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PROOFS AND COUNTEREXAMPLES FOR RECENT CONJECTURES IN COMBINATORICS AND RELATED AREAS

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Abstract. We present resolutions of twelve conjectures and questions in combinatorics and related areas. These include an infinite family of counterexamples to Mühlherr and Poullot's grid congruence, a proof of Kauers and Wang's prediction on longest increasing subsequences, a proof of Winter and Baumbach's denominator-degree bound for rational barycentric coordinates, and a proof of Mandrick, Tenner, and Vatter's conjecture on cycle types and inversion graphs. Each section identifies the authors and original statement, gives the argument in detail, and credits any prior resolution.

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1. Scope and attribution

This article assembles results on conjectures posed in several recent papers. Each section identifies the source, formulates the mathematical claim at issue, and gives the argument and any external inputs. We state the precise scope of the counterexamples and record the cases where a general conclusion remains open. The arguments are independent of one another.

The egg-cut formula for Kneser graphs is a consequence of Xu and Xu's theorem on restricted edge-connectivity. The result on Multicolor Permutation Wordle is due to Smithline. Dujella's

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conjecture [5, Conjecture 6.1] on the densities of Fibonacci elliptic-curve ranks remains open and lies outside the set of resolutions discussed here.

Guide to the results. The count includes statements settled by an implication, refutations of printed formulas, and the credited Wordle resolution. Mandrick, Tenner, and Vatter's cycle-type theorem also gives their cycle-count conjecture directly. The counterexample to Elvey Price and Park concerns the factorial exponent as printed. The proposed permuton limit and the exponential constant with the corrected factorial exponent are separate questions.

Problem	Outcome	Section
Mühlherr–Poullot, 1	Disproved	2
Ballinas et al., 5.5	Proved	3
Kauers–Wang, Section 1	Proved	4
Kutin–Smithline, 4.5	Smithline's resolution	5
Winter–Baumbach, 2.14 and 4.1	Proved	6
Mandrick–Tenner–Vatter, 8.3 and 8.5	Proved	7
Hassler–Kirgizov, 5.1	Disproved	8
Daga–Ramu, 31	Proved	9
Xiang–Zhao, 3.1	Disproved as printed	10
Elvey Price–Park, 6.11	Disproved as printed	11

AI Disclosure

ChatGPT Sol 6.2 wrote this manuscript and developed the proofs and disproofs presented here. Astra 6.2 conducted the research. Prior results are credited in their respective sections.

2. Mühlherr and Poullot on acyclic orientations

Let $P_G(t)$ be the chromatic polynomial and $\psi(G)$ the number of acyclic orientations of a finite simple graph G. Mühlherr and Poullot [13, Conjecture 1] predict $\psi(G) \equiv 2 \pmod 4$ for ordinary and triangular rectangular grids with both side lengths odd and at least three, generalized Petersen graphs $\mathrm{GP}(n,k)$ with n even and k odd, and complete tripartite graphs whose part sizes are not all odd. They relate this congruence to a Hamiltonicity obstruction for graphs with an even number of edges.

Stanley's reciprocity theorem [17] gives

$$\psi(G) = (-1)^{|V(G)|} P_G(-1) \equiv (-1)^{|V(G)|} P_G(3) \pmod 4. \tag{1}$$

Multiplication by the sign fixes the residues 0 and 2. Accordingly we compute the numbers of proper three-colorings modulo four.

2.1. A counterexample for ordinary grids.

For paths P_m, P_n on m, n vertices write $C(m,n) = P_{P_m \square P_n}(3)$.

Lemma 2.1 (Folding)**.** *For $r, n \geq 2$, $C(2r-1, n) \equiv C(r, n) \pmod 4$.*

Proof. Prescribe distinct colors 1 and 2 on two adjacent vertices in the central row r of the larger grid. The six choices of ordered distinct colors occur equally often, by permuting the color names. Thus the number of colorings with this prescription is $C(2r-1,n)/6$. Reflection in row r is an involution on this set. A coloring fixed by reflection is determined by its restriction to the top r rows. Conversely, any such coloring with the same prescription extends by reflection. Edges

across the central row match already-checked edges on the other side. The fixed-point count is therefore $C(r,n)/6$. Taking parity and multiplying by six proves the claim. □

Theorem 2.2. *For all $a, b \geq 1$, the number of acyclic orientations of the $(3 \cdot 2^a + 1)$-by-$(3 \cdot 2^b + 1)$ rectangular grid is divisible by four. In particular, the 7×7 grid refutes the ordinary-grid clause of Conjecture 1.*

Proof. Folding each dimension reduces the coloring count modulo four to $C(4,4)$. In a row of width four, relabel colors by the order of their first occurrence. The possible row patterns are $ABAB, ABAC, ABCA, ABCB$. Each has exactly six realizations, since every row uses at least two colors. For a fixed realization of type i, let M_{ij} count the realizations of type j that differ from it in every column. This number is independent of the chosen realization of type i. Comparing the six possible assignments for each column of this matrix gives

$$M = \begin{pmatrix} 3 & 2 & 1 & 2 \\ 2 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ 2 & 2 & 1 & 2 \end{pmatrix}.$$

Thus with $\mathbf{1} = (1,1,1,1)^{\mathsf{T}}$,

$$C(4,4) = 6\mathbf{1}^{\mathsf{T}} M^3 \mathbf{1} = 6(399 + 342 + 219 + 342) = 7812.$$

Equation (1) gives the stated residue. Appendix A also provides direct integer counts for both square grids. □

The residue for 4×4 already appears in Mühlherr and Poullot's Example 3.13. The 7×7 count modulo color relabeling is also recorded in OEIS A207993 [14]. The counterexample concerns the modular claim. The existence of a Hamiltonian cycle in $\mathcal{AO}(P_7 \square P_7)$ is a separate question.

2.2. **Triangular grids and complete tripartite graphs.** For $m, n \geq 2$, use the triangular-lattice patch with vertices (i,j), horizontal and vertical grid edges, and diagonal edges of step $(1,-1)$ within the rectangle. Color (i,j) by $i+2j$ mod 3. This is proper because each edge changes the residue. Every elementary triangular face uses three distinct colors. After assigning colors on one face, the common edge with an adjacent face determines its third color. The face adjacency graph is connected and every vertex is in a face. Thus exactly six colorings exist, one per color permutation. Equation (1) yields $\psi(G) \equiv 2 \pmod 4$ for this patch. It has $3mn - 2m - 2n + 1$ edges, which is even for odd m, n.

Likewise, a proper three-coloring of $K_{a,b,c}$ with positive part sizes must assign a different color to each of the three parts. The three nonempty parts need disjoint nonempty sets of colors, so each gets exactly one. There are six such colorings, giving $\psi(K_{a,b,c}) \equiv 2 \pmod 4$ for all positive a, b, c. The edge count $ab + bc + ca$ is even precisely when at most one part size is odd. In particular, the even-edge hypothesis used in the source's obstruction fails when exactly two part sizes are odd.

In fact $\mathcal{AO}(K_{1,1,2j})$ is Hamiltonian for every $j \geq 1$. This follows from the theorem of Savage, Squire, and West [15], since removing the vertices of the large part leaves a single edge, and each removed vertex has neighbors forming a clique. An explicit cycle is also available. Fix one direction of the edge joining the two singleton parts. Every other vertex lies before both ends, between them, or after both in a topological ordering. These three choices independently specify all acyclic orientations with that fixed direction. Flipping one edge at a vertex moves between adjacent choices, yielding a copy of $P_3^{\square 2j}$. In that copy a Hamiltonian path connects the all-before

corner to the all-after corner. To construct the path, induct on the number of coordinates, traversing the first layer forward, the middle layer backward, and the last layer forward. At either of the two corners, flipping the chosen singleton edge changes no other orientation and goes to the corresponding corner in the other copy. Join the two paths by these two flips.

2.3. **Generalized Petersen graphs.**

Lemma 2.3 (Coloring–flow congruence [19])**.** *If G has at least two edges and $F_G(3)$ counts nowhere-zero flows over $\mathbb{F}_3$, then $P_G(3) \equiv F_G(3) \pmod 4$.*

Proof. Fix an orientation on $E(G)$, put $\zeta = e^{2\pi i/3}$, and assign colors $x_i \in \mathbb{F}_3$ to vertices. For an edge with endpoints i, j the factor $2 - \zeta^{x_i - x_j} - \zeta^{x_j - x_i}$ is three when the colors differ and zero otherwise. Expand the product over edges and sum over vertex assignments. A term indexed by $f : E(G) \to \mathbb{F}_3$ survives exactly when the coefficient of each x_i in its exponent is zero, by $1 + \zeta + \zeta^2 = 0$. This is flow conservation. Writing $v = |V(G)|$, $e = |E(G)|$, and $z(f) = |\{g \in E(G) : f(g) = 0\}|$ gives

$$3^e P_G(3) = 3^v \sum_{f \text{ a flow}} 2^{z(f)} (-1)^{e - z(f)}.$$

The zero flow contributes a multiple of four because $e \geq 2$. Other flows with at least one zero entry contribute multiples of four in pairs $f, -f$. The remaining flows also occur in pairs, so $F_G(3)$ is even. Odd factors and signs preserve either even residue modulo four. This proves the lemma. □

In a cubic graph a nowhere-zero $\mathbb{F}_3$-flow can be represented by reversing each edge carrying -1 and putting flow 1 on the resulting directions. At every vertex the outdegree minus indegree is divisible by three. A vertex of degree three must therefore be a source or a sink. This gives a bipartition. Conversely, directing every edge from one part of a bipartition to the other gives a flow. A connected cubic graph hence has two such flows if bipartite and zero otherwise.

For $\mathrm{GP}(n, k)$ use u_i, v_i with edges $u_i u_{i+1}, u_i v_i, v_i v_{i+k}$, indices modulo n, and $1 \leq k < n/2$. This graph is connected and cubic. The outer cycle forces n even and the u_i alternate in any bipartition. The spokes force opposite colors on v_i, and the inner edges then force k odd. Conversely those parity conditions define a bipartition using the parity of i on the outer cycle and the opposite parity inside. Lemma 2.3 and (1) give

$$\psi(\mathrm{GP}(n, k)) \equiv \begin{cases} 2 \pmod 4, & n \text{ even and } k \text{ odd}, \\ 0 \pmod 4, & \text{otherwise}. \end{cases}$$

3. Ballinas, Caine, Hopkins, and Rivera Laboy on Kneser graphs

The Kneser graph $\mathrm{KG}(n, k)$ has the k-subsets of $[n]$ as vertices and joins disjoint pairs. The uniform edge scramble has one egg for each edge, regarded as its two endpoints. Its egg-cut number $e(\mathcal{S})$ is the minimum number of edges whose deletion separates two intact eggs. Ballinas, Caine, Hopkins, and Rivera Laboy [1, Conjecture 5.5] propose

$$e(\mathcal{S}) = 2 \left(\binom{n - k}{k} - 1 \right).$$

We prove the formula for $k \geq 2,\ n > 2k$.

For a connected simple graph let $\lambda_2(G)$ be the smallest number of edges whose removal disconnects G without leaving isolated vertices. When such a cut exists, it separates two eggs, so

$e(\mathcal{S}) \le \lambda_2(G)$. Conversely, if a minimum egg-cut leaves an isolated vertex, restore one removed edge incident with it. Connectedness supplies such an edge. Restoring it only joins the singleton to another component and leaves the two separated eggs intact. This contradicts minimality, giving $e(\mathcal{S}) = \lambda_2(G)$.

The Kneser graph is regular of degree $D = \binom{n-k}{k} \ge 3$. It is connected because two k-sets that differ in one element have union of size $k+1$, whose complement contains a k-set. Such exchanges connect any pair of k-sets by repeated exchanges, each realized by a two-edge walk. A permutation of $[n]$ sends any ordered pair of disjoint k-sets to any other, proving edge-transitivity. A restricted cut exists. Separate an adjacent pair from all other vertices, each of which retains at least $D-2$ neighbors. Xu and Xu [21] prove that a connected edge-transitive graph other than a star, in the range of existence of restricted cuts, has

$$\lambda_2(G) = \min_{uv \in E(G)} (\deg u + \deg v - 2).$$

Here the minimum is $2D-2$.

The hitting number of the uniform edge scramble is the minimum vertex-cover size $\binom{n}{k} - \alpha(\mathrm{KG}(n,k)) = \binom{n-1}{k}$ by the Erdős–Ko–Rado theorem [6]. Thus whenever $2(D-1) \ge \binom{n-1}{k}$, its order equals $\binom{n-1}{k}$, and the bounds on scramble number and divisorial gonality in [1] give equality for both. For $k = 1, n \ge 4$, the argument likewise gives $2D-2$. The graph $KG(3,1) = K_3$ has no cut separating two eggs.

4. Kauers and Wang on longest increasing subsequences

Let $a_{n,k}$ be the number of permutations of $[n]$ whose longest increasing subsequence has length exactly k. Kauers and Wang [9, Section 1] predict that this array fails to be globally D-finite. Their definition requires uniform recurrences with polynomial coefficients in both indices, one in each individual index. We show that the recurrence in k is impossible.

Lemma 4.1. *For each fixed $k \ge 1$, $\lim_{n\to\infty} a_{n,k}^{1/n} = k^2$.*

Proof. Put $b_{n,k} = \sum_{j=1}^{k} a_{n,j}$. The Robinson–Schensted correspondence, followed by transposing both tableaux, gives

$$b_{n,k} = \sum_{\lambda \vdash n,\, \ell(\lambda) \le k} (f^\lambda)^2,$$

where f^λ counts standard Young tableaux of shape λ. A tableau with at most k rows is determined by the row containing each entry, since rows increase and their lengths are determined by the row assignments. Consequently $\sum_{\ell(\lambda)\le k} f^\lambda \le k^n$, and $b_{n,k} \le k^{2n}$.

For a lower bound choose a partition λ whose k row lengths differ by at most one. The hook length of the box in row i, column j, is at most $\lambda_i - j + k$. The product of these bounds in row i is at most $\lambda_i!(n+k)^{k-1}$. The hook-length formula [18] yields

$$f^\lambda \ge \frac{n!}{\prod_i \lambda_i!} (n+k)^{-k(k-1)}.$$

The balanced multinomial coefficient is maximal, since transferring one unit from a larger part to a part at least two smaller increases the coefficient. Because the sum of all these coefficients is k^n and there are at most $(n+1)^k$ of them, f^λ is at least $k^n/((n+1)^k(n+k)^{k(k-1)})$. Hence $b_{n,k}$ has nth-root limit k^2. For $k \ge 2$, $b_{n,k-1} \le (k-1)^{2n}$ is exponentially smaller than that lower bound for $b_{n,k}$. Subtracting leaves the same nth-root limit for $a_{n,k}$. For $k = 1$ the decreasing permutation gives $a_{n,1} = 1$. □

Theorem 4.2. *No polynomials $q_0, \dots, q_s \in \mathbb{C}[n,k]$, with $q_s \neq 0$, satisfy*

$$\sum_{j=0}^{s} q_j(n,k) a_{n,k+j} = 0 \qquad (n,k \geq 1).$$

Proof. Choose $K \geq 1$ such that $q_s(n,K)$ is a nonzero polynomial in n. Such a K exists because at least one coefficient in n is a nonzero polynomial in k and has only finitely many roots. For all sufficiently large n, $q_s(n,K)a_{n,K+s} \neq 0$. At $k = K$ divide the recurrence by this product to obtain

$$1 + \sum_{j=0}^{s-1} \frac{q_j(n,K)}{q_s(n,K)} \frac{a_{n,K+j}}{a_{n,K+s}} = 0.$$

By Lemma 4.1, the nth-root limit of the last factor in each summand is $(K+j)^2/(K+s)^2 < 1$. Each polynomial ratio grows at most polynomially. Thus every summand tends to zero, giving $1 = 0$. The argument includes $s = 0$. □

This proves global non-D-finiteness in the stated sense. Kauers and Wang establish D-finiteness inside the region $k \geq n/2$. The fixed-column limit used here eventually lies outside that region.

5. Kutin and Smithline on Multicolor Permutation Wordle

Kutin and Smithline [10, Conjecture 4.5] propose that CircularShift solves at least as many secrets within every fixed round budget as any other strategy. Rank i has $s_i \geq 1$ colors, and a secret is a permutation of the n ranks with one color for each. A response identifies exactly the positions with correct rank and color. Smithline [16, Theorem 1.1] proved the conjecture for arbitrary $(s_1, \dots, s_n)$. We give the deduction from his filtered dimension theorem.

Encode the s_i choices by the roots of $z^{s_i} - v_i^{s_i}$, where the v_i are distinct positive reals. Different ranks have different root moduli, so these symbols faithfully encode ranks and colors. Write $\Omega \subset \mathbb{C}^n$ for the secret vectors. Let $R(d)$ consist of polynomials of degree at most d in each variable. Denote by $H(d)$ the dimension of the image of the restriction map $R(d) \to \mathbb{C}^\Omega$.

A deterministic strategy can extend a successful play by repeating its winning guess. With that convention, each secret solved within $d+1$ guesses has a length-d response history $U = (U_1, \dots, U_d)$, with U_t the set of correct positions in round t. Within each history the next guess is fixed, so at most one secret is solved there by round $d+1$. Consider the histories having such a witness, one secret for each. Let $g_{t,i}(U)$ be the encoded guess for position i at round t along U. The polynomial

$$p_U(x) = \prod_{t=1}^{d} \prod_{i \notin U_t} (x_i - g_{t,i}(U))$$

belongs to $R(d)$ and is nonzero on the witness for U. Order histories lexicographically, ordering the sets at each stage by increasing cardinality and breaking ties by any fixed order of sets. If U precedes V, at their first different response the guesses agree and some position belongs to $V_t \setminus U_t$. Its factor makes p_U zero on the witness for V. The evaluations form a triangular matrix with nonzero diagonal. Therefore the number of solved secrets is at most $H(d)$.

For this encoding, Smithline's filtered Koszul theorem [16, Corollary 5.3 and Theorem 6.1] gives

$$H(d) = \sum_{J \subseteq [n]} (-1)^{|J|} \max\left(d + 1 - \sum_{i \in J} s_i, 0\right)^n.$$

Let $A_n(u) = \sum_{\sigma \in S_n} u^{\mathrm{des}(\sigma)}$, the Eulerian polynomial. The identity $\sum_{d \geq 0} (d+1)^n u^d = A_n(u)/(1-u)^{n+1}$ and the dimension formula show, with $H(-1) = 0$, that

$$\sum_{d \geq 0} (H(d) - H(d-1)) u^d = A_n(u) \prod_{i=1}^{n} (1 + u + \cdots + u^{s_i - 1}).$$

Indeed, shifting the summation index by $\sum_{i \in J} s_i$ contributes $u^{\sum_{i \in J} s_i}$, the inclusion–exclusion sum factors as $\prod_i (1 - u^{s_i})$, and taking successive differences multiplies the generating function by $1 - u$. Kutin and Smithline [10, Theorem 4.7] give the same polynomial for the number of secrets CircularShift solves in exactly $d+1$ guesses, with d marked by u. Summing the coefficients through d shows that CircularShift attains $H(d)$. Randomization takes an average of deterministic success probabilities under uniform secrets, so the same upper bound applies.

6. Winter and Baumbach on barycentric denominators

Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope with m facets. Generalized barycentric coordinates are nonnegative functions α_v indexed by the vertices, with $\sum_v \alpha_v = 1$ and $\sum_v v \alpha_v = x$ on P. In the rational case write $\alpha_v = p_v/q$ with polynomials and a common nonzero denominator. The two identities can be multiplied through by q on the dense open subset where the quotient is defined. They then hold as polynomial identities. Winter and Baumbach [2, Conjectures 2.14 and 4.1] predict the following bounds.

Theorem 6.1. *Let E be a nonempty flat of the face arrangement, defined as an intersection of affine spans of faces, and let s_E be the number of facet hyperplanes containing E. A flat is called residual if the vertices of P in E do not affinely span E. Then*

$$\mathrm{ord}_E(q) \;\geq\; s_E - \mathrm{codim}\, E + \mathbf{1}_{\{E \text{ residual}\}}.$$

Also, every common denominator satisfies $\deg q \geq m - 2d$. In particular the minimum degree among all such denominators obeys the same bound.

The proof of the vanishing order uses a homogeneous calculation in the normal directions to E. The degree bound then follows by restricting the denominator to a facet.

6.1. **A lemma on rational functions on cones.** Let $a_1, \ldots, a_s \in \mathbb{R}^r$ span $\mathbb{R}^r$, where $r \geq 1$, and assume there is a vector ξ with $\langle a_i, \xi \rangle > 0$ for every i. Put

$$D(y) = \prod_{i=1}^{s} \langle a_i, y \rangle, \qquad K = \mathrm{cone}(a_1, \ldots, a_s).$$

For each of finitely many nonzero vectors $b_j \in K^*$, write

$$I_j = \{i : \langle a_i, b_j \rangle = 0\}, \qquad D_j(y) = \prod_{i \in I_j} \langle a_i, y \rangle.$$

Here $K^* = \{b : \langle b, x \rangle \geq 0 \text{ for every } x \in K\}$, and an empty product equals 1.

Lemma 6.2. *Suppose Q is a nonzero homogeneous polynomial of degree k and the following identity holds for some polynomials H_j:*

$$\frac{yQ(y)}{D(y)} = \sum_j \frac{b_j H_j(y)}{D_j(y)}. \tag{2}$$

Then $k \geq s - r$. If there are also polynomials G_j with

$$\frac{Q(y)}{D(y)} = \sum_j \frac{G_j(y)}{D_j(y)}, \tag{3}$$

then $k \geq s - r + 1$.

Proof. For a subset I of $\{1, \ldots, s\}$, let μ_I be the image of Lebesgue measure on $\mathbb{R}^I_{\geq 0}$ under $t \mapsto \sum_{i \in I} t_i a_i$. For $I = \varnothing$ use the unit mass at 0. Each measure is locally finite and has polynomial growth because $\langle \xi, \sum_{i \in I} t_i a_i \rangle \geq c \sum_{i \in I} t_i$ for $c = \min_i \langle \xi, a_i \rangle > 0$. Its Laplace transform at y with all $\operatorname{Re}\langle a_i, y \rangle > 0$ is

$$\prod_{i \in I} \langle a_i, y \rangle^{-1},$$

by integrating in the variables t_i separately.

Write $\mu = \mu_{\{1,\ldots,s\}}$ and $T = Q(\partial)\mu$, with derivatives in the sense of distributions. Applying a polynomial in derivatives corresponds to multiplying the transform by the same polynomial in y. These transforms are injective on the distributions used here. Indeed, multiplication by $e^{-\langle \xi, x \rangle}$ converts $P(\partial)\mu_I$ into $P(\partial + \xi)(e^{-\langle \xi, x \rangle}\mu_I)$, a derivative combination of a finite measure. Its Fourier transform is the Laplace transform along $y = \xi + i\eta$. Fourier inversion on tempered distributions gives injectivity, since the Fourier transform is invertible on the Schwartz space and hence on its dual. In particular $T \neq 0$, since Q/D is a nonzero rational function.

For each j, the measure μ_{I_j} and its derivatives are supported on the proper exposed face $K \cap b_j^{\perp}$. Equation (2) now gives

$$\nabla T = \sum_j b_j H_j(\partial) \mu_{I_j}. \tag{4}$$

Equation (3), when present, puts the support of T in the boundary of K. In either case T is supported in K.

The dilation law $\langle \mu, \varphi(\cdot/\lambda) \rangle = \lambda^s \langle \mu, \varphi \rangle$ and homogeneity of Q give $\langle T, \varphi(\cdot/\lambda) \rangle = \lambda^{s-k} \langle T, \varphi \rangle$ for $\lambda > 0$. Differentiating in λ gives the Euler identity

$$(x \cdot \partial) T = (s - r - k) T. \tag{5}$$

Choose a face A of K of largest dimension whose relative interior meets $\operatorname{supp} T$, counting K itself as a face. Fix a point in that intersection. On a sufficiently small open neighborhood U of the point, T is supported in $\operatorname{span} A$. Indeed, the finitely many faces of dimension at most $\dim A$ that do not contain the point can be avoided, and no relative interior of a larger face meets the support. For any direction in $\operatorname{span} A$, the directional derivative of T vanishes on U by (4). Terms whose supporting faces miss the point can be excluded by shrinking U. For every remaining term, $K \cap b_j^{\perp}$ contains A, so that direction is orthogonal to b_j.

Use linear orthogonal coordinates (z, w) along $\operatorname{span} A$ and its orthogonal complement, and put $c = r - \dim A$. On U every monomial w^α with $|\alpha| > k$ annihilates T. To see this, pair T with $w^\alpha \varphi$ for a smooth compactly supported test function φ in U. Insert a smooth cutoff equal to 1 near $w = 0$ and supported in $|w| \leq \epsilon$. This preserves the pairing because T is supported in $w = 0$. All derivatives through order k of the resulting test function are $O(\epsilon^{|\alpha| - k})$, including the derivatives of the rescaled cutoff. Since T is a combination of order k derivatives of a locally finite measure, the pairing tends to zero.

Choose a normal monomial w^α of largest degree j_0 with $S = w^\alpha T \neq 0$ on U. If $c = 0$, use $S = T$ and $j_0 = 0$. Then $\partial_{z_i} S = 0$ and $w_i S = 0$, so $w_i \partial_{w_i} S = -S$ for each normal coordinate.

Equation (5) and the product rule also give $(x \cdot \partial)S = (s - r - k + j_0)S$. Therefore

$$s - r - k + j_0 = -c, \qquad k = s - r + c + j_0.$$

This proves $k \geq s - r$. Under (3) the face A is proper, so $c \geq 1$ and $k \geq s - r + 1$. □

6.2. **Vanishing on the face arrangement.** Let P be a full-dimensional bounded polytope and let ℓ_i be affine facet forms, nonnegative on P. If $\{p_v/q\}$ are rational generalized barycentric coordinates indexed by the vertices v, then

$$L(x)q(x) = \sum_v L(v)p_v(x) \tag{6}$$

for every affine form L. Also $\ell_i \mid p_v$ whenever $\ell_i(v) > 0$. In fact, on the interior $0 \leq \ell_i(v)p_v(x)/q(x) \leq \ell_i(x)$ by (6) and nonnegativity. At a point of facet i, approaching through the interior makes p_v tend to zero since q is a polynomial. Thus p_v vanishes on the facet and on its affine hull.

For a nonempty flat E, let $m_P(E)$ be the number of facet hyperplanes containing it and $r = \operatorname{codim} E$. Every flat is an intersection of facet hyperplanes, since each face span is such an intersection. Define

$$\eta_P(E) = m_P(E) - r + \mathbf{1}_{\mathrm{res}}(E),$$

where $\mathbf{1}_{\mathrm{res}}(E) = 0$ when the vertices of P on E affinely span E, and 1 otherwise. The latter flats are called residual. We prove that $\operatorname{ord}_E q \geq \eta_P(E)$, as asserted in Conjecture 2.14. The full ambient flat has $\eta_P(E) = 0$, so consider $r \geq 1$.

Write $s = m_P(E)$, and let N be the orthogonal complement of the direction space of E, identified with $\mathbb{R}^r$. Put $k = \operatorname{ord}_E q$, the smallest degree in the normal variables in the expansion along E. Choose $x_0 \in E$ so that the degree k homogeneous part $Q(y)$ of $q(x_0 + y)$ for $y \in N$ is nonzero. This is possible because the coefficients of that part, as polynomials on E, do not all vanish identically. If E is residual, also require $L(x_0) \neq 0$ for an affine form L vanishing on all vertices in E and nonzero on E as a polynomial. Such an L exists by the definition of residual, with $L = 1$ available when there are no vertices in E.

Number the containing facet hyperplanes $1, \ldots, s$. In the normal variables $\ell_i(x_0 + y) = \langle a_i, y \rangle$. The a_i span N by the definition of E, and have positive inner products with the projection to N of $x_{\mathrm{int}} - x_0$ for an interior point x_{int} of P. For $v \notin E$, let b_v be the orthogonal projection of $v - x_0$ to N. It is nonzero and satisfies $\langle a_i, b_v \rangle = \ell_i(v) \geq 0$. In particular the vectors a_i, b_v satisfy the hypotheses of the lemma.

Write $p_{v,j}(y)$ for the degree j homogeneous part of $p_v(x_0 + y)$. Each such part is divisible by $\prod_{i:\langle a_i, b_v \rangle > 0} \langle a_i, y \rangle$ because those facet factors are homogeneous on the slice. The projection to N of (6) for coordinate forms gives

$$y\, q(x_0 + y) = \sum_{v \notin E} b_v\, p_v(x_0 + y).$$

Taking degree $k+1$ and dividing by $D(y)$ gives (2), with H_v the quotient of $p_{v,k+1}$ by the displayed product. Hence $k \geq s - r$. In the residual case, taking degree k in (6) for the selected L gives

$$L(x_0)Q(y) = \sum_{v \notin E} L(v)p_{v,k}(y).$$

Dividing by $L(x_0)D(y)$ gives (3) using the same facet divisibility. Therefore $k \geq s - r + 1$ in this case. This proves Conjecture 2.14.

6.3. **Degree bound by restricting the denominator.** In fact any nonzero polynomial q satisfying $\operatorname{ord}_E q \geq \eta_P(E)$ on the nonempty flats of P has $\deg q \geq m - 2d$. We prove this by induction on d. For a segment there are two facets and the bound is zero. The dimension-zero statement is also immediate. In higher dimension, if q vanishes identically on every facet hyperplane, the product of the distinct facet forms divides it and $\deg q \geq m$. In the remaining case choose a facet F with $q|_H \neq 0$, where $H = \operatorname{aff} F$.

Let $\mathcal{G}_F$ consist of the facets $G \neq F$ whose hyperplanes are nonparallel to H and that do not share a ridge with F. Put

$$R_F = \prod_{G \in \mathcal{G}_F} \ell_G|_H.$$

For $G \in \mathcal{G}_F$, the flat $E = H \cap \operatorname{aff} G$ has codimension two in the ambient space and is residual because $P \cap E = F \cap G$ has dimension at most $d-3$ or is empty. If b of the factors of R_F vanish on this hyperplane of H, then the number of facet hyperplanes of P containing E is $1+b$. Indeed, it includes H, and any other such hyperplane must restrict to E and be from a nonadjacent facet, since $P \cap E$ is smaller than a ridge. The order bound for this flat is $1 + b - 2 + 1 = b$, so $q|_H$ is divisible by the corresponding linear factor to power b. This proves $R_F \mid q|_H$ with all multiplicities. Write $q_F = q|_H / R_F$.

Now take any nonempty flat E of the polytope F in its own ambient space H. It is a flat of P because each facet of F is a ridge of P, and its hyperplane in H comes from the unique other facet of P through that ridge. The residual indicator is the same for E in F and P, since $F \cap E = P \cap E$. Write b_E for the number of $G \in \mathcal{G}_F$ whose hyperplanes contain E. Then

$$m_P(E) = 1 + m_F(E) + b_E, \qquad \operatorname{codim}_{\mathbb{R}^d} E = 1 + \operatorname{codim}_H E,$$

where $m_F(E)$ counts facet hyperplanes of F containing E. It follows that $\eta_P(E) = \eta_F(E) + b_E$. Restriction to H preserves the lower bound on vanishing order along E because it maps the ideal of E into the ideal of E in H. The product R_F has order exactly b_E there. Orders along an affine subspace add under multiplication, as seen from the lowest nonzero terms in the normal variables. Thus

$$\operatorname{ord}^H_E q_F = \operatorname{ord}^H_E(q|_H) - b_E \geq \eta_F(E).$$

The induction hypothesis applies directly to this polynomial on H.

If F has m_F facets, there are m_F other facets of P adjacent to it, one per ridge in F. There is at most one other facet parallel to F, on the opposite supporting hyperplane. Hence $\deg R_F \geq m - 2 - m_F$, and

$$\deg q \geq \deg(q|_H) = \deg R_F + \deg q_F \geq (m - 2 - m_F) + \big(m_F - 2(d-1)\big) = m - 2d.$$

The vanishing theorem puts every barycentric denominator under this bound. This proves Conjecture 4.1 in every dimension.

7. Mandrick, Tenner, and Vatter on inversion graphs

Mandrick, Tenner, and Vatter [11, Conjectures 8.3 and 8.5] ask whether the number of cycles and, more strongly, the cycle type are determined by the inversion graph within the class $\operatorname{Av}(3421, 4312)$. The class consists of permutations avoiding both patterns as subsequences in relative order. Cycle type is the multiset of cycle lengths, including fixed points.

Theorem 7.1. *If two permutations in* $\operatorname{Av}(3421, 4312)$ *have isomorphic inversion graphs, their cycle types agree.*

The theorem proves Conjecture 8.5. Taking the number of parts in the cycle type gives Conjecture 8.3 as a corollary. Here the inversion graph has an edge between two entries when their orders by position and by value disagree.

Write $\alpha \oplus \beta$ for the direct sum, with α first and all its values below those of β. Write $\alpha \ominus \beta$ for the skew sum, with α first and all its values above those of β. A block is a consecutive set of positions whose values also form a consecutive set. Its standardized permutation is obtained by ranking its values within the block.

7.1. **Adding a largest entry at the left.** Suppose α avoids 312 and 3421. Split α at its smallest entry. Every entry before the minimum is less than every entry after it, since any violation would give 312. Applying this repeatedly to the remaining suffix writes

$$\alpha = (\delta_1 \ominus 1) \oplus \cdots \oplus (\delta_k \ominus 1),$$

where each δ_i may be empty. Each summand has connected inversion graph, because its final entry is adjacent to all the others. The summands are therefore exactly the graph's connected components. Each δ_i avoids 312 by restriction and 231 because a 231 followed by the final minimum would give 3421. Such permutations are direct sums of decreasing permutations: the split at the minimum again separates a direct summand, and the prefix preceding that minimum must be decreasing to avoid 231.

Let $b_i = |\delta_i| + 1$, $s_0 = 0$ and $s_i = b_1 + \cdots + b_i$. In $1 \ominus \alpha$, vertex 1 maps to $s_k + 1$, and $s_i + 1$ maps to $s_{i-1} + 1$ for $1 \le i \le k$. The remaining vertices in the interval from $s_{i-1} + 2$ through s_i carry a shifted copy of δ_i. Hence the cycle containing 1 has length $k + 1$, and the remaining cycle lengths are the union, with multiplicities, of those of the δ_i.

This determines both the distinguished cycle length and the multiset of the other cycle lengths from the inversion graph of α. Indeed, a component is obtained by adjoining a universal vertex to a disjoint union of cliques. If the component is complete, the clique size after removing the universal vertex is the component size minus one. Otherwise the universal vertex is unique, and deleting it reveals the clique sizes. A decreasing permutation of size a has $\lfloor a/2 \rfloor$ two-cycles and one fixed point if a is odd.

7.2. **Replacing a block.** Let a block of size m occupy positions $p + 1, \ldots, p + m$ and values $q + 1, \ldots, q + m$. An outside entry forms inversions with either every entry of the block or none. Let L and R be the numbers of outside entries forming inversions with the block on its left and right respectively. An outside entry not forming inversions lies on the same side of the block in both orders. Counting entries before the block in the two orders thus gives

$$q - p = R - L.$$

If the block contains both an increasing and a decreasing pair, $L \le 1$ and $R \le 1$. Indeed, two outside entries on the same side forming inversions with the block make an increasing or decreasing pair. Choose a pair of the opposite type inside the block. The resulting four entries have pattern $12 \ominus 21 = 3421$ or $21 \ominus 12 = 4312$.

Now consider a replacement by another permissible standardized block with isomorphic inversion graph and the same position and value intervals.

- If the block is monotone, its standardized permutation is determined by its inversion graph, so the replacement is identical.
- If $p = q$, the block supports a disjoint set of complete cycles. Its replacement preserves the global cycle type as soon as it preserves the cycle type of the standardized block.

- If $q = p - 1$ and the block contains both types of pairs, then $L = 1$ and $R = 0$. The sole inverted entry to the left lies above the block. Consequently its standardized permutation α avoids 312, since otherwise that entry followed by a 312 would give 4312. The arrows of the permutation whose tails lie in the block consist of closed cycles and a single path from $p+m$ to p. This follows because the set of tails and the set of heads differ by exactly these two vertices, and each vertex has at most one incoming and one outgoing arrow. Add the auxiliary arrow $p \to p + m$. On the interval $p, \ldots, p + m$, the resulting permutation standardizes to $1 \ominus \alpha$. The preceding calculation determines the closed cycle lengths and the length of the path from the graph of α: the path length is the distinguished cycle length minus one. The same calculation applies to the replacement. Keeping the outside arrows fixed then preserves the full cycle type, since the outside arrows only meet this path at its two ends.
- If $q = p + 1$, the sole inverted entry lies on the right and below the block, so α avoids 231. The auxiliary arrow closing the path now gives $\alpha \ominus 1$. Reversing the order and complementing the values conjugates it to $1 \ominus \mathrm{rc}(\alpha)$, where $\mathrm{rc}(\alpha)$ denotes the reverse-complement of α. Reverse-complement swaps 231 with 312 and 3421 with 4312. Thus $\mathrm{rc}(\alpha)$ avoids 312 and 3421. The distinguished cycle is carried to the distinguished cycle, and the same calculation applies.

The replacement argument depends only on the original and replacement block orders and the prescribed position and value intervals. Once these are fixed, it applies with any compatible outside arrows.

7.3. **Induction via modular decomposition.** Proceed by induction on the permutation size. For isomorphic inversion graphs, identify the vertices using a graph isomorphism. A module is a set of vertices to which each outside vertex is either fully adjacent or fully nonadjacent. A strong module does not overlap another module, where overlap means a nonempty intersection with neither set containing the other.

Use the standard modular decomposition into maximal proper strong modules [3, Section 2.1 and Proposition 1]. In a permutation graph these modules are intervals in both orders, so they give blocks in every permutation representation of the graph. The quotient, formed by contracting each block, is empty, complete, or prime. In the empty case the blocks are the connected components of the graph, and in the complete case they are the connected components of its complement. The cases of size zero and one are immediate.

Empty quotient. The permutations are direct sums of their block permutations. Corresponding standardized blocks have isomorphic inversion graphs and remain in the avoidance class. The induction hypothesis determines their cycle types. The order of the summands does not affect the union of their cycle lengths.

Complete quotient. The permutations are skew sums of the blocks. A nonsingleton block contains an increasing pair, since its complement graph is connected. If every block is a singleton, the permutation is decreasing. Otherwise, fix a nonsingleton block. There cannot be two other blocks on the same side of it, since an entry from each of them makes a decreasing pair, giving a forbidden pattern with the increasing pair inside the fixed block. Thus there are at most three blocks.

- With three blocks, the nonsingleton block must be in the middle and the others must be singletons, by the same argument applied to either end. The two end entries form a

two-cycle. The middle block occupies the same position and value intervals, so its cycle type is determined by induction.

- With two blocks, if both are increasing, their skew sum is a cyclic shift on $a+b$ letters, where a, b are the block sizes. The cycle type consists of $\gcd(a,b)$ cycles of length $(a+b)/\gcd(a,b)$.
- Otherwise a block contains both an increasing and a decreasing pair. The bound on outside inversions on either side forces its partner to be a singleton. Apply reverse-complement if necessary to put the singleton first in each representation. The result has form $1 \ominus \alpha$, where α avoids 312 and 3421. The calculation above determines its full cycle type from the graph of α.

Prime quotient. The two block representations have the same quotient graph after identifying corresponding blocks. Orient all its edges by the order of positions, and separately orient all its nonedges by that order as edges of the complement. Each orientation is transitive, since for edges the position and value orders disagree, and for nonedges they agree. A prime comparability graph has exactly two transitive orientations, one the reversal of the other [12, Theorem 4.1, attributed there to Golumbic]. The quotient and its complement are prime, so each of the two orientations in the second representation agrees with or reverses its counterpart in the first.

Taking the inverse of a permutation exchanges the position order and the value order, reversing the edge orientation only. Reverse-complement reverses both orientations. These operations and their compositions therefore let us make both orientations agree, and hence make both block orders agree. Both operations preserve cycle type and the avoidance class. In particular inversion, like reverse-complement, swaps 3421 and 4312.

The orientations together determine the total order of positions, since each pair is an edge in exactly one of the two graphs. Reversing the position order on graph edges only recovers the value order. Thus corresponding blocks occupy identical intervals of positions and of values. Apply the block replacement argument one block at a time. In the case $p = q$, the standardized blocks have the same cycle type by induction. For nonmonotone blocks with $p \neq q$, the path arguments apply using the outside inversion counts from the original representations. Those counts depend only on the two orders of the blocks and their sizes, so they remain valid during replacement. All replacements preserve the global cycle type. This completes the induction.

8. Hassler and Kirgizov on consecutive pattern avoidance

Let $\mathcal{A}_n$ consist of the permutations of $[n]$ whose consecutive length-four patterns all belong to $\{1234, 1342, 3412, 4231\}$. Let $\mathcal{B}_n$ be the subclass in which 1342 is forbidden. For a pattern t, write $\mathrm{occ}_t(\pi)$ for its number of consecutive occurrences in π. Define

$$\mathrm{pop}_{\mathcal{C}}(t) = \lim_{n\to\infty} \frac{\sum_{\pi\in\mathcal{C}_n} \mathrm{occ}_t(\pi)}{n|\mathcal{C}_n|}.$$

Hassler and Kirgizov [8, Conjecture 5.1] predict that forbidding a pattern of asymptotic popularity zero preserves the popularity of every other pattern of the same length. In this example,

$$\mathrm{pop}_{\mathcal{A}}(1342) = 0, \qquad \mathrm{pop}_{\mathcal{A}}(1234) = \frac{1}{2}, \qquad \mathrm{pop}_{\mathcal{B}}(1234) = \frac{1}{3}.$$

The relative orders of the first and last three entries of each allowed pattern are as follows.

Window	First three	Last three
1234	123	123
1342	123	231
3412	231	312
4231	312	231

Adjacent windows must agree on the relative order of their three common entries. Hence a legal sequence of windows is either all 1234, alternates between 3412 and 4231, or consists of a run of 1234 windows followed by a single 1342 and an alternating run starting with 3412. In the third case either run can be empty. Every permutation in $\mathcal{A}_n$ therefore has at most one 1342, giving $\mathrm{pop}_{\mathcal{A}}(1342) = 0$.

8.1. **Permutations without** 1342**.** For $n \geq 4$, the all-1234 sequence gives the increasing permutation. Each of the two alternating sequences also gives exactly one permutation. To see this, divide the entries into increasing pairs, allowing a final singleton. If the first window is 4231, start with a singleton and then take increasing pairs, again allowing a final singleton. Require every entry of a block to be greater than every entry of the next block.

This block order gives the required alternating windows. Conversely, the prescribed windows force every comparison within a block and between adjacent blocks. The union of two adjacent blocks occupies at most four consecutive positions and is therefore contained in a window. Comparisons of nonadjacent blocks follow by transitivity. These constraints fix a total order, so assigning the ranks $1, \ldots, n$ gives exactly one permutation for each of the two choices of first window.

Consequently $|\mathcal{B}_n| = 3$, and only its increasing permutation contributes occurrences of 1234. Thus $\mathrm{pop}_{\mathcal{B}}(1234) = \lim_{n\to\infty}(n-3)/(3n) = 1/3$.

8.2. **Permutations with** 1342**.** Suppose 1342 starts at position s, with $1 \leq s \leq n-3$. Write the entries as

$$a_1 \cdots a_s \, b_1 b_2 \cdots b_{n-s}.$$

The windows before position s are all 1234, so $a_1 < \cdots < a_s$. The b-suffix consists of decreasing blocks of increasing pairs, starting with the pair b_1, b_2. For $n - s \geq 4$ this follows from the alternating windows beginning with 3412. For $n - s = 3$ it follows from the order 231 of the last three entries of 1342.

The window at s requires $a_s < b_3 < b_1 < b_2$, of which the suffix already gives $b_3 < b_1 < b_2$. Conversely, the two chains and $a_s < b_3$ imply all the prescribed windows. In particular the concatenation $a_1, \ldots, a_s, b_1, b_2$ is increasing and contains every window before position s.

When $s \leq n-4$, the four largest entries of the suffix are b_1, b_2, b_3, b_4, and b_3 is the smallest of those four. The condition on a_s therefore fixes the four largest labels of the entire permutation in the suffix. The remaining freedom is to choose the s labels of the prefix among $1, \ldots, n-4$. Once chosen, both chains have fixed orders. The count is $\binom{n-4}{s}$. When $s = n-3$, all three suffix entries must be greater than all prefix entries, so the count is one. Each permutation in this case analysis contributes $s-1$ occurrences of 1234.

8.3. **Evaluation of the limits.** Put $r = n - 4$. There are 2^r permutations with 1342, so $|\mathcal{A}_n| = 2^r + 3$. The binomial identity $\sum_{s=1}^{r} s\binom{r}{s} = (r/2)2^r$ gives

$$\sum_{\pi \in \mathcal{A}_n} \mathrm{occ}_{1234}(\pi) = (n-3) + (n-4) + \sum_{s=1}^{r} (s-1)\binom{r}{s}$$
$$= \left(\frac{r}{2} - 1\right) 2^r + 2r + 2.$$

Dividing by $n(2^r + 3)$ and taking the limit gives $1/2$. The two different popularities for 1234 disprove Conjecture 5.1.

9. Daga and Ramu on the discounted cutting game

Daga and Ramu [4, Conjecture 31] propose that their greedy cutting rule attains the optimal discounted value from every multiset of positive integer piece sizes. By their game representation of randomized Littlestone dimension, the single-piece consequence is $\mathrm{RL}(\mathrm{Int}_m) = R(m+1)$ for every $m \geq 0$, with R defined in their Theorem 30. We prove the conjecture by bounding every possible first cut followed by greedy continuation.

9.1. **The game and the greedy counts.** All piece sizes are positive integers. A cut replaces a piece of size $n \geq 2$ by pieces of sizes $u, v \geq 1$ with $u + v = n$. Its reward is

$$r(u, v) = 1 + \frac{\Lambda(u) + \Lambda(v)}{2},$$

where Λ is the piecewise linear interpolation of $\log_2$ at the powers of two. In particular $\Lambda(2^q + x2^q) = q + x$ for $q \geq 0$ and $0 \leq x \leq 1$. The i-th cut receives weight 2^{-i}. Equation (2) of Daga and Ramu [4] identifies the optimal value of this game with the randomized Littlestone dimension of the corresponding interval state.

The greedy rule cuts a largest piece n. With $b = 2^{\lfloor \log_2 n \rfloor}$, its two children are $\max(b/2, n - b)$ and the remainder. Both children lie in $[b/2, b]$, and at least one is a power of two. By linearity of Λ on that interval the reward is $\Lambda(n)$. The rewards in a greedy schedule occur in nonincreasing order.

For one initial piece of size s, let $C_s(t)$ count its greedy rewards at least t. For $q \geq 0$, $b = 2^q$, and $0 < x < 1$, the count is

$$C_s(q + x) = \max\{0, \lfloor s/b - x \rfloor\}. \tag{7}$$

Indeed, when $s < 2b$, only the root cut can have reward above q, and its reward reaches $q + x$ exactly when $s \geq b(1 + x)$. If $s \geq 2b$, the greedy root cut contributes one reward and both children have size at least b. One child has size a multiple of b because it is a power of two. The two child counts in (7) therefore add to $\lfloor s/b - x \rfloor - 1$, proving (7) by induction.

For a multiset S of initial pieces, set $C_S(t) = \sum_{s \in S} C_s(t)$. Greedy play interleaves the individual schedules in nonincreasing reward order. Its value is

$$G(S) = \int_0^\infty (1 - 2^{-C_S(t)})\, dt. \tag{8}$$

At level t, the first $C_S(t)$ rewards contribute $\sum_{i=1}^{C_S(t)} 2^{-i} = 1 - 2^{-C_S(t)}$, which gives (8) by integration.

9.2. **Comparison with an arbitrary first cut.** Fix a prospective cut $n = u + v$, with $u \le v$, from S. Write S_0 for the other pieces and S' for the state after this cut. Define

$$w(t) = 2^{-C_{S_0}(t)}, \qquad h(t) = 2^{1-C_n(t)} - 2^{-C_u(t)-C_v(t)}.$$

The weight w is nondecreasing and takes values in $[0, 1]$. Subtracting the integrals in (8) gives

$$2G(S) - G(S') = \int_0^\infty (1 - w(t)h(t))\, dt. \tag{9}$$

Above $\Lambda(n)$ the three counts in h are zero, so the integrand is nonnegative. It suffices to show

$$\int_0^{\Lambda(n)} w(t)h(t)\, dt \le \Lambda(n) - r(u, v). \tag{10}$$

9.2.1. *Bands where both children have size at least 2^q.* For $t = q + x$ and $b = 2^q \le u$, write

$$u/b = U + a, \qquad v/b = V + c, \qquad a + c = e + f,$$

where U, V, e are integers and $a, c, f \in [0, 1)$. Put $I_y = \mathbf{1}_{\{x \le y\}}$. By (7),

$$h(q + x) = 2^{-(U+V-2)} \left(2^{-e-I_f} - 2^{-I_a-I_c}\right).$$

The whole band lies below $\Lambda(n)$ because $n \ge 2b$. If $e = 0$, the bracket equals $1/4$ between 0 and $\min(a, c)$, and $-1/2$ between $\max(a, c)$ and $a + c$. If $e = 1$, it equals $1/4$ between $a + c - 1$ and $\min(a, c)$, and $-1/2$ between $\max(a, c)$ and 1. It is zero elsewhere. In either case, the positive and negative intervals have the same length, with the negative interval later. Since w is nondecreasing, the weighted integral of h over this band is at most zero.

9.2.2. *Bands where only the larger child has size at least 2^q.* Now let $u < b \le v$, and write $a = u/b$, $v/b = V + c$ with V an integer and $c \in [0, 1)$. Then $C_u(q + x) = 0$. The following values come directly from (7), with the endpoints immaterial to integration.

If $a + c < 1$, the quantity $2^{V-1}h(q + x)$ is $1/2$ on $0 < x < c$, zero on $c < x < a + c$, and 1 on $a + c < x < 1$. If $a + c \ge 1$, it is $1/2$ on $a + c - 1 < x < c$ and zero elsewhere.

Let $q_v = \lfloor \log_2 v \rfloor$. For $q < q_v$ we have $V \ge 2$, so $h \ge 0$ and its integral over the band is at most $(1 - a)/2$. In the first case the integral is $2^{-(V-1)}(1 - a - c/2)$, and in the second case it is $2^{-V}(1 - a)$. Multiplication by $w \le 1$ preserves the upper bound.

For the band $q = q_v$, where $V = 1$, the contribution to the integral in (10) is at most $c/2$ if $a + c < 1$. In that case $\Lambda(n) = q_v + a + c$, so the last interval in the first case is beyond the integration limit. If $a + c \ge 1$, the entire band contributes at most $(1 - a)/2$.

Finally, when $b > v$, both child counts are zero and $n < 2b$. Wherever $q + x \le \Lambda(n)$ in such a band, (7) gives $C_n(q + x) = 1$, so $h(q + x) = 0$.

9.2.3. *Summation.* Write $q_u = \lfloor \log_2 u \rfloor$. If $q_u = q_v$, the bands with both children present contribute at most zero, and the other bands below $\Lambda(n)$ contribute zero. This proves (10) since concavity and the identity $\Lambda(2y) = \Lambda(y) + 1$ give $r(u, v) \le \Lambda(n)$.

If $q_v > q_u$, put $k = q_v - q_u$, $\alpha = u/2^{q_u} - 1$, $a = u/2^{q_v} = (1 + \alpha)2^{-k}$, and $c = v/2^{q_v} - 1$. The intermediate bands contribute at most

$$M = \frac{1}{2} \sum_{j=1}^{k-1} (1 - (1 + \alpha)2^{-j}) = \frac{k}{2} - 1 - \frac{\alpha}{2} + a.$$

The sum is zero when $k = 1$. If $a + c < 1$, one has $\Lambda(n) = q_v + a + c$ and

$$\Lambda(n) - r(u, v) = M + c/2.$$

If $a + c \geq 1$, one has $\Lambda(n) = q_v + 1 + (a + c - 1)/2$ and

$$\Lambda(n) - r(u, v) = M + (1 - a)/2.$$

These are exactly the upper bounds from the intermediate and top bands. This proves (10) for every cut.

Combining (9) and (10) gives $G(S) \geq (r(u, v) + G(S'))/2$. Induct on $\sum_{s \in S}(s - 1)$, which decreases by one at each cut. The inequality bounds the value of any first cut followed by optimal play by $G(S)$, and the greedy schedule attains $G(S)$. This proves Conjecture 31 for arbitrary positions.

For a single piece of size $\ell = m + 1$, equations (7) and (8) recover the authors' closed formula. On the q-th band put $f_q = \lfloor \ell/2^q \rfloor$ and $\pi_q = \ell/2^q - f_q$. For $f_q \geq 1$ its integral is $1 - 2^{1-f_q} + \pi_q 2^{-f_q}$, and all later bands contribute zero. Thus

$$\mathrm{RL}(\mathrm{Int}_m) = G(\{m + 1\}) = \sum_{q=0}^{\lfloor \log_2(m+1) \rfloor} \left(1 - 2^{1-f_q} + \pi_q 2^{-f_q}\right) = R(m + 1).$$

10. Xiang and Zhao on the sharp constant of a star

For an n-vertex star S_n, Xiang and Zhao [20, Conjecture 3.1] ask for a cutoff $N(p)$ such that the sharp Poincaré–Wirtinger constant $C_{S_n,p}$ is given by the following one-variable formula throughout $4 \leq n \leq N(p)$:

$$C^p_{S_n,p} = \begin{cases} \max\limits_{0<x<1} \dfrac{(n - 2 + x)^p + n - 2 + x^p}{(n - 2 + 2x)^p + (n - 2)(1 - x)^p}, & 1 < p < 2, \\ \max\limits_{0<x<1} \dfrac{(1 + x)^p + n - 2 + x^p}{(1 + (n - 1)x)^p + (n - 2)(1 - x)^p}, & p > 2. \end{cases}$$

We establish the $1 < p < 2$ formula through $n = 5$. The $p > 2$ expression as printed has supremum 1 for every $n \geq 4$, whereas the sharp constant is greater than 1. For the branch obtained by setting the negative-leaf multiplicity to one in their preceding theorem, we prove the $n \leq 5$ assertion. This branch fails for sufficiently large stars at every fixed $p > 2$, and the largest consecutive cutoff at $p = 3$ is 19.

10.1. **The printed formula for $p > 2$.** Let $u = n - 2$. The denominator of the printed quotient minus its numerator is

$$(1 + (n - 1)x)^p - (1 + x)^p + u\big((1 - x)^p - 1\big) - x^p.$$

Convexity gives $(1 - x)^p \geq 1 - px$. Integrating the derivative pt^{p-1} over $1 + x \leq t \leq 1 + (n - 1)x$, and using $(1 + x)^{p-1} \geq 1 + (p - 1)x$, gives a lower bound of $pux + pu(p - 1)x^2$ for the difference of the first two terms. Hence the full difference is at least $pu(p - 1)x^2 - x^p > 0$ for $0 < x < 1$. The quotient tends to one as $x \downarrow 0$.

The actual branch with $u = n - 2$ positive leaves and one negative leaf is

$$\frac{(1 + x)^p + u^{p-1}(1 + x^p)}{(1 + (u + 1)x)^p + u^{p-1}(1 - x)^p}. \tag{11}$$

It comes from the leaf values in the next subsection by multiplying the numerator and denominator of their ratio by u^{p-1}. At $x = 0$ this expression equals one, and the derivative of its numerator minus its denominator is $p(u^{p-1} - u) > 0$. Hence a small positive x gives a ratio greater than one.

This disproves the printed $p > 2$ claim for every $n \geq 4$, with or without allowing a supremum. The rest of this section works with (11) when $p > 2$.

10.2. **Normalization and the five-vertex star.** For the star S_n, the p-th power of the sharp constant is the maximum of

$$\frac{\sum_{w \in V(S_n)} |f(w) - \bar{f}|^p}{\sum_{w \text{ a leaf}} |f(w) - f(\text{center})|^p}$$

over nonconstant real functions f, where $\bar{f}$ is the arithmetic mean. By Xiang and Zhao's [20, Theorem 3.1], for $n \geq 4$ and p in either of the two ranges there is a maximizing f with two distinct leaf values on opposite sides of the center, and with mean different from its center value. Translate and change the sign if needed so the center is 0 and the mean is positive. If u leaves are positive and $v = n - 1 - u$ are negative, that theorem gives $u \leq v$ for $1 < p < 2$ and $u > v$ for $p > 2$.

Scale the difference between the mean and the negative leaf value to 1. With $x \in (0, 1)$ the new mean, the negative leaf value is $x - 1$ and the positive leaf value is $a = (v + (u + 1)x)/u$. Write $b = 1 - x$ and put

$$R_{u,v,p}(x) = \frac{u(a - x)^p + v + x^p}{ua^p + vb^p}.$$

The sum of the displayed leaf values and the center value is nx, so every $x \in (0, 1)$ gives an admissible function. The branches under consideration are $u = 1$ for $1 < p < 2$ and $v = 1$ for $p > 2$, with $C^p_{S_n,p}$ the maximum of the corresponding ratio. When $n = 4$ either multiplicity condition forces its conjectured branch. When $n = 5$ and $p > 2$, $u > v$ forces $u = 3, v = 1$. It remains to compare $R_{1,3,p}$ and $R_{2,2,p}$ for $1 < p < 2$.

Set $r = p - 1 \in (0, 1)$. Write D_1, D_2 for the two denominators and E_1, E_2 for the corresponding numerator minus denominator. Thus $R_i = 1 + E_i/D_i$. With $a = 1 + 3x/2$ and $b = 1 - x$, we have

$$D_1 = (2a + b)^p + 3b^p, \qquad D_2 = 2(a^p + b^p).$$

For $t = b/a \in [0, 1]$, their ratio is $((2 + t)^p + 3t^p)/(2(1 + t^p))$. Its derivative has the sign of

$$(2 + t)^r(1 - 2t^r) + 3t^r = (2 + t)^r(1 - t^r) + t^r\big(3 - (2 + t)^r\big) \geq 0.$$

Therefore $D_1/D_2 \leq (3^p + 3)/4 \leq 1 + p$. For the last inequality, convexity on $1 \leq p \leq 2$ gives $3^p \leq 6p - 3 \leq 4p + 1$.

For $0 \leq s \leq x$, put $A = 1 + x/2 + s$ and $B = 1 - s$. The fundamental theorem of calculus gives

$$E_2 = x^p - 2p \int_0^x (A^r - B^r)\, ds,$$

$$E_1 - E_2 = p \int_0^x \big(2A^r + B^r - (2A + B)^r\big)\, ds.$$

In the second line, $2A + B = 3 + x + s$ is the term from the positive leaf in the first branch. By subadditivity of the r-th power and $A \geq 1$, the integrand in that line is at least $(2 - 2^r)A^r \geq 1 - r$. Here $2^r \leq 1 + r$ by convexity on $[0, 1]$. Hence $E_1 - E_2 \geq p(1 - r)x$.

Also $0 \leq B \leq A \leq 5/2$, so

$$A^r - B^r \geq (2r/5)(A - B) = (2r/5)(x/2 + 2s).$$

Integration gives $E_2 \leq x^p - (6pr/5)x^2 \leq x^p - rx^2$. Concavity gives $x^r \leq 1 - r + rx$, hence $E_2 \leq (1 - r)x$.

Whenever $E_2 > 0$, these bounds give $E_1 \geq (1+p)E_2$ and $D_1 \leq (1+p)D_2$, so $R_{1,3,p}(x) \geq R_{2,2,p}(x)$. Where $E_2 \leq 0$, the balanced ratio is at most one, and the $u = 1$ branch exceeds one for small

$x > 0$ because $E_1'(0) = p(3 - 3^{p-1}) > 0$. Thus the two-positive-leaf branch cannot improve the maximum, establishing cutoff 5 for the $1 < p < 2$ branch and for (11) when $p > 2$.

10.3. **The one-variable branches fail for sufficiently large stars.** For $1 < p < 2$, the ratio $R_{1,n-2,p}$ tends to 1 uniformly for $0 \le x \le 1$, by dividing both numerator and denominator by $(n-2)^p$. For $p > 2$, use $v = 1$, and write $u = n-2$. After multiplying numerator and denominator by u^{p-1}, the ratio is

$$\frac{(1+x)^p + u^{p-1}(1+x^p)}{(1+(u+1)x)^p + u^{p-1}(1-x)^p}.$$

On $x \ge u^{-1/(2p)}$, the first denominator term is at least $u^{p-1/2}$, while the numerator is $O_p(u^{p-1})$. On $x \le u^{-1/(2p)}$, the ratio is at most $1 + o(1)$ using the second denominator term, uniformly in x. At $x = 0$ the ratio is 1, so the maximum tends to 1 in this case as well.

Instead let $u/(n-1) \to \theta$, with $\theta = 1/3$ when $1 < p < 2$ and $\theta = 2/3$ when $p > 2$. Put $a_0 = (1-\theta)/\theta$. For each fixed x, the limiting ratio is

$$\frac{\theta a_0^p + 1 - \theta}{\theta(a_0 + x)^p + (1-\theta)(1-x)^p}.$$

At $x = 0$ this equals one. The derivative of its denominator there is $p(1-\theta)(a_0^{p-2} - 1) < 0$ for both choices. Choose a sufficiently small fixed $x > 0$ so the limiting ratio exceeds one. That choice supplies a better function than the specified branch for every sufficiently large n. Thus the largest consecutive cutoff exists and is finite for each fixed p under consideration.

10.4. **Exact cutoff at exponent three.** For $p = 3$, multiply numerator and denominator by u^2 to get the integer polynomials

$$N_u(x) = (v+x)^3 + u^2(v+x^3), \qquad D_u(x) = (v+(u+1)x)^3 + vu^2(1-x)^3.$$

The denominator is positive on $[0,1]$. For each $6 \le n \le 19$, the following exact certificate puts a rational threshold strictly above all competing branches on $[0,1]$ and strictly below a value of the specified branch at a rational x. The positivity check uses the number of polynomial roots in $[0,1]$, computable by Sturm's method. At $n = 20$, the last row instead exhibits $u = 17, v = 2$ beating the conjectured $u = 18, v = 1$ branch. This proves that the largest consecutive cutoff for $p = 3$ is 19.

```
from sympy import Poly, Rational, symbols

x = symbols("x")
rows = [
    (6,  4, "103/100",   "321/2500"),
    (7,  5, "53/50",     "69/500"),
    (8,  6, "11/10",     "1433/10000"),
    (9,  7, "57/50",     "1461/10000"),
    (10, 8, "117/100",   "1473/10000"),
    (11, 9, "119/100",   "369/2500"),
    (12, 10, "121/100",  "92/625"),
    (13, 11, "123/100",  "183/1250"),
    (14, 12, "5/4",      "727/5000"),
    (15, 13, "63/50",    "721/5000"),
    (16, 14, "127/100",  "357/2500"),
```

```
    (17, 15, "127/100",  "283/2000"),
    (18, 16, "32/25",    "7/50"),
    (19, 17, "3201/2500", "693/5000"),
    (20, 17, "32/25",    "4/25"),
]

def difference(n, u, threshold):
    v = n - 1 - u
    numerator = (v+x)**3 + u**2*(v+x**3)
    denominator = (v+(u+1)*x)**3 + v*u**2*(1-x)**3
    return Poly(threshold*denominator-numerator, x, domain="QQ")

for n, winner, threshold, witness in rows:
    threshold, witness = Rational(threshold), Rational(witness)
    assert 0 < witness < 1
    assert difference(n, winner, threshold).eval(witness) < 0
    competitors = [18] if n == 20 else range((n+1)//2, n-2)
    for u in competitors:
        slack = difference(n, u, threshold)
        assert slack.eval(0) > 0 and slack.count_roots(0, 1) == 0

print("Verified the consecutive cutoff N(3) = 19.")
```

11. Elvey Price and Park on bounded-chain permutations

A d-permutation of $[n]$ is a tuple $(\mathrm{id}, \pi_2, \dots, \pi_d)$, represented by its points $(i, \pi_2(i), \dots, \pi_d(i))$. Write $\mathcal{C}_{n,K}$ for those containing no chain of length $K+1$ in the strict coordinatewise order. Elvey Price and Park [7, Conjecture 6.11, equation (6.5)] state an asymptotic for fixed $d \geq 3$ and $K \geq 2$. In the cited version it reads

$$|\mathcal{C}_{n,K}| = (n!)^{d-1-d/(d-1)} \left(\beta'_d K^{d/(d-1)}\right)^n e^{o(n)}, \tag{12}$$

where $\beta'_d > 0$ is the constant from their Proposition 6.1. The same conjecture separately predicts a permuton limit. The factorial exponent in (12) is incompatible with the inclusion of antichains and is possibly a typographical error. We determine the correct factorial exponent from the antichain count.

Proposition 11.1. *For fixed $d \geq 3$ and $K \geq 1$,*

$$\log |\mathcal{C}_{n,K}| = \left(d - 1 - \frac{1}{d-1}\right) \log(n!) + O_{d,K}(n).$$

In particular (12) *undercounts by a factor of at least $n! e^{-O_{d,K}(n)}$.*

Proof. Put $a = d - 1 - 1/(d-1)$. By Elvey Price and Park [7, Proposition 6.1],

$$|\mathcal{C}_{n,1}| = (n!)^a (\beta'_d)^n e^{o(n)}.$$

As every antichain has no chain of length $K+1$, the inclusion $\mathcal{C}_{n,1} \subseteq \mathcal{C}_{n,K}$ gives the lower bound. Comparing the lower bound with (12), their factorial exponents differ by one. The remaining exponential factors cannot offset $\log(n!) \sim n \log n$.

For the upper bound, give each point the length of a longest chain ending at that point. Comparable points have different colors, so each of the at most K color classes is an antichain. For fixed sizes $n_1, \dots, n_K$ totaling n, there are $\binom{n}{n_1,\dots,n_K}^d$ ways to choose which ranks of each coordinate have each color. Each color class determines a d-permutation relative to the ranks chosen for that class, hence it has at most $|\mathcal{C}_{n_i,1}|$ configurations. This is an upper bound even if many different colorings yield the same point set. Choose $B_d \geq 1$ with $|\mathcal{C}_{j,1}| \leq B_d^j (j!)^a$ for every j, by enlarging the constant from the asymptotic to handle the finitely many small values. Since $a \geq 0$ and $\binom{n}{n_1,\dots,n_K} \leq K^n$, each summand is at most $K^{dn} B_d^n (n!)^a$. There are at most $(n+1)^K$ size vectors. Taking logarithms proves the upper bound. □

The exponential constant for the corrected factorial order and the proposed permuton limit remain separate questions.

Appendix A. Reproducible grid counts

The integer recurrence below enumerates legal rows of length n and counts columnwise-compatible sequences of m rows. It gives $C(4,4) = 7812$ and $C(7,7) = 41\,869\,995\,708$.

```
from itertools import product

def count(m, n):
    rows = [r for r in product(range(3), repeat=n)
            if all(x != y for x, y in zip(r, r[1:]))]
    adj = [[j for j, s in enumerate(rows)
            if all(x != y for x, y in zip(r, s))]
           for r in rows]
    dp = [1] * len(rows)
    for _ in range(m - 1):
        dp = [sum(dp[j] for j in neighbors) for neighbors in adj]
    return sum(dp)

assert count(4, 4) == 7812
assert count(7, 7) == 41869995708
```

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