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THE DUAL SMALE MEAN VALUE CONJECTURE FOR DEGREE NINE: A COMPLETE PURE LEAN 4 KERNEL PROOF VIA HYBRID TWO-VARIABLE GRACE–WALSH–SZEGO COLLAPSE

IVAN BESEVIC

Abstract. Let $P(z) = z + a_2 z^2 + \cdots + a_n z^n \in \mathbb{C}[z]$ be a complex polynomial of degree $n \geq 2$ with $a_n \neq 0$. The Dual Smale Mean Value Conjecture (Dubinin–Sugawa and Ng, 2009) asserts that

$$\max_{P'(\zeta)=0} \left| \frac{P(\zeta)}{\zeta} \right| \geq \frac{1}{n},$$

with equality if and only if $P(z) = \frac{1-(1-cz)^n}{nc}$ for some $c \in \mathbb{C} \setminus \{0\}$. Beyond the single-point barrier at degree $n = 8$, degree $n = 9$ introduces a sharper qualitative obstruction: the single-variable Grace–Walsh–Szego (GWS) envelope $H_9(a, v)$ satisfies $H_9(0, v) = G_8(v)$ with $\min_{|v|\leq 1} 9|H_9(0, v)| \approx 0.7794 < 1$, while $9|G_9(w)|$ plunges to ≈ 0.0173 near $|w| = 1$, $\arg(w) \approx \pm 79.1^\circ$ in 14 real dimensions ($\overline{\mathbb{D}}^7$).

In Version 1 of this work, the conjecture at $n = 9$ was proved by combining a Lean 4 formalization of the analytic framework with an external outward-rounded Rust branch-and-bound certifier over 107,256,687 nodes in 14 real dimensions. In the present paper, we *eliminate the external Rust verifier completely* and establish a 100% self-contained, unconditional proof verified entirely inside the Lean 4.36.0 C++ kernel (`DualSmale.dual_smale_degree_9_complete`), with zero `sorry`, zero `admit`, zero `native_decide`, and strictly the three standard axioms `[propext, Classical.choice, Quot.sound]`.

The complete elimination of external code rests on five mathematical and algorithmic reductions: (i) a *Hybrid Two-Variable GWS Collapse* that fixes the three largest-reciprocal critical points $(1, w_2, w_3)$ and collapses the remaining five critical points simultaneously across either complex linear combinations $\alpha_1 Q_1 + \alpha_2 Q_2 + \alpha_3 Q_3$ on $\overline{D}(0, |w_3|)$ or an exact unit-disk *pure-Q_3 change of variables* $u_i = z_i/w_3 \in \overline{\mathbb{D}}$ (quadratic in w_3^{-1}), reducing the global search space from 14D to 4D and compressing the certificate by a factor of 1,072× (from 107,256,687 nodes to 100,006 exact-integer leaves); (ii) an exact *Stage 1d inner-radius cut* $|w_3| \leq 161/200$ proved via 6-variable GWS on $\overline{D}(0, 161/200)$ in just 26 leaves, bypassing the $H_9(0, v) = G_8(v)$ singularity barrier without any inversion-GWS stage; (iii) an expanded analytic Stage 0 local theorem proving $9|H_9(a, v)| > 1$ on the asymmetric polydisk $|a - 1| \leq 1/30$, $|v - 1| \leq 1/5$, coupled with affine homotopy and the strict Maximum Modulus Principle to close the entire neighborhood $|a-1| \leq 1/30$, $|v| \leq 1$ in 2 leaves; (iv) an *adaptive zero-trigonometry stereographic Pythagorean unit-circle bisector* using exact rational points $u = \pm((q^2 - p^2) + 2ipq)/(q^2 + p^2)$ with zero precomputed grid tables, fused polynomial-and-derivative evaluation, and exact integer Schur–Cohn zero-freeness up to degree 7; and (v) a zero-Mathlib kernel architecture (`DualSmale9.PureKernel`) splitting all 100,006 leaves across 520 pure-`Init` certificate modules checked in parallel via `by decide +kernel`.

1. Introduction

In 1981, Smale [9] posed his mean value conjecture: for every complex polynomial $P(z) = z + a_2 z^2 + \cdots + a_n z^n$ of degree $n \geq 2$, does there exist a critical point ζ with $|P(\zeta)/\zeta| \leq 1 - 1/n$? In 2026, the primal conjecture was disproved with sharp global supremum $K^* = 4$ [1, 5].

By contrast, the *Dual Smale Mean Value Conjecture*, formulated independently by Dubinin–Sugawa [4] and Ng [8], asserts the reverse sharp lower bound on the *maximum* over all critical points. The conjecture was established for $n \leq 7$ by Hinkkanen, Kayumov, and Khammatova [6], for odd polynomials of arbitrary degree by Tang [10], up to the asymptotic lower bound

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$(n-1/2)^{1/n}/n^2$ by Jatar and Ng [7], and for degree $n=8$ via a 100% pure Lean 4 kernel proof in our companion work [2].

Theorem 1.1 (Main Theorem — Unconditional Pure Lean 4 Proof for $n=9$)**.** *Let* $P(z) = z + a_2z^2 + \cdots + a_9z^9 \in \mathbb{C}[z]$ *be any complex polynomial of degree* 9 *with* $a_9 \neq 0$*. Then*

$$\max_{P'(\zeta)=0} \left| \frac{P(\zeta)}{\zeta} \right| \geq \frac{1}{9}. \tag{1}$$

Moreover, equality holds in (1) *if and only if* $P(z) = \frac{1-(1-cz)^9}{9c}$ *for some* $c \in \mathbb{C} \setminus \{0\}$.

Writing the eight nonzero critical points of P in reciprocal form $w_j = 1/\zeta_j \in \mathbb{C} \setminus \{0\}$ ($j \in \{0,\ldots,7\}$) so that $P'(z) = \prod_{k=0}^{7}(1-w_kz)$, Theorem 1.1 is formalized verbatim in Lean 4.36.0 (`DualSmale9/DualSmaleNineComplete.lean`) as:

```
theorem dual_smale_degree_9_complete
    (w : Fin 8 -> C) (hw : forall j, w j != 0) :
    (exists j : Fin 8, 1 < ||(9 : C) * dualSmaleQuotient 9 w j||) \/
    (forall j : Fin 8, w j = w 0)
```

This theorem is verified *entirely* inside the Lean 4.36.0 C++ kernel, with zero external Rust, C, or Python code in the trusted computing base, zero `native_decide` (`Lean.ofReduceBool`), and depending strictly on the three standard axioms `[propext, Classical.choice, Quot.sound]`.

1.1. **Degree $n=9$: The Qualitative Singularity Barrier.** Normalizing by a minimal-modulus critical point $\zeta_0 = 1$, the remaining seven reciprocal critical points $w = (w_1,\ldots,w_7)$ lie in $\overline{\mathbb{D}}$, and

$$280 \cdot 9 \cdot P(1) = 1260 - 420e_1 + 210e_2 - 126e_3 + 84e_4 - 60e_5 + 45e_6 - 35e_7,$$

where $e_m = e_m(w_1,\ldots,w_7)$ is the m-th elementary symmetric polynomial. Two sharp obstructions separate $n=9$ from $n=8$:

(1) *Deep single-point collapse*: the 7-variable GWS diagonal $9|G_9(v)| = |280{\cdot}9P(1)|_{(v,\ldots,v)}/280$ plunges to ≈ 0.0173 near $|v|=1$, $\arg(v) \approx \pm 79.1^\circ$ (versus 0.6928 at $n=8$).
(2) *Origin singularity of H_9*: fixing one critical point a and collapsing the other six to $v \in \overline{\mathbb{D}}$ yields the degree-6 envelope $H_9(a,v)$, which satisfies $H_9(0,v) = 280 \cdot 9 \cdot G_8(v)$ and therefore dips to $\min_{|v|\leq 1} |H_9(0,v)|/280 \approx 0.7794 < 1$. Consequently, unlike $n=8$ where $C_+ \subset \overline{\mathbb{D}}$ was bounded away from the origin by $y \geq 0.422$, at $n=9$ the danger region of $H_9(\cdot,\overline{\mathbb{D}})$ engulfs the entire disk $|a| \leq 0.805$!

Table 1 summarizes the architectural leap from our Version 1 hybrid proof [3] to the present Version 2 pure Lean 4 kernel proof.

TABLE 1. Comparison between Version 1 (hybrid Lean 4 + Rust) and Version 2 (100% pure Lean 4.36.0 kernel) for degree $n=9$.

Dimension	Version 1 (Hybrid)	Version 2 (100% Pure Lean 4)
Trusted base	Lean 4 + 1,927-line Rust binary	**Lean 4.36.0 kernel only**
Search space	14 real dims ($w_1,\ldots,w_7 \in \overline{\mathbb{D}}^7$)	**4 real dims** ($(w_2,w_3) \in C_+^{(9)} \times C_\pm^{(9)}$)
Origin barrier	2-Block Inversion GWS	**Stage 1d cut** $\lvert w_3\rvert \leq 161/200$ **(26 leaves)**
Arithmetic	IEEE-754 `f64` outward rounding	**Exact $\mathbb{Z}$ fixed-point** ($S = 2^{20}$)
Circle engine	Fixed polar boxes + trig	**Adaptive stereographic bisection (zero trig)**
Stage 0 ball	$\max(\lvert a-1\rvert, \lvert v-1\rvert) \leq 1/100$	$\lvert a-1\rvert \leq 1/30$, $\lvert v-1\rvert \leq 1/5$ **(2 leaves)**
Cert. leaves	107,256,687 nodes	100,006 **exact-$\mathbb{Z}$ leaves (1,072× less)**
Proof engine	2,000 lines (28 theorems)	**9,350 lines + 520 pure-kernel cert modules**
Axioms	`propext, Classical.choice, Quot.sound`	`propext, Classical.choice, Quot.sound`

2. Hybrid Two-Variable Grace–Walsh–Szego Collapse (14D → 4D)

As in degree $n = 8$, we order the eight reciprocal critical points so that $w_0 = 1 \geq |w_2| \geq |w_3| \geq \max_{0\leq i\leq 4} |z_i|$ (with $\mathrm{Im}(w_2) \geq 0$ by conjugation invariance) and collapse the five remaining critical points $(z_0, z_1, z_2, z_3, z_4)$ simultaneously into a single complex variable v. At degree $n = 9$, clearing denominators by $2520 = 280 \cdot 9$ yields three exact-integer degree-5 polynomials

$$\begin{aligned} J_0^{(9)}(v) &= 1260 - 2100v + 2100v^2 - 1260v^3 + 420v^4 - 60v^5, \\ J_1^{(9)}(v) &= -420 + 1050v - 1260v^2 + 840v^3 - 300v^4 + 45v^5, \\ J_2^{(9)}(v) &= 210 - 630v + 840v^2 - 600v^3 + 225v^4 - 35v^5. \end{aligned} \tag{2}$$

For $w_2, w_3 \in \mathbb{C} \setminus \{0\}$ and $v \in \mathbb{C}$, define the three diagonalized leading quotients

$$\begin{aligned} Q_1(w_2, w_3, v) &:= J_0^{(9)}(v) + (w_2 + w_3)J_1^{(9)}(v) + w_2 w_3 J_2^{(9)}(v), \\ Q_2(w_2, w_3, v) &:= Q_1\left(\frac{1}{w_2}, \frac{w_3}{w_2}, \frac{v}{w_2}\right), \\ Q_3(w_2, w_3, v) &:= Q_2(w_3, w_2, v). \end{aligned} \tag{3}$$

Every linear combination $\alpha_1 Q_1 + \alpha_2 Q_2 + \alpha_3 Q_3$ is symmetric multiaffine of degree 5 in $(z_0, \ldots, z_4)$ before diagonalization, giving two complementary certificate modes on the same 4D box (w_2, w_3):

Theorem 2.1 (Hybrid Two-Variable GWS Collapse for $n = 9$)**.** *Let $w_2, w_3 \in \mathbb{C} \setminus \{0\}$ and $z \in \mathbb{C}^5$ satisfy $|z_i| \leq |w_3|$ for all $i \in \{0, \ldots, 4\}$, and set $w = (1, w_2, w_3, z_0, z_1, z_2, z_3, z_4)$.*

(1) ***Simultaneous Linear-Combination Mode (`.leaf`):*** *Let $R_3 \geq |w_3|$ with $R_3 > 0$. If there exist weights $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ such that*

$$\left|\alpha_1 Q_1(w_2, w_3, v) + \alpha_2 Q_2(w_2, w_3, v) + \alpha_3 Q_3(w_2, w_3, v)\right| > 280\,(|\alpha_1| + |\alpha_2| + |\alpha_3|)$$

for all $|v| \leq R_3$, then $\max_{j\in\{0,1,2\}} |9q_j(w)| > 1$.

(2) ***Pure-Q_3 Unit-Disk Mode (`.leafQ3`):*** *Under the exact rescaling $u_i := z_i/w_3 \in \overline{\mathbb{D}}$ $(i = 0, \ldots, 4)$, the third quotient $2520\, q_2(w)$ collapses to the degree-5 polynomial*

$$\widetilde{Q}_3(w_2, w_3, u) := Q_3(w_2, w_3, w_3 u) = \sum_{k=0}^{5} \Big(A_0[k] + A_1[k](1 + w_2)w_3^{-1} + A_2[k]\, w_2 w_3^{-2}\Big)u^k,$$

which is only quadratic *in w_3^{-1} (versus degree 7 in w_3^{-1} before rescaling!). If $|\widetilde{Q}_3(w_2, w_3, u)| > 280$ for all $u \in \overline{\mathbb{D}}$, then $|9q_2(w)| > 1$.*

Proof. Part (1) applies 5-variable GWS (`grace_walsh_szego_disk`) on $\overline{D}(0, R_3)$ to the symmetric multiaffine form $2520(\alpha_1 q_0 + \alpha_2 q_1 + \alpha_3 q_2)$ (`two_var_gws_exceeds_one` in `DualSmale9/TwoVarGWS.lean`). Part (2) observes that $q_2(1, w_2, w_3, z)$ equals $P_1^{(9)}(w_3^{-1}, w_2 w_3^{-1}, u_0, \ldots, u_4)$ by scale and permutation invariance; applying 5-variable GWS directly to $(u_0, \ldots, u_4) \in \overline{\mathbb{D}}^5$ yields $u \in \overline{\mathbb{D}}$ with $2520 q_2(w) = \widetilde{Q}_3(w_2, w_3, u)$ (`pure_q3_unit_disk_exceeds_one` in `DualSmale9/TwoVarGWS.lean`). □

Part (2) is decisive at degree $n = 9$: in the critical near-conjugate and moderate-$|w_3|$ regions of Stage 3, expanding $Q_3(w_2, w_3, v)$ in powers of w_3^{-1} up to w_3^{-7} suffers severe Taylor remainder growth, whereas the rescaled form $\widetilde{Q}_3(w_2, w_3, u)$ depends only on w_3^{-1} and w_3^{-2} and never loses the sharp $|z_i| \leq |w_3|$ constraint when w_3 varies across a box! In our final certificate, 16,780 leaves (1,406 in Stage 2 and 15,374 in Stage 3) close directly via `.leafQ3`.

3. Stage 1d Inner-Radius Cut and Expanded Stage 0 Local Ball

3.1. **Eliminating the Origin Singularity via Stage 1d.** Because $H_9(0, v) = 280 \cdot 9 \cdot G_8(v)$ drops below 280 on $|v| = 1$, we cannot prune small $|w_2|$ using $H_9(w_2, \overline{\mathbb{D}})$ alone. However, by modulus ordering $|w_3| \geq \max_i |z_i|$, whenever $|w_3| \leq 161/200 = 0.805$, *all six* critical points $(w_3, z_0, \ldots, z_4)$ lie in the smaller disk $\overline{D}(0, 161/200)$. Applying 6-variable GWS on $\overline{D}(0, 161/200)$ reduces $2520\, P(1)$ to $H_9\big(w_2, \frac{161}{200}u\big)$ for $u \in \overline{\mathbb{D}}$. Across the *entire* upper half

unit disk $w_2 \in \overline{\mathbb{D}}_+ := \{a \in \overline{\mathbb{D}} : \mathrm{Im}(a) \geq 0\}$, this rescaled degree-6 polynomial satisfies $|H_9(w_2, \frac{161}{200}u)| > 280$ uniformly, certified in just **26 exact-integer leaves** (`stage1d_cert` in `DualSmale9/Stage1Certs.lean`)!

Consequently, for every configuration surviving Stage 1b ($|w_2| > 167/200$) and Stage 1d ($|w_3| > 161/200$), both w_2 and w_3 are automatically bounded well away from zero ($|w_2| > 0.835$, $|w_3| > 0.805$), completely eliminating the need for Version 1's 2-Block Inversion GWS!

3.2. **Expanded Stage 0 Analytic Ball and Two-Leaf Homotopy Closure.**

Theorem 3.1 (Expanded Stage 0 Analytic Ball for $n = 9$)**.** *For all* $(a, v) \in \overline{\mathbb{D}}^2 \setminus \{(1, 1)\}$ *satisfying*

$$|a - 1| \leq \frac{1}{30}, \qquad |v - 1| \leq \frac{1}{5},$$

we have strictly $|H_9(a, v)| > 280$ *(i.e.,* $9|P(1)| > 1$*).*

Proof. Setting $a = 1 + d_1$, $v = 1 + d_2$ with $d_j = x_j + iy_j$, we decompose $9H_9(1 + d_1, 1 + d_2)/280 = Q_2 + C_3 + R_4$, where

$$Q_2 = 1 - \frac{1}{8}d_1 - \frac{3}{4}d_2 + \frac{3}{14}d_1d_2 + \frac{15}{28}d_2^2,$$
$$C_3 = -\frac{5}{14}d_2^3 - \frac{15}{56}d_1d_2^2,$$

and every monomial of R_4 has degree ≥ 4 and is divisible by d_2^2. Retaining the nonnegative unit-disk slack variables $u_j := -x_j - (x_j^2 + y_j^2)/2 \geq 0$ ($j = 1, 2$), we prove $|R_4| \leq \frac{13}{1000}|d_2|^2$ and $|Q_2+C_3|^2 > \left(1+\frac{13}{1000}|d_2|^2\right)^2$ (`stage0_H9_wide_ball` in `DualSmale9/Stage0Expanded.lean`). □

Combining Theorem 3.1 with two single-leaf degree-6 circle certificates (`stage0A1_cert` for $a = 1$, $|v - 1| \geq 1/5$, and `stage0Box_cert` for $|a - 1| \leq 1/30$, $|v - 1| \geq 1/5$ in disk perturbation mode) via affine homotopy and the strict Maximum Modulus Principle closes the entire neighborhood $|a - 1| \leq 1/30$, $|v| \leq 1$, $(a, v) \neq (1, 1)$ (`checkStage0Box_sound` in `DualSmale9/CertSoundness.lean`).

4. Adaptive Zero-Trigonometry Stereographic Circle Engine

At degree $n = 8$, the pure Lean 4 checker relied on five precomputed static Pythagorean grids of 16 to 256 points (`Grids.lean`). At degree $n = 9$, where certain near-boundary leaves require local angular resolution equivalent to 512–2048 points on a single short arc while remaining smooth everywhere else on $\partial\mathbb{D}$, static grids would waste 95% of their work on easy arcs.

We replace static grids completely by an *adaptive dyadic stereographic unit-circle bisector* (`checkCircle5` for degree 5 and `checkCircleGen` for degrees $5, 6, 7$ in `DualSmale9/PureKernel.lean`):

(1) **Exact Stereographic Parametrization**: Every unit-circle point $z \in \partial\mathbb{D} \setminus \{-1\}$ is uniquely represented as $\Phi(t) = \frac{(1-t^2)+2it}{1+t^2}$ for $t \in \mathbb{R}$, and the complement of $\{1\}$ by $-\Phi(t)$. Covering $t \in [-1, 1]$ on both semicircles partitions $\partial\mathbb{D}$ into four quarters with *zero trigonometry* and *zero precomputed tables.*

(2) **Dyadic Exact-Integer Arcs**: A dyadic interval $I_{p,q} = [(p - 1)/q, (p + 1)/q] \subset [0, 1]$ (with odd $p < q$ and $q = 2^m$) has rational midpoint $t = p/q$, giving the exact Pythagorean point

$$u = \pm\frac{(q^2 - p^2) \pm 2ipq}{q^2 + p^2} \in \partial\mathbb{D},$$

whose denominator is $c = q^2 + p^2 \in \mathbb{N}_{>0}$. Since $|\Phi'(s)| = \frac{2}{1+s^2} \leq \frac{2q^2}{q^2+(p-1)^2}$ for all $s \in I_{p,q}$, every point $z \in \Phi(I_{p,q})$ satisfies $z = ue^{i\phi}$ with exact rational angular radius

$$|\phi| \leq \frac{2q}{q^2 + (p - 1)^2} \leq \frac{d_u}{S}, \qquad d_u := \left\lceil \frac{2qS}{q^2 + (p - 1)^2} \right\rceil$$

(`stereographic_arc_angle_bound` in `DualSmale9/CertEngine.lean`).

(3) **Adaptive Subdivision + Fast-Fail**: If an arc fails the curvature-aware tangent bound at its midpoint u because the midpoint itself falls below the target threshold, the checker aborts immediately (`abort = true`); otherwise it bisects into $(2p-1, 2q)$ and $(2p+1, 2q)$ up to a per-leaf depth budget `fuel` (≤ 8).
(4) **Fused Native-`Int`/`Nat` Evaluation**: At each arc midpoint, `checkArcPt5` evaluates all three polynomials $(P_0, uP_0', u^2P_0'', G_2, uG_2', G_3, uG_3')$ in a single pass sharing $u^2, \ldots, u^5$ and c^5, uses native GMP `Nat` ceiling division and L^1 derivative bounds (only *one* self-checked Newton `isqrtU` per circle point), and verifies zero-freeness at the center via exact integer Schur–Cohn (`schurCohnI5..I7`) plus clopen homotopy (`DualSmale9/SchurCohn.lean`).

5. Global Case Decomposition and Pure Lean 4 Kernel Verification

Define the exact dyadic upper danger region $C_+^{(9)} \subset \overline{\mathbb{D}}$ (at scale $S = 2^{20} = 1{,}048{,}576$) by

$$C_+^{(9)} := \left\{ a = x + iy \in \overline{\mathbb{D}} : \ |a| > \frac{161}{200}, \ \ y \geq 0, \ \ x - y \leq \frac{5}{8} \right\},$$

and its conjugate $C_-^{(9)} := \overline{C_+^{(9)}}$. Observe that every point $a \in C_+^{(9)}$ satisfies $|a-1| > 1/30$ automatically (since $|a-1| \leq 1/30 \implies x - y \geq 29/30 - 1/30 = 14/15 > 5/8$).

Given any $w : \mathrm{Fin}\, 8 \to \mathbb{C} \setminus \{0\}$, the master theorem `dual_smale_degree_9_complete` (`DualSmale9/DualSmaleNineComplete.lean`) permutes and scales w so that $w_0 = 1 \geq |w_1| \geq |w_2| \geq |w_3|, \ldots, |w_7|$ and (via conjugation) $\mathrm{Im}(w_1) \geq 0$, then exhausts all non-extremal configurations across seven pure-kernel stages:

(1) **Stage 1b** ($|w_1| \leq 167/200$): 7-variable GWS on $\overline{D}(0, 167/200)$ reduces $2520P(1)$ to $2520G_9(\frac{167}{200}u)$, certified > 280 in **1 leaf** (`stage1b_cert`).
(2) **Stage 1d** ($|w_2| \leq 161/200$): 6-variable GWS on $\overline{D}(0, 161/200)$ reduces to $H_9(w_1, \frac{161}{200}u)$, certified > 280 across all of $\overline{\mathbb{D}}_+$ in **26 leaves** (`stage1d_cert`).
(3) **Double-Crit** ($w_0 = w_1 = 1$, $|w_2 - 1| > 1/30$, $\mathrm{Im}(w_2) \geq 0$): 5-variable GWS ($\alpha = (1, 0, 0)$) reduces to $Q_1(1, w_2, v)$, certified > 280 in **90 leaves** (`doubleCrit_cert`).
(4) **Stage 0** ($|a-1| \leq 1/30$, $a \neq 1$): closed analytically by Theorem 3.1 plus **2 leaves** (`stage0Box_cert`, `stage0A1_cert`).
(5) **Stage 1c** ($w_1 \notin C_+^{(9)}$, $|w_1| > 161/200$, $|w_1 - 1| > 1/30$, $\mathrm{Im}(w_1) \geq 0$): closed by 6-variable GWS ($H_9(w_1, v)$) in **102 leaves** (`stage1c_cert`). The same certificate also eliminates any surviving configuration with $w_1 \in C_+^{(9)}$ and $w_2 \notin C_+^{(9)} \cup C_-^{(9)} \cup \{1\}$ after swapping (w_1, w_2) and conjugating if $\mathrm{Im}(w_2) < 0$.
(6) **Stage 2** ($(w_1, w_2) \in C_+^{(9)} \times C_-^{(9)}$, $|w_2| \leq |w_1|$, $|w_1| > 167/200$): closed by Theorem 2.1 in **20,782 leaves** (19,376 linear-combination + 1,406 pure-Q_3) across 120 balanced shards (`S2_0..S2_119`).
(7) **Stage 3** ($(w_1, w_2) \in C_+^{(9)} \times C_+^{(9)}$, $|w_2| \leq |w_1|$, $|w_1| > 167/200$): closed by Theorem 2.1 in **79,003 leaves** (63,629 linear-combination + 15,374 pure-Q_3) across 399 balanced shards (`S3_0..S3_398`).

Every shard imports *only* `DualSmale9.PureKernel` (zero Mathlib imports, ~ 50 MB RAM per worker), chunks its subtree into ≤ 8-leaf theorems checked directly via `by decide +kernel`, and recombines them in $O(1)$ via `combine_splitX2/Y2/X3/Y3`. Table 2 summarizes all verified stages and Table 3 lists all proof-engine modules.

6. Formal Verification Architecture and Reproducibility

The entire proof can be independently audited from source in Lean 4.36.0 with three commands (on cached Mathlib, the 10-module proof engine compiles in under 45 s and all 520 pure-kernel certificate modules verify in parallel in 3 h 30 m wall-clock time at 801% CPU across 10 cores on an Apple M5 machine):

TABLE 2. Complete 100% pure Lean 4.36.0 kernel verification breakdown for degree $n = 9$ (zero external verifiers, zero `sorry`, zero `native_decide`).

Stage	Certified Region	Leaves	Pure-Q_3	Shards	Lean 4 Module(s)
0	$\lvert a-1\rvert \le 1/30,\ \lvert v\rvert \le 1$	2	—	—	`Stage0Expanded`, `Stage1Certs`
1b	$\lvert w_k\rvert \le 167/200$	1	—	—	`Stage1Certs` (`stage1b_cert`)
1d	$\lvert w_2\rvert \le 161/200,\ w_1 \in \overline{\mathbb{D}}_+$	26	—	—	`Stage1Certs` (`stage1d_cert`)
DC	$w_0 = w_1 = 1,\ \lvert w_2 - 1\rvert > 1/30$	90	—	—	`Stage1Certs` (`doubleCrit_cert`)
1c	$w_1 \in \overline{\mathbb{D}}_+ \setminus C_+^{(9)},\ \lvert w_1\rvert > 161/200$	102	—	—	`Stage1Certs` (`stage1c_cert`)
2	$(w_1, w_2) \in C_+^{(9)} \times C_-^{(9)}$	20,782	1,406	120	`Stage2Cert` (`S2_0..S2_119`)
3	$(w_1, w_2) \in C_+^{(9)} \times C_+^{(9)}$	79,003	15,374	399	`Stage3Cert` (`S3_0..S3_398`)
Total	**All of** $(\mathbb{C} \setminus \{0\})^8$	**100,006**	**16,780**	**519**	`DualSmaleNineComplete` **(PASS)**

TABLE 3. Summary of Lean 4.36.0 proof-engine modules for degree $n = 9$ (zero `sorry`, zero `native_decide`).

Module (`DualSmale9/.*`)	Lines	Role
`GraceWalshSzego`	542	Laguerre polar deriv., GWS on $\overline{D}(0, r)$, `symMultiaffine_succ`
`PolynomialReduction`	353	Exact 2520-identities for $P(1)$, G_9, H_9, reciprocal Q_j
`SchurCohn`	1,076	Min. modulus, clopen homotopy, Schur–Cohn (deg. 4–7), arc tangent
`Stage0Expanded`	334	Analytic wide ball $\lvert a-1\rvert \le 1/30$, $\lvert v-1\rvert \le 1/5$
`TwoVarGWS`	415	Scale/conj/perm invariance, Q_1, Q_2, Q_3, hybrid 2-var + pure-Q_3 GWS
`PureKernel` (zero Mathlib)	923	Pure-`Init` exact-$\mathbb{Z}$ engine, stereographic bisector, tree combiners
`CertEngine`	2,133	Soundness of stereographic bisection, fused eval, Schur–Cohn
`CertSoundness`	970	Soundness for Stage 0, 1b, 1d, Double-Crit, Stage 1c
`TwoVarSoundness`	2,147	Soundness for `buildP1_9`, `buildP2_9`, `buildQ3Pure_9`, Stages 2–3
`DualSmaleNineComplete`	457	Unconditional theorem `dual_smale_degree_9_complete`
Proof engine total	**9,350**	**+ 520 pure-kernel cert modules (zero Mathlib)**

```
cd lean
lake exe cache get
lake build DualSmale9
lake env lean check_axioms.lean
```

The axiom audit confirms that `DualSmale.dual_smale_degree_9_complete` and all certificate theorems depend strictly on the standard axioms `[propext, Classical.choice, Quot.sound]` (with `stage0Box_cert`, `stage0A1_cert`, and `stage1b_cert` depending solely on `[propext]`), with zero `sorryAx` and zero `Lean.ofReduceBool`.

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ESIEE-IT, France
Email address: ivan.besevic@edu.esiee-it.fr