}} \leq \frac{56(3+\varepsilon)}{\varepsilon}\, M.$$

(iii) Double-logarithmic partial ℓ^2 growth: *For every integer $N \geq 1$,*

$$\sum_{n=1}^{N} |\hat{\mu}(n)|^2 \leq 57\, M + 112\, M\, \ln_+\!\big(M \log_2(2N)\big),$$

and the left-hand side is $o(\ln\ln(4N))$ whenever $s_k \to 0$.

(iv) Exact if-and-only-if monotone block criterion: *Let $(b(k))_{k \geq 0}$ be any bounded non-increasing sequence of non-negative real numbers. Every Borel probability measure $\nu \in M^+(\mathbb{T})$ with $\hat{\nu} \geq 0$ and $s_k(\nu) \leq b(k)$ for all $k \geq 0$ satisfies $\hat{\nu} \in \ell^2(\mathbb{Z})$ if and only if*

$$\sum_{k=0}^{\infty} \frac{b(k)}{k+1} < \infty.$$

The companion expanded version [8] develops further extensions of Theorems 1.1–1.3 in Sections 6–10: a signed solution to Problem 8.13 on a compact Lebesgue-null set K_∞ and an obstruction ruling out $\hat{\mu} \geq 0$ on Kahane's set E_∞^{Kah} [8, Theorems 6.1–6.2]; an extension of Theorem 1.2 past the $q = 3$ Hausdorff–Young barrier [8, Theorem 7.2] to all $q > 5/2$ on $\mathbb{T}$ [8, Theorem 8.1] and on $\mathbb{T}^D$ [8, Theorem 9.1(b)]; and sharp critical exponents for stopped-martingale measures, multidimensional Besov–L^p classes, and sharpened local-collar constants [8, Theorem 8.2, Sections 9–10]. The proofs in the present paper are entirely self-contained and independent of [8].

2. Preliminaries: Herglotz representation and the dyadic criterion

Identify the unit circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ with $[-\pi, \pi)$ equipped with normalized Lebesgue measure $dm(\theta) = d\theta/(2\pi)$, and let $M(\mathbb{T})$ denote the space of finite real Borel measures on $\mathbb{T}$, with $M^+(\mathbb{T})$ the cone of positive Borel measures. For $\mu \in M(\mathbb{T})$, its Fourier–Stieltjes coefficients are

$$\hat{\mu}(n) := \int_{\mathbb{T}} e^{-in\theta}\, d\mu(\theta) \qquad (n \in \mathbb{Z}).$$

When μ is real and $\hat{\mu}(n) \geq 0$ for all $n \in \mathbb{Z}$, reality forces $\hat{\mu}(-n) = \overline{\hat{\mu}(n)} = \hat{\mu}(n) \geq 0$, so μ is even. For each integer $k \geq 0$, let $B_k := \{n \in \mathbb{Z} : 2^k \leq n < 2^{k+1}\}$ denote the k-th positive dyadic block, and set

$$s_k(\mu) := \sum_{n \in B_k} |\hat{\mu}(n)|, \qquad M(\mu) := \sup_{k \geq 0} s_k(\mu).$$

A measure $\mu \in M(\mathbb{T})$ is *atomless* if $\mu(\{\theta\}) = 0$ for every $\theta \in \mathbb{T}$. In that case its normalized periodic primitive

$$W_\mu(\theta) := \mu([0,\theta]) - \hat{\mu}(0)\frac{\theta}{2\pi} \qquad (\theta \in [0, 2\pi], \text{ extended } 2\pi\text{-periodically})$$

is continuous and of bounded variation on $\mathbb{T}$, with $W_\mu(0) = W_\mu(2\pi) = 0$. Writing $\Delta_h^2 W(\theta) := W(\theta + h) + W(\theta - h) - 2W(\theta)$ for the second symmetric difference (noting that $\Delta_h^2 W_\mu = \Delta_h^2 G_\mu$, since Δ_h^2 annihilates linear functions), we say that μ belongs to *Zygmund's class* Λ_* (denoted A^* in Problem 8.13 of [10]) if μ is atomless and

$$\|W_\mu\|_{\Lambda_*} := \sup_{\theta \in \mathbb{R},\, h > 0} \frac{|\Delta_h^2 W_\mu(\theta)|}{h} < \infty.$$

We say that μ belongs to the *little Zygmund class* λ_* if μ is atomless and $\sup_{\theta \in \mathbb{R}} |\Delta_h^2 W_\mu(\theta)| = o(h)$ as $h \to 0^+$.

On the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, a holomorphic function $F : \mathbb{D} \to \mathbb{C}$ lies in the *Bloch space* $\mathcal{B}$ if its Bloch seminorm

$$\|F\|_{\mathcal{B}} := \sup_{z \in \mathbb{D}} (1 - |z|^2)|F'(z)|$$

is finite, and in the *little Bloch space* $\mathcal{B}_0$ if $(1-|z|^2)|F'(z)| \to 0$ as $|z| \to 1^-$. By Herglotz's theorem [11], holomorphic functions $F(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ on $\mathbb{D}$ with $\operatorname{Re} F > 0$ correspond bijectively to Borel probability measures μ on $\mathbb{T}$ via

$$F(z) = \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta), \qquad a_n = 2\hat{\mu}(n) \quad (n \geq 1). \tag{1}$$

For measures with non-negative Fourier–Stieltjes coefficients, membership in Λ_* and $\mathcal{B}$ (as well as λ_* and $\mathcal{B}_0$) is governed directly by the dyadic block sums $s_k(\mu)$. This equivalence goes back to Zygmund [19] and Holland–Twomey [13]; for completeness we record a short self-contained proof.

Lemma 2.1 (Dyadic characterization for non-negative Fourier coefficients)**.** *Let $\mu \in M(\mathbb{T})$ be a finite real Borel measure with $c_n := \hat{\mu}(n) \geq 0$ for all $n \in \mathbb{Z}$.*

(a) *One has $\mu \in \Lambda_*$ if and only if $M(\mu) < \infty$, with*

$$\frac{\pi}{12}\, \|W_\mu\|_{\Lambda_*} \leq M(\mu) \leq 20\pi^2 e\, \|W_\mu\|_{\Lambda_*}.$$

Moreover, $\mu \in \lambda_*$ if and only if $\lim_{k \to \infty} s_k(\mu) = 0$.

(b) *If μ is a probability measure and F is its Herglotz function (1), then $F \in \mathcal{B}$ if and only if $M(\mu) < \infty$ (hence if and only if $\mu \in \Lambda_*$), with*

$$\frac{1}{40}\, \|F\|_{\mathcal{B}} \leq M(\mu) \leq e\, \|F\|_{\mathcal{B}}.$$

Moreover, $F \in \mathcal{B}_0$ if and only if $\lim_{k \to \infty} s_k(\mu) = 0$.

Proof. We first note that if $M(\mu) < \infty$, then $\sum_{n=1}^{2^K - 1} c_n^2 \leq \sum_{k=0}^{K-1} s_k(\mu)^2 \leq K M(\mu)^2$, so $(2N+1)^{-1} \sum_{|n| \leq N} c_n^2 = O(N^{-1} \ln N) \to 0$; by Wiener's lemma ($\sum_{\theta \in \mathbb{T}} \mu(\{\theta\})^2 = \lim_{N \to \infty} (2N+1)^{-1} \sum_{|n| \leq N} |\hat{\mu}(n)|^2$), μ is atomless. Thus in both directions of (a), W_μ is continuous and of bounded variation with $W_\mu(0) = W_\mu(2\pi) = 0$. Integration by parts gives $\widehat{W_\mu}(n) = c_n/(2\pi i n)$ for $n \neq 0$, so $W_\mu(\theta) \sim \widehat{W_\mu}(0) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{c_n}{n} \sin(n\theta)$.

Part (a), sufficiency and λ_*. Assume $M(\mu) < \infty$. Then $\sum_{n=1}^{\infty} c_n/n \leq \sum_{k=0}^{\infty} 2^{-k} s_k(\mu) \leq 2M(\mu) < \infty$, so the sine series converges uniformly and absolutely to $W_\mu(\theta) - \widehat{W_\mu}(0)$. Using $\Delta_h^2(\sin(n\theta)) = -4\sin^2(nh/2)\sin(n\theta)$, we have for every $\theta \in \mathbb{R}$ and $h > 0$,

$$|\Delta_h^2 W_\mu(\theta)| \leq \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{c_n}{n} \sin^2\left(\frac{nh}{2}\right). \tag{2}$$

For $h > 1$, the right-hand side is at most $\frac{8}{\pi} M(\mu) < \frac{12}{\pi} M(\mu) h$. For $0 < h \leq 1$, set $K := \lfloor \log_2(1/h) \rfloor \geq 0$ (so $2^K \leq 1/h < 2^{K+1}$) and split (2) at 2^{K+1}: using $|\sin(nh/2)| \leq nh/2$ on $n < 2^{K+1}$ and $\sin^2(nh/2) \leq 1$ on $n \geq 2^{K+1}$,

$$\sum_{k=0}^{K} \sum_{n \in B_k} \frac{c_n}{n} \sin^2\left(\frac{nh}{2}\right) \leq \frac{h^2}{4} \sum_{k=0}^{K} 2^{k+1} s_k(\mu) \leq \frac{h^2}{4}\, 2^{K+2} M(\mu) \leq M(\mu) h,$$

$$\sum_{k=K+1}^{\infty} \sum_{n \in B_k} \frac{c_n}{n} \sin^2\left(\frac{nh}{2}\right) \leq \sum_{k=K+1}^{\infty} 2^{-k} s_k(\mu) \leq 2^{-K} M(\mu) < 2M(\mu) h.$$

Combining both pieces yields $\|W_\mu\|_{\Lambda_*} \le \frac{12}{\pi} M(\mu)$. If $s_k(\mu) \to 0$, given any $\varepsilon > 0$ pick K_0 with $s_k(\mu) \le \varepsilon$ for $k \ge K_0$; for $0 < h \le 2^{-K_0}$, the prefix $k < K_0$ contributes at most $\frac{h^2}{4}\, 2^{K_0+1} M(\mu) = O_{K_0}(h^2)$ while the ranges $K_0 \le k \le K$ and $k > K$ contribute at most εh and $2\varepsilon h$, respectively, giving $\limsup_{h\to 0^+} \sup_\theta |\Delta_h^2 W_\mu(\theta)|/h \le \frac{12}{\pi}\varepsilon \to 0$, so $\mu \in \lambda_*$.

Part (a), necessity. Suppose $\mu \in \Lambda_*$. Define the holomorphic function $H(z) := \frac{1}{\pi}\sum_{n=1}^\infty \frac{c_n}{n} z^n$ on $\mathbb{D}$, and set $U(\theta) := W_\mu(\theta) - \widehat{W_\mu}(0)$ and $\Phi(z) := (z\frac{d}{dz})^2 H(z) = \frac{1}{\pi}\sum_{n=1}^\infty n c_n z^n$. For $z = re^{i\theta} \in \mathbb{D}$, $\operatorname{Im} H(re^{i\theta}) = (P_r * U)(\theta)$, where $P_r(t) = (1-r^2)/D(t)$ is the Poisson kernel with $D(t) = (1-r)^2 + 4r\sin^2(t/2)$. Since P_r'' is even with mean zero,

$$-\operatorname{Im}\Phi(re^{i\theta}) = \partial_{\theta\theta}(P_r * U)(\theta) = \frac{1}{4\pi}\int_{-\pi}^{\pi} P_r''(t)\, \Delta_t^2 U(\theta)\, dt.$$

Direct differentiation gives $P_r''(t) = (1-r^2)\big(-2r\cos t/D(t)^2 + 8r^2\sin^2 t/D(t)^3\big)$. Because $\sin^2 t \le 4\sin^2(t/2)$ and $4r\sin^2(t/2) \le D(t)$, we have $|P_r''(t)| \le 20r(1-r)/D(t)^2$. Using $\sin(t/2) \ge t/\pi$ on $[0,\pi]$ yields, for every $r \in (0,1)$,

$$\int_0^\pi t|P_r''(t)|\, dt \le 20r(1-r)\int_0^\infty \frac{t\, dt}{\big((1-r)^2 + \frac{4r}{\pi^2}t^2\big)^2} = \frac{5\pi^2}{2(1-r)},$$

so $|\operatorname{Im}\Phi(re^{i\theta})| \le \frac{5\pi}{4}\frac{\|W_\mu\|_{\Lambda_*}}{1-r}$ on $\mathbb{D}$ (holding trivially at $r = 0$ since $\Phi(0) = 0$). Applying the Schwarz integral formula to $-i\Phi$ (with $\Phi(0) = 0$) on $|z| < R := (1+\rho)/2$, and using the Poisson-kernel normalization $\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2-\rho^2}{|Re^{it}-\rho|^2}\, dt = 1$, gives

$$|\Phi'(\rho)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2Re^{it}}{(Re^{it}-\rho)^2} \operatorname{Im}\Phi(Re^{it})\, dt\right| \le \frac{5\pi}{4}\frac{\|W_\mu\|_{\Lambda_*}}{1-R}\cdot\frac{2R}{R^2-\rho^2} \le \frac{10\pi\|W_\mu\|_{\Lambda_*}}{(1-\rho)^2},$$

and integrating from 0 to $r \in (0,1)$ yields $\frac{1}{\pi}\sum_{n=1}^\infty n c_n r^n = \Phi(r) \le \frac{10\pi\|W_\mu\|_{\Lambda_*}}{1-r}$. Evaluating at $r = 1 - 2^{-(k+1)}$ (where every $n \in B_k$ satisfies $n \le 2^{k+1}-1$ and hence $r^n \ge (1-2^{-(k+1)})^{2^{k+1}-1} > e^{-1}$) gives $\frac{2^k}{e} s_k(\mu) \le 20\pi^2\|W_\mu\|_{\Lambda_*} 2^k$, hence $M(\mu) \le 20\pi^2 e\, \|W_\mu\|_{\Lambda_*}$. If in addition $\mu \in \lambda_*$, then for any $\varepsilon > 0$ we may pick $\eta \in (0,\pi)$ with $|\Delta_t^2 U(\theta)| \le \varepsilon|t|$ for all $|t| \le \eta$; splitting the P_r''-integral at $|t| \le \eta$ (contributing at most $\frac{5\pi}{4}\frac{\varepsilon}{1-r}$) versus $|t| > \eta$ (where $D(t) \ge 4r\sin^2(\eta/2)$ yields $O_{\eta,\mu}(1-r) = o((1-r)^{-1})$) shows $(1-r)\sup_\theta |\operatorname{Im}\Phi(re^{i\theta})| \to 0$ as $r \to 1^-$. The same Schwarz-integral and radial-integration step (splitting the radial integral at a fixed threshold $\rho_0 \in (0,1)$) then gives $(1-r)\Phi(r) \to 0$, and evaluating at $r_k = 1 - 2^{-(k+1)}$ yields $s_k(\mu) \to 0$.

Part (b). Because $a_n = 2c_n \ge 0$, $|F'(re^{i\theta})| \le F'(r) = 2\sum_{n=1}^\infty n c_n r^{n-1}$ with equality at $\theta = 0$. Taking $r_k = 1-2^{-(k+1)} \in [1/2, 1)$ gives $n r_k^{n-1} > 2^k/e$ on B_k and $1 - r_k^2 \ge 2^{-(k+1)}$, so $(1-r_k^2)F'(r_k) \ge e^{-1} s_k(\mu)$; this simultaneously yields $M(\mu) \le e\|F\|_{\mathcal{B}}$ and shows that $F \in \mathcal{B}_0$ implies $s_k(\mu) \to 0$. Conversely, for any $r \in [0,1)$,

$$(1-r^2)|F'(re^{i\theta})| \le 4(1-r)\sum_{k=0}^\infty s_k(\mu)\, 2^{k+1} r^{2^k-1}. \tag{3}$$

For $r \in [0, 1/2]$, $\sum_{k=0}^\infty 2^{k+1} r^{2^k-1} \le 2 + \sum_{k=1}^\infty 2^{k+1} 2^{-(2^k-1)} < 6$, giving $(1-r^2)|F'| \le 24 M(\mu)$. For $r \in [1/2, 1)$, $r^{2^k-1} \le 2r^{2^k}$ and $2^k r^{2^k} \le 2\int_{2^{k-1}}^{2^k} r^x\, dx$ $(k \ge 1)$, so

$$\sum_{k=0}^\infty 2^{k+1} r^{2^k-1} \le 2 + 8\int_1^\infty r^x\, dx = 2 + \frac{8r}{\ln(1/r)} \le \frac{10}{1-r}$$

(since $\ln(1/r) \ge 1 - r$). Substituting into (3) gives $\|F\|_{\mathcal{B}} \le 40 M(\mu)$. Finally, if $s_k(\mu) \to 0$, splitting (3) at any fixed K_0 with $\sup_{k \ge K_0} s_k(\mu) \le \varepsilon$ shows $\limsup_{r\to 1^-} \sup_\theta (1-r^2)|F'(re^{i\theta})| \le 40\varepsilon \to 0$, so $F \in \mathcal{B}_0$. □

3. A purely singular little-Zygmund measure with non-negative Fourier coefficients

In this section we prove Theorem 1.1, which simultaneously refutes the first part of Holland's Problem 5.69 and answers Holland's Problem 8.13 affirmatively in the strongest form: inside the little Zygmund class λ_* (equivalently, the little Bloch space $\mathcal{B}_0$). We also record two short related constructions: localization to an arbitrary interval $[-\delta, \delta]$ with *strictly* positive Fourier–Stieltjes coefficients (Corollary 3.1), and an absolutely continuous counterexample to Problem 5.69 given by a single analytic square (Proposition 3.2).

Why do classical Riesz products $\prod_k (1 + a_k\cos(n_k\theta))$ fail here? By Duren [5] and Holland–Twomey [13], a classical lacunary Riesz product $(n_{k+1}/n_k \ge 3)$ belongs to Λ_* only when $\sum a_k^2 < \infty$, which forces an L^2 density. The key idea of Theorem 1.1 is to replace single cosine factors by *non-flat lacunary analytic squares* $Q_R = \big|\sum_{j=1}^R a_j e^{i16^j\theta}\big|^2$ with non-uniform, harmonic-decay weights $a_j = (jH_R)^{-1/2}$. Each factor has dyadic block sums bounded by $2/H_R \to 0$ yet maintains a second moment $\int Q_R^2\, dm \approx 2$ uniformly in R and a uniformly bounded third moment $\int Q_R^3\, dm \le 6$. Iterating widely dilated copies $X_t(\theta) = Q_{R_t}(L_t\theta)$ drives the L^2 norm to infinity geometrically while keeping all dyadic block sums decaying to zero and separating all moments up to order three, which forces pure singularity via Borel–Cantelli.

Proof of Theorem 1.1. We construct the measure μ in five elementary steps.

Step 1: The lacunary analytic-square building block Q_R. For any integer $R \geq 2$, let $H_R := \sum_{j=1}^{R} j^{-1}$ be the R-th harmonic number, set $a_j := (jH_R)^{-1/2} > 0$ and $d_j := 16^j$ for $1 \leq j \leq R$, and define the non-negative even trigonometric polynomial

$$Q_R(\theta) := \left| \sum_{j=1}^{R} a_j e^{id_j\theta} \right|^2 = 1 + 2 \sum_{1 \leq i < j \leq R} a_i a_j \cos\big((d_j - d_i)\theta\big). \tag{4}$$

By construction, $Q_R \geq 0$, $\widehat{Q_R}(0) = \sum_{j=1}^{R} a_j^2 = 1$, and $\widehat{Q_R}(n) \geq 0$ for every $n \in \mathbb{Z}$.

For $1 \leq i < j \leq R$, the positive differences $d_j - d_i = 16^j - 16^i$ are pairwise distinct: indeed, if $j = j'$ and $i \neq i'$ then $|(d_j - d_i) - (d_j - d_{i'})| = |16^{i'} - 16^i| \geq 240$, while if $j > j'$ then $(d_j - d_i) - (d_{j'} - d_{i'}) \geq 15 \cdot 16^{j-1} - 16^{j-1} = 14 \cdot 16^{j-1} \geq 3584$. Thus distinct frequencies in $\operatorname{supp}(\widehat{Q_R})$ are separated by at least 240. Moreover, each difference with top index $j \geq 2$ satisfies

$$2^{4j-1} < \frac{15}{16}\, 16^j \leq d_j - d_i \leq 16^j - 16 < 2^{4j},$$

so it lies strictly inside the single dyadic block B_{4j-1}, with distinct j occupying distinct blocks. Since $t \mapsto t^{-1/2}$ is decreasing on $(0, \infty)$, we have $\sum_{i=1}^{j-1} i^{-1/2} < \int_0^{j-1} t^{-1/2}\, dt = 2\sqrt{j-1} < 2\sqrt{j}$, and hence each occupied positive block has sum

$$\sum_{n \in B_{4j-1}} \widehat{Q_R}(n) = a_j \sum_{i=1}^{j-1} a_i = \frac{1}{H_R \sqrt{j}} \sum_{i=1}^{j-1} \frac{1}{\sqrt{i}} < \frac{2}{H_R}. \tag{5}$$

Because all nonzero frequencies $\pm(d_j - d_i)$ are distinct and $\max_j a_j^2 = 1/H_R$, Parseval's identity gives

$$\int_{\mathbb{T}} Q_R^2\, dm = 1 + 2 \sum_{1 \leq i < j \leq R} a_i^2 a_j^2 = 2 - \sum_{j=1}^{R} a_j^4 \geq 2 - \frac{1}{H_R}. \tag{6}$$

For the third moment, expand $\big(\sum_{j=1}^{R} a_j e^{i16^j\theta}\big)^3 = \sum_{j_1, j_2, j_3 = 1}^{R} a_{j_1} a_{j_2} a_{j_3} e^{i(16^{j_1} + 16^{j_2} + 16^{j_3})\theta}$. In base 16, every sum $16^{j_1} + 16^{j_2} + 16^{j_3}$ has digit sum $3 < 16$ and hence no carries; the exponent therefore determines the unordered multiset $S = \{j_1, j_2, j_3\}$ uniquely. Letting $m_S \in \{1, 3, 6\}$ be the number of ordered triples realizing S, Parseval's identity yields

$$\int_{\mathbb{T}} Q_R^3\, dm = \sum_S m_S^2 \prod_{j \in S} a_j^2 \leq 6 \sum_S m_S \prod_{j \in S} a_j^2 = 6 \left(\sum_{j=1}^{R} a_j^2 \right)^3 = 6. \tag{7}$$

Step 2: Inductive dilated products and decaying dyadic block sums. Set $P_0 \equiv 1$ (with degree $D_0 := 0$). Inductively for $t = 1, 2, \ldots$, let $D_{t-1} := \deg(P_{t-1})$, choose an integer $R_t \geq 2$ large enough that

$$H_{R_t} \geq \max\big(4,\ 2t\, P_{t-1}(0)\big),$$

and choose a power of two $L_t = 2^{\ell_t} > 8(D_{t-1} + 1)$. Define

$$X_t(\theta) := Q_{R_t}(L_t\theta), \qquad P_t(\theta) := P_{t-1}(\theta) X_t(\theta).$$

Each P_t is a non-negative even trigonometric polynomial with $\widehat{P_t} = \widehat{P_{t-1}} * \widehat{X_t} \geq 0$. Every frequency in $\operatorname{supp}(\widehat{P_t})$ has the form $L_t n + h$ with $n \in \operatorname{supp}(\widehat{Q_{R_t}})$ and $|h| \leq D_{t-1} < L_t/8$. Since distinct frequencies of Q_{R_t} differ by at least 240, the translates $L_t n + [-D_{t-1}, D_{t-1}]$ are pairwise disjoint:

- For $n = 0$, the frequencies stay in $[-D_{t-1}, D_{t-1}]$ and $\widehat{P_t}(h) = \widehat{P_{t-1}}(h)$ identically for all $|h| \leq D_{t-1}$.
- For $n < 0$, we have $n \leq -240$ and hence $L_t n + h \leq -240 L_t + D_{t-1} < 0$, so negative frequencies of X_t never contribute to positive frequencies of P_t.
- For $n = d_j - d_i > 0$ $(1 \leq i < j \leq R_t)$, since $d_j \geq 16^2 = 256$ and $|h| < L_t/8$,

$$2^{\ell_t + 4j - 1} = \frac{1}{2} L_t d_j < L_t \left(\frac{15}{16} d_j - \frac{1}{8} \right) \leq L_t (d_j - d_i) + h \leq L_t \left(d_j - 16 + \frac{1}{8} \right) < L_t d_j = 2^{\ell_t + 4j}.$$

Because $2^{\ell_t + 4j - 1} \geq 128 L_t > D_{t-1}$, each $j \in \{2, \ldots, R_t\}$ populates the single fresh dyadic block $B_{\ell_t + 4j - 1}$ strictly above all blocks of P_{t-1} (see Figure 2). Since $\sum_{|h| \leq D_{t-1}} \widehat{P_{t-1}}(h) = P_{t-1}(0)$, inequality (5) gives

$$\sum_{n \in B_{\ell_t + 4j - 1}} \widehat{P_t}(n) = P_{t-1}(0) \cdot a_j \sum_{i=1}^{j-1} a_i < P_{t-1}(0) \cdot \frac{2}{H_{R_t}} \leq \frac{1}{t}. \tag{8}$$

Thus every positive dyadic block sum of P_t is at most 1, and every block populated at stage t has sum at most $1/t$.

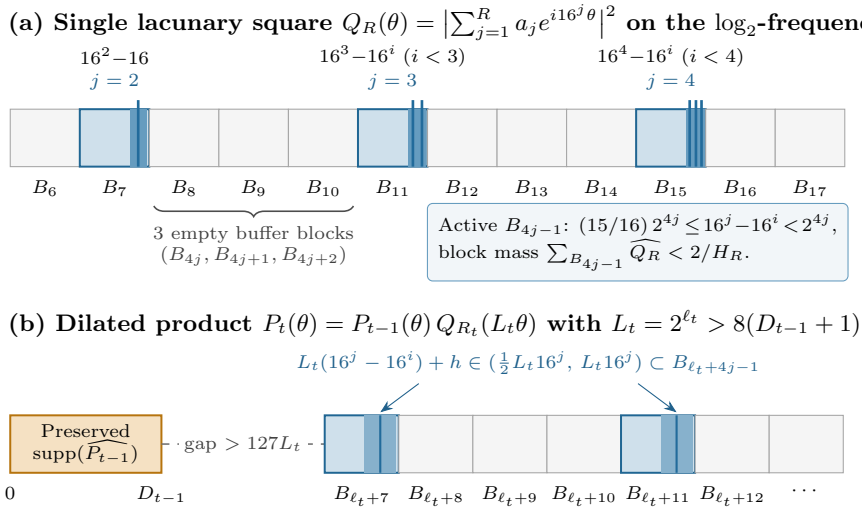


FIGURE 2. Dyadic block isolation in the proof of Theorem 1.1 (where $H_R := \sum_{j=1}^{R} j^{-1}$ and $D_{t-1} := \deg(P_{t-1})$). **(a)** In the building block Q_R, every positive difference $16^j - 16^i$ $(1 \le i < j \le R)$ falls in $[\frac{15}{16}\cdot 2^{4j}, 2^{4j}) \subset B_{4j-1}$ (shaded dark blue), leaving three empty buffer blocks between consecutive active blocks and capping each block sum at $< 2/H_R$. **(b)** Multiplying P_{t-1} by $Q_{R_t}(L_t\theta)$ with $L_t = 2^{\ell_t} > 8(D_{t-1}+1)$ broadens each frequency by $|h| \le D_{t-1} < L_t/8$ yet keeps every new positive frequency strictly inside its single fresh dyadic block B_{ℓ_t+4j-1}, preserving all earlier coefficients and driving block sums to 0.

Step 3: Moment separation up to power 3*, the limit measure* μ*, and* $\hat{\mu} \notin \ell^2$. We claim that for any $T \ge 1$ and any integers $0 \le e_1, \dots, e_T \le 3$,

$$\int_{\mathbb{T}} \prod_{i=1}^{T} X_i^{e_i}\, dm = \prod_{i=1}^{T} \int_{\mathbb{T}} X_i^{e_i}\, dm. \tag{9}$$

Indeed, if all $e_i = 0$ this is trivial. Otherwise let i be the largest index with $e_i \ge 1$. Non-negativity of $\widehat{X_s}$ prevents cancellation of top frequencies, so $\deg(P_{i-1}) = \sum_{s=1}^{i-1} \deg(X_s)$ and hence $\deg\big(\prod_{s<i} X_s^{e_s}\big) \le 3\sum_{s<i}\deg(X_s) = 3D_{i-1} < L_i$. On the other hand, every nonzero frequency of $X_i^{e_i}$ is a nonzero multiple of L_i and thus has magnitude at least L_i. Only the constant term $\int_{\mathbb{T}} X_i^{e_i}\, dm$ of $X_i^{e_i}$ can pair with $\prod_{s<i} X_s^{e_s}$ to produce frequency 0, and downward induction on i yields (9).

Taking $e_1 = \cdots = e_t = 1$ shows that $P_t\, dm$ is a Borel probability measure on $\mathbb{T}$. For every $n \in \mathbb{Z}$, $\widehat{P_t}(n)$ is eventually constant (once $D_t \ge |n|$). By weak-$*$ compactness of Borel probability measures on $\mathbb{T}$ and density of trigonometric polynomials in $C(\mathbb{T})$, there is a unique even Borel probability measure μ on $\mathbb{T}$ such that $P_t\, dm \xrightarrow{*} \mu$, with $\hat{\mu}(n) = \widehat{P_t}(n) \ge 0$ for $|n| \le D_t$. By (8), $s_k(\mu) \le 1$ for all $k \ge 0$ and $\lim_{k\to\infty} s_k(\mu) = 0$.

Applying (9) with $e_1 = \cdots = e_t = 2$ and (6) (noting $H_{R_s} \ge 4$) yields

$$1 + 2\sum_{n=1}^{D_t} \hat{\mu}(n)^2 = \int_{\mathbb{T}} P_t^2\, dm = \prod_{s=1}^{t} \int_{\mathbb{T}} Q_{R_s}^2\, dm \ge \left(\frac{7}{4}\right)^t \xrightarrow{t\to\infty} +\infty,$$

proving $\sum_{n=1}^{\infty} \hat{\mu}(n)^2 = +\infty$. Conversely, for every $q > 2$, the pairwise disjointness of the translates $L_s n + [-D_{s-1}, D_{s-1}]$ gives

$$\sum_{n\in\mathbb{Z}} \widehat{P_t}(n)^q = \prod_{s=1}^{t} \left(1 + 2 \sum_{1\le i<j\le R_s} a_{s,i}^q a_{s,j}^q\right) \le \prod_{s=1}^{t} \left(1 + \frac{\zeta(q/2)^2}{H_{R_s}^q}\right),$$

where $a_{s,j} := (jH_{R_s})^{-1/2}$. Since $H_{R_s} \ge 2sP_{s-1}(0) \ge 2s$ (as $P_{s-1}(0) \ge \widehat{P_{s-1}}(0) = 1$) and $q > 2$, the infinite product converges, so $\hat{\mu} \in \ell^q(\mathbb{Z})$ for every $q > 2$.

Step 4: Pure singularity $\mu \perp m$. Under m, (9) gives $\mathbb{E}_m[X_i] = 1$, $\operatorname{Cov}_m(X_i, X_j) = 0$ for $i \ne j$, and $\operatorname{Var}_m(X_i) = \int Q_{R_i}^2\, dm - 1 \le 1 \le 6$. Under μ, because each X_i is continuous, testing X_i, X_iX_j $(i \ne j)$, and X_i^2 against $P_T\, dm$ with $T \ge \max(i,j)$ (all exponents in $\{1,2,3\}$) and letting $T \to \infty$ via (9), (6), and (7) gives

$$\mathbb{E}_\mu[X_i] = \int_{\mathbb{T}} Q_{R_i}^2\, dm \ge \frac{7}{4}, \qquad \operatorname{Cov}_\mu(X_i, X_j) = 0 \ \ (i \ne j), \qquad \operatorname{Var}_\mu(X_i) \le \int_{\mathbb{T}} Q_{R_i}^3\, dm \le 6.$$

Define the averages $A_k := 2^{-k}\sum_{i=1}^{2^k} X_i$ ($k \geq 1$). By pairwise uncorrelatedness, $\mathrm{Var}_\nu(A_k) \leq 6\cdot 2^{-k}$ for both $\nu \in \{m, \mu\}$. Chebyshev's inequality gives $\nu\big(|A_k - \mathbb{E}_\nu[A_k]| \geq k^{-1}\big) \leq 6k^2 2^{-k}$, which is summable in k, so the Borel–Cantelli lemma yields $A_k - \mathbb{E}_\nu[A_k] \to 0$ ν-almost everywhere. Consequently $A_k \to 1$ m-a.e., whereas $\liminf_{k\to\infty} A_k \geq 7/4$ μ-a.e. Thus m and μ concentrate on disjoint Borel sets, proving $\mu \perp m$.

Step 5: Little Zygmund and little Bloch regularity. Since $s_k(\mu) \to 0$, Lemma 2.1 implies immediately that μ is atomless, $\mu \in \lambda_* \subset \Lambda_*$, and its Herglotz function $F(z) = 1 + 2\sum_{n=1}^\infty \hat\mu(n) z^n$ satisfies $\mathrm{Re}\, F > 0$ on $\mathbb{D}$ and $F \in \mathcal{B}_0 \subset \mathcal{B}$, with $\sum_{n=1}^\infty a_n^2 = 4\sum_{n=1}^\infty \hat\mu(n)^2 = +\infty$. (In addition, $\mu \perp m$ implies $F \notin H^1(\mathbb{D})$ by the F. and M. Riesz theorem.) □

Corollary 3.1 (Full support and arbitrary $[-\delta,\delta]$ localization with $\hat\mu > 0$ strictly)**.** *The measure μ of Theorem 1.1 has full topological support* $\mathrm{supp}(\mu) = \mathbb{T}$. *Moreover, for every $\delta \in (0,\pi)$, there exists an even, positive, purely singular, atomless Borel probability measure $\mu_\delta \in \lambda_*$ supported exactly on $[-\delta,\delta]$ such that $\widehat{\mu_\delta}(n) > 0$ strictly for every $n \in \mathbb{Z}$ and $\widehat{\mu_\delta} \notin \ell^2(\mathbb{Z})$.*

Proof. For any fixed $t \geq 1$, the tail products $\prod_{s=t+1}^T X_s\, dm$ converge weak-$*$ to a probability measure ν_t satisfying $\mu = P_t\,\nu_t$. Applying Steps 2–5 starting at stage $t+1$ shows that ν_t is atomless, and since $L_{t+1} \mid L_s$ for every $s > t$ (all L_s being increasing powers of two), ν_t is invariant under rotation by $2\pi/L_{t+1}$. Any open arc $I \subset \mathbb{T}$ of length greater than $2\pi/L_{t+1}$ therefore has $\nu_t(I) \geq 1/L_{t+1} > 0$; since P_t is a nonzero trigonometric polynomial (hence has finitely many zeros), $\mu(I) > 0$. As $L_{t+1} \to \infty$, $\mathrm{supp}(\mu) = \mathbb{T}$ (that is, every open arc meets the dense Lebesgue-null Borel set carrying the full mass of μ).

Now fix $\delta \in (0,\pi)$ and let $\psi_\delta(x) := e^{-x/\delta}\mathbf{1}_{[0,\delta]}(x)$ on $[-\pi,\pi)$. Its periodic autocorrelation $H_\delta := \psi_\delta * \widetilde{\psi_\delta}$ (where $\widetilde{\psi_\delta}(x) = \psi_\delta(-x)$) is continuous, even, strictly positive on $(-\delta,\delta)$, and supported exactly on $[-\delta,\delta]$ (no wraparound occurs since $2\delta < 2\pi$). A direct integration gives

$$h_\ell := \widehat{H_\delta}(\ell) = |\widehat{\psi_\delta}(\ell)|^2 = \frac{\delta^2|1 - e^{-1-i\ell\delta}|^2}{4\pi^2(1+(\ell\delta)^2)} \qquad (\ell \in \mathbb{Z}).$$

Because $|e^{-1-i\ell\delta}| = e^{-1} < 1$, we have $0 < h_\ell \leq C_\delta/(1+\ell^2)$ for every $\ell \in \mathbb{Z}$. Since $\sum_{\ell\in\mathbb{Z}} h_\ell < \infty$, the Fourier series of H_δ converges uniformly, so $Z := \int_{\mathbb{T}} H_\delta\, d\mu = \sum_{\ell\in\mathbb{Z}} h_\ell \hat\mu(-\ell) \geq h_0 > 0$. Setting $d\mu_\delta := Z^{-1} H_\delta\, d\mu$ produces an even, positive, purely singular, atomless probability measure with $\mathrm{supp}(\mu_\delta) = [-\delta,\delta]$,

$$\widehat{\mu_\delta}(n) = Z^{-1}\sum_{\ell\in\mathbb{Z}} h_\ell \hat\mu(n-\ell) \geq Z^{-1}\max\big(h_n,\ h_0\hat\mu(n)\big) > 0 \qquad (\forall n \in \mathbb{Z}),$$

and hence $\widehat{\mu_\delta} \notin \ell^2(\mathbb{Z})$. To check $\mu_\delta \in \lambda_*$, fix $k \geq 1$ and split the convolution sum for $\sum_{n\in B_k} \widehat{\mu_\delta}(n)$ at $|\ell| \leq 2^{k-1}$ versus $|\ell| > 2^{k-1}$. For $|\ell| \leq 2^{k-1}$, we have $n - \ell \in B_{k-1} \cup B_k \cup B_{k+1}$, contributing at most $Z^{-1}(\sum_\ell h_\ell)(s_{k-1}(\mu) + s_k(\mu) + s_{k+1}(\mu)) \to 0$. For any fixed $|\ell| > 2^{k-1}$, the shifted interval $B_k - \ell$ has length 2^k, so it meets at most $2k+2$ signed dyadic blocks $\pm B_j$ together with $\{0\}$, giving $\sum_{n\in B_k} \hat\mu(n-\ell) \leq 1 + (2k+2)M(\mu)$. Summing against $h_\ell \leq C_\delta/(1+\ell^2)$ over $|\ell| > 2^{k-1}$ adds $O_\delta(k2^{-k}) \to 0$. Hence $s_k(\mu_\delta) \to 0$, and Lemma 2.1 yields $\mu_\delta \in \lambda_*$. □

Finally, the failure of Problem 5.69 is not caused solely by singularity: an absolutely continuous probability density in λ_* can also fail to lie in $L^2(\mathbb{T})$.

Proposition 3.2 (Absolutely continuous single-square counterexample to Problem 5.69(a))**.** *There exists $G \in H^2(\mathbb{D})$ with non-negative Taylor coefficients $\hat G(n) \geq 0$ for all $n \geq 0$ such that the probability density $P := |G|^2/\|G\|_2^2 \in L^1(\mathbb{T}) \setminus L^2(\mathbb{T})$ satisfies $\hat P(n) \geq 0$ for all $n \in \mathbb{Z}$ and $\lim_{k\to\infty} s_k(P) = 0$ (hence $P\,dm \in \lambda_* \subset \Lambda_*$).*

Proof. We first note a finite building block: given any $T > 0$ and $\eta > 0$, take a finite analytic factor $A(\theta) = \prod_{s=1}^t\big(\sum_{j=1}^{R_s} a_{s,j} e^{iL_s 16^j\theta}\big)$ from Theorem 1.1 with $P_t = |A|^2$, $\hat A \geq 0$, $\|A\|_2 = 1$, $\int |A|^4\, dm \geq T$, and $s_k(|A|^2) \leq 1$ for all k. Choose a power of two $L = 2^\ell > 8(\deg A + \deg(|A|^2) + 1)$ and $R \geq 2$ with $H_R \geq \max\big(4,\ 2A(0)^2,\ (A(0)/\eta)^2\big)$, and set

$$\Phi(\theta) := A(\theta)\sum_{j=1}^R (jH_R)^{-1/2} e^{iL16^j\theta}.$$

Then $\hat\Phi \geq 0$, $\hat\Phi(0) = 0$, $\|\Phi\|_2 = 1$, $\int |\Phi|^4\, dm = \int |A|^4\, dm \cdot \int Q_R^2\, dm \geq \frac{7}{4}T > T$ (by (9)), and $s_k(|\Phi|^2) \leq 1$ for all k. Moreover, every frequency of Φ takes the form $L16^j + h$ ($0 \leq h \leq \deg A < L/8$), which lies in $B_{\ell+4j}$; thus distinct j occupy distinct blocks, and every dyadic block sum of Φ itself satisfies $\sum_{n\in B_k} \hat\Phi(n) \leq A(0)/\sqrt{H_R} \leq \eta$.

Now construct Φ_s and powers of two D_s inductively for $s = 1, 2, \ldots$. Put $\lambda_s := 2^{-s}$ and $G_{s-1} := \sum_{r=1}^{s-1} \lambda_r \Phi_r(D_r\,\cdot)$ (with $G_0 \equiv 0$). Choose Φ_s as above with $T = 2^{4s}$ and $\eta_s := \big(2\max(1, G_{s-1}(0))\big)^{-1}$, and choose a power of two $D_s > 4(\deg G_{s-1} + 1)$. Because $\hat\Phi_s(0) = 0$, the summands $\lambda_s\Phi_s(D_s\,\cdot)$ have pairwise disjoint positive frequency supports, so $G_s \to G := \sum_{s=1}^\infty \lambda_s\Phi_s(D_s\,\cdot)$ in $H^2(\mathbb{D})$ with $\|G\|_2^2 = \sum_{s=1}^\infty 4^{-s} = 1/3$ and $\hat G \geq 0$. Hence $|G_s|^2 \to |G|^2$ in $L^1(\mathbb{T})$ with $\widehat{|G|^2} \geq 0$.

For each $s \geq 1$, set $Q_s := |\Phi_s(D_s\,\cdot)|^2$ and write $|G_s|^2 - |G_{s-1}|^2 = \lambda_s^2 Q_s + \lambda_s\big(\Phi_s(D_s\,\cdot)\overline{G_{s-1}} + \overline{\Phi_s(D_s\,\cdot)}G_{s-1}\big)$. The self-term contributes at most $\lambda_s^2 = 4^{-s}$ to any positive dyadic block. For the cross-term, positive frequencies arise solely from $\lambda_s\Phi_s(D_s\,\cdot)\overline{G_{s-1}}$ (with coefficient λ_s) at frequencies $D_s n - h > 0$ where $n \in \operatorname{supp}(\hat{\Phi}_s)$ ($n \geq 1$) and $0 \leq h \leq \deg G_{s-1} < D_s/4$; since $h < D_s n/4$, each dyadic block B_k meets translates from at most two dyadic blocks of Φ_s, adding at most $2\lambda_s\eta_s G_{s-1}(0) \leq 2^{-s}$ to $s_k(|G_s|^2 - |G_{s-1}|^2)$. By subadditivity of s_k on the L^1-convergent telescoping series $|G|^2 = \sum_{s=1}^{\infty}(|G_s|^2 - |G_{s-1}|^2)$, the fact that $\sum_{s=1}^{\infty}(2^{-s} + 4^{-s}) < \infty$, and the finite degree of each G_s, we obtain $\lim_{k\to\infty} s_k(|G|^2) = 0$. Finally, non-negativity of all correlations gives $\widehat{|G|^2}(n) \geq \sum_{s=1}^{\infty}\lambda_s^2\widehat{Q_s}(n)$ for every $n \in \mathbb{Z}$, so by Parseval's identity,

$$\int_{\mathbb{T}} |G|^4\,dm \geq \sum_{s=1}^{\infty}\lambda_s^4\int_{\mathbb{T}}|\Phi_s|^4\,dm \geq \sum_{s=1}^{\infty} 2^{-4s}\cdot 2^{4s} = +\infty.$$

Setting $P := |G|^2/\|G\|_2^2$ completes the proof. □

4. General Bloch functions with positive real part: proof of Theorem 1.2

Even though Problem 5.69 has a negative answer when $a_n \geq 0$, its general consequence asked by Holland [3, 10]—whether every Bloch function $F(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\operatorname{Re} F > 0$ on $\mathbb{D}$ satisfies $\sum_{n=1}^{\infty}|a_n|^4 < \infty$—turns out to be true. In fact, we obtain an explicit quadratic ℓ^4 bound together with ℓ^q summability for all $q > 3$ and an iterated-logarithm weighted estimate at the Hausdorff–Young endpoint $q = 3$.

The key device is to apply Green's identity directly to convex and concave functions of the positive harmonic real part $u = \operatorname{Re} F$, rather than to powers of $|F|$. Because u is harmonic and strictly positive on $\mathbb{D}$, its Laplacian identities require no control of the harmonic conjugate $\operatorname{Im} F$ and avoid all trigonometric blow-up factors as $q \to 3^+$.

Lemma 4.1 (Real-part harmonic Green–Hardy–Stein identity)**.** *Let $F = u + iv$ be holomorphic on $\mathbb{D}$ with $u = \operatorname{Re} F > 0$ on $\mathbb{D}$ and $u(0) = 1$. For every $R \in (0,1)$, the following identities hold:*

(i) *For every $\alpha \in (0,1)$,*

$$\alpha(1-\alpha)\int_0^R\left(\int_{\mathbb{T}}\frac{|F'(re^{i\theta})|^2}{u(re^{i\theta})^{2-\alpha}}\,dm(\theta)\right)\ln\!\left(\frac{R}{r}\right)r\,dr = 1 - \int_{\mathbb{T}} u(Re^{i\theta})^{\alpha}\,dm(\theta) < 1, \tag{10}$$

and consequently

$$\int_0^1\left(\int_{\mathbb{T}}\frac{|F'(re^{i\theta})|^2}{u(re^{i\theta})^{2-\alpha}}\,dm(\theta)\right)\ln\!\left(\frac{1}{r}\right)r\,dr \leq \frac{1}{\alpha(1-\alpha)}. \tag{11}$$

(ii) *At the endpoint $\alpha = 1$,*

$$\int_0^R\left(\int_{\mathbb{T}}\frac{|F'(re^{i\theta})|^2}{u(re^{i\theta})}\,dm(\theta)\right)\ln\!\left(\frac{R}{r}\right)r\,dr = \int_{\mathbb{T}} u(Re^{i\theta})\ln u(Re^{i\theta})\,dm(\theta). \tag{12}$$

Proof. Let $g \in C^2(\mathbb{D})$ and define the circle mean $M_g(r) := \int_{\mathbb{T}} g(re^{i\theta})\,dm(\theta)$ for $r \in [0,1)$. Writing the Laplacian in polar coordinates,

$$\Delta g = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 g}{\partial\theta^2},$$

and integrating over $\theta \in [0, 2\pi)$ against $dm(\theta) = d\theta/(2\pi)$, the periodic θ-derivative vanishes and we obtain $(rM_g'(r))' = r\int_{\mathbb{T}}(\Delta g)(re^{i\theta})\,dm(\theta)$ for every $r \in (0,1)$. Integrating by parts against $\ln(R/r)$ on $[\varepsilon, R]$ $(0 < \varepsilon < R < 1)$ gives

$$\int_{\varepsilon}^{R}\left(\int_{\mathbb{T}}(\Delta g)(re^{i\theta})\,dm(\theta)\right)\ln\!\left(\frac{R}{r}\right)r\,dr = -\varepsilon M_g'(\varepsilon)\ln\!\left(\frac{R}{\varepsilon}\right) + M_g(R) - M_g(\varepsilon).$$

Since $g \in C^2(\mathbb{D})$, Δg is bounded on $\overline{B(0,R)}$, M_g' is bounded on $[0,R]$, and $r\ln(R/r) \in L^1(0,R)$, so letting $\varepsilon \to 0^+$ sends $\varepsilon M_g'(\varepsilon)\ln(R/\varepsilon) \to 0$ and $M_g(\varepsilon) \to g(0)$, yielding

$$\int_0^R\left(\int_{\mathbb{T}}(\Delta g)(re^{i\theta})\,dm(\theta)\right)\ln\!\left(\frac{R}{r}\right)r\,dr = M_g(R) - g(0). \tag{13}$$

Now $u = \operatorname{Re} F > 0$ is harmonic ($\Delta u = 0$), and the Cauchy–Riemann equations give $|\nabla u|^2 = u_x^2 + u_y^2 = |F'|^2$. Hence, for any $\varphi \in C^2((0,\infty))$, the chain rule gives $\Delta(\varphi(u)) = \varphi''(u)|\nabla u|^2 + \varphi'(u)\Delta u = \varphi''(u)|F'|^2$. Taking $\varphi(t) = -t^{\alpha}$ $(\alpha \in (0,1))$ gives $\Delta(-u^{\alpha}) = \alpha(1-\alpha)|F'|^2/u^{2-\alpha}$ and $g(0) = -u(0)^{\alpha} = -1$, so (13) yields (10); since $u > 0$, the right-hand side is strictly less than 1, and letting $R \uparrow 1^-$ via the monotone convergence theorem gives (11). Taking $\varphi(t) = t\ln t$ gives $\Delta(u\ln u) = |F'|^2/u$ and $g(0) = 1\cdot\ln 1 = 0$, so (13) yields (12). □

Proof of Theorem 1.2. Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be holomorphic on $\mathbb{D}$ with $\operatorname{Re} F > 0$ on $\mathbb{D}$ and finite Bloch seminorm $M_{\mathcal{B}} := \|F\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|F'(z)| < \infty$. If $M_{\mathcal{B}} = 0$, then F is constant and $a_n = 0$ for all $n \geq 1$, so all three inequalities hold trivially. Assume henceforth that $M_{\mathcal{B}} > 0$. Replacing F by $(F - i\operatorname{Im} F(0))/\operatorname{Re} F(0)$ rescales a_n

$(n \geq 1)$ and $M_{\mathcal{B}}$ by $(\operatorname{Re} F(0))^{-1}$ while preserving all three asserted inequalities; we may therefore assume without loss of generality that $F(0) = 1$ (so $u(0) = 1$). For $r \in (0,1)$, write $u_r(\theta) := u(re^{i\theta})$ and $F_r'(\theta) := F'(re^{i\theta}) = \sum_{n=1}^{\infty} n a_n r^{n-1} e^{i(n-1)\theta}$.

Parts (i) and (ii) ($q > 3$, including $q = 4$). Fix $q > 3$ and set $\alpha := \frac{2}{q-1} \in (0,1)$ and $p := \frac{q}{q-1} \in (1, \frac{3}{2})$, so that p and q are conjugate exponents with $q/p = q - 1$. For each $r \in (0,1)$, the Hausdorff–Young inequality on $\mathbb{T}$ applied to F_r' gives

$$\sum_{n=1}^{\infty} n^q |a_n|^q r^{q(n-1)} \leq \left(\int_{\mathbb{T}} |F_r'(\theta)|^p \, dm(\theta) \right)^{q-1}.$$

Factor the integrand as

$$|F_r'|^p = |F_r'|^{\frac{q-2}{q-1}} \left(\frac{|F_r'|^2}{u_r^{2-\alpha}} \right)^{\frac{1}{q-1}} u_r^{\frac{2-\alpha}{q-1}}.$$

Pulling out the Bloch bound $|F_r'| \leq M_{\mathcal{B}}/(1-r^2)$ and applying Hölder's inequality with conjugate exponents $q-1$ and $(q-1)/(q-2)$, we note that $\frac{2-\alpha}{q-2} = \frac{2-2/(q-1)}{q-2} = \frac{2}{q-1} = \alpha$, and obtain

$$\int_{\mathbb{T}} |F_r'|^p \, dm \leq \left(\frac{M_{\mathcal{B}}}{1-r^2} \right)^{\frac{q-2}{q-1}} \left(\int_{\mathbb{T}} \frac{|F_r'|^2}{u_r^{2-\alpha}} \, dm \right)^{\frac{1}{q-1}} \left(\int_{\mathbb{T}} u_r^{\alpha} \, dm \right)^{\frac{q-2}{q-1}}.$$

Since $u_r > 0$ is harmonic with $\int_{\mathbb{T}} u_r \, dm = u(0) = 1$ and $\alpha \in (0,1)$, Jensen's inequality gives $\int_{\mathbb{T}} u_r^{\alpha} \, dm \leq 1$. Raising both sides to the $(q-1)$-th power yields

$$\sum_{n=1}^{\infty} n^q |a_n|^q r^{q(n-1)} \leq \left(\frac{M_{\mathcal{B}}}{1-r^2} \right)^{q-2} \int_{\mathbb{T}} \frac{|F_r'|^2}{u_r^{2-\alpha}} \, dm. \tag{14}$$

Multiplying (14) by $(1-r^2)^{q-2} \ln(1/r)\, r \, dr$, integrating over $r \in (0,1)$ by Tonelli's theorem, and applying (11) with $\frac{1}{\alpha(1-\alpha)} = \frac{(q-1)^2}{2(q-3)}$ gives

$$\sum_{n=1}^{\infty} |a_n|^q I_{n,q} \leq \frac{(q-1)^2}{2(q-3)} M_{\mathcal{B}}^{q-2}, \tag{15}$$

where, upon substituting $t = r^2$ ($r\,dr = \frac{1}{2}\,dt$ and $\ln(1/r) = \frac{1}{2}\ln(1/t)$),

$$I_{n,q} := n^q \int_0^1 r^{q(n-1)} (1-r^2)^{q-2} \ln\left(\frac{1}{r}\right) r \, dr = \frac{n^q}{4} \int_0^1 t^{\frac{q(n-1)}{2}} (1-t)^{q-2} \ln\left(\frac{1}{t}\right) dt.$$

At $q = 4$, using the elementary integral $\int_0^1 t^m \ln(1/t)\, dt = (m+1)^{-2}$ $(m > -1)$ on $t^{2n-2}(1-t)^2 = t^{2n-2} - 2t^{2n-1} + t^{2n}$, we evaluate $I_{n,4}$ explicitly for every $n \geq 1$:

$$I_{n,4} = \frac{n^4}{4} \left(\frac{1}{(2n-1)^2} - \frac{2}{(2n)^2} + \frac{1}{(2n+1)^2} \right) = \frac{n^2(12n^2-1)}{8(4n^2-1)^2} = \frac{3}{32} + \frac{20n^2 - 3}{32(4n^2-1)^2} > \frac{3}{32}.$$

Substituting $I_{n,4} > \frac{3}{32}$ into (15) at $q = 4$ (where $\frac{(q-1)^2}{2(q-3)} = \frac{9}{2}$) yields

$$\sum_{n=1}^{\infty} |a_n|^4 \leq \frac{32}{3} \cdot \frac{9}{2} M_{\mathcal{B}}^2 = 48\, M_{\mathcal{B}}^2,$$

proving Part (i).

For general $q > 3$, we show that $I_{n,q} > \frac{1}{4(2\sqrt{e})^q}$ for every $n \geq 1$. For $n \geq 2$, restrict the integral to $[1 - \frac{1}{n}, 1 - \frac{1}{2n}]$ (of length $\frac{1}{2n}$), where $\ln(1/t) \geq 1 - t \geq \frac{1}{2n}$ and

$$t^{\frac{q(n-1)}{2}} \geq \left(1 - \frac{1}{n}\right)^{\frac{q(n-1)}{2}} = \left(1 + \frac{1}{n-1}\right)^{-\frac{q(n-1)}{2}} > e^{-q/2}.$$

Hence

$$I_{n,q} > \frac{n^q}{4} \cdot \frac{1}{2n} \cdot e^{-q/2} \cdot (2n)^{-(q-1)} = \frac{1}{4(2\sqrt{e})^q}.$$

For $n = 1$, using $\ln(1/t) \geq 1 - t$ on $(0,1)$ gives

$$I_{1,q} \geq \frac{1}{4} \int_0^1 (1-t)^{q-1} \, dt = \frac{1}{4q} > \frac{1}{4(2\sqrt{e})^q},$$

since $(2\sqrt{e})^q > e^q > q$ for all $q > 0$. Inserting $I_{n,q} > \frac{1}{4(2\sqrt{e})^q}$ into (15) yields

$$\sum_{n=1}^{\infty} |a_n|^q \leq \frac{2(q-1)^2}{q-3} (2\sqrt{e})^q M_{\mathcal{B}}^{q-2},$$

which proves Part (ii).

Part (iii) (Critical endpoint $q = 3$). Integrating $|F'(te^{i\theta})| \le \frac{M_{\mathcal{B}}}{1-t^2}$ radially from 0 to $R \in (0,1)$ gives

$$u(Re^{i\theta}) \le |F(Re^{i\theta})| \le 1 + M_{\mathcal{B}} \int_0^R \frac{dt}{1-t^2} = 1 + \frac{M_{\mathcal{B}}}{2} \ln\!\left(\frac{1+R}{1-R}\right).$$

Define

$$\widetilde{L}(R) := 1 + \ln\!\left(1 + \frac{M_{\mathcal{B}}}{2} \ln\!\left(\frac{1+R}{1-R}\right)\right) \qquad (0 \le R < 1),$$

which is continuously differentiable and strictly increasing from $\widetilde{L}(0) = 1$ to $\lim_{R\to 1^-} \widetilde{L}(R) = +\infty$. Since $u_R > 0$ and $\int_{\mathbb{T}} u_R \, dm = 1$, Lemma 4.1(ii) yields

$$J(R) := \int_0^R \left(\int_{\mathbb{T}} \frac{|F_r'|^2}{u_r} \, dm\right) \ln\!\left(\frac{R}{r}\right) r \, dr = \int_{\mathbb{T}} u_R \ln u_R \, dm \le \ln\!\left(1 + \frac{M_{\mathcal{B}}}{2} \ln\!\left(\frac{1+R}{1-R}\right)\right) < \widetilde{L}(R).$$

Fix $\varepsilon > 0$ and set $\phi(R) := \widetilde{L}(R)^{-(1+\varepsilon)}$, so $-\phi'(R) = (1+\varepsilon)\widetilde{L}(R)^{-(2+\varepsilon)}\widetilde{L}'(R) > 0$. Integrating $(-\phi'(R))J(R)$ over $R \in (0,1)$ gives

$$\int_0^1 \big(-\phi'(R)\big) J(R) \, dR < \int_0^1 (1+\varepsilon)\widetilde{L}(R)^{-(1+\varepsilon)}\widetilde{L}'(R) \, dR = (1+\varepsilon) \int_1^\infty y^{-(1+\varepsilon)} \, dy = \frac{1+\varepsilon}{\varepsilon}.$$

Applying Tonelli's theorem to the left-hand side and restricting the inner R-integral from $[r,1)$ to $[\sqrt{r},1)$, where $\ln(R/r) \ge \ln(\sqrt{r}/r) = \frac{1}{2}\ln(1/r)$ and $\int_{\sqrt{r}}^1 (-\phi'(R)) \, dR = \phi(\sqrt{r}) = \widetilde{L}(\sqrt{r})^{-(1+\varepsilon)}$, we deduce

$$\int_0^1 \left(\int_{\mathbb{T}} \frac{|F_r'|^2}{u_r} \, dm\right) \frac{\ln(1/r)}{\widetilde{L}(\sqrt{r})^{1+\varepsilon}} \, r \, dr < \frac{2(1+\varepsilon)}{\varepsilon}. \tag{16}$$

On each circle $|z| = r$, applying Hausdorff–Young at $(p,q) = (\frac{3}{2}, 3)$, pulling out $(M_{\mathcal{B}}/(1-r^2))^{1/2}$ from $|F_r'|^{3/2} = |F_r'|^{1/2}(|F_r'|/\sqrt{u_r})\sqrt{u_r}$, and applying Cauchy–Schwarz against $\sqrt{u_r}$ (using $\int_{\mathbb{T}} u_r \, dm = 1$) gives

$$\sum_{n=1}^\infty n^3 |a_n|^3 r^{3(n-1)} \le \left(\int_{\mathbb{T}} |F_r'|^{3/2} \, dm\right)^2 \le \frac{M_{\mathcal{B}}}{1-r^2} \int_{\mathbb{T}} \frac{|F_r'|^2}{u_r} \, dm.$$

Multiplying by $\frac{(1-r^2)\ln(1/r)\, r}{\widetilde{L}(\sqrt{r})^{1+\varepsilon}}$, integrating over $r \in (0,1)$, and substituting (16) yields

$$\sum_{n=1}^\infty |a_n|^3 W_n < \frac{2(1+\varepsilon)}{\varepsilon} M_{\mathcal{B}}, \qquad W_n := \frac{n^3}{4} \int_0^1 t^{\frac{3(n-1)}{2}} (1-t) \frac{\ln(1/t)}{\widetilde{L}(t^{1/4})^{1+\varepsilon}} \, dt, \tag{17}$$

where we again substituted $t = r^2$ (so $\sqrt{r} = t^{1/4}$). For every $t \in [0, 1 - \frac{1}{2n}]$ $(n \ge 1)$, the factorization $1 - t = (1-t^{1/4})(1+t^{1/4})(1+t^{1/2}) < 4(1-t^{1/4})$ implies $1 - t^{1/4} > \frac{1-t}{4} \ge \frac{1}{8n}$, whence

$$\frac{1+t^{1/4}}{1-t^{1/4}} < \frac{2}{1/(8n)} = 16n,$$

and therefore $\widetilde{L}(t^{1/4}) < 1 + \ln\big(1 + \frac{M_{\mathcal{B}}}{2}\ln(16n)\big) =: D_n$ throughout $[0, 1 - \frac{1}{2n}]$. For $n \ge 2$, restricting the integral defining W_n to $[1 - \frac{1}{n}, 1 - \frac{1}{2n}]$ and using $\ln(1/t) \ge 1 - t \ge \frac{1}{2n}$ and $t^{\frac{3(n-1)}{2}} > e^{-3/2}$ gives

$$W_n > \frac{n^3}{4} \cdot \frac{1}{2n} \cdot e^{-3/2} \cdot \left(\frac{1}{2n}\right)^2 \cdot D_n^{-(1+\varepsilon)} = \frac{e^{-3/2}}{32\, D_n^{1+\varepsilon}}.$$

For $n = 1$, restricting to $[0, \frac{1}{2}]$ and using $\ln(1/t) \ge 1 - t$ gives

$$W_1 > \frac{1}{4\, D_1^{1+\varepsilon}} \int_0^{1/2} (1-t)^2 \, dt = \frac{7}{96\, D_1^{1+\varepsilon}} > \frac{e^{-3/2}}{32\, D_1^{1+\varepsilon}}.$$

Substituting $W_n > \frac{e^{-3/2}}{32\, D_n^{1+\varepsilon}}$ into (17) yields

$$\sum_{n=1}^\infty \frac{|a_n|^3}{\Big(1 + \ln\big(1 + \frac{M_{\mathcal{B}}}{2}\ln(16n)\big)\Big)^{1+\varepsilon}} \le \frac{64\, e^{3/2}(1+\varepsilon)}{\varepsilon} M_{\mathcal{B}},$$

completing the proof of Theorem 1.2. □

Remark 4.2. Qualitatively, $(a_n) \in \ell^q$ for $q > 3$ can also be deduced by combining Smirnov's theorem ($F \in H^{p_0}(\mathbb{D})$ for all $p_0 < 1$ whenever $\operatorname{Re} F > 0$ [6]) with Hardy–Littlewood/Littlewood–Paley derivative bounds, Bloch interpolation, and Hausdorff–Young; working directly with the real-part Green–Hardy–Stein identity (Lemma 4.1) avoids harmonic conjugates and $\sec(\pi p_0/2)$ blow-up factors, yields the explicit constant 48 at $q = 4$, and reaches the critical $q = 3$ endpoint. In (14), the exponent $q = 3$ is the exact threshold accessible via the Hausdorff–Young inequality on radial derivatives F_r': as shown in the expanded version [8, Theorem 7.2], the weighted L^p-integral $\int_0^1 \|F_r'\|_{L^p(\mathbb{T})}^q (1-r)^{q-1}\, dr$ ($p = q/(q-1)$) diverges on Kahane's singular Zygmund measure [14] for every $q \in [2, 3]$. In the expanded version [8, Theorems 8.1 and 9.1(b)], we bypass Hausdorff–Young completely via Bohr-set projections and weighted Rudin–Chang estimates to prove that $\sum_{n=1}^{\infty} |a_n|^q < \infty$ for all $q > \frac{5}{2}$ on $\mathbb{T}$, and the analogous $\hat{\mu} \in \ell^q(\mathbb{Z}^D)$ bound on $\mathbb{T}^D$ for every $D \geq 1$.

5. Sharp critical exponent $q = 2$ and exact block-decay dichotomy

We now prove Theorem 1.3. Throughout this section, let μ be any positive Borel probability measure on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ (not necessarily even, with arbitrary complex Fourier–Stieltjes coefficients $\hat{\mu}(n) = \int_{\mathbb{T}} e^{-in\theta}\, d\mu(\theta)$ satisfying $\hat{\mu}(0) = 1$ and $\hat{\mu}(-n) = \overline{\hat{\mu}(n)}$). For each integer $k \geq 0$, write $B_k := [2^k, 2^{k+1}) \cap \mathbb{Z}$, and set

$$s_k := \sum_{n \in B_k} |\hat{\mu}(n)|, \qquad \tilde{s}_k := \max(s_{k-1}, s_k, s_{k+1}) \quad (s_{-1} := 0), \qquad E_k := \sum_{n \in B_k} |\hat{\mu}(n)|^2.$$

When $\hat{\mu}(n) \geq 0$ for all $n \in \mathbb{Z}$, Lemma 2.1 shows that $M := \sup_{k \geq 0} s_k < \infty$ holds if and only if $\mu \in \Lambda_*$, and $\lim_{k \to \infty} s_k = 0$ holds if and only if $\mu \in \lambda_*$.

5.1. **The local-collar weak-ℓ^2 inequality and its consequences.** The core engine is an elementary Riesz-product test on a single residue class of dyadic blocks modulo 4. Its key feature is that all frequencies exceeding threshold δ inside a block are tested simultaneously with one block-independent weight $\alpha = \delta/(8w)$, while modulo-4 spacing isolates every cross-term inside a disjoint three-block collar (Figure 3).

Proposition 5.1 (Local-collar weak-ℓ^2 inequality)**.** *Let μ be a positive Borel probability measure on $\mathbb{T}$, let $I \subset \mathbb{Z}_{\geq 0}$ be any (finite or infinite) set of block indices, and suppose $\tilde{s}_k \leq w$ for all $k \in I$, where $w > 0$. Then for every $\delta > 0$,*

$$N_I(\delta) \;:=\; \#\Big\{ n \in \bigcup_{k \in I} B_k : |\hat{\mu}(n)| \geq \delta \Big\} \;\leq\; \frac{56\, w}{\delta^2}, \tag{18}$$

with strict inequality when I is finite.

Proof. Fix a residue class $a \in \{0, 1, 2, 3\}$ and a finite subset

$$A \subset \{k \in I : k \equiv a \pmod 4,\ F_k \neq \emptyset\}, \qquad F_k := \{n \in B_k : |\hat{\mu}(n)| \geq \delta\}.$$

Put $N_k := |F_k| \geq 1$ for $k \in A$. If $A = \emptyset$, then $\sum_{k \in A} N_k = 0 < 14w/\delta^2$ trivially; assume $A \neq \emptyset$. For every $k \in A$, we have $N_k \delta \leq \sum_{n \in F_k} |\hat{\mu}(n)| \leq s_k \leq w$, so setting $\alpha := \delta/(8w) > 0$ gives

$$\alpha N_k \;\leq\; \frac{1}{8} \qquad (\forall\, k \in A).$$

For each $n \in F_k$, write $\hat{\mu}(n) = |\hat{\mu}(n)| e^{i\phi_n}$ with $\phi_n \in \mathbb{R}$, and define the real trigonometric polynomial

$$v_k(\theta) \;:=\; \sum_{n \in F_k} \cos(n\theta + \phi_n) \;=\; \frac{1}{2} \sum_{n \in F_k} \big(e^{i\phi_n} e^{in\theta} + e^{-i\phi_n} e^{-in\theta} \big).$$

Then $|v_k(\theta)| \leq N_k$ everywhere on $\mathbb{T}$, $\|\widehat{\alpha v_k}\|_{\ell^1(\mathbb{Z})} = \alpha N_k \leq 1/8$, and $\sup_{m \in \mathbb{Z}} |\widehat{\alpha v_k}(m)| = \alpha/2$. Since $\hat{\mu}(-n) = \overline{\hat{\mu}(n)} = |\hat{\mu}(n)| e^{-i\phi_n}$, integrating v_k against μ gives

$$\int_{\mathbb{T}} v_k(\theta)\, d\mu(\theta) \;=\; \frac{1}{2} \sum_{n \in F_k} \big(e^{i\phi_n} \hat{\mu}(-n) + e^{-i\phi_n} \hat{\mu}(n) \big) \;=\; \sum_{n \in F_k} |\hat{\mu}(n)| \;\geq\; \delta N_k.$$

Consider the strictly positive trigonometric polynomial

$$R(\theta) \;:=\; \prod_{k \in A} \big(1 - \alpha v_k(\theta) \big) \;\geq\; \Big(\frac{7}{8} \Big)^{|A|} \;>\; 0 \qquad (\theta \in \mathbb{T}).$$

For any subset $A' \subset A$ and integer $r \geq 1$, put

$$W_r^{(A')}(\theta) \;:=\; \sum_{\substack{S \subset A' \\ |S| = r}} \prod_{k \in S} \big(\alpha v_k(\theta) \big)$$

(interpreted as 0 if $r > |A'|$), and write $W_r := W_r^{(A)}$, so that $R = 1 - W_1 + \sum_{r=2}^{|A|} (-1)^r W_r$.

For each $k \geq 0$, define the symmetric local collar $C_k := \pm(B_{k-1} \cup B_k \cup B_{k+1})$, with $B_{-1} := \emptyset$ (so $0 \notin C_k$). Take any nonempty subset $S \subset A'$ with top index $k := \max(S)$. Any frequency m in the Fourier support of $\prod_{j \in S} v_j$ has the form

$$m \;=\; \varepsilon_k n_k + \sum_{j \in S \setminus \{k\}} \varepsilon_j n_j, \qquad \varepsilon_j \in \{-1, +1\}, \quad n_j \in F_j \subset B_j.$$

If $S = \{k\}$, then $|m| = n_k \in B_k \subset C_k$. If $|S| \geq 2$, then $k \geq 4$ and every $j \in S \setminus \{k\}$ satisfies $j \leq k - 4q$ for distinct integers $q \geq 1$; since $n_j < 2^{j+1}$, the lower-block sum satisfies

$$\Big| \sum_{j \in S \setminus \{k\}} \varepsilon_j n_j \Big| \;<\; \sum_{q=1}^{\infty} 2^{k-4q+1} \;=\; \frac{2}{15}\, 2^k.$$

Because $2^k \leq n_k < 2^{k+1}$, we obtain

$$2^{k-1} \;<\; \frac{13}{15}\, 2^k \;<\; |m| \;<\; \frac{32}{15}\, 2^k \;<\; 2^{k+2},$$

so $m \in C_k$ in all cases. Moreover, any two distinct indices $k_1 < k_2$ in A satisfy $k_2 \geq k_1 + 4$, hence $k_2 - 1 \geq k_1 + 3 > k_1 + 1$, so the collars $\{C_k : k \in A\}$ are pairwise disjoint (separated by at least one empty buffer block; Figure 3). Consequently $\operatorname{supp}(\widehat{W_r^{(A')}}) \subset \bigsqcup_{k \in A'} C_k$ for every $A' \subset A$ and $r \geq 1$.

We now prove by induction on $r \geq 1$ that, for every $A' \subset A$,

$$\sup_{m \in \mathbb{Z}} \big|\widehat{W_r^{(A')}}(m)\big| \;\leq\; \frac{\alpha}{2}\, 8^{-(r-1)}, \tag{19}$$

and that for $r \geq 2$, every $k \in A'$, and every $m \in C_k$,

$$\big|\widehat{W_r^{(A')}}(m)\big| \;\leq\; \frac{\alpha^2}{2}\, N_k\, 8^{-(r-2)}. \tag{20}$$

For $r = 1$, $W_1^{(A')} = \sum_{k \in A'} \alpha v_k$ has pairwise disjoint supports $\pm F_k \subset C_k$, so $|\widehat{W_1^{(A')}}(m)| \leq \alpha/2$, proving (19). For $r \geq 2$, fix $A' \subset A$ and any $m \in C_k$ with $k \in A'$. By collar disjointness, only subsets $S \subset A'$ of size r with $\max(S) = k$ can contribute to the m-th Fourier coefficient of $W_r^{(A')}$. Writing $A'_k := \{j \in A' : j < k\}$, those terms equal $(\alpha v_k) \cdot W_{r-1}^{(A'_k)}$, so the induction hypothesis gives

$$\big|\widehat{W_r^{(A')}}(m)\big| \;=\; \big|(\widehat{\alpha v_k} * \widehat{W_{r-1}^{(A'_k)}})(m)\big| \;\leq\; \|\widehat{\alpha v_k}\|_{\ell^1(\mathbb{Z})} \sup_{x \in \mathbb{Z}} \big|\widehat{W_{r-1}^{(A'_k)}}(x)\big| \;\leq\; (\alpha N_k) \cdot \frac{\alpha}{2}\, 8^{-(r-2)}.$$

This establishes (20), and since $\alpha N_k \leq 1/8$, it also yields (19), completing the induction.

For each $k \in A$, symmetry $|\hat{\mu}(-m)| = |\hat{\mu}(m)|$ gives

$$\sum_{m \in C_k} |\hat{\mu}(-m)| \;=\; 2(s_{k-1} + s_k + s_{k+1}) \;\leq\; 6\tilde{s}_k \;\leq\; 6w.$$

Integrating $R > 0$ against the probability measure μ, noting that

$$\Big| \int_{\mathbb{T}} W_r \, d\mu \Big| \;=\; \Big| \sum_{m \in \mathbb{Z}} \widehat{W_r}(m)\, \hat{\mu}(-m) \Big| \;\leq\; \sum_{k \in A} \sum_{m \in C_k} |\widehat{W_r}(m)|\, |\hat{\mu}(-m)|,$$

and applying (20) together with $\sum_{r=2}^{\infty} 8^{-(r-2)} = 8/7$, we deduce

$$0 \;<\; \int_{\mathbb{T}} R \, d\mu \;\leq\; 1 - \alpha\delta \sum_{k \in A} N_k + \frac{8}{7} \sum_{k \in A} \Big(\frac{\alpha^2}{2}\, N_k \Big)(6w) \;=\; 1 - \Big(\frac{1}{8} - \frac{3}{56} \Big) \frac{\delta^2}{w} \sum_{k \in A} N_k \;=\; 1 - \frac{\delta^2}{14w} \sum_{k \in A} N_k.$$

Hence $\sum_{k \in A} N_k < 14w/\delta^2$. When I is finite, taking $A = \{k \in I : k \equiv a \pmod 4,\ F_k \neq \emptyset\}$ and summing over $a \in \{0, 1, 2, 3\}$ yields $N_I(\delta) < 56w/\delta^2$. For infinite I, taking the supremum over all finite subsets $I' \subset I$ gives $N_I(\delta) \leq 56w/\delta^2$. □

From Proposition 5.1 we immediately obtain the global ℓ^q ($q > 2$) and log-weighted ℓ^2 bounds, the single-log level-set energy bound, the finite-N ceiling $O_M(\ln\ln(4N))$, and the convergence half of the exact monotone block criterion. Write $\ln_+(x) := \max(0, \ln x)$ for $x > 0$ and $\ln_+(0) := 0$.

Corollary 5.2 (Global ℓ^q for $q > 2$ and log-weighted ℓ^2)**.** *Let μ be a positive Borel probability measure on $\mathbb{T}$ with $M := \sup_{k \geq 0} s_k < \infty$. Then for every $q > 2$ and every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} |\hat{\mu}(n)|^q \;\leq\; \frac{56\, q}{q-2}\, M \, \big(\min(1, M)\big)^{q-2}, \tag{21}$$

$$\sum_{\substack{n \geq 1 \\ \hat{\mu}(n) \neq 0}} \frac{|\hat{\mu}(n)|^2}{\big(1 + \ln(1/|\hat{\mu}(n)|)\big)^{1+\varepsilon}} \;\leq\; \frac{56(3+\varepsilon)}{\varepsilon}\, M. \tag{22}$$

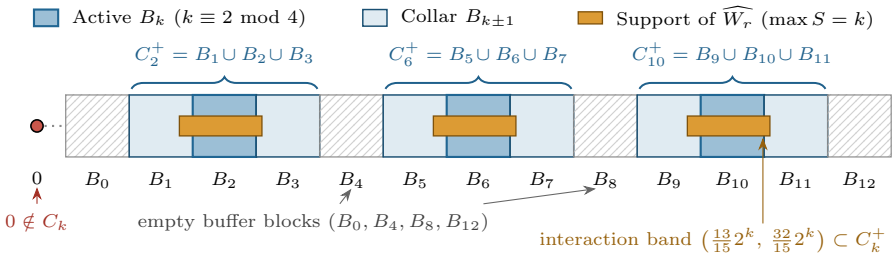


FIGURE 3. Modulo-4 local-collar disjointness in Proposition 5.1 (shown for the residue class $k \equiv 2 \pmod 4$ on the positive $\log_2$-frequency axis; the negative axis is symmetric). Combining a top frequency $n_k \in B_k$ ($k \in A$) with arbitrary lower-block frequencies $n_j \in B_j$ ($j \in A$, $j \le k-4$) shifts $|m|$ by strictly less than $\frac{2}{15}2^k$, confining every product term with $\max S = k$ to $(\frac{13}{15}2^k, \frac{32}{15}2^k) \subset C_k^+ = B_{k-1} \cup B_k \cup B_{k+1}$ (amber bar). Empty hatched buffer blocks (B_0, B_4, B_8, B_{12}) separate consecutive collars and keep $0 \notin C_k$, yielding the per-collar induction $|\widehat{W_r}(m)| \le \frac{\alpha^2}{2} N_k 8^{-(r-2)}$ ($r \ge 2$).

Proof. If $M = 0$, then $\hat\mu(n) = 0$ for all $n \ge 1$ and both bounds are trivial. Assume $M > 0$, and apply Proposition 5.1 with $I = \mathbb{Z}_{\ge 0}$ and $w = M$ to get $N(\delta) := \#\{n \ge 1 : |\hat\mu(n)| \ge \delta\} \le 56M/\delta^2$ for all $\delta > 0$. Since $|\hat\mu(n)| \le \hat\mu(0) = 1$ and $|\hat\mu(n)| \le s_{\lfloor \log_2 n \rfloor} \le M$, every $n \ge 1$ satisfies $|\hat\mu(n)| \le u := \min(1, M)$. By Tonelli's theorem,

$$\sum_{n=1}^{\infty} |\hat\mu(n)|^q \;=\; q \int_0^u \delta^{q-1} N(\delta)\, d\delta \;\le\; 56\, q\, M \int_0^u \delta^{q-3}\, d\delta \;=\; \frac{56\, q}{q-2}\, M\, u^{q-2},$$

proving (21). For (22), set $\phi(0) := 0$ and $\phi(t) := t^2/(1 + \ln(1/t))^{1+\varepsilon}$ on $(0, 1]$. Then ϕ is continuous on $[0, 1]$ and satisfies on $(0, 1)$

$$0 \;<\; \phi'(t) \;=\; \frac{2t}{(1+\ln(1/t))^{1+\varepsilon}} + \frac{(1+\varepsilon)t}{(1+\ln(1/t))^{2+\varepsilon}} \;\le\; \frac{(3+\varepsilon)t}{(1+\ln(1/t))^{1+\varepsilon}}.$$

Writing $\phi(x) = \int_0^x \phi'(\delta)\, d\delta$, Tonelli's theorem and the substitution $y = \ln(1/\delta)$ give

$$\sum_{\substack{n\ge 1 \\ \hat\mu(n)\ne 0}} \phi(|\hat\mu(n)|) \;=\; \int_0^1 \phi'(\delta) N(\delta)\, d\delta \;\le\; 56(3+\varepsilon) M \int_0^\infty \frac{dy}{(1+y)^{1+\varepsilon}} \;=\; \frac{56(3+\varepsilon)}{\varepsilon}\, M. \qquad \square$$

Lemma 5.3 (Single-log level-set energy bound)**.** *For every finite block set $I \subset \mathbb{Z}_{\ge 0}$ with $\tilde s_k \le w$ for all $k \in I$ ($w > 0$),*

$$\sum_{k\in I} E_k \;\le\; 57\, w \;+\; 112\, w\, \ln_+(w|I|). \tag{23}$$

Proof. If $w|I| \le 1$, then $E_k \le s_k^2 \le w^2$ for each $k \in I$, so $\sum_{k\in I} E_k \le w^2|I| \le w \le 57w$. Suppose now that $w|I| > 1$, and set $\tau := 1/|I| \in (0, w)$. Coefficients with $|\hat\mu(n)| \le \tau$ contribute at most $\tau \sum_{k\in I} s_k \le \tau w|I| = w$. For the remaining coefficients, note that $|\hat\mu(n)| \le s_k \le w$ for $n \in B_k$, and every $x \in [0, w]$ obeys the identity

$$x^2\, \mathbf{1}_{x>\tau} \;=\; \tau^2\, \mathbf{1}_{x>\tau} + 2 \int_\tau^w \delta\, \mathbf{1}_{x\ge\delta}\, d\delta.$$

Summing over $n \in \bigcup_{k\in I} B_k$ and applying Tonelli's theorem together with Proposition 5.1 yields

$$\sum_{k\in I} \sum_{\substack{n\in B_k \\ |\hat\mu(n)|>\tau}} |\hat\mu(n)|^2 \;\le\; \tau^2 N_I(\tau) + 2\int_\tau^w \delta\, N_I(\delta)\, d\delta \;\le\; 56\, w + 112\, w \ln\!\Big(\frac{w}{\tau}\Big) \;=\; 56\, w + 112\, w \ln(w|I|).$$

Adding the two pieces establishes (23). $\square$

Proposition 5.4 (Double-logarithmic finite-N ceiling and $\sum s_m^*/m$ criterion)**.** *Let μ be a positive Borel probability measure on $\mathbb{T}$.*

(a) *If $M := \sup_{k\ge 0} s_k < \infty$, then for every integer $N \ge 1$,*

$$\sum_{n=1}^{N} |\hat\mu(n)|^2 \;\le\; 57\, M + 112\, M\, \ln_+\!\big(M \log_2(2N)\big). \tag{24}$$

If moreover $\lim_{k\to\infty} s_k = 0$, then $\sum_{n=1}^N |\hat\mu(n)|^2 = o(\ln\ln(4N))$ as $N \to \infty$.

(b) *If $\lim_{k\to\infty} s_k = 0$ and $(s_m^*)_{m\ge 1}$ denotes the non-increasing rearrangement of $(s_k)_{k\ge 0}$, then*

$$\sum_{k:\, \tilde s_k \le 1} E_k \;\le\; 361 + 448 \sum_{m=1}^{\infty} \frac{s_m^*}{m} \tag{25}$$

(with strict inequality whenever the right-hand side is finite). In particular, $\sum_{m=1}^{\infty} s_m^*/m < \infty$ implies $\hat{\mu} \in \ell^2(\mathbb{Z})$; consequently, $\sum_{k=0}^{\infty} s_k^p < \infty$ implies $\hat{\mu} \in \ell^2(\mathbb{Z})$ for every finite $p \in (0, \infty)$.

Proof. Part (a). For $M = 0$ the claim is trivial. For $M > 0$ and $N \geq 1$, set $K := \lfloor \log_2 N \rfloor$ and $I := \{0, 1, \ldots, K\}$. Then $\{1, \ldots, N\} \subset \bigcup_{k=0}^{K} B_k$, $|I| = K + 1 \leq \log_2(2N)$, and $\tilde{s}_k \leq M$ on I, so Lemma 5.3 gives (24). If $s_k \to 0$, fix any integer $K_0 \geq 1$ and put $\eta_{K_0} := \sup_{k \geq K_0 - 1} s_k$. If $\eta_{K_0} = 0$, the partial sums are bounded; if $\eta_{K_0} > 0$, applying Lemma 5.3 to $I_{K_0} := \{K_0, \ldots, \lfloor \log_2 N \rfloor\}$ with $w = \eta_{K_0}$ for $N \geq 2^{K_0}$ yields

$$\sum_{n=1}^{N} |\hat{\mu}(n)|^2 \;\leq\; \sum_{n=1}^{2^{K_0}-1} |\hat{\mu}(n)|^2 + 57\,\eta_{K_0} + 112\,\eta_{K_0}\,\ln_+\!\big(\eta_{K_0} \log_2(2N)\big).$$

Dividing by $\ln\ln(4N)$ (since $\ln\log_2(2N)/\ln\ln(4N) \to 1$) and letting $N \to \infty$ gives

$$\limsup_{N \to \infty} \frac{\sum_{n=1}^{N} |\hat{\mu}(n)|^2}{\ln\ln(4N)} \;\leq\; 112\,\eta_{K_0},$$

and letting $K_0 \to \infty$ completes (a).

Part (b). Assume $\lim_{k\to\infty} s_k = 0$. For each integer $j \geq 0$, set $L_j := \{k \geq 0 : 2^{-(j+1)} < \tilde{s}_k \leq 2^{-j}\}$. Because $s_k \to 0$, each L_j is finite, and $\{k \geq 0 : \tilde{s}_k > 1\}$ is finite as well. Since $\tilde{s}_k = 0$ forces $E_k = 0$, summing Lemma 5.3 over $I = L_j$ with $w = 2^{-j}$ $(j \geq 0)$ gives

$$\sum_{k:\,\tilde{s}_k \leq 1} E_k \;=\; \sum_{j=0}^{\infty} \sum_{k \in L_j} E_k \;\leq\; \sum_{j=0}^{\infty} 57 \cdot 2^{-j} + 112 \sum_{j=0}^{\infty} 2^{-j} \ln_+(2^{-j}|L_j|) \;=\; 114 + 112 \sum_{j=0}^{\infty} 2^{-j} \ln_+(2^{-j}|L_j|).$$

Put $m_j := \#\{m \geq 1 : s_m^* > 2^{-(j+1)}\} = \#\{r \geq 0 : s_r > 2^{-(j+1)}\} < \infty$. If $k \in L_j$, then $\max(s_{k-1}, s_k, s_{k+1}) > 2^{-(j+1)}$, so at least one index $r \in \{k-1, k, k+1\}$ satisfies $s_r > 2^{-(j+1)}$; hence $|L_j| \leq 3m_j$. When $m_j = 0$, $\ln_+(2^{-j}|L_j|) = 0$; when $m_j \geq 1$, $2^{-j}|L_j| \leq 3m_j$ implies

$$\ln_+(2^{-j}|L_j|) \;\leq\; \ln(3m_j) \;=\; \ln 3 + \ln m_j \;\leq\; \ln 3 + \sum_{r=1}^{m_j} \frac{1}{r}.$$

Because (s_m^*) is non-increasing, $r \leq m_j$ holds if and only if $s_r^* > 2^{-(j+1)}$, i.e., $2^{-j} < 2s_r^*$. Whenever $s_r^* > 0$, letting $j_0 \geq 0$ be the least integer with $2^{-j_0} < 2s_r^*$ gives $\sum_{j \geq 0:\, 2^{-j} < 2s_r^*} 2^{-j} = 2 \cdot 2^{-j_0} < 4s_r^*$ (and the sum is $0 = 4s_r^*$ when $s_r^* = 0$). By Tonelli's theorem,

$$\sum_{j=0}^{\infty} 2^{-j} \ln_+(2^{-j}|L_j|) \;\leq\; \sum_{j=0}^{\infty} 2^{-j} \ln 3 + \sum_{r=1}^{\infty} \frac{1}{r} \sum_{\substack{j \geq 0 \\ 2^{-j} < 2s_r^*}} 2^{-j} \;\leq\; 2\ln 3 + 4 \sum_{r=1}^{\infty} \frac{s_r^*}{r}.$$

Since $114 + 224\ln 3 < 360.09 < 361$, (25) follows in $[0, +\infty]$, with strict inequality whenever $\sum_{m=1}^{\infty} s_m^*/m < \infty$. Because $\{k \geq 0 : \tilde{s}_k > 1\}$ is finite, $\sum_{m=1}^{\infty} s_m^*/m < \infty$ implies $\sum_{k=0}^{\infty} E_k < \infty$. Finally, if $\sum_{k=0}^{\infty} s_k^p = D^p < \infty$ for some $p \in (0, \infty)$, then $s_k \to 0$ and $m(s_m^*)^p \leq D^p$ for all $m \geq 1$, so $\sum_{m=1}^{\infty} s_m^*/m \leq D \sum_{m=1}^{\infty} m^{-1-1/p} < \infty$, giving $\hat{\mu} \in \ell^2(\mathbb{Z})$. □

5.2. **The matching converse: exact necessary and sufficient block criterion.** We now show that the convergence condition $\sum_{k=0}^{\infty} b(k)/(k+1) < \infty$ is not only sufficient, but literally necessary across every bounded non-increasing block envelope $b(k)$, with no regularity or slow-variation assumption and no rescaling of b.

Proposition 5.5 (Sharp monotone-envelope converse)**.** *Let $(b(k))_{k \geq 0}$ be a bounded non-increasing sequence of non-negative real numbers satisfying*

$$\sum_{k=0}^{\infty} \frac{b(k)}{k+1} \;=\; +\infty.$$

Then there exists an even, positive, purely singular, atomless Borel probability measure μ on $\mathbb{T}$ such that $\hat{\mu}(n) \geq 0$ for all $n \in \mathbb{Z}$, $\sum_{n=1}^{\infty} \hat{\mu}(n)^2 = +\infty$, and

$$s_k(\mu) \;:=\; \sum_{n \in B_k} \hat{\mu}(n) \;\leq\; b(k) \qquad \text{for every } k \geq 0. \tag{26}$$

In particular, $\mu \in \Lambda_*$, and if $\lim_{k\to\infty} b(k) = 0$ then $\mu \in \lambda_*$.

Proof. Observe first that $b(k) > 0$ for every $k \geq 0$ (otherwise monotonicity would force $b(k) = 0$ eventually, contradicting divergence of $\sum b(k)/(k+1)$), and $b(0) = \sup_{k \geq 0} b(k) > 0$.

Fix any integer offset $\ell \geq 0$, and set $u_j := b(\ell + 4j) > 0$ and $U_j := \sum_{i=1}^{j} u_i > 0$ for $j \geq 1$ (with $U_0 := 0$). By monotonicity of b, for each $j \geq 1$ and $r \in \{0, 1, 2, 3\}$ we have $b(\ell + 4j + r) \leq u_j$ and $\ell + 4j + r + 1 \geq 4j$, whence

$$\sum_{j=1}^{\infty} \frac{u_j}{j} \;=\; \sum_{j=1}^{\infty} 4 \cdot \frac{u_j}{4j} \;\geq\; \sum_{j=1}^{\infty} \sum_{r=0}^{3} \frac{b(\ell + 4j + r)}{\ell + 4j + r + 1} \;=\; \sum_{k=\ell+4}^{\infty} \frac{b(k)}{k+1} \;=\; +\infty.$$

Next, for every integer $i \geq 1$, telescoping gives $\sum_{j=i}^{\infty} j^{-2} \leq 2/i$ (for $i = 1$: $1 + \sum_{j=2}^{\infty} (\frac{1}{j-1} - \frac{1}{j}) = 2$; for $i \geq 2$: $\sum_{j=i}^{\infty} j^{-2} \leq \sum_{j=i}^{\infty} (\frac{1}{j-1} - \frac{1}{j}) = \frac{1}{i-1} \leq \frac{2}{i}$). Hence for every integer $R \geq 1$, reversing the order of summation yields

$$\sum_{j=1}^{R} \frac{U_j}{j^2} \;=\; \sum_{i=1}^{R} u_i \sum_{j=i}^{R} \frac{1}{j^2} \;\leq\; 2 \sum_{i=1}^{R} \frac{u_i}{i}.$$

Applying Cauchy–Schwarz,

$$\Big(\sum_{j=1}^{R} \frac{u_j}{j}\Big)^2 \;=\; \Big(\sum_{j=1}^{R} \frac{u_j}{\sqrt{U_j}} \cdot \frac{\sqrt{U_j}}{j}\Big)^2 \;\leq\; \Big(\sum_{j=1}^{R} \frac{u_j^2}{U_j}\Big)\Big(\sum_{j=1}^{R} \frac{U_j}{j^2}\Big) \;\leq\; 2\Big(\sum_{j=1}^{R} \frac{u_j^2}{U_j}\Big)\Big(\sum_{j=1}^{R} \frac{u_j}{j}\Big).$$

Dividing by $\sum_{j=1}^{R} u_j/j > 0$ gives $\sum_{j=1}^{R} u_j^2/U_j \geq \frac{1}{2} \sum_{j=1}^{R} u_j/j$, and therefore

$$\sum_{j=1}^{\infty} \frac{u_j^2}{U_j} \;=\; +\infty. \tag{27}$$

We now construct the dilated product $P_t = P_{t-1} X_t$ inductively just as in Section 3, starting from $P_0 \equiv 1$ ($D_0 := \deg(P_0) = 0$). Given P_{t-1} (even, non-negative, $\widehat{P_{t-1}} \geq 0$, $\int_{\mathbb{T}} P_{t-1}\, dm = 1$, degree D_{t-1}), choose a power of two $L_t = 2^{\ell_t} > 8(D_{t-1} + 1)$. With offset $\ell = \ell_t$, define $u_j := b(\ell_t + 4j)$ and $U_j := \sum_{i=1}^{j} u_i$. By (27), we may pick a finite integer R_t such that

$$H_t \;:=\; \sum_{j=1}^{R_t} \frac{u_j^2}{U_j} \;\geq\; \max\big(4\, b(0),\; 2\, P_{t-1}(0)\big)$$

(necessarily $R_t \geq 2$, since the $j = 1$ term equals $u_1 \leq b(0) < 4b(0)$). Put $a_j := u_j/\sqrt{U_j H_t} > 0$ for $1 \leq j \leq R_t$, and form the weighted lacunary analytic square

$$Q^{(t)}(\theta) \;:=\; \Big|\sum_{j=1}^{R_t} a_j e^{i 16^j \theta}\Big|^2 \;=\; 1 + 2 \sum_{1 \leq i < j \leq R_t} a_i a_j \cos\big((16^j - 16^i)\theta\big),$$

setting $X_t(\theta) := Q^{(t)}(L_t \theta)$ and $P_t(\theta) := P_{t-1}(\theta) X_t(\theta)$.

Check the moment and block-mass bounds at stage t. First, $\sum_{j=1}^{R_t} a_j^2 = 1$, and because $U_j \geq u_j$, every $a_j^2 = u_j^2/(U_j H_t) \leq u_j/H_t \leq b(0)/(4b(0)) = 1/4$. Therefore, exactly as in Step 1 of Section 3 (using pairwise distinctness of $16^j - 16^i$ and zero base-16 carries for triple sums),

$$\int_{\mathbb{T}} Q^{(t)}\, dm = 1, \qquad \int_{\mathbb{T}} \big(Q^{(t)}\big)^2 dm = 2 - \sum_{j=1}^{R_t} a_j^4 \geq \frac{7}{4}, \qquad \int_{\mathbb{T}} \big(Q^{(t)}\big)^3 dm \leq 6.$$

Second, since $L_t > 8(D_{t-1} + 1)$, the frequencies of P_{t-1} in $[-D_{t-1}, D_{t-1}]$ remain unchanged, no negative frequency of X_t reaches a positive frequency of P_t, and for each $j \in \{2, \ldots, R_t\}$ all new positive frequencies $L_t(16^j - 16^i) + h$ ($1 \leq i < j$, $|h| \leq D_{t-1}$) lie strictly inside the single fresh dyadic block $B_{\ell_t + 4j - 1}$, strictly above $\deg(P_{t-1})$. Moreover, for every $i \geq 1$,

$$\frac{u_i}{\sqrt{U_i}} \;=\; \frac{U_i - U_{i-1}}{\sqrt{U_i}} \;=\; \frac{\sqrt{U_i} + \sqrt{U_{i-1}}}{\sqrt{U_i}} \big(\sqrt{U_i} - \sqrt{U_{i-1}}\big) \;\leq\; 2\big(\sqrt{U_i} - \sqrt{U_{i-1}}\big),$$

so telescoping yields $\sum_{i=1}^{j-1} u_i/\sqrt{U_i} \leq 2\sqrt{U_{j-1}} < 2\sqrt{U_j}$. Hence the total mass of P_t in block $B_{\ell_t + 4j - 1}$ satisfies

$$\begin{aligned} s_{\ell_t + 4j - 1}(P_t) \;&=\; P_{t-1}(0)\, a_j \sum_{i=1}^{j-1} a_i \;<\; P_{t-1}(0)\, \frac{u_j}{\sqrt{U_j H_t}} \cdot \frac{2\sqrt{U_j}}{\sqrt{H_t}} \\ &=\; \frac{2\, P_{t-1}(0)}{H_t}\, u_j \;\leq\; u_j \;=\; b(\ell_t + 4j) \;\leq\; b(\ell_t + 4j - 1), \end{aligned}$$

using $H_t \geq 2P_{t-1}(0)$ and monotonicity of b. All other dyadic blocks either keep their value from P_{t-1} or remain empty ($0 \leq b(k)$), so $s_k(P_t) \leq b(k)$ for every $k \geq 0$ and every $t \geq 1$.

Because the moment bounds $\int Q^{(t)}\, dm = 1$, $\int (Q^{(t)})^2\, dm \geq 7/4$, and $\int (Q^{(t)})^3\, dm \leq 6$ match those of Section 3 verbatim, Steps 3–4 of Section 3 apply without change: $P_t\, dm \overset{*}{\rightharpoonup} \mu$ for an even, positive, purely singular Borel

probability measure μ on $\mathbb{T}$ with $\hat{\mu}(n) \geq 0$ for all $n \in \mathbb{Z}$, $\sum_{n=1}^{\infty} \hat{\mu}(n)^2 = +\infty$, and $s_k(\mu) \leq b(k)$ for every $k \geq 0$. Since $s_k(\mu) \leq b(0)$ for all k, we have $\sum_{n=1}^{N} \hat{\mu}(n)^2 \leq b(0)^2(\log_2 N + 1) = o(N)$, so μ is atomless by Wiener's lemma and $\mu \in \Lambda_*$ (and $\mu \in \lambda_*$ whenever $b(k) \to 0$). □

Proof of Theorem 1.3. Parts (i), (ii), and (iii) of Theorem 1.3 are given directly by Proposition 5.1, Corollary 5.2, and Proposition 5.4. For the equivalence in Part (iv), let $(b(k))_{k\geq 0}$ be any bounded non-increasing non-negative sequence. If $\sum_{k=0}^{\infty} b(k)/(k+1) < \infty$, then monotonicity forces $b(k) \to 0$ (since $b(2k) \sum_{j=k+1}^{2k} (j+1)^{-1} \geq \frac{1}{3} b(2k)$ for $k \geq 1$). For any positive probability measure μ with $s_k(\mu) \leq b(k)$ for all $k \geq 0$, we therefore have $s_k(\mu) \to 0$, and for every $m \geq 1$ and $t > b(m-1)$ the set $\{k \geq 0 : s_k(\mu) > t\}$ is contained in $\{0, \ldots, m-2\}$ (of size at most $m-1$), forcing $s_m^* \leq b(m-1)$. Hence $\sum_{m=1}^{\infty} s_m^*/m \leq \sum_{k=0}^{\infty} b(k)/(k+1) < \infty$, and Proposition 5.4(b) yields $\hat{\mu} \in \ell^2(\mathbb{Z})$. Conversely, if $\sum_{k=0}^{\infty} b(k)/(k+1) = +\infty$, Proposition 5.5 produces a purely singular probability measure μ with $\hat{\mu} \geq 0$, $s_k(\mu) \leq b(k)$ for all $k \geq 0$, and $\hat{\mu} \notin \ell^2(\mathbb{Z})$. □

Figure 4 summarizes both the exact monotone block-decay threshold of Theorem 1.3 (iv) and the compression of partial ℓ^2 growth from $O(\ln N)$ for general Bloch functions [17, 2] and $\Theta(\sqrt{\ln N})$ for Kahane's positive singular Λ_* measure with sign-changing coefficients [14] (proved in [8, Theorem 7.1]) down to $O_M(\ln\ln(4N))$ whenever $\sup_{k\geq 0} s_k \leq M$—in particular, for every $\mu \in \Lambda_*$ with $\hat{\mu} \geq 0$.

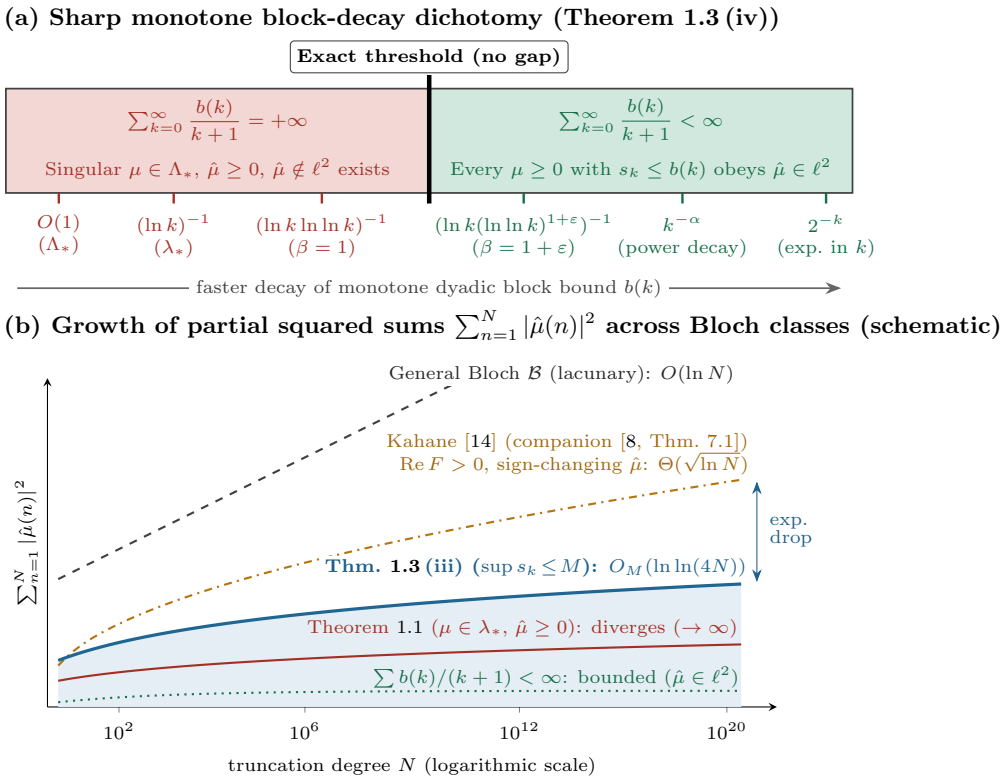


FIGURE 4. Sharp quantitative boundaries of Theorem 1.3. **(a)** Across all bounded non-increasing dyadic block envelopes $b(k)$, the condition $\sum_{k=0}^{\infty} b(k)/(k+1) < \infty$ is an exact if-and-only-if dichotomy separating forced ℓ^2 convergence (green, right, including every ℓ^p block class $p < \infty$ and every iterated-log profile $b(k) = (\ln k\,(\ln\ln k)^{\beta})^{-1}$ with $\beta > 1$) from purely singular non-ℓ^2 counterexamples (red, left, reaching $\beta = 1$). **(b)** Schematic comparison of partial ℓ^2 growth $\sum_{n=1}^{N} |\hat{\mu}(n)|^2$ (or $\frac{1}{4}\sum_{n=1}^{N} |a_n|^2$ for general Bloch functions): partial squared sums grow up to $O(\ln N)$ for general Bloch functions [17, 2] and as $\Theta(\sqrt{\ln N})$ for Kahane's positive singular Λ_* measure with sign-changing coefficients [14] (proved in the companion [8, Theorem 7.1]), whereas $\sup_{k\geq 0} s_k \leq M$ (in particular, $\mu \in \Lambda_*$ with $\hat{\mu} \geq 0$) compresses the upper envelope to $O_M(\ln\ln(4N))$ (blue shaded corridor), inside which Theorem 1.1's counterexample still diverges slowly.

6. Discussion

The results proved here settle both halves of Holland's Problem 5.69 (1984) and Holland's Problem 8.13 (1977) [1, 3, 10], separating the non-negative ℓ^2 question from the general ℓ^4 question that Holland linked via convolution squares. Requiring all Fourier–Stieltjes coefficients of a Zygmund probability measure to be non-negative does not

force $\hat{\mu} \in \ell^2(\mathbb{Z})$ and does not preclude pure singularity (Theorem 1.1), nor even absolute continuity outside $L^2(\mathbb{T})$ (Proposition 3.2). At the same time, positivity of $\operatorname{Re} F$ in the Bloch space is far stronger than general Bloch regularity: whereas lacunary Bloch series $\sum z^{2^k}$ have Taylor coefficients outside ℓ^q for every finite q, every Bloch–Herglotz function satisfies $\sum_{n=1}^{\infty} |a_n|^4 \le 48\,\|F\|_{\mathcal{B}}^2\,(\operatorname{Re} F(0))^2$ with $(a_n) \in \ell^q$ for all $q > 3$ (Theorem 1.2). For non-negative coefficients, the summability threshold lowers to the exact boundary $q = 2$ with the sharp dyadic-envelope criterion $\sum b(k)/(k+1) < \infty$ (Theorem 1.3).

These resolutions also sharpen the frontier of what remains unknown in the area.

The optimal ℓ^q exponent for general Bloch–Herglotz functions. Theorem 1.2 establishes $(a_n) \in \ell^q$ for all $q > 3$ (leaving unweighted $(2, 3]$ open in the present paper), and the companion expanded version [8, Theorem 8.1] bypasses the $q = 3$ Hausdorff–Young barrier [8, Theorem 7.2] for arbitrary complex coefficients to reach every $q > 5/2$ (while proving the exact critical exponent $q_c = 2$ for every balanced stopped-martingale measure, including Kahane's [14], in [8, Theorem 8.2]). Whether every Bloch function with positive real part has $(a_n) \in \ell^q$ for all $q > 2$ remains open across $(2, 3]$ here (and across the narrow strip $q \in (2, 5/2]$ jointly with [8]).

Compact Lebesgue-null support for positive measures. Corollary 3.1 localizes our purely singular little-Zygmund probability measure to any interval $[-\delta, \delta]$ with strictly positive Fourier–Stieltjes coefficients, while the companion expanded version constructs a *signed* measure in Λ_* with $\hat{\mu} \ge 0$ supported on a compact Lebesgue-null set K_∞ [8, Theorem 6.1] and proves that no nonzero real distribution with $\hat{\mu} \ge 0$ can live on Kahane's classic set E_∞^{Kah}, and none with bounded dyadic sums on a related 4-adic Cantor set [8, Theorem 6.2]. (Recent constructions of signed singular measures with $\hat{\mu}(n) \to 0$ taking prescribed non-negative Fourier–Stieltjes values [15] impose neither measure positivity nor the Zygmund condition Λ_*.) Does there exist a *positive* probability measure $\mu \in \Lambda_*$ with $\hat{\mu} \ge 0$ whose topological support has Lebesgue measure zero?

Exact finite-degree growth for $a_n \ge 0$. Theorem 1.3(iii) caps $\sum_{n=1}^{N} \hat{\mu}(n)^2$ by $O_M(\ln\ln(4N))$, whereas the iterated-product construction in [8, Proposition 6.6] achieves $\Omega(3^{\log^* N})$ (where $\log^* N$ denotes the iterated logarithm) and [8, Theorem 10.3] proves that 3 is the exact asymptotic base for separated analytic-square products. Determining the true extremal rate of growth of $\sup \sum_{n=1}^{N} \hat{P}(n)^2$ over all mean-1, degree-N non-negative trigonometric polynomials P with $\hat{P} \ge 0$ and dyadic block sums $s_k(P) \le 1$ invites further investigation.

Acknowledgments

This work was done overwhelmingly by AI models, steered by the author in a continuous critical dialogue, variously with Anthropic's Claude, OpenAI's ChatGPT and Google's Gemini. The author takes full responsibility for the paper.

References

[1] J. M. Anderson, K. F. Barth, and D. A. Brannan, *Research problems in complex analysis*, Bull. London Math. Soc. **9** (1977), no. 2, 129–162.
[2] J. M. Anderson, J. Clunie, and Ch. Pommerenke, *On Bloch functions and normal functions*, J. Reine Angew. Math. **270** (1974), 12–37.
[3] K. F. Barth, D. A. Brannan, and W. K. Hayman, *Research problems in complex analysis*, Bull. London Math. Soc. **16** (1984), no. 5, 490–517.
[4] D. A. Brannan and W. K. Hayman, *Research problems in complex analysis*, Bull. London Math. Soc. **21** (1989), no. 1, 1–35.
[5] P. L. Duren, *Smoothness of functions generated by Riesz products*, Proc. Amer. Math. Soc. **16** (1965), 1263–1268.
[6] P. L. Duren, *Theory of H^p Spaces*, Pure and Applied Mathematics, Vol. 38, Academic Press, New York–London, 1970.
[7] P. L. Duren, H. S. Shapiro, and A. L. Shields, *Singular measures and domains not of Smirnov type*, Duke Math. J. **33** (1966), 247–254.
[8] D. Goel, *Resolution of Holland's (1977, 1984) problems on Bloch functions and singular Zygmund measures, and other miscellanea*, companion expanded preprint, 2026; https://doi.org/10.5281/zenodo.24642945
[9] W. K. Hayman, *Research Problems in Function Theory*, The Athlone Press, University of London, 1967.
[10] W. K. Hayman and E. F. Lingham, *Research Problems in Function Theory: Fiftieth Anniversary Edition*, Problem Books in Mathematics, Springer, Cham, 2019; DOI https://doi.org/10.1007/978-3-030-25165-9 and preprint https://arxiv.org/abs/1809.07200
[11] G. Herglotz, *Über Potenzreihen mit positivem, reellen Teil im Einheitskreis*, Ber. Verh. Königl. Sächs. Ges. Wiss. Leipzig Math.-Phys. Kl. **63** (1911), 501–511.
[12] F. Holland, *The extreme points of a class of functions with positive real part*, Math. Ann. **202** (1973), 85–87.
[13] F. Holland and J. B. Twomey, *Fourier–Stieltjes series of measures in Zygmund's class Λ_**, Proc. Roy. Irish Acad. Sect. A **76** (1976), no. 26, 289–299.
[14] J.-P. Kahane, *Trois notes sur les ensembles parfaits linéaires*, Enseign. Math. (2) **15** (1969), 185–192.
[15] P. Ohrysko, *Fourier–Stieltjes palettes on compact abelian groups*, preprint (November 2026), https://arxiv.org/abs/2611.21340
[16] G. Piranian, *Two monotonic, singular, uniformly almost smooth functions*, Duke Math. J. **33** (1966), 255–262.
[17] Ch. Pommerenke, *On Bloch functions*, J. London Math. Soc. (2) **2** (1970), 689–695.
[18] H. S. Shapiro, *Monotonic singular functions of high smoothness*, Michigan Math. J. **15** (1968), 265–275.
[19] A. Zygmund, *Smooth functions*, Duke Math. J. **12** (1945), 47–76.
[20] A. Zygmund, *Trigonometric Series*, Vols. I and II, 2nd ed., Cambridge University Press, Cambridge, 1959.