Integer {2}-domination on five-row cylinders

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Abstract

We determine the integer {2}-domination number of $P_5 \square C_n$ for every $n \geq 3$. The minimum density 23/10 per column is attained precisely at multiples of twenty. At those circumferences all minimizers are translates of one periodic word. For an arbitrary feasible labeling, the number of three-column windows absent from this word is at most five times the excess $10W - 23n$, with sharp factor five. On an open strip an optimal labeling has at most 46 such windows, with equality only at width 48. The proofs use finite integer transfer recurrences with explicit period and potential certificates.

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1 The recurring word

An integer {2}-dominating function on a graph assigns a value in $\{0, 1, 2\}$ to each vertex, with sum at least two on every closed neighbourhood. Write W for the sum of all labels and $\gamma(n)$ for its minimum on $P_5 \square C_n$. The rows are ordered and the columns cyclically indexed. Lee and Liu study integer {2}-domination on rectangular grids [1]. Garzón et al. use min-plus powers for set-valued 2-domination on cylinders, where a vertex outside the set must have two neighbours in it [2]. For the five-row cylinder their parameter equals $\lceil 15n/7 \rceil$. At circumference twenty it equals 43, whereas the closed-neighbourhood integer problem has value 46.

Encode a column as a base-three integer with the top entry most significant. Let B be the following cyclic word, introduced in the open-strip calculation:

$$(4, 90, 91, 28, 30, 84, 91, 10, 108, 7,\\ 108, 10, 91, 84, 30, 28, 91, 90, 4, 135).$$

For a cyclic labeling x, let $D(x)$ count the centres i for which (x_{i-1}, x_i, x_{i+1}) is absent from the periodic repetition of B. Set $\mathcal{R} = \{3, 6, 9, 10, 13, 19\}$.

Theorem 1. *For $n \geq 3$,*

$$\gamma(n) = \lceil 23n/10 \rceil + \mathbf{1}_{\{n \bmod 20 \in \mathcal{R}\}} - \mathbf{1}_{\{n \in \{6,9\}\}}.$$

Every feasible cyclic labeling satisfies $D(x) \leq 5(10W - 23n)$. Equality with $D(x) > 0$ occurs. The labelings of weight $23n/10$ exist precisely for $20 \mid n$ and are exactly the twenty translates of B repeated around the cylinder.

For instance, $\gamma(9) = 21$, $\gamma(10) = 24$, $\gamma(19) = 45$ and $\gamma(20) = 46$. The formula and the stability inequality bound the number of exceptional windows in any minimizing cylinder by 65.

2 A finite certificate for all circumferences

For three consecutive columns a, b, c, with $b_0 = b_6 = 0$, define

$$d_i(a, b) = \max(0, 2 - a_i - b_{i-1} - b_i - b_{i+1}), \quad 1 \leq i \leq 5.$$

The constraint on b is exactly $c \geq d(a, b)$ coordinatewise. The states are the pairs $(b, d(a, b))$ induced by all a, b. From (b, d), reading $c \geq d$ gives state $(c, d(b, c))$ and weight $w(c) = \sum c_i$. Each rooted closed walk of length n reads a cyclically indexed labeling. Its stored column is the one just read, so every stored deficit is the deficit of the actual two preceding columns, including at the join. This proves feasibility and recovers the original walk from the labeling. Conversely, reading a feasible cyclic labeling gives the states prescribed by its adjacent pairs. The correspondence preserves weight. For $n \geq 3$ the two horizontal neighbours are distinct.

Let $A_j(s, t)$ be the minimum weight of a walk from s to t in exactly j steps. Start with zeros on the diagonal and infinity elsewhere. Extending by one directed edge computes A_{j+1}, and $\gamma(j) = \min_s A_j(s, s)$

for $j \geq 3$. The supplied C++ calculation constructs all 1595 states and 161572 edges directly from the neighbourhoods. It checks the stated values for $3 \leq j \leq 70$ and the identity

$$A_{70}(s,t) = A_{50}(s,t) + 46$$

at all 1595^2 entries, which are finite at both lengths. Extending a walk commutes with adding a common constant to the matrix. Hence the same period holds after every further step. The right side of Theorem 1 has this period for $n \geq 50$. This proves the formula.

3 Stability and the open strip

To count exceptional windows, add to state $(b, d(a,b))$ the successor of the ordered pair (a,b) in B, or the marker -1 if the pair does not occur. The twenty ordered pairs in B are distinct. On reading c, let $e = 0$ if c equals the marker and $e = 1$ otherwise. Marked states and transitions are generated from real column pairs as above, giving 1615 states and 163588 edges. Assign integer cost $p(s,c) = 5(10w(c) - 23) - e$.

Define $h_0 = 0$ and let $h_{j+1}(t)$ be the minimum of $h_j(t)$ and all $h_j(s) + p(s,c)$ on edges ending at t. The independent integer program constructs $h_{28} = h_{29}$, with entries in $[-232, 0]$. It checks $h(t) - h(s) \leq p(s,c)$ on every edge. Its direct enumeration also verifies the pulled-back inequality on all 11929864 feasible raw triples. Summing around a closed walk telescopes and proves the bound on D. One explicit 50-column labeling in the certificate has weight 116 and $D = 50$. The word B is feasible and has weight 46. If $10W = 23n$, all triples are triples of B and each ordered pair determines the next column uniquely. The closed labeling must therefore traverse the single twenty-cycle an integral number of times, proving the classification.

For $P_5 \square P_n$ count only triples centred in columns $2, \ldots, n-1$. Let $M(n)$ be the largest such count amongst minimizing labelings.

Theorem 2. *$M(n) \leq 46$, with equality exactly at $n = 48$. For $n \geq 49$, in residue order $n \bmod 10 = 0, 1, \ldots, 9$, the values are*

$$(13, 4, 33, 17, 7, 36, 20, 10, 42, 23).$$

Proof. Minimize the ordered cost $(W, -D)$ lexicographically in the marked graph. Start with all pairs a, b for which the centre a is dominated with zero exterior left column, with cost $(w(a) + w(b), 0)$. Each further column contributes $(w(c), -e)$. A terminal state is allowed exactly when its required right contribution is zero. These conventions test both boundary columns and count each interior triple once. Write F_n for the full vector of optimal ordered costs before the terminal restriction. Exact comparison in the supplied program gives $F_{78} = F_{68} + (23, 0)$ entrywise, interpreting infinity as absorbing. Lexicographic order commutes with translation, propagating the identity to all larger lengths. Direct values through length 78 give the maximum and the residue table. At width one the count is zero. □

Proof and computation. The C++ and Python programs reconstruct their graphs from the local definition. The Lean file proves the general propagation and telescoping implications. The finite matrix and potential identities above use the integer programs. Chengyi Fang initiated, funded and directed the work and is the contact for corrections. AI systems developed the arguments and code and checked them in separate contexts. The sponsor has not independently checked the proofs.

References

[1] J.-Y. Lee and C.-A. Liu, *The integer {2}-domination number of grids*, arXiv:2502.00134.

[2] E. M. Garzón et al., *HPC acceleration of large* $(\min, +)$ *matrix products to compute domination-type parameters in graphs*, J. Supercomput. doi:10.1007/s11227-022-04574-5.