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When multiplicity bounds make surjective labeling hard

In valued constraint satisfaction, Matl and Živný [2, Theorem 32] proved that on two labels, surjective tractability is equivalent to tractability under every fixed system of label-count lower bounds. In §6 they asked whether this holds on larger domains.

A labeling ϕ assigns a label to each variable in a finite set V. Under a binary cost f, each constraint on (u, v) of nonnegative rational weight w adds $w\, f(\phi(u), \phi(v))$ to the cost. A surjective labeling uses every label.

Theorem. There is a symmetric binary cost $f\colon \{0, \dots, 9\}^2 \to \{0, 1\}$ with zero diagonal and connected zero-cost graph for which minimum-cost surjective labeling is solvable in polynomial time. For the same f, impose lower bounds of six on $0, 1, 2, 3$, two on 6, and one elsewhere. Deciding whether some feasible labeling costs at most a rational K is NP-complete.

This answers their question negatively if P $\neq$ NP. Write H for the graph joining labels i, j when $f(i, j) = 0$, including all loops. Dropping self-pairs and summing weights on unordered pairs uv gives a simple weighted graph on V. Symmetry and the zero diagonal preserve labeling costs. A labeling respects an edge when its endpoint labels are adjacent in H.

Among $0, \dots, 4$, only $\{0, 1\}$ and $\{2, 3\}$ are nonadjacent in H. For $|V| \geq 10$, some optimal surjection uses 1 at most three times and $5, \dots, 9$ once each, so guessing those vertices and representatives at $0, 2, 3, 4$ leaves the rest in $\{0, 2, 3, 4\}$ and reduces $\{2, 3\}$ to a minimum cut. The higher bounds force at least six vertices at 1 and pin added vertices so $\{0, 1\}$ and $\{2, 3\}$ encode Max Cut.

Ordinary surjectivity. The table defines f by listing the nonneighbors $I_i = \{j \mid f(i, j) = 1\}$ in H and replacements R_i for each label i. Since $i \notin I_i$, $j \in I_i$ is symmetric, and 0 reaches its only nonneighbor 1 via 2, f is symmetric with $f(i, i) = 0$ and H is connected. Column B_i lists the old labels j for which some pair in $R_i \times R_j$ is a nonedge of H. For example, the nonneighbors of the labels in R_1 form $\{1, 2, 4, 7\}$, met only by R_0, R_7, R_8, R_9. Since $B_i \subseteq I_i$ in every row, replacing all labels simultaneously, with independent choices at each vertex, preserves every respected edge.

i	I_i	R_i	B_i	i	I_i	R_i	B_i
0	$\{1\}$	$\{4\}$	$\{1\}$	5	$\{2, 4, 7\}$	$\{0\}$	$\{7\}$
1	$\{0, 6, 7, 8, 9\}$	$\{5, 6, 8, 9\}$	$\{0, 7, 8, 9\}$	6	$\{1, 4, 7\}$	$\{0\}$	$\{7\}$
2	$\{3, 5, 8, 9\}$	$\{3\}$	$\{8, 9\}$	7	$\{1, 3, 5, 6, 8\}$	$\{1\}$	$\{1, 3, 5, 6, 8\}$
3	$\{2, 7\}$	$\{0\}$	$\{7\}$	8	$\{1, 2, 4, 7\}$	$\{7\}$	$\{1, 2, 4, 7\}$
4	$\{5, 6, 8, 9\}$	$\{3\}$	$\{8, 9\}$	9	$\{1, 2, 4\}$	$\{2\}$	$\{1, 2, 4\}$

In an optimal surjection, keep one vertex at each of $5, \dots, 9$ and move surplus at 5 to 3 and at $6, \dots, 9$ to 0. Since $I_3 \subseteq I_5$ and $I_0 \subseteq I_j$ for $6 \leq j \leq 9$, moving vertices one by one enlarges closed neighborhoods. If 1 occurs at least four times, split its vertices across all four labels in R_1 and send every other label to its unique replacement. All ten labels stay covered and only the singleton at 7 becomes 1, so repeating the surplus moves leaves one to three vertices at 1 and one at each of $5, \dots, 9$.

For $n = |V| < 10$, report infeasibility. Otherwise guess one to three vertices at 1, singletons at $5, \dots, 9$, and one representative each at $0, 2, 3, 4$, fixing 10 to 12 vertices in $O(n^{12})$ branches and leaving free vertices in $\{0, 2, 3, 4\}$. Two fixed endpoints give constant cost, one gives unary costs, and only $\{2, 3\}$ conflicts between free vertices.

At each free u, merge $0, 4$ into a cheaper $m_u \in \{0, 4\}$ and encode $2, m_u, 3$ as $(x_u, y_u) = (0, 0), (1, 0), (1, 1)$. In an s-t cut with bit 1 on the s-side, arcs $y_u \to x_v$ and $y_v \to x_u$ of capacity w charge w exactly on $\{2, 3\}$ for each free edge uv of weight w. A unary cost (α, β, γ) on

$2, m_u, 3$ equals $\alpha + (\beta - \alpha)x_u + (\gamma - \beta)y_u$, matched up to a constant on each bit z of coefficient δ by $z \to t$ of capacity δ when $\delta > 0$ and $s \to z$ of capacity $-\delta$ when $\delta < 0$. Arcs $y_u \to x_u$ of capacity 1 plus the preceding capacity sum enforce $y_u \le x_u$. Capacities have polynomial bit length, and the cheapest decoded labeling over all branches finds a minimum-cost surjection in polynomial time.

Fixed lower bounds. Reducing from classical Max Cut [1] on a simple m-edge graph G of optimum $\mathrm{MaxCut}(G)$, add twelve isolates to get F. Each $u \in V(F)$ gets a path $p_u q_u r_u$. Its middle label records the shore of u. We shall realize the two shores by the triples $(2, 0, 2)$ and $(0, 3, 1)$ in path order. Write P, Q, R for the sets of p_u, q_u, r_u, respectively.

Add vertices h, a, b, t and $C = \{c_1, \dots, c_5\}$. Join h to every other vertex outside C, Q to C, P to b, c_1, c_2, and R to a, b. Also join b to a, c_1, c_2 and a to c_3. Give these edges and all path edges weight $4m + 1$ and call them protected. At each u, give $aq_u, r_u c_1, r_u c_3$ weight $\deg_F(u)$. For each $uv \in E(F)$ add unit edges $p_u r_v$ and $p_v r_u$. Dropping weight-0 pairs leaves a simple graph with integer weights at most $4m + 1$.

Consider a feasible labeling respecting all protected edges. Since $|C| = 5$ while $0, 1, 2, 3$ each have quota six, each occurs outside C, so by reflexivity $\phi(h)$ sees $0, 1, 2, 3$ and equals 4. Its nonneighbors $I_4 = \{5, 6, 8, 9\}$ lie in C, whose five vertices must carry $\{5, 6, 6, 8, 9\}$. Every q_u sees h and C, giving $\phi(Q) \subseteq \{0, 3\}$. Outside $P \cup R$, only a, b, t could take 1 or 2, whereas each has quota six, so $1, 2 \in \phi(P \cup R)$. Every p_u sees c_1, c_2, while 1 sees only the single 5 on C, so $1 \notin \phi(P)$ and $1 \in \phi(R)$.

Vertex b sees h and $P \cup R$, giving $\phi(b) \in \{1, 2, 4\}$, where only 2 sees two entries of $\{5, 6, 6, 8, 9\}$, forcing $\phi(b) = 2$ and $\phi(c_1) = \phi(c_2) = 6$. Vertex a sees h, b, and R, so $\phi(a) \in \{1, 2, 4\}$, and edge ac_3 into $\{5, 8, 9\}$ forces $\phi(a) = 1$ and $\phi(c_3) = 5$. Edges to h, b, c_1 and h, a, b then give $\phi(p_u) \in \{0, 2\}$ and $\phi(r_u) \in \{1, 2, 4\}$.

The labels $\phi(q_u) \in \{0, 3\}$ define a cut of G, and $E(F) = E(G)$. Split each degree weight into one unit per incident edge, and on each $uv \in E(F)$ charge endpoint u for its units on $aq_u, r_u c_1, r_u c_3$ and edge $p_v r_u$, partitioning all non-protected weight. Since each of $1, 2, 4$ misses 5 or 6, endpoint u pays at least 1 on $r_u c_1, r_u c_3$, plus 1 on aq_u when $\phi(q_u) = 0$. Thus a crossing edge costs at least 3, and an edge with $\phi(q_u) = \phi(q_v) = 0$ costs at least 4. When $\phi(q_u) = \phi(q_v) = 3$, the path edges forbid 2, forcing $\phi(p_u) = \phi(p_v) = 0$ and $\phi(r_u), \phi(r_v) \in \{1, 4\}$. At endpoint u, edges $p_v r_u, r_u c_3, r_u c_1$ then meet labels $0, 5, 6$, and both 1 and 4 miss two of them, so each endpoint pays 2 and uv costs 4. A labeling whose cut has size s therefore incurs cost at least $3s + 4(m - s) = 4m - s \ge 4m - \mathrm{MaxCut}(G)$.

Extend a maximum cut of G with six added isolates on each shore. Use the two shore triples, put $(4, 1, 2, 7)$ on (h, a, b, t) and $(6, 6, 5, 8, 9)$ on $(c_1, \dots, c_5)$. The isolates supply at least six vertices at each of $0, 1, 2, 3$, and all quotas and protected edges hold. Per incident edge, each r_u pays one unit to c_1, c_3 and aq_u pays one unit on $(2, 0, 2)$ only, while $p_u r_v$ and $p_v r_u$ fail only between two $(0, 3, 1)$ paths. Hence edges cost 3 across shores and 4 within a shore, attaining $4m - \mathrm{MaxCut}(G) \le 4m$. Every optimal labeling respects the protected edges, matching the lower bound. A Max Cut threshold k thus maps to $4m - k$, and a labeling certifies any rational cost bound in polynomial time. □

References

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[2] G. Matl and S. Živný. Using a min-cut generalisation to go beyond Boolean surjective VCSPs. *Algorithmica* 82 (2020), 3492–3520, Theorem 32 and §6. DOI.