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CUBIC PLETHYSM IS SURPRISINGLY EXPRESSIVE

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Abstract. We give an explicit realization of every Kronecker coefficient in symmetric cubic plethysm, with the outer degree depending only on the common size of the indexing partitions. The construction isolates tensor products on extremal faces of Schur modules. We develop this approach further to realize graded invariant algebras of mixed tensors in Kronecker theory and rectangular cubic plethysm. Applications include quiver semi-invariants, Ehrhart algebras, counterexamples to polynomial bounds on saturation and positivity indices, and exact counting-table realizations on single stretching rays with one staircase label.

1. Introduction

Let $g(\alpha,\beta,\gamma)$ be the Kronecker coefficient of three partitions of m, and let $a^{\lambda}_{\mu,\nu} = \langle s_\lambda, s_\mu[s_\nu]\rangle$ denote a Schur plethysm coefficient. The former govern tensor products of symmetric-group representations; the latter govern compositions of Schur functors. Finding positive combinatorial rules for them is the subject of Problems 10 and 9 of Stanley [26]. We begin with the Schur expansion of $h_D[h_3]$, where the inner degree is fixed at three.

The starting observation is that the representation which carries Kronecker multiplicities is already present in a cubic functor. For $V = A \oplus B \oplus C$, the exterior cube contains the summand $A \otimes B \otimes C$. Taking an exterior power of this summand gives the Kronecker coefficients by Schur–Weyl duality. But the same exterior power of the whole exterior cube contains many other summands, and there is no reason for their multiplicities to separate. How can we retain just the contribution of the triple tensor product?

Give A, B, C weights $3, 1, 0$. The triple tensor product then has weight four, and no other cubic multidegree has this weight. We take an exterior power large enough to fill every sector of greater weight and leave precisely the desired number of factors at weight four. The filled sectors contribute a known determinant character. This is the reason for the shifts in the following formula.

Theorem 1.1. *For $\alpha,\beta,\gamma \vdash m \geq 1$, pad α,β,γ' to length m and put*

$$D_m = \binom{m}{3} + 3m\binom{m}{2} + m,$$

$$(u,v,w) = \left(3m^2 - 4m + 1,\ 3\binom{m}{2},\ \binom{m}{2}\right),$$

$$\Theta = ((u+\alpha_i)_{i=1}^m, (v+\beta_i)_{i=1}^m, (w+\gamma'_i)_{i=1}^m).$$

Then $\Theta \vdash 3D_m$ and

$$g(\alpha,\beta,\gamma) = \langle s_{\Theta'}, h_{D_m}[h_3]\rangle = \langle s_\Theta, e_{D_m}[e_3]\rangle. \tag{1}$$

The map $(\alpha,\beta,\gamma) \mapsto (D_m, \Theta')$ is injective and computable in polynomial time in unary encoding. In the family $h_D[h_b]$, the smallest inner degree for such a realization is $b = 3$.

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Thus a single Schur expansion contains the entire Kronecker table of S_m. A positive combinatorial rule for this cubic family would give one for Kronecker coefficients in the counting-complexity sense. Fischer and Ikenmeyer [10, Theorems 3.1, 3.2 and 7.1], extending the discrete-tomography methods of Ikenmeyer, Mulmuley and Walter [13], proved hardness of cubic plethysm and obtained exact identities for a special class of Kronecker coefficients. The additional assertion here is an unrestricted, uniform coefficient identity. A closely related precedent is Ikenmeyer and Panova's realization of ordinary Kronecker coefficients as reduced ones [14].

The mechanism behind the formula is a classical observation about Schur modules: the maximal cocharacter face is an irreducible module for a block Levi. Its highest weight consists of consecutive blocks of the original partition. This has a useful converse for our purposes. By choosing the weight sectors and the blocks of the partition, we can arrange for particular representations to survive on the face, with known determinant twists. Section 2 records the precise form of this observation, and Section 3 applies it to the cubic functors. Reduction on faces of branching cones has a general formulation in [25]; in Kronecker theory, additive constructions and faces also feature in the stability results of Stembridge and Manivel [27, 18]. Here the weights are chosen to isolate entire Schur factors, whose labels we can then vary.

There are two ways to extend the construction. First, a rectangle on a sector consisting of two summands relates their Schur labels by complementation. This lets us prescribe some labels indirectly and, when a common label is left free, sum over it by Schur's lemma. Section 4 uses these two operations to fix arguments of Kronecker coefficients and to obtain other classical multiplicities. An equivariant version of the face construction also realizes every Kronecker coefficient as the difference of a symmetric and an exterior square multiplicity (Proposition 4.3).

Second, the extremal faces can be chosen compatibly with Cartan multiplication. This changes the question: which graded algebras, rather than which individual dimensions, occur in cubic plethysm? Section 5 gives the algebraic counterpart of the opening theorem. Its natural intermediate algebra is

$$\mathcal{B}(A,B,C) = \mathrm{Sym}(A \otimes B \otimes C)^{U_A \times U_B \times U_C},$$

where the U's are upper maximal unipotent subgroups. This is graded by triples of partitions of equal size, with homogeneous dimensions given by Kronecker coefficients. We show that any linear graded slice of this algebra can be transported to rectangular cubic plethysm, including its multiplication.

We are then led to ask which algebras occur as such slices. A rectangle on a direct sum $Y \oplus Z$ has an affine chart with coordinates in $Y \otimes Z^*$. Thus the very operation that relates two Schur labels also gives a way to introduce dual tensor factors. In Section 6 we use these charts to build mixed tensors of arbitrary order inside a tensor grid. Contraction of auxiliary factors recovers the original representation, while determinant padding makes the construction linear in the degree. Theorem 6.2 realizes positively graded highest-weight slices of polynomial algebras on direct sums of mixed tensors of general linear groups. Two of the Kronecker labels have fixed shapes up to stretching, one square at the base scale. On a single grading ray we obtain isomorphisms

$$R \cong \mathcal{K}_{\alpha,\beta,\gamma} \cong \mathcal{P}^{(3)}_{\kappa,\Lambda},$$

where the algebras on the right have the stretched Kronecker and rectangular-outer cubic plethysm coefficients as their Hilbert functions.

One consequence is that rational Ehrhart algebras occur in these families. In the torus case, however, we can dispense with large auxiliary tensor spaces: the variables are scalar exponents and the constraints are linear equations. This observation leads to the arithmetic cycle of Section 8. Besides a direct realization of Ehrhart algebras, it gives quantitative size bounds: the polynomial-height bounds in the relaxed saturation hypotheses of Mulmuley [4, Appendix] fail (Corollary 8.4). A final modification realizes matching polytopes with a prescribed staircase label. Binary degree

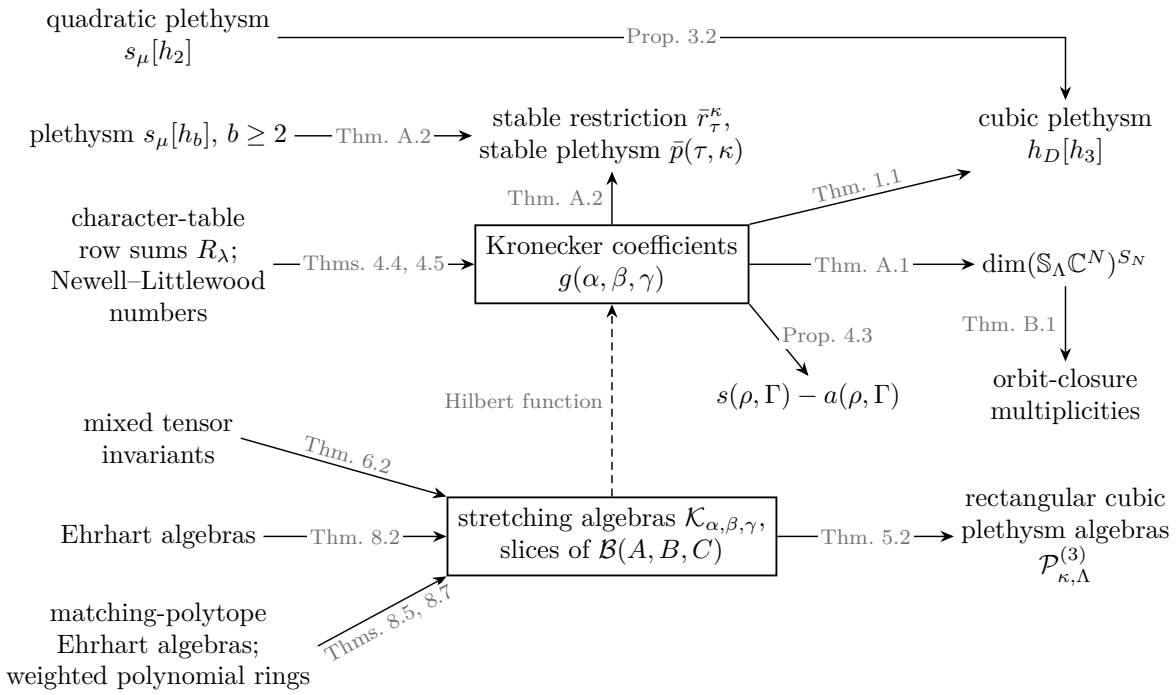

FIGURE 1. The realizations proved in this paper. Arrows into and out of $g(\alpha, \beta, \gamma)$ identify multiplicities; arrows at $\mathcal{K}_{\alpha,\beta,\gamma}$ are isomorphisms of graded algebras. The dashed arrow takes Hilbert functions.

equations then give weighted polynomial rings without regrading, and an entire counting table can occur at consecutive degrees of a single stretching ray (Corollary 8.8). The remaining questions concern the limitations of the construction, in particular the restrictions on source representations and on target Young diagrams (Section 9). Two further applications of extremal face extraction, to stable restriction and to orbit closures, are developed in Appendices A and B; they require no part of the graded-algebra argument. Figure 1 shows how all the realizations connect.

We work over $\mathbb{C}$. Complexity statements use unary partition encoding unless specified otherwise. The all-empty Kronecker instance has value 1. Plethysm with an empty outer or inner partition is immediate, so we consider positive degrees.

2. Extremal Schur faces

The cubic example requires two facts: a maximal weight space of a Schur functor separates into Schur functors of the weight sectors, and a highest-weight vector on this face for the block group is already highest for the whole group. We will also need the compatibility of this separation with products. We give all three statements together, so that subsequent constructions use the same convention for their faces.

Write $\mathbb{S}_\lambda V$ for the Schur module, λ' for the conjugate partition, and (k^r) for r parts equal to k. We use the Hall inner product and the notation $[U : X]$ for multiplicity of an irreducible U in X. All partitions in concatenations are padded to the indicated block lengths. We use Schur–Weyl duality and the Cauchy identities (see [17, Chapter I]) in the forms

$$\mathbb{S}_\gamma(A \otimes B) \cong \bigoplus_{\alpha,\beta} g(\alpha, \beta, \gamma) \mathbb{S}_\alpha A \otimes \mathbb{S}_\beta B, \tag{2}$$

$$\bigwedge^m (A \otimes B \otimes C) \cong \bigoplus_{\alpha,\beta,\gamma} g(\alpha, \beta, \gamma) \mathbb{S}_\alpha A \otimes \mathbb{S}_\beta B \otimes \mathbb{S}_{\gamma'} C, \tag{3}$$

$$\mathrm{Sym}^m(A\otimes B)\cong\bigoplus_{\xi\vdash m}\mathbb{S}_\xi A\otimes\mathbb{S}_\xi B, \tag{4}$$

$$\bigwedge^m(A\otimes B)\cong\bigoplus_{\xi\vdash m}\mathbb{S}_\xi A\otimes\mathbb{S}_{\xi'} B. \tag{5}$$

The sign twist $[\gamma']=[\gamma]\otimes\mathrm{sgn}$ gives (3). The symmetric-function involution $\omega(s_\lambda)=s_{\lambda'}$ satisfies

$$\omega(s_\mu[s_\nu])=\begin{cases}s_\mu[s_{\nu'}], & |\nu| \text{ even},\\ s_{\mu'}[s_{\nu'}], & |\nu| \text{ odd}.\end{cases} \tag{6}$$

This follows by applying $\omega(p_j)=(-1)^{j-1}p_j$ to the power-sum expansion: on $p_r[s_\nu]$ the extra sign relative to $p_r[s_{\nu'}]$ is $(-1)^{(r-1)|\nu|}$.

Let $V=\bigoplus_i V_i$ with ordered bases, and let $L=\prod_i \mathrm{GL}(V_i)$. Choose a central cocharacter ζ acting on V_i by t^{c_i}, with $c_1>\cdots>c_k$. For an L-module with weights bounded above write $X_{\max}$ for its top weight space.

Lemma 2.1 (Face extraction)**.** *(i) In a polynomial* $\mathrm{GL}(V)$*-module, the* L*-highest-weight vectors on a maximal* ζ*-face are precisely the* $\mathrm{GL}(V)$*-highest-weight vectors there. The assertion holds on the weight space at any upper bound for the weights of the module, even if that space is zero.*

(ii) Suppose $Y=\bigoplus_{j=1}^s Y_j$ *is an* L*-module with strictly decreasing cocharacter weights on its weight spaces, of dimensions* d_j*. For a partition* κ *with consecutive blocks* $\kappa^{[j]}$ *of lengths* d_j*,*

$$(\mathbb{S}_\kappa Y)_{\max}\cong\bigotimes_j \mathbb{S}_{\kappa^{[j]}}Y_j. \tag{7}$$

In particular, if $Y=Y_>\oplus Y_0\oplus Y_<$ *is split at a threshold,* $H=\dim Y_>$ *and* $k\geq\mu_1$*,* $\ell(\mu)\leq\dim Y_0$*, then*

$$(\mathbb{S}_{(k^H,\mu)}Y)_{\max}\cong(\det Y_>)^k\otimes\mathbb{S}_\mu Y_0. \tag{8}$$

(iii) The identifications in (7)*, over all polynomial Schur labels* κ*, may be chosen compatibly with Cartan multiplication. Under these identifications, the product of top faces is the blockwise Cartan product.*

Proof. Raising operators between distinct V_i have positive cocharacter weight and annihilate the top face, proving (i). For (ii), regard $\mathbb{S}_\kappa Y$ as an irreducible $\mathrm{GL}(Y)$-module. By the Poincaré–Birkhoff–Witt theorem its top face is generated from its highest-weight line by lowering operators of weight zero, namely those within the Y_j. This is the irreducible module for $\prod_j\mathrm{GL}(Y_j)$ with highest weight given by the padded blocks of κ. Restriction to L proves (7); filling the first H rows with k proves (8). If there are infinitely many weight spaces but only finitely many at or above any threshold, the same argument applies to a finite initial segment containing the blocks of κ; lower weights cannot contribute to the top face. For (iii), realize the polynomial Cartan algebra of Y in the coordinate ring of a matrix space, taking highest-weight vectors on the second factor. Choose its highest-weight vectors in the fundamental Schur degrees, and use their products in every other degree. The product on the top faces is equivariant for $\prod_j\mathrm{GL}(Y_j)$, and on their highest-weight vectors it is nonzero and adds the Schur degrees. In the product of the block Cartan algebras choose highest-weight vectors in the fundamental degrees as well, and use their products. Normalize each face isomorphism by sending its chosen highest-weight vector to the corresponding product vector on the block side. These vectors multiply exactly in both algebras. The two equivariant maps from the tensor product of any two faces to the next face then agree on the tensor of highest-weight vectors; multiplicity one of the Cartan component implies that they agree everywhere. This assertion applies to finite-dimensional Y; for the infinite-dimensional use of (ii) only the finite initial segment is needed. □

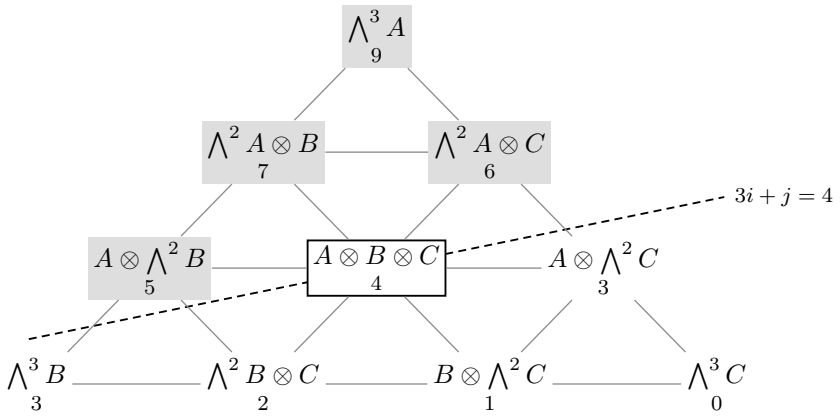

FIGURE 2. The ten multidegrees of $\bigwedge^3(A \oplus B \oplus C)$ with their weights $3i + j$. The shaded sectors lie above the level line through $A \otimes B \otimes C$ and are saturated in the proof of Theorem 1.1.

In (7) we can prescribe the Schur factors on finitely many weight levels, up to determinant twists. Given dominant integral weights $\eta^{[j]}$ of $\mathrm{GL}(Y_j)$ (negative entries allowed), choose nonnegative integers k_j from the last block upward so that $\eta^{[j]} + (k_j^{d_j})$ are nonnegative and their concatenation is dominant. Formula (7) then realizes $\bigotimes_j \mathbb{S}_{\eta^{[j]}} Y_j$ (with rational Schur modules allowed), twisted by $\prod_j (\det Y_j)^{k_j}$. The same choice can be made with zero weights on all unused sectors. We use this observation in Section 4.

For a dominant integral weight $\lambda = (\lambda_1, \dots, \lambda_d)$, allowing negative entries, we use $\mathbb{S}_\lambda E$ for the corresponding rational Schur module and write $\lambda^\vee = (-\lambda_d, \dots, -\lambda_1)$, so $\mathbb{S}_{\lambda^\vee} E \cong (\mathbb{S}_\lambda E)^*$. We use HW for a highest-weight space.

For $k = 1, \mu = (1^m)$, saturation says $(\bigwedge^{H+m} Y)_{\max} = \det(Y_>) \otimes \bigwedge^m Y_0$. Exterior powers thus turn an interior sector into a highest-weight face. Finally, determinants are evaluated using $\det(\bigwedge^j E) = (\det E)^{\binom{d-1}{j-1}}$, $\det(\mathrm{Sym}^j E) = (\det E)^{\binom{d+j-1}{j-1}}$ for $d = \dim E$, and the usual tensor determinant rule. These identities follow by counting the total exponent of any basis vector.

3. Cubic plethysm

We first apply the face lemma with an exterior outer power. In this case every block of the outer label consists of ones or zeros, and saturating a weight sector simply means taking its determinant. The only sector that is not saturated will carry the multiplicity we seek.

Proof of Theorem 1.1. Take $V = A \oplus B \oplus C$, all three blocks of dimension m, with cocharacter $(3, 1, 0)$. In $Y = \bigwedge^3 V$ the multidegree (i, j, k) has weight $3i + j$. The threshold 4 singles out $Y_0 = A \otimes B \otimes C$, with exactly four higher sectors:

$$Y_> = \bigwedge^3 A \oplus (\bigwedge^2 A \otimes B) \oplus (\bigwedge^2 A \otimes C) \oplus (A \otimes \bigwedge^2 B).$$

Their dimension is $D_m - m$. Figure 2 shows all ten sectors with their weights. To compute the determinant use $\det(\bigwedge^2 A) = (\det A)^{m-1}$, $\det(\bigwedge^3 A) = (\det A)^{\binom{m-1}{2}}$, and $\det(X \otimes Z) = (\det X)^{\dim Z} (\det Z)^{\dim X}$. For $m \geq 2$ the resulting exponents are

$$u = \binom{m-1}{2} + 2m(m-1) + \binom{m}{2}, \qquad v = 3\binom{m}{2}, \qquad w = \binom{m}{2}.$$

For $m = 1$ all higher exterior sectors are zero and the shifts vanish, as in the theorem.

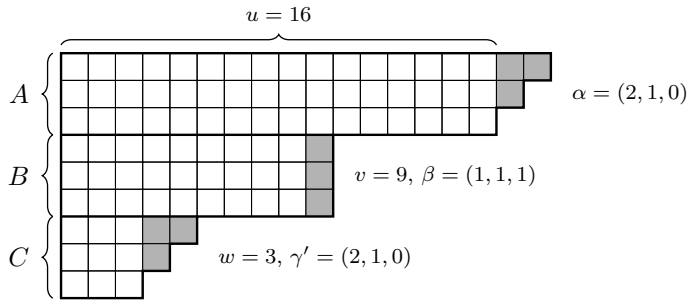

FIGURE 3. The partition $\Theta = (18, 17, 16, 10, 10, 10, 5, 4, 3)$ of Theorem 1.1 for $m = 3$, $\alpha = \gamma = (2, 1)$, $\beta = (1, 1, 1)$. White cells are the determinant shifts u, v, w; shaded cells are the three inputs.

By (8), the top face of $\bigwedge^{D_m} Y$ is

$$(\det A)^u (\det B)^v (\det C)^w \otimes \bigwedge^m (A \otimes B \otimes C).$$

Using (3) gives the weight Θ with multiplicity $g(\alpha, \beta, \gamma)$. The gaps $u - v \geq m$ and $v - w \geq m$ for $m \geq 2$ ensure dominance; for $m = 1$, $\Theta = (1, 1, 1)$. Its size is $m(u + v + w) + 3m = 3D_m$, and its cocharacter weight is $3mu + mv + 4m$, exactly the weight of the filled sectors and the m threshold factors. Lemma 2.1(i) and (6) give (1). The three padded blocks recover the input, and $D_m = O(m^3)$. Finally, $h_D[h_1] = h_D$ and $h_D[h_2] = \sum_{\lambda \vdash D} s_{2\lambda}$ are multiplicity-free [17], whereas $g((4, 2), (4, 2), (3, 2, 1)) = 2$. □

For example, when $m = 2$ the outer degree is 8 and the shifts are $(5, 3, 1)$. The triple $((2), (2), (2))$ gives $\Theta = (7, 5, 5, 3, 2, 2)$, hence $\langle s_{(6,6,4,3,3,1,1)}, h_8[h_3] \rangle = 1$. All other triples of partitions of two are realized in the same Schur expansion, including those with zero multiplicity. Figure 3 draws Θ for a triple of partitions of three.

Replacing the exterior cube by the symmetric cube gives the other two cubic families. Unequal block dimensions can also reduce the size of the construction.

Corollary 3.1 (Dual and adaptive realizations)**.** *Set* $D_m^+ = \binom{m+2}{3} + 3m\binom{m+1}{2} + m$ *and*

$$(u^+, v^+, w^+) = \left(3m^2 + 4m + 1,\ 3\binom{m+1}{2},\ \binom{m+1}{2}\right).$$

With Θ^+ *defined using these shifts in Theorem 1.1,*

$$g(\alpha, \beta, \gamma) = \langle s_{\Theta^+}, e_{D_m^+}[h_3] \rangle = \langle s_{(\Theta^+)'}, h_{D_m^+}[e_3] \rangle.$$

If $\ell(\alpha), \ell(\beta), \ell(\gamma) \leq L$ *for* $L \geq 2$*, there is also a realization in* $h_D[h_3]$ *with* $D = \binom{L}{3} + 2L\binom{L}{2} + (\binom{L}{2} + 1)m$ *and output Durfee rank at most* $\max(2L, \binom{L+1}{2})$*. It realizes every simultaneous stretching value along an affine outer-degree ray.*

Proof. For the first assertion replace every inner exterior power in the preceding proof by the corresponding symmetric power, retaining the outer exterior power. The higher sectors have total dimension $D_m^+ - m$. Using $\det(\mathrm{Sym}^j A) = (\det A)^{\binom{m+j-1}{j-1}}$ gives $u^+ = \binom{m+2}{2} + 2m(m+1) + \binom{m+1}{2} = 3m^2 + 4m + 1$ and the stated v^+, w^+. Both successive gaps are at least m, so the same dominant-weight argument applies. For the second assertion take block dimensions (L, L, m). The shifts become $u = (L-1)(m + 2L - 1)$, $v = 3w$, $w = \binom{L}{2}$, and the output is the conjugate of the same padded concatenation, now of lengths (L, L, m). The gaps $u - v \geq m$ and $v - w \geq L$ ensure dominance since $\gamma_1' \leq L$. All rows past the first $2L$ have length at most $w + L$, proving the Durfee bound. Replacing m by tm with L fixed proves the assertion on stretching for $t > 0$. At $t = 0$

the same construction uses a zero-dimensional C and fills all the higher sectors, giving a single determinant character and multiplicity one. □

For any outer Schur functor of degree D, take $\dim E$ sufficiently large. The maximal face of $\bigwedge^{b+1}(\mathbb{C} \oplus E)$ for weights $(1,0)$ is $\mathbb{C} \otimes \bigwedge^b E$. Applying Lemma 2.1 to its Schur power prepends a row of length D to the output partition of a coefficient of $s_\mu[e_b]$: its width is at most D since each tensor factor is exterior. Iteration and (6) give the same realizations in all four symmetric/exterior families at every fixed $b \geq 3$.

In the cubic construction the unsaturated sector had three tensor factors. If two of the factors are drawn from the same block, we instead obtain $\bigwedge^2 B \otimes A$. The auxiliary space A can carry an outer Schur label through the exterior Cauchy identity. This gives a way to absorb the outer label of a quadratic plethysm into the output partition.

Proposition 3.2 (Quadratic plethysm in cubic plethysm)**.** *For $\mu \vdash d \geq 1$, $\lambda \vdash 2d$, set $N = 2d+1$, $u = \binom{N-1}{2}$, $D = \binom{N}{3} + d$. Then*

$$\langle s_\lambda, s_\mu[h_2] \rangle = \langle s_{((N+\mu_i)_{i=1}^u, \lambda)}, h_D[h_3] \rangle.$$

The analogous formula for $s_\mu[e_2]$ replaces the final λ by λ'.

Proof. For $V = B \oplus A$ of dimensions N, μ_1 and weights $(1,0)$, the top two sectors of $\bigwedge^3 V$ are $\bigwedge^3 B$ and $\bigwedge^2 B \otimes A$. Saturate the first and apply (5) to the second. The resulting top face of $\bigwedge^D(\bigwedge^3 V)$ is

$$(\det B)^u \otimes \bigoplus_{\xi \vdash d} \mathbb{S}_\xi(\bigwedge^2 B) \otimes \mathbb{S}_{\xi'} A.$$

The dominant weight $((u+\tau_i)_{i=1}^N, \mu')$ has multiplicity $\langle s_\tau, s_\mu[e_2] \rangle$. Its conjugate is $((N+\mu_i)_{i=1}^u, \tau')$, since $u \geq \ell(\mu)$. Apply (6), with $\tau = \lambda'$ or λ, respectively. □

For inner degree two, Proposition 3.2 eliminates the outer Schur label entirely. For general inner degree we can still eliminate that label, at the expense of changing the inner representation.

Proposition 3.3 (Outer absorption)**.** *For $\mu \vdash d \geq 1$, $\nu \vdash b \geq 1$, $\lambda \vdash bd$, let $P \geq \max(\ell(\lambda), \ell(\nu))$. Pad λ, ν to length P and set $\eta = (\nu + (1^P), 1)$, $\Lambda = (\lambda + (d^P), \mu)$. Then*

$$\langle s_\lambda, s_\mu[s_\nu] \rangle = \langle s_\Lambda, h_d[s_\eta] \rangle.$$

Proof. On $A \oplus B$ with $\dim A = P$ and $\dim B \geq \ell(\mu)$, the top face of $\mathbb{S}_\eta(A \oplus B)$ for weights $(1,0)$ is $\det(A) \otimes \mathbb{S}_\nu A \otimes B$. Its dth symmetric power has top face $(\det A)^d \otimes \mathrm{Sym}^d(\mathbb{S}_\nu A \otimes B)$. Apply Cauchy to select $\mathbb{S}_\mu(\mathbb{S}_\nu A) \otimes \mathbb{S}_\mu B$. The shift d ensures dominance of Λ, and Lemma 2.1 gives the identity. □

4. Fixed labels and contractions

The outer absorption construction leaves the choice of a Schur label to an auxiliary highest weight. There is a related question internal to Kronecker theory: can the variation of all three input partitions be carried by just one output partition? A rectangle on a two-summand sector is suited to this purpose, since its restriction pairs a Schur label with its uniquely determined complement. We first use the pairing to prescribe labels, then use the same mechanism with free labels to obtain sums of multiplicities. For related results on rectangular Kronecker coefficients, see [15].

The pairing is rectangular branching (Figure 4):

$$\mathbb{S}_{(K^r)}(U \oplus W) \cong \bigoplus_{\eta \subseteq (K^r)} \mathbb{S}_\eta U \otimes \mathbb{S}_{\eta^c} W, \qquad \eta^c = (K - \eta_r, \ldots, K - \eta_1). \tag{9}$$

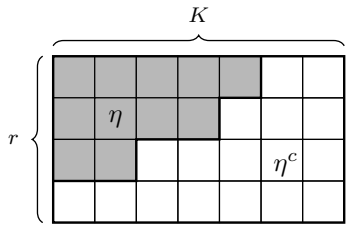

FIGURE 4. Rectangular branching (9) for $K = 7$, $r = 4$: the label $\eta = (5, 4, 2, 0)$ on U forces its complement $\eta^c = (7, 5, 3, 2)$ on W, read from the opposite corner.

Indeed, the Littlewood–Richardson coefficient $c_{\eta,\theta}^{(K^r)}$ vanishes unless both labels fit inside the rectangle. In dimension r it is the multiplicity of $(\det \mathbb{C}^r)^K$ in $\mathbb{S}_\eta \mathbb{C}^r \otimes \mathbb{S}_\theta \mathbb{C}^r$. By Schur's lemma this multiplicity is one exactly when $\theta = \eta^c$, and is zero otherwise. Together with (7), this allows us to fix two arguments of the Kronecker coefficient.

Proposition 4.1 (Fixed tensor products)**.** *For each m there exist fixed partitions R_m, T_m of a common size polynomial in m, with R_m a square and T_m having at most three distinct parts, such that every $g(\alpha, \beta, \gamma)$ of S_m equals $g(R_m, T_m, F_m(\alpha, \beta, \gamma))$ for an explicit injective map. There is also a homogeneous fixed-rectangle version preserving all simultaneous stretching values.*

Proof. We first fix both arguments, then make one square. Fix $L \geq \max(\ell(\alpha), \ell(\beta), \ell(\gamma), 1)$ and let $P = (3^{L^2-1}, 2^{L^2+2L+1}, 1^{3L^2+1})$, $Q = ((5L+2)^L, (3L+2)^L)$, each of size $C_L = 8L^2 + 4L$. Put

$$F = \big(5Lm - \alpha_L, \ldots, 5Lm - \alpha_1,\ (4L-1)m,$$
$$(2Lm + \beta_i)_{i=1}^{L}, (Lm + \gamma_i)_{i=1}^{L}\big).$$

Take $A = X \oplus E \oplus Y \oplus Z$, $B = U \oplus W$, with E a line and all other blocks of dimension L. For weights $(2, 1, 0, -1)$ and $(1, 0)$, the sectors of $A \otimes B$ in decreasing order, and their Schur factors after removal of determinant twists, are

weight	sector	shift	factor
3	$X \otimes U$	$3m$	$(\mathrm{Sym}^m)^*$
2	$(X \otimes W) \oplus (E \otimes U)$	$2m$	1
1	$(Y \otimes U) \oplus (E \otimes W)$	m	$\mathbb{S}_{(m^L)}$
0	$(Y \otimes W) \oplus (Z \otimes U)$	m	1
-1	$Z \otimes W$	0	Sym^m

These are the blocks of mP in (7) (Figure 5); the Schur functor in the middle is on $(Y \otimes U) \oplus (E \otimes W)$. After the indicated twists, the target on X, Y, Z is $(\mathbb{S}_\alpha X)^*, \mathbb{S}_\beta Y, \mathbb{S}_\gamma Z$. Cauchy on the first and last factors therefore selects $(\mathbb{S}_\alpha U)^*$ and $\mathbb{S}_\gamma W$. The target on U is trivial and that on W is $(\det W)^m$. Rectangular branching forces the $E \otimes W$ label to be the complement of γ in (m^L), leaving $\mathbb{S}_\gamma(Y \otimes U)$ on the other summand and degree $(L-1)m$ on E. The required multiplicity is $g(\alpha, \beta, \gamma)$ by (2). Restoring the twists gives (F, mQ), a dominant pair of the top weight. Thus $g(\alpha, \beta, \gamma) = g(mP, mQ, F)$ by Lemma 2.1 and (2).

For the second step, given $a, b, c \vdash n$, choose $r \geq \ell(b)$ and $q \geq \ell(c)$. For $K \geq \max(b_1, c_1, a_1)$ with $qK \geq n$, rectangular saturation in $(C \oplus A) \otimes (B \oplus E)$ with weights $(1, 0)$ on each factor gives

$$g(a, b, c) = g((K^{q(r+1)}),\ B_0,\ A_0), \tag{10}$$
$$A_0 = ((r+1)K - c_q, \ldots, (r+1)K - c_1, a),$$
$$B_0 = (qK + b_1, \ldots, qK + b_r, qK - n).$$

	U	W
X	$X \otimes U$ $(\mathrm{Sym}^m)^*$ 3	$X \otimes W$ 2
E	$E \otimes U$ 2	$E \otimes W$ 1
Y	$Y \otimes U$ 1	$Y \otimes W$ 0
Z	$Z \otimes U$ 0	$Z \otimes W$ Sym^m -1

FIGURE 5. The weight sectors of $A \otimes B$ in the proof of Proposition 4.1, with weights in the corners. The rectangle (m^L) is placed on the shaded sector of weight one.

Indeed, $\dim C = q$, $\dim B = r$ and E is a line. The sector $C \otimes B$ is filled with K; at the threshold remains $\mathbb{S}_{(K^q)}((A \otimes B) \oplus (C \otimes E))$. Its complementary C-label forces $\mathbb{S}_c(A \otimes B)$, giving (10). The C-block of A_0 is bounded below by $rK \geq a_1$, and $qK - n \geq 0$ ensures that B_0 is a partition. Their sizes are $q(r+1)K$ and their combined cocharacter weight is $2qrK + qK$, the top weight.

Apply (10) to $(a, b, c) = (F, mQ, mP)$, using $r = 2L, q = \ell(P)$. When $L = m$, take $K = q(r+1)$; since $q = 5L^2+2L+1$, this is at least $(5L+2)m$, which bounds $F_1, (mQ)_1, (mP)_1$. Also $qK \geq m|P|$ since $q = \ell(P)$ and $K \geq 3m$. The first argument is square and B_0 has at most three distinct parts. Alternatively take $K = (5L+2)m$ for fixed L, which satisfies the same bounds; all labels then scale linearly with m and the construction commutes with simultaneous stretching. The formulas recover F and then the three input partitions. □

The rectangular absorption in (10) works because the other summand of the middle sector supplies a complementary label. The same two-summand sector can be produced in an exterior square, now with $\bigwedge^2 A$ in place of $A \otimes B$. It follows that we can fix an outer square for an entire range of quadratic plethysms.

Proposition 4.2 (Square outer partitions)**.** *For every $M \geq 1$ put $Q_M = (5M^2 + M)/2$. Every coefficient of $s_\mu[h_2]$ with $|\mu| \leq M$ occurs in $s_{(Q_M^{Q_M})}[h_2]$ by an explicit polynomial-time map. Consequently a single square-outer quadratic plethysm contains the entire Kronecker table of S_m for each m.*

Proof. Let A, C, E have dimensions $2M, M, 1$ and weights $0, 1, -1$. In $\bigwedge^2(C \oplus A \oplus E)$ the positive sector is $\bigwedge^2 C \oplus (C \otimes A)$ of dimension $Q_M - M$; the zero sector is $\bigwedge^2 A \oplus (C \otimes E)$. With $Q = Q_M$, the square Schur functor therefore leaves $\mathbb{S}_{(Q^M)}$ on the zero sector. Rectangular branching selects $\mathbb{S}_\mu(\bigwedge^2 A)$ by the C-label complementary to μ. More explicitly, for $d = |\mu|$, $\lambda \vdash 2d$, the target concatenation in order C, A, E is

$$((3MQ - \mu_{M+1-i})_{i=1}^{M},\ (MQ + \lambda'_i)_{i=1}^{2M},\ MQ - d).$$

Here the determinant shifts before complementation are $(3M-1)Q$ on C and MQ on A. The concatenation is dominant, since $2MQ \geq \mu_1 + \lambda'_1$, and has the top weight. Its conjugate gives the desired coefficient by (6).

For the last assertion, apply (8) to $\bigwedge^2(A \oplus C \oplus E \oplus B)$, where A, B, C have dimension m, E is a line, and the weights are $(2, 1, -1, -2)$. The positive sector has dimension $2m^2$ and determinant $(\det A)^{2m}(\det C)^{2m-1}E^m$. The threshold is $(A \otimes B) \oplus (C \otimes E)$. With outer shape (m^{2m^2+m}), the top face leaves the rectangle (m^m) on the threshold. Rectangular branching selects $\mathbb{S}_\gamma(A \otimes B)$,

giving

$$g(\alpha,\beta,\gamma) = \langle s_{\Psi'}, s_{(m^{2m^2+m})}[h_2]\rangle, \quad \Psi = ((2m^2+\alpha_i)_i, (2m^2-\gamma_{m+1-i})_i, 2m^2-m, \beta). \tag{11}$$

All blocks are dominant and have the top weight. Apply the first assertion with $M = m(2m^2+m)$. □

Proposition 4.3 (Splitting tensor squares)**.** *For $\rho,\Gamma \vdash N$ write $s(\rho,\Gamma)$ and $a(\rho,\Gamma)$ for the multiplicities of $[\Gamma]$ in $\mathrm{Sym}^2[\rho]$ and $\bigwedge^2[\rho]$, respectively. For every $\alpha,\beta,\gamma \vdash m \geq 1$ there are ρ,Γ of common size $O(m^3)$, constructible in polynomial time in unary encoding, such that*

$$g(\alpha,\beta,\gamma) = s(\rho,\Gamma) - a(\rho,\Gamma).$$

Proof. Let $V = A \oplus B$ with $\dim A = p \geq m$, $p \equiv 2 \pmod 4$, $p \leq m+3$, and $\dim B = m$. Give the two blocks weights $1, 0$. Choose $\eta \vdash 2m$ with empty 2-core and 2-quotient $(\gamma, \varnothing)$. The classical 2-quotient specialization gives $s_\eta(X,-X) = \varepsilon s_\gamma(X^2)$, with $\varepsilon \in \{1,-1\}$; here X^2 denotes the alphabet of squared variables. Explicitly, for $r = pm$, take the decreasing rearrangement of the beta numbers $2(\gamma_i + r - i),\ 2(r-i)+1$ for $1 \leq i \leq r$, padding γ to length r. Subtract the staircase $(2r-1,\dots,0)$ to obtain η. Separating even and odd columns in the Schur alternant at $(X,-X)$ proves the specialization and computes ε by the resulting permutation sign relative to the denominator.

Choose $K \in \{2m, 2m+1\}$ with $(-1)^K\varepsilon = 1$, and put

$$\Gamma = (K^{p^2}, \eta), \qquad \rho = ((pK+\alpha_i)_{i=1}^p, \beta).$$

Both are partitions of $N = p^2K + 2m$. The maximal face of $\mathbb{S}_\Gamma(V \otimes V)$, of weight $2(p^2K+m)$, is by (8)

$$(\det(A\otimes A))^K \otimes \mathbb{S}_\eta((A\otimes B)\oplus(B\otimes A)).$$

The block isomorphism is equivariant for the tensor-factor swap τ, which preserves the sectors (Figure 6). For $h = (h_A, h_B)$ in the block Levi, take the trace of $\tau(h\otimes 1)$ on this face. On $A\otimes A$ its determinant is $(-1)^{\binom{p}{2}}(\det h_A)^p$; on the middle sector its eigenvalues occur in opposite pairs whose squares are the eigenvalues of $h_A \otimes h_B$. The trace is therefore

$$(-1)^{K\binom{p}{2}}\varepsilon(\det h_A)^{pK} s_\gamma(h_A\otimes h_B) = (\det h_A)^{pK} s_\gamma(h_A\otimes h_B).$$

Here $\binom{p}{2}$ is odd, so the two signs cancel.

In the decomposition (2), the twisted trace vanishes on pairs of distinct labels exchanged by τ. On the diagonal summand indexed by λ the swap on the Specht multiplicity space has trace $s(\lambda,\Gamma) - a(\lambda,\Gamma)$ (Schur–Weyl duality), and on the two Schur factors the twisted swap has trace $s_\lambda(h)$, by $\mathrm{tr}(\mathrm{swap}(T\otimes 1)) = \mathrm{tr}(T)$. The top weight of the diagonal Schur-factor pair is $2\sum_{i\leq p}\lambda_i$, which cannot exceed $2(p^2K+m)$ for an occurring summand. Thus only labels with $\sum_{i\leq p}\lambda_i = p^2K+m$ contribute on the maximal face; their twisted trace on the Schur factors is the product of the two block characters by (7). Comparing the coefficient of $s_{\alpha+(pK)^p}(h_A)s_\beta(h_B)$ on both sides using (2) gives the assertion. □

For further results on the two halves of tensor squares, see [2]. Here the trace on a saturated face extracts their imbalance.

4.1. **Contractions of classical multiplicities**

In Propositions 4.1 and 4.2, the auxiliary sectors transmit prescribed Schur labels. When a label is shared by two sectors but is not specified at the target, Schur's lemma sums over that label. This gives the following two identities.

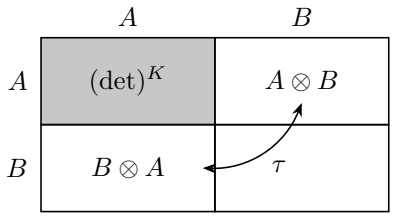

FIGURE 6. The top face in Proposition 4.3. The rectangle saturates the shaded sector $A \otimes A$, the label η lies on $(A \otimes B) \oplus (B \otimes A)$, and the swap τ exchanges these two cells.

Theorem 4.4 (Character-table row sums)**.** *For $\lambda \vdash m$, write $R_\lambda = \sum_{\rho \vdash m} \chi^\lambda(\rho)$. Define*

$$P = ((6m^2)^m, (2m^2)^m), \quad Q = ((5m^2)^m, (3m^2)^m),$$
$$\Gamma = ((4m)^{m^2-1}, 3m,\ 3m, (2m)^{m^2-1},$$
$$(m + \lambda_i)_{i=1}^m, m^{m^2-m},\ m^{m^2-1}, 0).$$

Then $R_\lambda = g(P, Q, \Gamma)$.

Proof. The character of the conjugation action on $\mathbb{C}[S_m]$ at an element of cycle type ρ is z_ρ, the order of its centralizer. Since the class has size $m!/z_\rho$, its contribution to the character inner product with χ^λ is exactly $\chi^\lambda(\rho)$. Decomposing $\mathbb{C}[S_m]$ as $\bigoplus_\eta [\eta] \otimes [\eta]^*$ gives $R_\lambda = \sum_{\eta \vdash m} g(\eta, \eta, \lambda)$. To contract this sum, take $A = U_1 \oplus U_2$, $B = V_1 \oplus V_2$ with all four blocks of dimension m and weights $(3, 0)$, $(1, 0)$. The sectors of $A \otimes B$ are in lexicographic order $11, 12, 21, 22$, of dimensions m^2. The four consecutive blocks of Γ select respectively

$$\mathrm{Sym}^m(U_1 \otimes V_1)^*, \quad \mathrm{Sym}^m(U_1 \otimes V_2), \quad \mathbb{S}_\lambda(U_2 \otimes V_1), \quad \mathrm{Sym}^m(U_2 \otimes V_2)^*,$$

up to determinant powers $4m, 2m, m, m$. The product of determinant characters has precisely the pair of weights (P, Q). The concatenations are dominant; all residual factors together have cocharacter weight zero. The multiplicity of (P, Q) on the face is therefore the invariant dimension of the displayed tensor product. By Cauchy on the three symmetric powers and Schur's lemma at the four blocks, all internal labels must coincide, leaving $\sum_\eta g(\eta, \eta, \lambda)$. Lemma 2.1 and (2) finish the proof. □

Replacing some of the Cauchy decompositions by Littlewood–Richardson branching on tied weight sectors gives the stable tensor-product multiplicities of the classical orthogonal and symplectic groups [11].

Theorem 4.5 (Newell–Littlewood numbers)**.** *For arbitrary partitions α, β, γ, there are explicitly constructible partitions P, Q, Γ of a common size polynomial in $|\alpha| + |\beta| + |\gamma|$ such that*

$$N_{\alpha,\beta,\gamma} := \sum_{\delta,\epsilon,\eta} c^\alpha_{\delta,\epsilon} c^\beta_{\delta,\eta} c^\gamma_{\epsilon,\eta} = g(P, Q, \Gamma).$$

When $|\alpha| + |\beta| + |\gamma|$ is even, the construction can be chosen to satisfy $N_{t\alpha,t\beta,t\gamma} = g(tP, tQ, t\Gamma)$ for all positive integers t.

Proof. Let d_1, d_2, d_3 be half of $|\alpha| + |\beta| - |\gamma|$, $|\alpha| + |\gamma| - |\beta|$, $|\beta| + |\gamma| - |\alpha|$, respectively. If any is negative, all stretches vanish, so take the triple $((2), (2), (1, 1))$, which remains zero on stretching. If the total size is odd, use the same zero coefficient for the unstretched input. For the all-empty triple take $((1), (1), (1))$, which stays one on stretching. In the remaining case the d_i are nonnegative integers; set $r = \max(1, \ell(\alpha), \ell(\beta), \ell(\gamma))$, $M = \max(|\alpha|, |\beta|, |\gamma|) > 0$.

Take $A = U_1 \oplus U_2 \oplus U_3 \oplus L_4$, $B = V \oplus L_6 \oplus L_2 \oplus L_3 \oplus L_1 \oplus L_5$, where the U_i, V have dimension r and the L_i are lines. The respective cocharacter weights are $(40, 30, 10, 0)$ and $(50, 22, 15, 10, 5, 2)$. The following table gives the levels in $A \otimes B$; the four marked levels are the active ones.

	V	L_6	L_2	L_3	L_1	L_5
U_1	90	62	55	**50**	**45**	42
U_2	80	52	**45**	40	35	**32**
U_3	**60**	**32**	25	20	15	12
L_4	**50**	22	15	10	5	2

Write Y_w for the full sector of weight w. To construct Γ, assign each sector the following constant row shift (omitted sectors have shift zero):

$6M$	$90, 80, 62$
$4M$	$60, 55, 52, 50$
$2M$	45
M	$42, 40, 35, 32.$

Add to these constant blocks, at levels $60, 50, 45, 32$ respectively, the padded highest weights of

$$\mathrm{Sym}^{d_3} Y_{60}, \quad (\mathbb{S}_\beta Y_{50})^*, \quad \mathbb{S}_\alpha Y_{45}, \quad (\mathbb{S}_\gamma Y_{32})^*.$$

For a dual weight use the negative reversed parts. Concatenate in decreasing weight order. This defines Γ explicitly. Since the active labels have size at most M, the blocks are nonnegative and dominant, including at all adjoining constant blocks. The resulting face of $\mathbb{S}_\Gamma(A \otimes B)$ is the product of the four active factors, twisted by the determinant character specified by the table. Its two target weights are

$$\begin{aligned} P &= (((6r+17)M)^r, ((6r+9)M)^r, ((4r+1)M)^r, 4rM - d_3), \\ Q &= (((16r+4)M)^r, 11rM - d_3, 6rM + d_2, 5rM - d_1, \\ &\qquad\qquad 3rM + d_1, 2rM - d_2). \end{aligned}$$

Indeed the determinant exponents are obtained by summing the table along the row and column blocks with the opposite block dimensions. The line excesses in the order $L_4, L_6, L_2, L_3, L_1, L_5$ are $-d_3, -d_3, d_2, -d_1, d_1, -d_2$. The weights are dominant since $0 \le d_i \le M$; the sum of their parts is $(16r^2 + 31r)M - d_3$, the same as $|\Gamma|$. The excesses also match the cocharacter degree of the four active factors, so Lemma 2.1 applies.

After removing the determinant character, the four active factors have Schur labels on the following spaces:

$$\begin{aligned} &\mathrm{Sym}^{d_3}(U_3 \otimes V), \qquad \mathbb{S}_\beta(U_1 \otimes L_3 \oplus V \otimes L_4)^*, \\ &\mathbb{S}_\alpha(U_1 \otimes L_1 \oplus U_2 \otimes L_2), \quad \mathbb{S}_\gamma(U_2 \otimes L_5 \oplus U_3 \otimes L_6)^*. \end{aligned}$$

Cauchy on the first factor supplies a common label η of size d_3 on U_3, V. Branching the remaining three factors and contracting on U_1 and U_2 identifies their two other labels, say δ of size d_1 and ϵ of size d_2. Contraction on U_3, V identifies both remaining labels with η. The line characters impose exactly these sizes, so the product of the three branching multiplicities is $c^\alpha_{\delta,\epsilon} c^\beta_{\delta,\eta} c^\gamma_{\epsilon,\eta}$. Summing gives the claimed multiplicity (Figure 7). Since r is unchanged by stretching and M, d_i scale with t, so do all three output partitions.

□

When $|\alpha| + |\beta| = |\gamma|$, this is the usual Littlewood–Richardson coefficient $c^\gamma_{\alpha,\beta}$. Theorems 4.4 and 4.5 thus extend the domain of the cubic realization without changing its proof. Their constructions use different contractions of Schur labels in a single weight face. Section 6 develops a general contraction construction for mixed tensor invariant algebras.

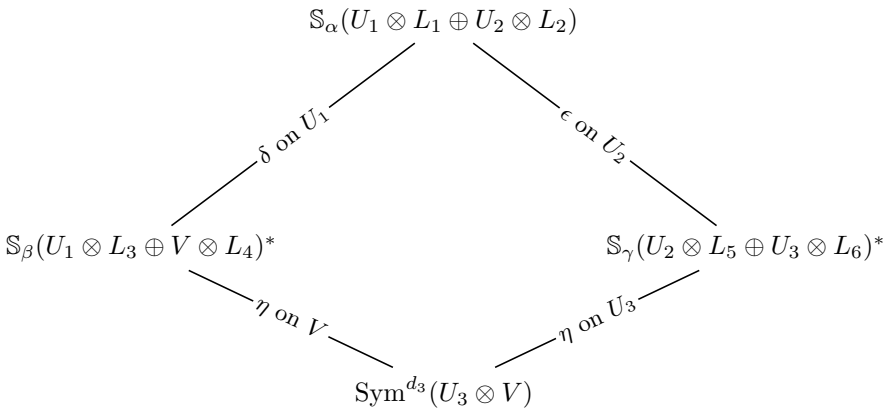

FIGURE 7. The contraction in the proof of Theorem 4.5. Each edge is a label shared by two active factors on the indicated space; summing over δ, ϵ, η gives $N_{\alpha,\beta,\gamma}$.

In particular, for Stanley's Problems 9, 10 and 12 [26], the reductions give the implications

$$h_D[h_3] \text{ coefficients in } \#\mathsf{P} \implies g \in \#\mathsf{P} \implies R \in \#\mathsf{P}.$$

All the maps so far have polynomial unary overhead. Exact computation of Kronecker coefficients is #P-hard under parsimonious many-one reductions, and testing whether a given coefficient is positive is NP-hard [13, 14]. The corresponding hardness statements transfer to the universal subclasses. The cubic case was already established in [10].

For fixed L, Proposition 4.1 also preserves the stretching functions in its rectangular version. Via the quantum marginal correspondence [5], it gives a reduction of asymptotic Kronecker positivity to spectra with a flat global spectrum and a prescribed marginal spectrum (for each input size and length bound).

5. From multiplicities to graded algebras

The contractions of Section 4 identify multiplicity spaces by matching Schur labels. Some of the identifications even hold after simultaneous stretching. It is natural to ask what happens to a product of vectors in two different stretching degrees. The determinant twists are additive in the labels, and Lemma 2.1(iii) makes the face identifications compatible with Cartan multiplication. The issue is to arrange that the weight sectors themselves stay fixed as the degree varies. We return to the opening cubic construction with this requirement.

For partitions of a common size, define the Kronecker stretching algebra

$$\mathcal{K}_{\alpha,\beta,\gamma} = \bigoplus_{t \geq 0} \mathrm{HW}_{t\alpha,t\beta}\, \mathbb{S}_{t\gamma}(A \otimes B),$$

where the dimensions of A, B are at least the lengths of α, β. Here HW denotes the indicated highest-weight space, and multiplication is the Cartan product. Its Hilbert function is $g(t\alpha, t\beta, t\gamma)$. For a Schur plethysm label pair (κ, Λ), write $\mathcal{P}^{(3)}_{\kappa,\Lambda} = \bigoplus_t \mathrm{HW}_{t\Lambda}\, \mathbb{S}_{t\kappa}(\mathrm{Sym}^3 V)$ with the analogous multiplication. The Cartan product can be realized in the polynomial Cartan algebra $\bigoplus_\lambda \mathbb{S}_\lambda E$ (the highest-weight algebra of a matrix coordinate ring), so it is associative and the algebra is a domain. Enlarging A, B does not change the stretching algebra: in the larger spaces the desired weight has zero coordinates in the added dimensions, so its vectors use only the original tensor factors; the extra raising operators annihilate them. This agrees with the Cartan product.

The exterior realization in Theorem 1.1 fills each higher sector once. To preserve the multiplication, fill each sector by a rectangle whose width is the common size of the three labels.

We give the construction for linear families, since we will need to keep track of several degrees simultaneously.

Proposition 5.1 (Simultaneous rectangular cubic realization)**.** *Let $(\alpha(\xi), \beta(\xi), \gamma(\xi))$ be any integral linear family of partition triples of equal size $n(\xi)$, with fixed length bounds. The corresponding slice of a three-tensor highest-weight algebra is isomorphic, respecting its grading, to a slice of the Cartan algebra of $\mathrm{Sym}^3 V$ whose outer Schur labels are rectangular.*

Proof. Pad the three factors to the same dimension $d \geq 1$; call them A, B, C. In $V = A \oplus B \oplus C$ take weights $(3, 1, 0)$. The weight-four sector of $Y = \mathrm{Sym}^3 V$ is $A \otimes B \otimes C$, and the higher sectors are

$$\mathrm{Sym}^3 A, \quad \mathrm{Sym}^2 A \otimes B, \quad \mathrm{Sym}^2 A \otimes C, \quad A \otimes \mathrm{Sym}^2 B.$$

Their sum $Y_>$ has dimension $H = \binom{d+2}{3} + 3d\binom{d+1}{2}$, with determinant exponents

$$(u, v, w) = \left(3d^2 + 4d + 1,\ 3\binom{d+1}{2},\ \binom{d+1}{2}\right)$$

on A, B, C. Put

$$\kappa(\xi) = (n(\xi)^{H+1}),$$
$$\Lambda(\xi) = ((un + \alpha_i)_{i=1}^d, (vn + \beta_i)_{i=1}^d, (wn + \gamma_i)_{i=1}^d).$$

These are partitions (with Λ of size $3|\kappa|$). The maximal face of $\mathbb{S}_\kappa Y$ is $(\det Y_>)^n \otimes \mathrm{Sym}^n(A \otimes B \otimes C)$ by Lemma 2.1. Removing the twist leaves precisely the original three highest weights. The maps are compatible with Cartan multiplication, and the padding is linear in ξ. This proves the assertion for the entire slice, including its zero components. Enlarging the tensor factors before the construction does not change the polynomial highest-weight spaces, since the labels have zero entries in the new dimensions and the corresponding vectors use only the original tensor cells. □

Theorem 5.2 (Rectangular cubic realization)**.** *For every $\alpha, \beta, \gamma \vdash m$, there are a rectangle κ and a partition Λ of $3|\kappa|$, constructible with polynomial unary overhead, such that*

$$\mathcal{K}_{\alpha,\beta,\gamma} \cong \bigoplus_{t \geq 0} \mathrm{HW}_{t\Lambda}\, \mathbb{S}_{t\kappa}(\mathrm{Sym}^3 V) \tag{12}$$

as graded algebras. In particular, $g(t\alpha, t\beta, t\gamma) = \langle s_{t\Lambda}, s_{t\kappa}[h_3] \rangle$ for every $t \geq 0$.

Proof. Apply Proposition 5.1 to the ray $(t\alpha, t\beta, t\gamma)$ with $d = \max(\ell(\alpha), \ell(\beta), \ell(\gamma), 1)$. By Cauchy applied to $(A \otimes B) \otimes C$, the C-highest-weight algebra of these degrees is $\bigoplus_t \mathbb{S}_{t\gamma}(A \otimes B)$ with Cartan multiplication: its degree products are the unique Cartan maps up to scalar, and the highest-weight lines multiply nontrivially in the symmetric algebra. Choosing the line in degree one and its powers fixes these scalars compatibly. Taking the remaining highest weights identifies this ray with $\mathcal{K}_{\alpha,\beta,\gamma}$. The formulas of the proposition give $|\kappa| = m(H + 1) = O(md^3)$, so the unary overhead is polynomial. □

The difference between the two cubic constructions is worth noting. In Theorem 1.1 the dimension of the filled space changes with m. Here the dimensions stay fixed and only the width of the rectangle varies. Consequently the determinant characters multiply in the degree, and the surviving symmetric powers carry their ordinary multiplication.

6. From mixed tensors to three tensors

The cubic construction has now reduced the algebra realization problem to a question about three-tensor highest-weight algebras. To go further, consider a general tensor product with standard and dual factors, and the polynomial algebra it generates. A tensor of this kind need not have just three factors. But a rectangular Schur module on a direct sum gives a polynomial

algebra on a tensor product involving a dual. If two such interactions have an auxiliary factor in common, one covariant and the other contravariant, taking invariants contracts that factor. Repeating this operation suggests that we can build a tensor of any order from two-cell sectors of a single tensor grid.

Two constraints must be met for this idea to give a graded algebra. The sectors must be weight spaces for one additive cocharacter, and the degree bounds of their rectangular charts must be uniform over the original grading. We address these constraints in turn, after recording the rectangular chart and stating the precise realization theorem.

Fei [8, 9] relates Kronecker multiplicities to semi-invariants and cluster structures for flagged Kronecker quivers. Meir constructs a universal ring that specializes to invariant rings of mixed tensor data as the underlying dimension varies [20], and studies symmetric-group descriptions of their Hilbert functions [19]. The question here is one of realization inside a specific branching algebra: for a given mixed tensor source, we seek an isomorphism onto a linear slice of an ordinary three-tensor Kronecker algebra. The prescribed grading, rather than just the dimensions of its components, is part of the data.

6.1. **Rectangular sections**

A Schur face is a tensor product of Schur modules of weight sectors, so dual factors do not appear on it directly. A rectangular Schur module on a direct sum of two sectors supplies them through a convenient affine chart.

Lemma 6.1 (Rectangular sections)**.** *Let Y, Z be vector spaces, $q = \dim Z > 0$. For every $b \geq 0$ there is a $\mathrm{GL}(Y) \times \mathrm{GL}(Z)$-equivariant injection*

$$J_b : \mathbb{S}_{(b^q)}(Y \oplus Z) \otimes (\det Z)^{-b} \longrightarrow \mathrm{Sym}(Y \otimes Z^*).$$

Its image contains every polynomial of degree at most b. Under the Cartan products and the indicated determinant twists, $J_{b+b'}(uv) = J_b(u)J_{b'}(v)$.

Proof. Use the rectangular subalgebra of the Cartan algebra of $Y \oplus Z$, generated by $\bigwedge^q(Y \oplus Z)$ with the Plücker relations. Let Δ be the Plücker coordinate of a basis wedge of Z. On the chart $\Delta \neq 0$, ratios of single replacements of a basis vector of Z by a vector of Y give independent matrix entries transforming as $Y \otimes Z^*$. All other normalized Plücker coordinates are minors of this matrix. Dividing a degree-b element by Δ^b therefore gives a polynomial in these entries. The chart is dense, so this map is injective. Every monomial of degree at most b is obtained by multiplying at most b single-replacement coordinates and filling the remaining factors with Δ. Division is multiplicative and removes the character $(\det Z)^b$. Equivalently, the image has Cauchy summands $\mathbb{S}_\lambda Y \otimes \mathbb{S}_\lambda Z^*$ with $\lambda \subseteq (b^q)$, which include all summands of degrees at most b. □

6.2. **The realization theorem**

All vector spaces and algebras are over $\mathbb{C}$. Let

$$G = \prod_{i=1}^{s} \mathrm{GL}(V_i), \qquad M = \bigoplus_{a=1}^{N} M_a, \qquad M_a = \bigotimes_{j=1}^{k_a} V_{i(a,j)}^{\varepsilon_{a,j}}, \quad V^+ = V, \quad V^- = V^*.$$

Repeated factors are allowed. A tensor of order zero means a trivial line. A commuting torus T is allowed to act on each M_a by a character. This includes separate polynomial degrees for the summands, as well as their linear combinations. We take $\mathrm{Sym}(M)$ rather than $\mathbb{C}[M]$ as our convention; the latter is $\mathrm{Sym}(M^*)$, which belongs to the same class.

Put $U = \prod_i U_{V_i}$. The algebra $\mathrm{Sym}(M)^U$ is graded by the lattice Λ of integral weights of the diagonal tori of G and the characters of T. We write H_λ for its component of weight λ; a nondominant general linear weight gives the zero space. Let L be a lattice, $\sigma \subset L_{\mathbb{R}}$ a pointed

rational polyhedral cone spanning $L_{\mathbb{R}}$, and $j : L \hookrightarrow \Lambda$ an integral linear injection. Assume that the general linear parts of $j(\xi)$ are dominant for $\xi \in \sigma$, and that

$$R = \bigoplus_{\xi \in \sigma \cap L} H_{j(\xi)}, \qquad R_0 = \mathbb{C},$$

with each component finite-dimensional. We refer to this as a positively graded slice. The lattice can be restricted to the span of the support if desired. For G-invariants one prescribes zero general linear weights; for SL-invariants one prescribes determinant weights and retains their character grading. For a grading on $\mathrm{Sym}(M)^G$ induced by T, take the cone generated by its support and its saturated lattice in the character lattice. Positivity means that a linear functional is strictly positive on the nonzero support. These conventions specify exactly which multigradings are retained.

Theorem 6.2. *For every positively graded slice R as above there exist a positive integral linear functional ℓ on $\sigma \setminus \{0\}$, fixed partitions P, Q of the same size with P a square, and an integral linear map Γ from L to a fixed partition-label lattice, such that for every $\xi \in \sigma \cap L$,*

$$F(\xi) = \big(\ell(\xi)P, \ell(\xi)Q, \Gamma(\xi)\big)$$

is a triple of partitions of equal size. The map F is injective, and for fixed finite-dimensional spaces A, B, C there is an isomorphism of L-graded algebras

$$R \;\cong\; \bigoplus_{\xi \in \sigma \cap L} \mathcal{B}(A, B, C)_{F(\xi)}. \tag{13}$$

In particular,

$$\dim R_\xi = g\big(\ell(\xi)P, \ell(\xi)Q, \Gamma(\xi)\big). \tag{14}$$

The formula holds also at every lattice hole of the support. The isomorphism preserves the given grading and multiplication, without dilation or passage to a Veronese algebra.

Here stretching a square refers to multiplying its parts; its height stays fixed. Thus $\ell(\xi)P$ is rectangular for every positive degree. Neither the ranks nor P, Q depend on ξ. The construction is explicit once a finite homogeneous generating set for the source slice, or a bound on its fine degrees, is supplied. No polynomial bound on these ranks in the tensor order is asserted.

Corollary 6.3. *Every compatibly positively singly graded mixed tensor invariant algebra $R = \mathrm{Sym}(M)^G$ with $R_0 = \mathbb{C}$ and finite-dimensional pieces is isomorphic to a Kronecker stretching algebra*

$$R \cong \mathcal{K}_{P',Q',\gamma} := \bigoplus_{t \geq 0} \mathrm{HW}_{tP',tQ'}\, \mathbb{S}_{t\gamma}(A \otimes B),$$

with a rectangular first label. For a primitive positive degree parameter on a ray, the rectangle may be chosen square at degree one by taking $\ell(t) = t$.

If the support consists only of degree zero, the same assertion follows, for example, from $g((2t, 2t), (4t), (3t, t)) = 0$ for $t > 0$. Otherwise use the theorem on the full degree ray (including its holes). The proof below also permits any chosen positive integral ℓ, so its value on the positive generator can be one.

6.3. **Contractions and additive cell weights**

The rectangular section lemma leaves a bound on the polynomial degree of each interaction. We will eventually choose this bound uniformly in the prescribed grading. For the moment, we work with the full polynomial algebras, so that the contractions can be described without reference to the target Schur labels.

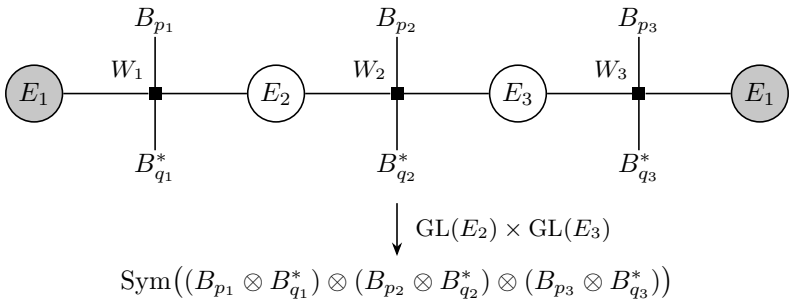

FIGURE 8. Lemma 6.4 for $r = 3$. Each interaction W_j joins E_j and E_{j+1} and carries the legs B_{p_j}, $B^*_{q_j}$. Taking invariants of the internal rows fuses the three interactions; the shaded row E_1 is a line.

6.3.1. *Contraction through a cycle*

The following elementary consequence of the first and second fundamental theorems for GL will be useful. For spaces D, E, F, the invariant algebra

$$\big(\mathrm{Sym}(D \otimes E^*) \otimes \mathrm{Sym}(E \otimes F)\big)^{\mathrm{GL}(E)} \tag{15}$$

is generated by the pairwise contractions, parametrized by $D \otimes F$. The relations are the minors of size $\dim E + 1$ of the contraction matrix. In particular, if $\dim E \geq \min(\dim D, \dim F)$, the natural contraction map from $\mathrm{Sym}(D \otimes F)$ is an isomorphism. One can see the absence of relations by factoring every matrix of the indicated size through E. The isomorphism respects the two polynomial degrees on the left: a degree-d polynomial on the right has degree d in both factors. We use this standard invariant-theoretic form of matrix multiplication [29, 24].

Let $B_1, \ldots, B_m$ be a list of spaces, with repetitions in subsequent tensor products allowed. Consider a cycle of r private row spaces E_j, with $E_{r+1} = E_1$, and two lists of column indices p_j, q_j. Form the interactions

$$W_j = (E_j \otimes B_{p_j}) \otimes (E_{j+1} \otimes B_{q_j})^*. \tag{16}$$

Choose E_1 a line. Choose the remaining row dimensions large enough for successive contractions (the product of all external dimensions $\dim B_{p_j} \dim B_{q_j}$ is a sufficient bound for each).

Lemma 6.4. *There is a column-group equivariant algebra isomorphism*

$$\left(\bigotimes_{j=1}^{r} \mathrm{Sym}(W_j)\right)^{\prod_j \mathrm{GL}(E_j)} \cong \mathrm{Sym}\left(\bigotimes_{j=1}^{r} (B_{p_j} \otimes B^*_{q_j})\right).$$

The degree on the right equals the degree in each interaction on the left (Figure 8).

Proof. Eliminate $E_2, E_3, \ldots, E_r$ in turn by (15), combining the two incident factors into a single polynomial algebra at each step. The dimension bound ensures there are no determinantal relations in these steps: the already combined prefix, apart from its last internal factor, has dimension at most the product bound since E_1 is a line. The final tensor has factors E_1 and E_1^*, which cancel canonically. The remaining row action is trivial. The degree assertion follows from the central scalar actions of the row groups. The construction is equivariant for the external groups before identifying any repeated column spaces, and hence also after such identification. □

6.3.2. *Balancing by lines*

The contraction lemma is independent of how the two cells of an interaction are situated in a weight diagram. To situate them on one cocharacter face, write the formal row weights as a_j and

the column weights as p_j, q_j. Equal weights of the two cells require $a_j + p_j = a_{j+1} + q_j$. Summing around the cycle forces $\sum_j p_j = \sum_j q_j$. An arbitrary mixed tensor need not satisfy this identity. We enforce it by adding lines, whose character twists can be recovered from the prescribed highest weights.

We now specify the column spaces and their formal weights. First include the standard spaces of G and the lines corresponding to a basis of characters of the extra torus T. Thus the tensor factors for a summand can incorporate its character of T using repeated lines and dual lines. Refer to these as the original columns B_i, and use independent formal weights b_i for them. Add a shadow line D_i of weight $b_i + \delta$ for every original column, and two lines S, S' of weights $s, s + \delta$. The symbols s, δ, b_i are independent. Replace each original factor B_i by

$$B_i \otimes S' \otimes D_i^* \otimes S^*, \tag{17}$$

and each dual factor by the dual of this expression. These replacements introduce only one-dimensional character twists. The sum of the formal weights of the covariant factors equals that of the contravariant factors in each replacement. An empty tensor can first be expressed as $B_0 \otimes B_0^*$ with B_0 a private line of prescribed weight zero in the representation (its formal cell weight is independent). If necessary, such a line and its shadow are included among the columns. It adds no nontrivial action to the source.

In a prescribed original highest weight $(\lambda_i)_i$, let $w_i = \sum_j (\lambda_i)_j$, where for a line this is simply its character exponent. The induced shadow weights are

$$D_i : -w_i, \qquad S : -\sum_i w_i, \qquad S' : \sum_i w_i. \tag{18}$$

These weights depend only on the original highest weights and characters, even when the separate tensor degrees are not prescribed. This follows because the scalar center acts on each mixed tensor by its net number of standard factors, and the extra characters in (17) use exactly those same numbers.

Write $\widetilde{\lambda}$ for the combined list of original and shadow weights, with zero weights on further auxiliary columns. It satisfies two identities, as a formal identity in the cell weights $b(B)$:

$$\sum_B |\widetilde{\lambda}_B| = 0, \qquad \sum_B b(B)|\widetilde{\lambda}_B| = 0, \tag{19}$$

where $|\lambda|$ means the sum of the entries, also for rational weights. Both follow immediately from (18).

6.3.3. *Two cells of the same weight*

The balance identity allows the row weights to be defined recursively. It does not yet rule out a third cell at the same weight, which would introduce an unwanted tensor factor into a rectangular section. We separate the intended pairs by introducing independent marker coordinates (Figure 9). The key is to use each marker once on either side of the contraction, so it disappears from the resulting tensor.

Lemma 6.5 (Marker cycle)**.** *Suppose $P_1, \dots, P_h$ and $Q_1, \dots, Q_h$ are lists from a finite set of distinct formal column weights, with $h \geq 2$, $\sum P_i = \sum Q_i$ and $P_1 \neq Q_h$. Enlarge the columns by independent marker lines of weights $z_1, \dots, z_h$. There exist $2h$ distinct row weights and two column lists p, q such that $a_j + p_j = a_{j+1} + q_j$ cyclically, every chosen color has exactly these two cells, and all chosen colors are distinct. The resulting tensor in Lemma 6.4 has precisely the original P factors and dual Q factors, with marker characters cancelling.*

Proof. Set

$$p = (z_1, \dots, z_h, P_1, \dots, P_h), \quad q = (Q_1, \dots, Q_h, z_1, \dots, z_h), \quad a_1 = 0, \quad a_{j+1} = a_j + p_j - q_j.$$

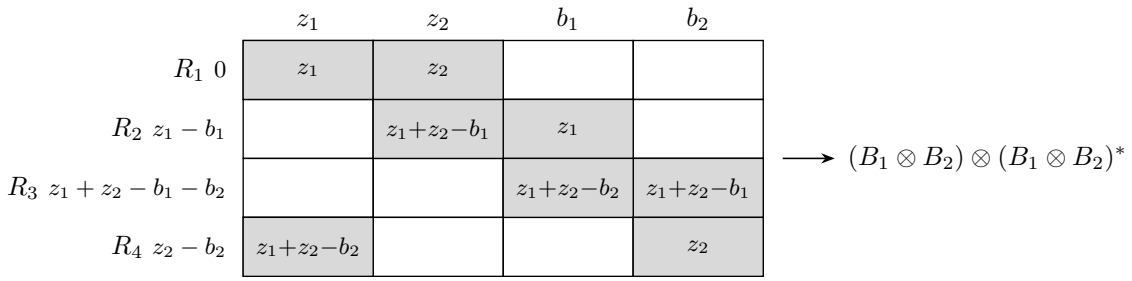

FIGURE 9. The marker cycle of Lemma 6.5 for $h = 2$ and $P = Q = (b_1, b_2)$. Each interaction weight occupies exactly the two shaded cells (R_j, p_j) and (R_{j+1}, q_j); contracting around the cycle leaves the tensor on the right.

The row cycle closes by the sum condition. The marker supports of the $2h$ rows are the proper prefixes of $\{1, \dots, h\}$ (including the empty prefix) and its nonempty suffixes (including the full suffix). They are distinct for $h \geq 2$. A selected color in the first half has marker support a nonempty prefix; one in the second half has support a nonempty suffix. For either type, a cell with a nonmarker column must be on the unique row with precisely that marker support, and its column is uniquely determined. A cell with a marker column must use a row whose support is obtained by deleting that marker, with no repeated marker coordinate. For a proper prefix or proper suffix there is exactly one such row, including the empty row for a singleton. At full support there are exactly two candidate marker cells, from the prefix of length $h - 1$ and the suffix of length $h - 1$. The formal weights of the two full-support selected colors differ by $P_1 - Q_h$, so these candidates belong to distinct sectors. This proves exact isolation, including distinctness of the selected colors. The markers occur once on each side of the contracted tensor and are lines, so cancel equivariantly. □

For each summand, the balanced tensor of (17) has at least two positive and two negative factors. Their weights can be ordered with $P_1 \neq Q_h$: in a single replacement the positive list contains two distinct column weights, so a positive entry different from a chosen negative entry exists. The same holds for dual replacements. Apply Lemma 6.5 separately to every summand, using private markers and row spaces. An independent formal shift added to every row of a given cycle separates its colors from those of all other cycles. Marker lines of other cycles cannot create an additional cell of a selected color, since they would introduce a foreign marker coordinate. Choose a final extra row E_* of dimension one, with an independent shift. Every cell of this row is isolated from every other cell in its own sector, since the column weights are distinct and the shift does not occur elsewhere. Specialize all formal parameters simultaneously to integers off the finite union of hyperplanes corresponding to forbidden coincidences. Such a specialization exists (one may substitute distinct sufficiently large powers of an integer). The resulting actual cocharacter has distinct row weights, distinct column weights, exactly two cells in each interaction sector, and isolated cells in the extra row. Order the rows and columns by decreasing weights.

6.4. **The graded algebra realization**

The cycles give the desired mixed tensor representation, and the marker construction places its interactions in separate sectors of one tensor grid. It remains to choose their rectangular labels, and to impose the original highest weights. Both steps must be uniform in the degree: a separate choice of padding at each degree would lose the algebra structure. The padding observation of Section 2 has a uniform form for this purpose.

If $L = 0$, Theorem 6.2 is the degree-zero realization of $\mathbb{C}$ by the empty partition triple, with $\ell = 0$ and any fixed square P and equal-size Q. Assume henceforth $L \neq 0$. Let the degrees ξ

range over the integral points of the pointed rational polyhedral cone σ, and choose an integral linear functional ℓ strictly positive on its nonzero points. Suppose rational Schur factors $\mathbb{S}_{\eta^{[c]}(\xi)}E_c$ are desired, with dominant weights whose entries are linear in ξ. Let K_c be positive decreasing integers. Adding $K_c\ell(\xi)$ to every part of $\eta^{[c]}(\xi)$ makes their concatenation a polynomial partition if the gaps and the smallest shift are sufficiently large. Indeed, on the compact set $\{\xi \in \sigma : \ell(\xi) = 1\}$ all the entries are bounded. This realizes the desired factors up to the determinant character

$$\prod_c (\det E_c)^{K_c\ell(\xi)}. \tag{20}$$

The determinant character is known and linear in the degree. This observation applies equally to unused sectors by putting their rational labels equal to zero.

6.4.1. *Uniform capacity*

The algebra $\mathrm{Sym}(M)^U$ is finitely generated, by finite generation for maximal unipotent invariants [12]. The slice R is also finitely generated, with generators homogeneous in both its given degrees and the separate polynomial degrees of all tensor summands. Here is one way to see this: choose generators of the highest-weight algebra homogeneous for the full torus and fine degree grading. Their monomials whose full weights lie in $j(L) \cap j(\sigma)$ span the slice. The set of their exponent vectors is a rational polyhedral cone intersected with an integral lattice (allowing finite-index sublattices), and is a finitely generated monoid by Gordan's lemma. The images of its monomial generators generate R. The assumption $R_0 = \mathbb{C}$ ensures that any nonconstant homogeneous generator has positive ℓ-degree. Consequently there is an integer $M_0 \geq 1$ such that, for all $\xi \in \sigma \cap L$, every fine polynomial degree of a nonzero homogeneous vector of R_ξ is at most $M_0\ell(\xi)$. To obtain M_0 bound the ratios of the generator degrees in every summand to their positive ℓ-degrees; the inequalities extend to products and hence to the components they span. Set

$$b(\xi) = M_0\ell(\xi). \tag{21}$$

Here the bound concerns all fine homogeneous components, including when several fine degrees contribute to a single R_ξ. The same bound applies to every interaction in the corresponding cycles, by Lemma 6.4. Thus rectangular sections of capacity $b(\xi)$ suffice for every contributing fine degree, not just for a dense open set of parameters.

6.4.2. *Choosing the Schur factors*

Let A be the direct sum of the row spaces and B the direct sum of the column spaces constructed in Section 6.3. For each selected interaction sector write $Y_c \oplus Z_c$, with $Y_c = E_j \otimes B_{p_j}$, $Z_c = E_{j+1} \otimes B_{q_j}$, and put $q_c = \dim Z_c$. Prescribe the rectangular Schur factor

$$\mathbb{S}_{(b(\xi)^{q_c})}(Y_c \oplus Z_c).$$

Remove its twist $(\det Z_c)^{b(\xi)}$ using Lemma 6.1.

The extra row E_* serves a different purpose. In its cell $E_* \otimes B_j$ prescribe the rational Schur module of weight $\widetilde{\lambda}_{B_j}(\xi)^\vee$, where $\widetilde{\lambda}(\xi)$ is the weight list of (18) obtained from $j(\xi)$. The marker columns have zero weight, so their factors are trivial. On all other sectors prescribe the zero label. The sum of the weights in the extra row is zero by (19), so its general linear group acts trivially on the tensor product of its prescribed rational factors. The same two identities imply that the total column scalar weight and the column cocharacter weight contributed by these dual Schur modules are zero. The interaction factors after removal of their twists have total row and column scalar weights zero; each has cocharacter weight zero by the selected equal-cell condition.

Now choose positive decreasing constants K_c, one for every sector, with sufficiently large gaps, and add the shifts $K_c\ell(\xi)$ as in (20). The concatenated Schur labels give a polynomial partition $\gamma_0(\xi)$, linear in ξ. The signed entries of the dual Schur labels cause no problem, since all entries

are linear and bounded on the compact section $\ell = 1$. Combining the global padding and the reservoir twists gives a character of the product of all row and column groups,

$$\chi(\xi) = \prod_c (\det E_c^{\mathrm{sec}})^{K_c \ell(\xi)} \prod_{c\,\text{selected}} (\det Z_c)^{b(\xi)}, \tag{22}$$

where E_c^{sec} is the entire cell sector (not a row space). This character depends on ξ only through $\ell(\xi)$. Write its row and column exponents as constant blocks and concatenate in the order of the row and column cocharacters to obtain $\ell(\xi)P_0$ and $\ell(\xi)Q_0$.

We check that these labels are partitions. For the padding part alone, the exponent at a row of weight a_i is $\sum_j \dim(B_j)K(a_i + b_j)$, where $K(w) := K_c$ for the sector c of weight w. Thus $K(w)$ is strictly increasing in w. Hence these exponents strictly increase with a_i; the analogous column statement follows by interchanging rows and columns. By first choosing gaps for the Schur labels and then multiplying every padding constant by a common sufficiently large integer, the padding differences dominate the fixed reservoir corrections. All row and column exponents are positive. Thus P_0, Q_0 are fixed polynomial partitions. Scalar degree balance, or the action of the scalar matrices on the maximal face, gives

$$|\gamma_0(\xi)| = \ell(\xi)|P_0| = \ell(\xi)|Q_0|. \tag{23}$$

In particular the common size is linear in $\ell(\xi)$, although the individual parts of $\gamma_0(\xi)$ carry all the original weight data.

6.4.3. *Taking invariants after removing the twists*

Lemma 2.1 identifies the top cocharacter face of $\mathbb{S}_{\gamma_0(\xi)}(A \otimes B)$, after removing $\chi(\xi)$, with the product of the rectangular sections and the extra-row rational factors. Its row and column trivial-isotypic part is precisely the highest-weight space with labels $\ell(\xi)P_0, \ell(\xi)Q_0$ in the original module. To justify this, the residual highest-weight labels are zero on each general linear block, which requires the trivial representation. Their combined cocharacter weight is zero by the construction and (19). Thus the specified weight lies on the top face (whether or not the desired space is nonzero), and all raising operators between distinct blocks vanish there. Conversely a vector of this prescribed weight in the full Schur module must lie in that face. This gives the asserted identification, not merely an injection.

Dehomogenize the reservoirs and momentarily allow the full polynomial algebras of the interactions. Take invariants under the original row groups. Lemma 6.4 produces exactly $\mathrm{Sym}(M)$ with the one-dimensional twists (17). Taking the remaining column invariants against the extra-row factors extracts the prescribed highest weights. More precisely, for a rational $\mathrm{GL}(B_j)$-module V,

$$(V \otimes \mathbb{S}_{\lambda^\vee} B_j)^{\mathrm{GL}(B_j)} \cong \mathrm{Hom}_{\mathrm{GL}(B_j)}(\mathbb{S}_\lambda B_j, V) \cong V_\lambda^{U_{B_j}}.$$

For the last map evaluate at a chosen highest-weight vector. Complete reducibility of rational representations of each general linear group identifies the invariant space with the indicated multiplicity space; the evaluation map is an isomorphism onto the whole highest-weight space, not just a map into it. The shadow and marker character conditions are automatic by (18). The source is therefore exactly R_ξ. In each contributing fine degree every edge degree is at most $b(\xi)$ by (21), so each reservoir contains the full required polynomial degree. Under the injective chart maps, the invariant space of the reservoirs is thus a subspace of the full interaction invariant space R_ξ that contains every fine component of R_ξ. The two spaces coincide, even at a support hole. This uses containment of low-degree polynomials, not equality of the reservoir image with a degree truncation. We have proved

$$R_\xi \cong \mathrm{HW}_{\ell(\xi)P_0, \ell(\xi)Q_0}\, \mathbb{S}_{\gamma_0(\xi)}(A \otimes B). \tag{24}$$

6.4.4. *Products and injectivity*

Contractions are maps of polynomial algebras; dehomogenization is multiplicative in the rectangular section degree; every determinant shift is linear in the original degree. The highest-weight extraction by dual Schur factors is compatible with the Cartan product on the dual factors: choose the embedding $\mathbb{S}_{\lambda+\mu}B \hookrightarrow \mathbb{S}_\lambda B \otimes \mathbb{S}_\mu B$ carrying the chosen highest-weight vector to the product highest-weight vector. The dual projection is the Cartan product. For equivariant maps $f : \mathbb{S}_\lambda B \to V$ and $g : \mathbb{S}_\mu B \to V$ in the source algebra, their product is the composite of this embedding, $f \otimes g$, and multiplication in V. Evaluation on the chosen highest-weight vector therefore gives $f(v_\lambda)g(v_\mu)$. These choices are associative, by the compatible fundamental highest-weight vectors in the rational Cartan algebra. Finally the Schur-face identifications respect products by Lemma 2.1. Hence (24) assembles into a graded algebra isomorphism.

The label map is injective: its common size determines $\ell(\xi)$ by (23); the fixed consecutive sector blocks of $\gamma_0(\xi)$ then recover the rational Schur weights on the extra row by subtracting the known padding. Dualizing recovers every original highest weight and torus character, hence $j(\xi)$ and ξ. This proves the fixed-two-shape assertion before imposing the rectangular restriction.

6.5. **A square among the fixed shapes**

We have already encountered the numerical rectangular absorption identity in (10). Here we need its multiplicative version, since the first two labels must retain their fixed shapes throughout the degree cone.

Lemma 6.6 (Rectangular absorption)**.** *Let (α, β, γ) be a triple of partitions of common size n. Choose positive integers r, q bounding the lengths of β, γ respectively, and $K \geq n$. Define, with zero padding to the indicated lengths,*

$$\widehat{\alpha} = ((r+1)K - \gamma_q, \dots, (r+1)K - \gamma_1, \alpha),$$
$$\widehat{\beta} = (qK + \beta_1, \dots, qK + \beta_r, qK - n),$$
$$\widehat{\gamma} = (K^{q(r+1)}).$$

These are partitions of equal size and $g(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}) = g(\alpha, \beta, \gamma)$. For integral linear families of labels and K, with fixed ranks and $K \geq n$ throughout a degree cone, this equality is induced by a graded algebra isomorphism.

Proof. Choose spaces A_0, B_0, C_0 with $\dim B_0 = r$, $\dim C_0 = q$, and A_0 sufficiently large. In $(C_0 \oplus A_0) \otimes (B_0 \oplus \mathbb{C})$ assign cocharacter weights $(1, 0)$ to both block decompositions. The highest sector $C_0 \otimes B_0$ has dimension qr. The middle sector is $(A_0 \otimes B_0) \oplus C_0$. The maximal face of the Schur module with rectangular label (K^{qr+q}) is

$$(\det(C_0 \otimes B_0))^K \otimes \mathbb{S}_{(K^q)}((A_0 \otimes B_0) \oplus C_0).$$

In the second summand of the middle sector C_0 is tensored with the additional line of the second group. Remove its determinant twist and the determinant of the highest sector. Lemma 6.1 gives polynomial functions in the representation $A_0 \otimes B_0 \otimes C_0^*$ with the inverse character of that line. The residual weights prescribed by the displayed partitions are $\alpha, \beta, \gamma^\vee$, and line weight $-n$. The central weights force degree n, and the reservoir contains the full degree-n component since $K \geq n$. The prescribed weights have the maximal cocharacter value, and highest-weight extraction on this face gives the original Kronecker multiplicity by Cauchy and duality. All operations respect Cartan products and determinant characters as in Section 6.4; dualizing the third label here means highest-weight extraction for the contragredient action on C_0^*, equivalently a change of Borel in that factor. This is compatible with multiplication. Dominance follows from $K \geq n$, and size equality follows either by summing the parts or by scalar characters. □

To finish Theorem 6.2, use the symmetry of the triple tensor highest-weight algebra under permutation of its three factors and apply Lemma 6.6 to

$$(\alpha, \beta, \gamma) = (\gamma_0(\xi), \ell(\xi)Q_0, \ell(\xi)P_0).$$

Choose q, r large enough that $h = q(r+1) \geq |P_0|$, and set $K = h\ell(\xi)$. The common size of the input is $|P_0|\ell(\xi) \leq K$. The new rectangular label is $\ell(\xi)(h^h)$, the new second label is fixed up to $\ell(\xi)$, and only the remaining label varies with the original degree. Permute the three factors to put the rectangle first. The result is (13) with $P = (h^h)$. Injectivity is preserved because the new rectangle determines $\ell(\xi)$ and the appended block of $\widehat{\alpha}$ recovers $\gamma_0(\xi)$. This finishes the proof.

Remark 6.7. The use of the extra row is important: placing the dual Schur factors on distinct auxiliary rows would generally introduce different degree-dependent row characters. A single line row makes their total character zero by (19), so the two fixed shapes survive. The final construction uses neither a localization of the target graded algebra nor a limit of increasing auxiliary dimensions. Rectangular dehomogenization is an injective chart computation whose bounded image contains each full required component.

7. Some invariant algebras

The original question concerned the cubic functor, so we first compose the mixed tensor realization with the rectangular cubic construction. The two maps are linear in the grading parameter and compatible with products; in particular their composition remembers the algebra that gave rise to its multiplicities.

Corollary 7.1. *Every algebra covered by Theorem 6.2 admits a multiplicative realization in rectangular-outer cubic Schur plethysm, with a single integral linear encoding of the degree. In a singly graded instance there are partitions κ, Λ, with κ rectangular, such that*

$$\dim R_t = \langle s_{t\Lambda}, s_{t\kappa}[h_3] \rangle \qquad (t \geq 0).$$

Proof. Compose the isomorphism of Theorem 6.2 with Proposition 5.1. The rectangle has width $n(\xi) = |P|\ell(\xi)$, so the labels vary integrally and linearly, including in degree zero. □

7.1. Higher Kronecker coefficients

The most immediate test of the contraction construction is to start with a tensor product of more than three factors. The reduction should then recover all its highest-weight spaces simultaneously, including their products.

For $k \geq 2$ and spaces $V_1, \ldots, V_k$ define

$$\mathcal{B}_k = \mathrm{Sym}(V_1 \otimes \cdots \otimes V_k)^{\prod_i U_{V_i}}.$$

It is graded by k-tuples of partitions of the same size. By Schur–Weyl duality its dimension in the component $(\lambda_1, \ldots, \lambda_k)$ of size n is the higher Kronecker coefficient

$$g_k(\lambda_1, \ldots, \lambda_k) = \dim([\lambda_1] \otimes \cdots \otimes [\lambda_k])^{S_n}.$$

The size n is a positive linear functional on the cone of tuples of dominant polynomial weights of equal size. Therefore Theorem 6.2 gives the following simultaneous identity, for all tuples with the prescribed length bounds.

Corollary 7.2. *For every fixed k and fixed dimensions of the V_i, there exist a square P, a partition Q and an injective integral linear encoding of the degree into triples $(nP, nQ, \Gamma(\lambda_1, \ldots, \lambda_k))$ such that*

$$g_k(\lambda_1, \ldots, \lambda_k) = g(nP, nQ, \Gamma(\lambda_1, \ldots, \lambda_k)).$$

The identities arise from a multigraded algebra isomorphism between $\mathcal{B}_k$ and a slice of one ordinary Kronecker branching algebra.

For a single ray, all tensor orders are consequently reduced to three ordinary symmetric-group factors at the level of entire stretching algebras. The dimensions in the target depend on the source dimensions and tensor order. This statement does not assert a uniformly efficient reduction when the tensor order itself grows.

7.2. **Quiver semi-invariants and matrix invariants**

The previous example used one full tensor product. The freedom to take direct sums and to introduce dual factors also accommodates arrows of a quiver: an arrow carries one covariant and one contravariant factor. Taking invariants at the vertices imposes the usual contraction conditions.

For a finite quiver Q_{v} with dimension vector (d_i), use the representation

$$M = \bigoplus_{a:i\to j} V_j \otimes V_i^*, \qquad \dim V_i = d_i.$$

One may use its dual if working with the coordinate algebra of the representation space. Its semi-invariant algebra is the $\prod_i \mathrm{SL}(V_i)$-invariant algebra; its weight spaces under the centers are precisely the $\prod_i \mathrm{GL}(V_i)$-highest-weight spaces with determinant labels. Thus Theorem 6.2 applies to its positively graded character slices, with all arrow degrees retained or with their coarsenings together with the character weights. This permits cyclic quivers as well as acyclic quivers if the grading is positive. In particular, if θ is a character and $\mathcal{L}$ an additional positive polynomial-degree constraint when needed, the associated diagonal graded ring of semi-invariants has a Kronecker stretching realization. Whenever Proj of that ring is the relevant quiver quotient, the graded isomorphism identifies the projective schemes. This conclusion concerns the indicated section rings, not equality of the ambient representation spaces.

For m matrices of size d, take $M = \mathrm{End}(\mathbb{C}^d)^{\oplus m}$ with the simultaneous conjugation action. The ordinary polynomial degree is positive on the invariant ring, and a torus may instead record all matrix degrees. The theorem realizes the full graded trace algebra (with its relations, not merely its Hilbert function) inside ordinary Kronecker theory. No restriction on d, m is needed.

7.3. **Determinantal rings**

In the proof of the contraction lemma, the auxiliary dimensions were chosen large enough that the contractions had no relations. If one starts with an invariant algebra in which the dimension is smaller, the resulting determinantal relations become part of the algebra to be realized. This gives a simple test of the multiplicative, rather than just numerical, assertion of Theorem 6.2.

Let D, F have dimensions p, q and let E have dimension $r \le \min(p, q)$. Consider the mixed tensor invariant algebra (15). By the fundamental theorems of invariant theory (as in (15)) it is

$$\mathrm{Sym}(D \otimes F)/I_{r+1},$$

where I_{r+1} is generated by the $(r+1)$-minors of the generic $p \times q$ matrix. Give the contraction generators degree one, by giving the first summand in (15) degree one and the second degree zero. The equality of the two summand degrees in every invariant implies the degree-zero invariant part is $\mathbb{C}$ and each piece is finite-dimensional. Hence every such determinantal coordinate ring is a Kronecker stretching algebra. Its degree-t dimension is

$$\sum_{\substack{\lambda\vdash t\\ \ell(\lambda)\le r}} \dim \mathbb{S}_\lambda D \, \dim \mathbb{S}_\lambda F,$$

and this entire function equals one stretched ordinary Kronecker coefficient family. For instance the coordinate ring of the hypersurface of singular 3×3 matrices (take $r = 2$) has Hilbert series $(1 - z^3)/(1 - z)^9$, realized exactly in Kronecker theory. This example illustrates that the

construction retains equations of the invariant algebra, including nonmonomial determinantal relations.

7.4. **Torus invariants and geometric realizations**

At the other end of the dimension range, all factors may be lines. A contraction condition then becomes a linear relation between monomial exponents, and the invariant algebra is a semigroup algebra. In this setting the prescribed grading can record the dilation of a polytope.

Let P be a nonempty rational polytope. Its Ehrhart semigroup S_P consists of (t, x) with $t \in \mathbb{Z}_{\geq 0}$ and $x \in tP \cap \mathbb{Z}^d$. After an integral coordinate translation of the polytope it lies in a nonnegative orthant. Introducing nonnegative slack coordinates for the rational inequalities, after clearing denominators, identifies S_P with the nonnegative integral kernel of a fixed integer matrix, retaining the degree t. Such a semigroup algebra is a torus invariant algebra of a polynomial ring, with the original degree given by a commuting torus. Each character of the acting torus is a mixed tensor product of standard lines and their duals. Thus Theorem 6.2 already implies a qualitative realization of every rational Ehrhart algebra. In the next section we give a direct construction with much smaller auxiliary dimensions in this scalar case. The isomorphisms identify the affine spectra and, for any single positive grading ray, the corresponding projective schemes and graded section rings. In particular they preserve generators, relations and arithmetic holes in the support.

8. Scalar cycles and Ehrhart algebras

The existence of an Ehrhart-algebra realization follows already from the mixed tensor theorem. That proof, however, uses auxiliary spaces whose dimensions are bounded by products of tensor dimensions and an abstract capacity bound. For questions about the sizes or the shapes of the resulting partitions, a more direct construction is useful.

When all row and column spaces are lines, the symmetric powers of a weight sector have a monomial basis. Their weights are just row, column and color sums of a matrix of exponents. Thus the realization problem becomes one of encoding arbitrary bounded systems of integral linear equations by these three kinds of sums. We first explain the conversion from matrices to Schur faces, and then construct the required matrices. De Loera and Onn's transportation realization of linear and integer programs [7] is a precedent for this encoding problem; here all colors must come from one additive cocharacter. Transportation methods also enter Kronecker stability [27] and bounds on Kronecker coefficients [23]. The question of a lattice-counting description of general multiplicities is distinct: obstructions to such descriptions are known for plethysm [16]. We instead start with a given lattice-point algebra and construct its representation-theoretic host.

8.1. **Colored matrices**

Let $a_1 > \cdots > a_R$ and $b_1 > \cdots > b_C$ be integer weights, and call $a_i + b_j$ the color of cell (i, j). Write s for the number of colors and Y_ℓ $(1 \leq \ell \leq s)$ for the span of all cells at the ℓth color in decreasing order, of dimension d_ℓ. Prescribe nonnegative integral row, column and color margins u_i, v_j, m_ℓ. Assume the identities

$$\sum_i u_i = \sum_j v_j = \sum_\ell m_\ell, \qquad \sum_i a_i u_i + \sum_j b_j v_j = \sum_\ell c_\ell m_\ell, \tag{25}$$

where c_ℓ denotes the color. A sufficient condition is a real feasible matrix. The identities, however, also allow infeasible systems.

Lemma 8.1 (Colored-matrix realization)**.** *There exist partitions α, β, γ such that $\mathcal{K}_{\alpha,\beta,\gamma}$ has a monomial basis indexed in degree t by nonnegative integral matrices with row, column and color margins tu, tv, tm. The product adds matrices and degrees.*

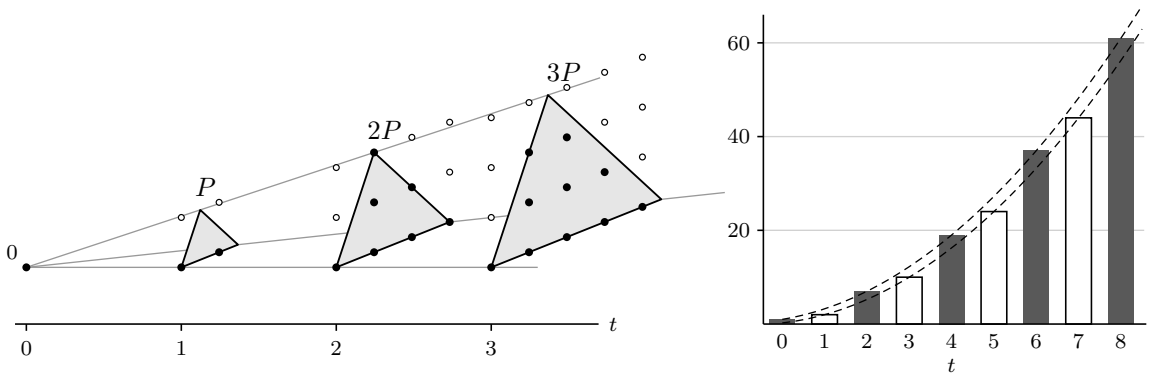

FIGURE 10. Theorem 8.2 for the triangle P with vertices $(0,0)$, $(\frac{3}{2},0)$, $(\frac{1}{2},1)$. Left: the lattice points of P, $2P$, $3P$ on slices of the cone over P. Right: $g(t\alpha,t\beta,t\gamma) = |tP\cap\mathbb{Z}^2| = 1,2,7,10,19,24,37,44,61$, a quasi-polynomial with constituents $\frac{1}{4}(3t^2+6t+4)$ for even t (shaded) and $\frac{1}{4}(3t^2+4t+1)$ for odd t.

Proof. Take $A=\mathbb{C}^R, B=\mathbb{C}^C$ with these cocharacters. Choose nonnegative shifts K_ℓ such that the blocks $(K_\ell+m_\ell, K_\ell^{d_\ell-1})$ concatenate to a partition γ. Set

$$\alpha_i = u_i + \sum_j K_{\mathrm{lev}(i,j)}, \qquad \beta_j = v_j + \sum_i K_{\mathrm{lev}(i,j)}.$$

Choose $K_\ell = K_{\ell+1} + m_{\ell+1} + B_0$ (with $K_s = 0$), where B_0 exceeds every u_i, v_j. The decrease in row and column weights then ensures dominance. When all row margins and all column margins are constant separately, one may take $B_0 = 0$. The first identity of (25) gives equality of partition sizes, and the second gives equality with the maximal cocharacter weight. By Lemma 2.1, after removing the determinant character the top face is $\bigotimes_\ell \mathrm{Sym}^{tm_\ell} Y_\ell$. Its monomial weight basis at the target consists of exactly the prescribed matrices. All positive-root operators of $\mathrm{GL}(A)\times\mathrm{GL}(B)$ strictly increase cocharacter weight and hence vanish there. The multiplicative assertion of Lemma 2.1 gives ordinary multiplication of these monomials, with determinant twists adding in t. □

8.2. **Arithmetic cycles**

A color margin can impose a linear equation provided that all copies of a variable appearing in it are tied together. The simplest way to transmit a scalar exponent is to give every vertex of an even cycle the same margin. Once one edge exponent is known, the remaining exponents alternate between that value and its complement. To impose signed linear equations, we assign colors to these two classes of edges. As in the mixed tensor proof, private markers make the specified colors additive and exclude unwanted cells. The following theorem implements this idea at all dilation degrees at once.

Theorem 8.2 (Ehrhart algebra realization)**.** *For every nonempty rational polytope $P\subset\mathbb{R}^d$, there are explicitly constructible partitions α,β,γ such that*

$$\mathcal{K}_{\alpha,\beta,\gamma} \cong \mathbb{C}[S_P], \qquad S_P = \{(t,x) : t\in\mathbb{Z}_{\geq 0},\ x\in tP\cap\mathbb{Z}^d\}, \tag{26}$$

with the grading given by t. In particular $g(t\alpha,t\beta,t\gamma) = |tP\cap\mathbb{Z}^d|$ for every $t\geq 0$ (with $0P=\{0\}$).

Figure 10 shows the statement for a rational triangle.

Proof. Translate P by an integral vector so its coordinates are nonnegative, and write it using integral inequalities $Ax\leq b$ (clear denominators in the inequalities, not in the lattice). Introduce

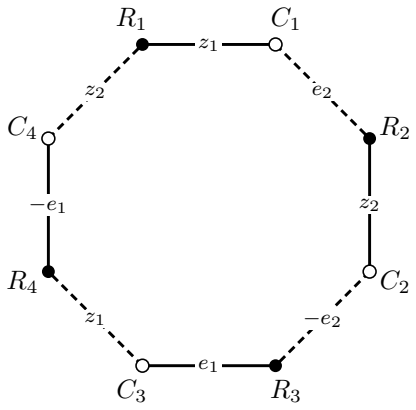

FIGURE 11. The cycle of Theorem 8.2 for a column $(1,-1)^{\mathsf{T}}$ of H. Solid edges carry the exponent x_j and dashed edges carry $Mt - x_j$; edge labels are the formal colors.

the slack vector $y = b - Ax$. Compactness gives a common positive integral upper bound M on the translated coordinates and slacks. At degree t the slack vector is $tb - Ax$, and the translation is multiplied by t. Thus the resulting system, which we denote by $Hx = tb$, $0 \le x_j \le Mt$, is in additive, degree-preserving bijection with S_P. We may enlarge the system by complement variables $x_j + y_j = Mt$ (at degree t) to ensure nonzero columns; we may add a fixed-zero variable for a zero-dimensional empty system. Append the negative sum of the rows as a redundant equation. Each column now sums to zero. Let $r_j = 2\sum_{h_{ij}>0} h_{ij} = 2\sum_{h_{ij}<0} |h_{ij}|$, a positive even integer.

Introduce independent formal colors e_i and put $c_i = e_i, d_i = -e_i$. For column j form two lists P_j, Q_j of length r_j: P_j contains h_{ij} copies of each of c_i, d_i for $h_{ij} > 0$, and Q_j contains $|h_{ij}|$ copies of each for $h_{ij} < 0$. Their color supports are disjoint and both lists sum to zero. With $r = r_j$ and private independent markers $z_1, \dots, z_r$, form the alternating edge lists of a bipartite cycle of length $4r$:

$$A = (z_1, \dots, z_r, P_j), \qquad B = (Q_j, z_1, \dots, z_r).$$

Write R_i, C_i for the row and column vertices. The edges are (R_i, C_i) of color A_i and (R_{i+1}, C_i) of color B_i, with cyclic indices. Every vertex has margin Mt. The edge exponents alternate between x_j on A-edges and $Mt - x_j$ on B-edges: at each vertex the second incident exponent is uniquely determined by the first. Figure 11 gives the smallest example, corresponding to a column with entries $1, -1$. Each marker has margin Mt; the contribution to each of c_i, d_i is $h_{ij}x_j + Mt\max(-h_{ij}, 0)$. Prescribe for these two colors the common margin $t(b_i + M\sum_j \max(-h_{ij}, 0))$, nonnegative by feasibility of P. This enforces exactly $Hx = tb$. All auxiliary exponents are determined uniquely and add under multiplication.

It remains to realize the selected colors as sums of vertex weights, with no extra cells. Within each cycle give the vertices formal weights ρ_i, χ_i by setting $\rho_1 = 0$, $\chi_i = A_i - \rho_i$ and $\rho_{i+1} = \rho_i + B_i - A_i$. Equality of the two list sums closes the cycle. Along any path from a row to a column, alternating the signs of the edge colors starting with a plus telescopes to the sum of the endpoint weights. For a nonedge (R_i, C_k) of weight $\rho_i + \chi_k$, a shortest path has odd length between 3 and $2r - 1$. The two occurrences of z_i are at positions $2i - 1$ and $2r + 2i$ in the cyclic edge list $A_1, B_1, \dots, A_{2r}, B_{2r}$. Their smaller cyclic distance is $2r - 1$, so a path with at most $2r - 1$ edges cannot contain both occurrences. The only adjacent pair of nonmarker edges is B_r, A_{r+1}, at the transition between the two halves of the lists. A path of length at least five therefore contains at least two independent markers, and cannot give a selected color. For length three, if there is just one marker, it either occurs with sign -1 in the middle, or the other two

edges contribute a nonzero difference of colors from the disjoint supports of P_j, Q_j. In neither case is the sum selected. Every path of length three contains a marker. Thus there are no extra selected cells.

Separate the cycles by adding an independent shift to the rows of each cycle and subtracting it from its columns. Every off-diagonal cell then has a nonzero shift coordinate, so is unselected. Distinct rows (or columns) have distinct weights: equality would give identical neighborhoods in the selected graph, a disjoint union of simple cycles of length at least eight, which has no such pair. Specialize the independent coordinates to powers of a sufficiently large integer to preserve the finite set of required equalities and inequalities. Order the rows and columns accordingly. Let $N = 2\sum_j r_j$ be the number of rows and also columns. All row and column margins are M at $t = 1$; all unselected colors have margin zero. A rational point of P provides a real feasible matrix, hence (25). Lemma 8.1 gives the desired algebra because the bijection of lattice points and matrices respects addition. □

The qualitative realization did not require any estimate on the auxiliary spaces. Here the cycle size is controlled directly by the coefficients of the linear system. We record the resulting bound, since it lets us distinguish the size of a Kronecker label from the arithmetic complexity of its stretching support. With $N = 2\sum_j r_j$, the sum of color margins is NM. The choice $K_\ell = \sum_{a>\ell} m_a$ in Lemma 8.1 gives $|\alpha| = |\beta| = |\gamma| \le M(N^3 + N)$ and lengths at most N^2. The integer cocharacters can be encoded with polynomially many bits in N and the coefficient list size: each formal difference has bounded integer coefficients, so radix specialization is explicit. No large cocharacter value enters the partition entries.

For a fixed augmented system $Hx = b$ and bound M, the cycles and row and column margins above are independent of b. Call an integral right-hand side admissible if all prescribed color margins are nonnegative, with the redundant negative-sum row included. We also require $b_i = 0$ for every zero row i of the augmented H, since such a row has no cells of colors $\pm e_i$. Both identities of (25) then hold without a feasibility assumption: the sum of the augmented right-hand sides is zero, and the two colors in each pair $(e_i, -e_i)$ cancel in the weighted sum. Over any bounded set of admissible right-hand sides, choose the shifts K_ℓ uniformly in Lemma 8.1. Then α, β are fixed and $\gamma(b)$ is affine in b; increasing one original coordinate of b by one, when both right-hand sides are admissible, moves two boxes from the two colors of the redundant row to the two colors of the changed row.

Corollary 8.3 (Every normal projective toric variety is a Kronecker quotient)**.** *Every normal projective toric variety occurs as* $\operatorname{Proj} \mathcal{K}_{\alpha,\beta,\gamma}$ *and as the projective spectrum of a rectangular cubic plethysm stretching algebra. Every positively graded polynomial ring occurs as a Kronecker stretching algebra. The possible sets* $\{t \ge 0 : g(t\alpha, t\beta, t\gamma) > 0\}$ *are exactly the additive submonoids of* $\mathbb{Z}_{\ge 0}$.

Proof. An ample torus-equivariant line bundle on a normal projective toric variety has section ring the Ehrhart algebra of its lattice polytope [6]. Apply Theorems 8.2 and 5.2. For the polynomial-ring assertion with at least one generator use the rational simplex $P = \{x \ge 0 : \sum_i a_i x_i = 1\}$ for positive integral degrees a_i, so its semigroup algebra is $\mathbb{C}[X_1, \dots, X_k]$ with $\deg X_i = a_i$. Every nonzero additive submonoid of $\mathbb{Z}_{\ge 0}$ is finitely generated (take its least positive element and the least member of each occupied residue class modulo the least positive element), so occurs as the degree support of such a ring. The ring $\mathbb{C}$ and support $\{0\}$ are realized by $((2), (2), (1, 1))$. Conversely the domain property of the Cartan algebra implies closure under addition, and degree zero contains the unit. □

Briand, Orellana and Rosas [4] disproved the original saturation assertion by examples with few rows. The revised hypotheses in Mulmuley's appendix allow nonzero saturation and positivity

indices, but require them to be polynomially bounded in the largest partition height. The size estimate above lets us test these quantitative requirements.

Corollary 8.4 (Superpolynomial saturation and positivity indices)**.** *For each $n \geq 1$ there are triples of partitions of common size $O(n^3)$ and lengths $O(n^2)$ with*

$$\sum_{t\geq 0} g(t\alpha, t\beta, t\gamma)z^t = \frac{1}{(1-z^{2^n})(1-z^{2^n+1})}. \tag{27}$$

Their saturation and positivity indices (in the sense of [4, Appendix]*) are both $2^{2n} - 2^n - 1$. In every rational presentation with polynomial numerator and denominator of the form in* [4, Appendix, (5.1)]*, the largest denominator exponent is at least $2^n + 1$. Thus the polynomial-height bounds of relaxed SH(a), PH2 and PH3 fail for Kronecker coefficients, even if height is replaced by unary input size.*

Proof. Encode $2^n u + (2^n + 1)v = 1$ by two doubling chains $x_0 = u, y_0 = v$, $x_{i+1} = 2x_i, y_{i+1} = 2y_i$ for $0 \leq i < n$, and $x_n + y_n + y_0 = 1$. Every variable lies in $[0, 1]$ and the absolute-entry sum of the augmented integer system is $O(n)$, hence the cycle construction uses $N = O(n)$ and $M = 1$. It preserves the graded polynomial ring with generator degrees $L = 2^n, L + 1$. This gives (27) and the size bounds. Figure 12 shows the case $n = 2$.

For coprime positive $a, b > 1$, the number of solutions of $au + bv = t$ increases by one when t increases by ab: choose the unique residue $v_0 \in [0, a - 1]$ with $bv_0 \equiv t \pmod a$, then count $v = v_0 + ka$ with $k \geq 0$ and $bv \leq t$. For t in a fixed residue class modulo ab, let $u_0 \in [0, b - 1]$ and $v_0 \in [0, a - 1]$ be the least nonnegative solutions of $au_0 \equiv t \pmod b$ and $bv_0 \equiv t \pmod a$. Then the constituent polynomial is $1 + (t - au_0 - bv_0)/(ab)$. This follows by counting $u = u_0 + bi$, $v = v_0 + aj$ with $i + j = (t - au_0 - bv_0)/(ab)$, which is at least -1 for $t \geq 0$. Its root in that residue class is the last missing integer there (possibly negative). The largest root is $ab - a - b$, attained for $u_0 = b - 1$ and $v_0 = a - 1$. Thus shifting the quasipolynomial by $ab - a - b$ makes all constant terms nonnegative and all slopes positive, and every smaller nonnegative integral shift leaves a positive integer root in a nonzero constituent. The saturation index (positivity at positive integers of each nonzero constituent after shifting) and the positivity index (nonnegative coefficients after shifting) both equal $ab - a - b$. Take $a = L, b = L + 1$. At a primitive $(L + 1)$-th root of unity the series has a pole, so any expression with denominator $\prod_i (1 - z^{a_i})^{d_i}$ must have at least one a_i divisible by $L + 1$. In particular its maximum is at least $L + 1$, independently of the choice of positive form. This also obstructs the modular index, defined from the minimal positive form. The density condition SH(b) is not addressed here. A recent unrefereed preprint [1] reports a modular-index obstruction for triple-rectangular Kronecker coefficients and related index obstructions for symmetric cubic plethysm; its Kronecker result does not address SH(a) or PH2. □

Theorem 5.2 transports the same series to $\langle s_{t\Lambda}, s_{t\kappa}[h_3]\rangle$ with κ rectangular. Indeed, $|\kappa| = O(md^3) = O(n^9)$, where $m = O(n^3)$ and $d = O(n^2)$ bound the original size and lengths. The ambient groups have ranks $3d$ and $\binom{3d+2}{3}$, and the label heights are $O(d^3)$. All three polynomial index bounds therefore fail for these rectangular cubic rays as well, with the inner label fixed as in [21, Sections 1.6, 3.3–3.4].

8.3. **One staircase suffices**

The arithmetic cycle used the freedom to prescribe capacities on individual colors. Suppose instead that the outer Schur label is fixed to be a full staircase. We can no longer choose those capacities independently, but a full staircase has a different useful property: in a two-dimensional color sector its consecutive parts differ by one, and in a one-dimensional sector the factor is a character. Thus a staircase can supply binary variables without any adjustable color margins.

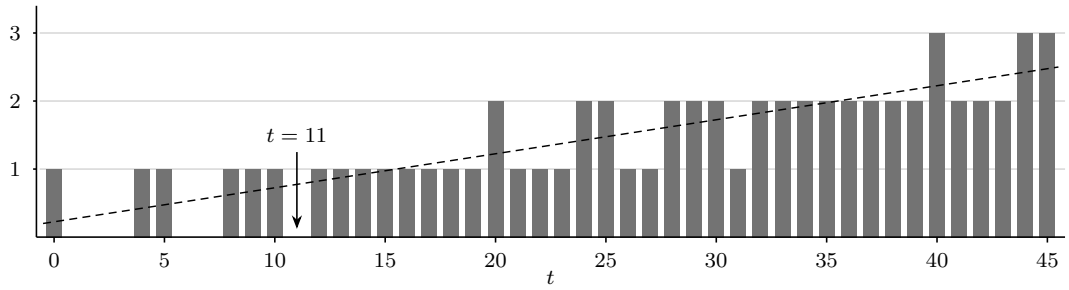

FIGURE 12. The Hilbert function of Corollary 8.4 for $n = 2$: the number of solutions of $4u + 5v = t$, realized as $g(t\alpha, t\beta, t\gamma)$. The last zero is at $t = 11 = ab - a - b$; the dashed line is the average $(t + 4.5)/20$.

Theorem 8.5 (Realization by a staircase)**.** *Let $E \subset X \times Y \times Z$, where the three vertex sets have equal finite cardinality. There are polynomial-unary-size strict partitions α, β and a staircase $\rho_D = (D, D-1, \ldots, 1)$ such that $g(\rho_D, \alpha, \beta)$ is the number of perfect three-dimensional matchings of E. More generally, if the fractional matching polytope*

$$P_E = \{x \in \mathbb{R}^E_{\geq 0} : \sum_{e \ni v} x_e = 1 \ (v \in X \cup Y \cup Z)\}$$

is nonempty, then $\mathcal{K}_{\alpha,\beta,\rho_D} \cong \mathbb{C}[S_{P_E}]$. Consequently positivity with one staircase argument and two strict arguments is NP*-hard, and exact computation is #*P*-hard under parsimonious reductions.*

Proof. Put $n = |X| = |Y| = |Z|$ and $m = |E|$. The empty instance with $n = 0$ gives one, realized by $((1), (1), (1))$; an isolated vertex gives zero, realized by $((3), (3), (2, 1))$. Otherwise $m \geq n \geq 1$. Use independent formal row weights R_v for all vertices and R_d for a dummy vertex, and a formal coordinate δ. Introduce primary and complementary rows $R_v, R_v + \delta$ for every vertex, and $R_d, R_d - 3\delta$ for the dummy. For $e = (i, j, k) \in E$ introduce independent coordinates p_e, q_e, s_e and four columns

$$p_e, \quad q_e + R_k - R_j + 2\delta, \quad q_e, \quad s_e.$$

In them select respectively the row pairs for i, j, k, d. Denote the primary cell colors by A, B, C, D_0 and the complementary colors by A', B', C', D_0'. Thus $B = C + 2\delta$, $A' = A + \delta$, $B' = B + \delta$, $C' = C + \delta$, $D_0' = D_0 - 3\delta$. We have $A + B = A' + C'$ and $D_0 + C = D_0' + B'$. For each triple add two private 2×2 blocks of colors

$$\begin{pmatrix} A & C' \\ A' & B \end{pmatrix}, \qquad \begin{pmatrix} D_0 & B' \\ D_0' & C \end{pmatrix}. \tag{28}$$

Their diagonal sum identities allow additive row and column weights: take row weights $(0, \delta)$ and $(0, -3\delta)$ respectively, and the displayed first-row entries as column weights. Add an independent shift to the rows of each private block and subtract it from its columns.

Every selected color now occurs exactly twice, and all other colors occur once. Indeed in the common block all cells are distinct: columns with different private coordinates have differences involving a private coordinate; the only pair sharing one coordinate has difference $R_k - R_j + 2\delta$, which is not a common row difference. Within each private square the four entries are distinct, and on the block diagonal the only coincidences with the common matrix are precisely the prescribed eight colors per triple, which are disjoint for different triples and between the two private squares. Off-diagonal blocks are distinguished by their independent shift differences. Within each such block a row difference cannot equal a nonzero column difference: private row differences are multiples of δ, whereas common column differences are never such multiples; private column differences contain private column coordinates that do not occur in any row difference. The same argument gives distinct row and column weights. Specialize to integers as before.

There are $R = 6n + 2 + 4m$ rows and $C_0 = 8m$ columns. Prescribe residual margins 1 on every column and each private row. On the common rows prescribe $1, \deg(v) - 1$ on v, v', and $n, m - n$ on d, d'. Every repeated color has residual margin one and each singleton margin zero. This is exactly the residual margin pattern of ρ_{RC_0}: in a two-dimensional sector the face factor is $(\det Y)^K \otimes Y$, where K is the smaller of the consecutive staircase parts. In a singleton it is $(\det Y)^K$. Define α, β by adding these determinant shifts to the residual row and column margins, as in Lemma 8.1. For any two successive rows the padding decreases by at least C_0, since the cell colors in each column strictly decrease and the smaller staircase part of each sector strictly decreases with the color order. Residual row margins differ by at most m. Hence α is strict; the same argument on columns gives strictness of β. The total residual mass on each side is $8m$, so the partition sizes agree.

We verify the maximal-weight identity even in infeasible instances. The private rows and columns, all of margin one, contribute half the sum of the eight selected colors per triple, by the two diagonal identities in (28). In the common block, the column contribution is the sum of its column weights. Its row contribution is the sum of primary row weights over the $3m$ incidences and m dummy incidences, plus the correction $\delta(\sum_v(\deg(v) - 1) - 3(m - n)) = 0$. Hence the common block also contributes half the sum of the eight colors. This gives (25), so the highest-weight vectors are precisely the monomials at the target.

At degree t the two-dimensional sectors give $\mathrm{Sym}^t Y$ after removing the determinant twists. Each private square has all four vertex margins t, so its diagonal exponents are a common value, with the off-diagonal exponents its complement to t. As a color is shared by exactly one common and one private cell, the common entries of colors A, B are equal to x_1, while A', C' equal $t - x_1$. Similarly D_0, C equal x_2, while D_0', B' equal $t - x_2$. The common column of B, B' forces $x_1 = x_2 = x_e$. The primary vertex margins enforce $\sum_{e \ni v} x_e = t$. Complementary and dummy margins follow, since $\sum_e x_e = nt$. Conversely any such x fixes every exponent. Their addition respects the Cartan product, giving the algebra statement and the count at $t = 1$. The staircase size is $O((n + m)^4)$ and the output is explicit in polynomial time. The parsimonious counting hardness follows from the reductions to perfect three-dimensional matching in [28, Propositions 2–3]; the decision assertion also follows from that reduction. □

Corollary 8.6 (Toric geometry with one staircase)**.** *Every normal projective toric variety occurs as* $\operatorname{Proj} \mathcal{K}_{\alpha,\beta,\rho_D}$ *with* α, β *strict.*

Proof. It suffices to represent a rational polytope P by a fractional perfect matching polytope, preserving the lattice after a Veronese regrading. As in Theorem 8.2, write P using nonnegative bounded variables and integral equations; introduce a complementary variable $M_0 - x_i$ with equation $x_i + y_i = M_0$ for a common integral bound M_0, and replace each negative term by its positive complementary term minus its constant. All right-hand sides are then nonnegative by feasibility. Choose an integer $M > 0$ at least as large as the right-hand sides and coordinate bounds. Divide the variables by M, introduce a variable z with $Mz = 1$, and pad each equation with copies of z until its right-hand side is 1. The resulting nonnegative system consists entirely of sums of occurrences equal to 1 (repetitions are allowed), and every variable is at most 1. At stretching degree Mt, its integral solutions are in additive bijection with $tP \cap \mathbb{Z}^d$, since $z = t$ and the rescaled variable exponents are precisely the original integral coordinates and uniquely determined auxiliaries.

To obtain a tripartite matching system, make three copies of every equation, one in each class. For a variable occurring k times, replace its occurrences by $x_{j,c}$ $(1 \le j \le k,\ c = 1, 2, 3)$. Add

auxiliary variables L_j, R_j and the following equations, with cyclic subscripts:

class	additional equations	
1	$x_{j,2} + R_j = 1$	$x_{j,3} + L_j = 1$
2	$x_{j,1} + L_j = 1$	$x_{j,3} + R_{j+1} = 1$
3	$x_{j,1} + R_j = 1$	$x_{j,2} + L_j = 1.$

Every new variable occurs once in each class; the equations force the occurrence variables to agree and determine the auxiliary variables as complements. The three classes have equal numbers of equations, and the incidence triples of the variables are distinct. Regard equations as vertices and variables as edges of a tripartite hypergraph. Its matching polytope P_E realizes the system, with all auxiliary coordinates additive after homogenization. Consequently the Mth Veronese subalgebra of $\mathbb{C}[S_{P_E}]$, regraded by t, is $\mathbb{C}[S_P]$. Apply Theorem 8.5 and invariance of Proj under Veronese subalgebras. This argument asserts existence; the repeated coefficients and padding are not claimed to give polynomial binary overhead. □

The Veronese step is unnecessary for weighted polynomial rings: binary chains put their defining degree equation directly into the matching format.

Theorem 8.7 (Weighted polynomial rings with one staircase)**.** *Let $a_1, \dots, a_k$ be positive integers, $k \geq 1$, of total binary length L. There are explicitly constructible strict partitions α, β and a full staircase ρ_D of common size $O(L^4)$ such that*

$$\mathcal{K}_{\alpha,\beta,\rho_D} \cong \mathbb{C}[X_1, \dots, X_k], \qquad \deg X_i = a_i,$$

with the original grading. The construction takes polynomial time in L.

Proof. The monomial semigroup is given by $\sum_i a_i x_i = t$, $x_i \geq 0$. Compute each $a_i x_i$ by a binary Horner chain starting at x_i (the leading digit is one); each step is $p' = 2p + \epsilon x_i$, $\epsilon \in \{0, 1\}$. Replace it by the two sum equations

$$p' + \bar{p}' = t, \qquad p + p + \epsilon x_i + \bar{p}' = t.$$

All intermediate values lie between 0 and $a_i x_i \leq t$, so complements are nonnegative and uniquely determined. The final sum equation and the chain equations have right-hand side t and $O(L)$ total variable occurrences. The occurrence-copy construction of Corollary 8.6 therefore applies directly at degree t, with no regrading. Its hypergraph has $O(L)$ vertices and edges and no isolated vertices or repeated edges. Its fractional matching polytope is nonempty, by taking $x_1 = 1/a_1$ and the other original variables zero at $t = 1$. All auxiliary coordinates are uniquely determined and additive in (t, x), giving the required graded semigroup isomorphism. Theorem 8.5 supplies the algebra realization and size bound. □

In particular, taking weights $2^n, 2^n + 1$ gives the series (27) even with one staircase label and two strict labels, of common size $O(n^4)$. Thus the same index obstructions hold with this additional shape restriction. Weighted polynomial rings also allow us to freeze all the base diagrams while varying the stretching degree alone.

Corollary 8.8 (A counting table on a single stretching ray)**.** *For every $f \in \#\mathsf{P}$ there are, for each input length $n \geq 1$, strict partitions α_n, β_n, a full staircase ρ_{D_n}, and an integer $T_n > 0$, constructible in polynomial time in n (the partitions in unary and T_n in binary), such that*

$$g(t\alpha_n, t\beta_n, t\rho_{D_n}) = f(\mathrm{bin}_n(t - T_n)) \qquad (T_n \leq t < T_n + 2^n).$$

Here $\mathrm{bin}_n(x)$ is $(x_1, \dots, x_n)$, where $x = \sum_{i=1}^{n} 2^{i-1} x_i$. The stretching algebra can be chosen to be a weighted polynomial ring with eventually positive Hilbert function. By Theorem 5.2, the same table occurs on one rectangular cubic plethysm stretching ray.

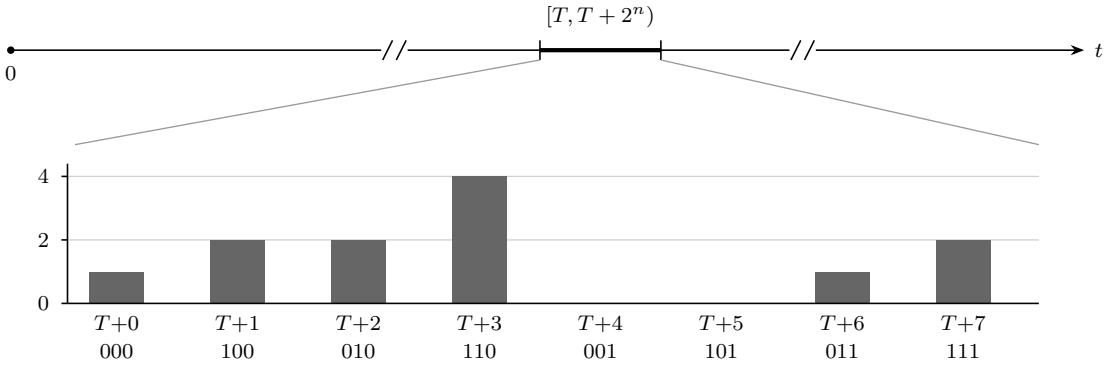

FIGURE 13. Corollary 8.8 for $n = 3$ and $f(x) = \#\{w \in \{0,1\}^2 : (x_1 \vee w_1) \wedge (\neg x_3 \vee w_2) \wedge (x_2 \vee \neg w_2)\}$. The bars are $g(t\alpha_3, t\beta_3, t\rho_{D_3})$ at $t = T + x$, with $T = 103\,192\,213\,607\,128\,131\,696$, computed from the weighted polynomial ring of the proof by enumerating all monomials of each degree; they agree with f. Inputs are written $x_1x_2x_3$, least significant bit first, so $x = x_1 + 2x_2 + 4x_3$.

Proof. Use a polynomial-size bounded-fan-in (at most two-input) verifier circuit for f with n input bits and a fixed polynomial number of witness bits. Its Tseitin clauses (including the true output) uniquely determine the gate bits. Remove repeated literals and tautologies. A clause with literal sum L_c is enforced by $L_c = 1$ for width one, $L_c + (1 - s_0) = 2$ for width two, or $L_c + (1 - s_0) + 2(1 - s_1) = 4$ for width three. The slack bits are unique for each satisfying clause. An empty clause, if present, can instead be expressed by an inconsistent pair of unit clauses on a new bit. The resulting system on $k \geq n$ Boolean variables is $\sum_i c_{ji}^{\epsilon_i} = b_j$ $(1 \leq j \leq q)$, with $0 \leq c_{ji}^{\epsilon} \leq 2$ and $1 \leq b_j \leq 4$, and its solutions at a fixed input count precisely f.

Put $B = 4k + 5$, $Q = (k + 1)2^n$, $r = k + q$. For each variable and choice $\epsilon \in \{0, 1\}$ define a positive weight

$$w_i^{\epsilon} = Q\Big(B^r + B^{i-1} + \sum_{j=1}^{q} c_{ji}^{\epsilon} B^{k+j-1}\Big) + \ell_i^{\epsilon},$$

where $\ell_i^{\epsilon} = \epsilon 2^{i-1}$ for the input variables $1 \leq i \leq n$ in least-significant-bit order and is zero otherwise. Set

$$T_n = Q\Big(kB^r + \sum_{i=0}^{k-1} B^i + \sum_{j=1}^{q} b_j B^{k+j-1}\Big).$$

For $0 \leq x < 2^n$, consider $\sum_{i,\epsilon} w_i^{\epsilon} z_i^{\epsilon} = T_n + x$ with nonnegative integral z_i^{ϵ}. The target is less than $(k+1)QB^r$ (all lower digits are at most four), so at most k weights can be selected with multiplicity. Their low-order contributions sum to less than Q, and every radix-B digit sum is at most $2k < B$. Reduction modulo Q and then digit comparison give the input x, the equations $z_i^0 + z_i^1 = 1$, and the verifier equations. Conversely each accepting witness yields exactly one solution. Thus the weighted polynomial ring on these $2k$ generators has Hilbert function $f(\mathrm{bin}_n(x))$ at degree $T_n + x$. All weights have polynomial total binary length. Adjoining two consecutive weights $T_n + 2^n, T_n + 2^n + 1$ leaves this interval unchanged and makes the degree support cofinite. Apply Theorem 8.7. □

Figure 13 carries out this construction for a small verifier.

For two identical staircase labels the corresponding positivity assertion is Saxl's conjecture, namely $g(\rho_k, \rho_k, \lambda) > 0$ for every $\lambda \vdash |\rho_k|$. Theorem 8.5 gives hardness when only one staircase is prescribed, independently of Saxl's conjecture. It does not assert hardness of exact computation for a staircase tensor square.

9. Further questions

We close with three questions, one for each end of the paper: the outer label in cubic plethysm, the source of the algebra realizations, and the shape of the target.

Theorem 1.1 shows that inner degree three suffices for all Kronecker multiplicities. By Proposition 3.3, arbitrary outer Schur labels can be absorbed into the inner representation; Proposition 3.2 already realizes quadratic plethysm in symmetric cubic plethysm. This suggests that inner degree three suffices for plethysm itself.

Conjecture 9.1 (Cubic plethysm)**.** *There is a polynomial-time map, in unary partition encoding, sending every Schur plethysm instance* (λ, μ, ν) *to* (D, ρ) *with* $\rho \vdash 3D$ *such that*

$$a^{\lambda}_{\mu,\nu} = \langle s_\rho, h_D[h_3] \rangle.$$

The conjecture would reduce the positive counting problems for both Kronecker coefficients and general Schur plethysm to the single family $h_D[h_3]$.

Theorem 6.2 treats arbitrary sums of full mixed tensor products of standard and dual general linear representations. An arbitrary rational representation of a general linear group is a direct summand of such a module, but the invariant ring of the summand is in general a quotient of the invariant ring of the larger representation. This does not provide an isomorphism of the required type. Likewise, embedding a reductive group in a general linear group does not identify its invariants with the general linear invariants of a full mixed tensor module. These are two reasons to distinguish the following question from the mixed tensor theorem.

Conjecture 9.2 (Reductive invariant algebras)**.** *Let* H *be a connected complex reductive algebraic group acting rationally on a finite-dimensional space* W*, and let a commuting algebraic torus action give a lattice grading* $R = \mathbb{C}[W]^H = \bigoplus_{\xi \in \Gamma} R_\xi$*. Assume* $R_0 = \mathbb{C}$*, each* R_ξ *is finite-dimensional, and an integral linear functional on* Γ *is strictly positive on every nonzero degree in the support of* R*. Put* $\sigma = \operatorname{cone}\{\xi : R_\xi \neq 0\}$ *and* $L = \Gamma \cap \operatorname{span}_{\mathbb{R}}(\sigma)$*. Then there exist finite-dimensional* A, B, C *and an injective integral linear map* F *on* L *taking* $\sigma \cap L$ *to triples of partitions of equal size, of lengths bounded by* $\dim A, \dim B, \dim C$*, respectively, such that*

$$R \cong \bigoplus_{\xi \in \sigma \cap L} \mathcal{B}(A, B, C)_{F(\xi)}$$

as L*-graded algebras, with the original multiplication on each side.*

In particular, the conjecture would realize every compatibly positively graded polynomial invariant algebra of a connected reductive group as a Kronecker stretching algebra. The cases of quiver semi-invariants and general linear mixed tensor invariants are covered by Theorem 6.2; the conjecture asks for arbitrary rational representations of arbitrary connected reductive groups. One can further ask whether the fixed-shape refinement of Theorem 6.2, with one rectangular label, persists in this setting. Proposition 5.1 would transport such a realization into rectangular cubic plethysm. One approach is through localization. Lemma 2.1(iii) identifies the whole algebra of top faces after inverting a single determinant vector.

Corollary 9.3 (Schur localization)**.** *For a finite-dimensional* $\mathrm{GL}(V)$*-module* Y *with ordered weight sectors* $Y_1, \dots, Y_s$ *as in Lemma 2.1, let* $\mathcal{C}(Y_j)$ *denote the rational Cartan algebra (the direct sum over all dominant integral weights of* $\mathrm{GL}(Y_j)$*), and let* U_G, U_L *be the upper unipotent subgroups of* $G = \mathrm{GL}(V)$ *and its block Levi* L*. Put* $\mathcal{F} = \bigoplus_\kappa ((\mathbb{S}_\kappa Y)_{\max})^{U_G}$*, with polynomial labels* κ*. Choose integers* $k_1 > \cdots > k_s > 0$*, and let* Δ *be a nonzero vector in the one-dimensional face belonging to the concatenation of* $(k_j^{d_j})$*. Then*

$$\mathcal{F}[\Delta^{-1}] \cong \left(\bigotimes_j \mathcal{C}(Y_j) \right)^{U_L},$$

with the grading on Schur labels identified by concatenation of the block weights.

Proof. By Lemma 2.1, $\mathcal{F}$ is the subalgebra on the right supported on nonnegative block labels whose concatenation is dominant. The element Δ maps to the product of the determinant characters of powers k_j. This product is invertible in the rational Cartan algebra. Any fixed collection of dominant integral block labels becomes polynomial and jointly dominant after adding $N(k_j^{d_j})$ for sufficiently large N, proving surjectivity after localization. Injectivity follows from the Cartan algebra embedding. The same proof works for a product of general linear groups with block-diagonal cocharacters. □

Corollary 9.3 suggests using invertible tensor frames, but an identification on that open locus alone does not control the boundary.

Even within the mixed tensor theorem, it is natural to ask whether the two fixed shapes may both be rectangular, or even equal, without changing the grading. Iterating the rectangular absorption argument does not establish this. The auxiliary dimensions in the contraction proof grow with prefix tensor products, and the markers trade efficiency for exact isolation. More economical choices could make the construction useful in computation as well.

The staircase construction of Theorem 8.5 uses consecutive unequal parts to provide independent two-dimensional sectors. Corollary 8.6 shows that even a staircase label suffices for arbitrary projective toric geometry. A partition ρ is a sum $\sum_{i\geq 1}(\rho_i - \rho_{i+1})(1^i)$ of fundamental weights. We conjecture that the number of nonzero summands, rather than their coefficients or positions, controls its universal realization capacity.

Conjecture 9.4 (Prescribed Young diagrams)**.** *Let H be a connected complex reductive algebraic group and W a finite-dimensional rational H-module. Let $R = \mathbb{C}[W]^H = \bigoplus_{t\geq 0} R_t$ have a nonnegative integral grading induced by a commuting algebraic $\mathbb{C}^\times$-action on W, with $R_0 = \mathbb{C}$ and finite-dimensional graded components. There is an integer $c(R) \geq 1$ such that, for every partition ρ having at least $c(R)$ distinct positive parts, there are partitions $\alpha, \beta \vdash |\rho|$ with*

$$R \cong \mathcal{K}_{\alpha,\beta,\rho}$$

as graded algebras.

Moreover, there is a universal polynomial P such that, for every $\lambda, \mu, \nu \vdash m \geq 1$ and every ρ with at least $P(m)$ distinct positive parts, one can explicitly construct $\alpha, \beta \vdash |\rho|$ satisfying

$$\mathcal{K}_{\lambda,\mu,\nu} \cong \mathcal{K}_{\alpha,\beta,\rho}$$

as graded algebras.

The prescribed diagram is unchanged, and the isomorphisms preserve the original grading, without localization, degeneration, dilation or passage to a Veronese subalgebra. The second assertion would make a single staircase of polynomial size universal for all Kronecker stretching algebras of a given input size. The first would give every sufficiently corner-rich diagram the same realization capacity for any fixed invariant problem of a connected reductive group. By Theorem 5.2, each such algebra would also occur in rectangular cubic plethysm.

Appendix A. Restriction and stable plethysm

The preceding constructions isolated sectors in a fixed homogeneous representation. There is another use of the same face lemma, independent of the mixed tensor construction: one can order the sectors of the whole symmetric algebra by a negative cocharacter. Its Schur powers have a direct interpretation as restriction from GL_N to permutation matrices. We record the resulting realizations here, starting with the cubic sector of Theorem 1.1.

For $\sigma \vdash N$, define $r_\lambda^\sigma = \dim \mathrm{Hom}_{S_N}([\sigma], \mathbb{S}_\lambda \mathbb{C}^N)$. With $\mathcal{H} = \sum_{j\geq 0} h_j$, the character identity is

$$r_\lambda^\sigma = \langle s_\lambda, s_\sigma[\mathcal{H}] \rangle. \tag{29}$$

Indeed $\mathrm{Sym}^\bullet(V \otimes \mathbb{C}^N) \cong (\mathrm{Sym}^\bullet V)^{\otimes N}$ as a $\mathrm{GL}(V) \times S_N$-module, where S_N permutes the factors. Cauchy on the left and Schur–Weyl on the right prove (29) degree by degree; in particular no stability hypothesis is needed. This argument also shows that its coefficients are zero when $\ell(\lambda) > N$.

Theorem A.1 (Permutation invariants)**.** *For each $m \geq 1$ put $N = 1 + 3m + \binom{3m+1}{2} + D_m^+$. For every $\alpha, \beta, \gamma \vdash m$, define*

$$\Xi = \Theta^+ + ((3m+2)^{3m}), \qquad \Lambda = \Xi + (1^N),$$

where Θ^+ is from Corollary 3.1 and Ξ is padded to length N. Then

$$g(\alpha, \beta, \gamma) = r_\Xi^{(1^N)} = r_\Lambda^{(N)} = \dim(\mathbb{S}_\Lambda \mathbb{C}^N)^{S_N}.$$

Proof. Take the same $V = A \oplus B \oplus C$ as in the cubic proof but assign weights $(-6, -8, -9)$. On $\mathrm{Sym}^\bullet V$ the threshold -23 isolates $A \otimes B \otimes C$. Above it lie all degrees $0, 1, 2$ and the four cubic sectors used in Corollary 3.1; degrees ≥ 4 have weight at most -24. The higher sector therefore has dimension $N - m$ and its determinant differs from the earlier cubic determinant by $(\det V)^{3m+2}$, from degrees $1, 2$. Apply exterior saturation, then (3) and Lemma 2.1. The coefficient of s_Ξ in $e_N[\mathcal{H}]$ is $g(\alpha, \beta, \gamma)$; the partition is dominant and has length $3m \leq N$. Equation (29) gives the sign-isotypic dimension. Twisting by $\det \mathbb{C}^N$, which restricts to sign, gives the invariant dimension. □

The negative cocharacter used here separates both degrees and multidegrees. If we separate only degrees, a rectangular reservoir again allows us to prescribe an arbitrary Schur label. The resulting restriction coefficient is already stable, and its GL label can be fixed to a rectangle.

In the notation of stable restriction (see Orellana–Zabrocki [22]), put

$$\bar{r}_\tau^\kappa = \lim_{N \to \infty} r_\kappa^{(N - |\tau|, \tau)}.$$

For stable plethysm (see Bowman–Paget–Wildon [3]), write $\bar{p}(\tau, \kappa)$ for the eventual value of

$$\langle s_{(MN - |\kappa|, \kappa)}, s_{(N - |\tau|, \tau)}[h_M] \rangle \quad (M, N \to \infty).$$

Theorem A.2 (Stable plethysm and restriction)**.** *Let $b \geq 2$, $\mu \vdash d \geq 1$, $\lambda \vdash bd$, integers $r \geq \ell(\lambda)$, $K \geq \mu_1$, $c \geq \lambda_1 + K$. Put*

$$h = \sum_{j=2}^{b-1} \binom{r+j-1}{j}, \qquad u = \sum_{j=2}^{b-1} \binom{r+j-1}{j-1},$$
$$T = c + Ku, \qquad \kappa = (T^r), \quad \tau = (c - \lambda_r, \ldots, c - \lambda_1, K^h, \mu).$$

Then

$$\langle s_\lambda, s_\mu[h_b] \rangle = \bar{r}_\tau^\kappa = \bar{p}(\tau, \kappa).$$

For fixed b, the map has polynomial unary overhead if r, K, c are chosen polynomially in the input size. In fact the restriction and plethysm stabilize to this value whenever $M \geq rT$ and $N \geq rT + |\tau|$ (with M irrelevant for restriction). For fixed b, r, taking $K = \mu_1$, $c = \lambda_1 + K$ gives a homogeneous realization of the stretching ray $(t\lambda, t\mu)$.

Proof. Let $L = N - |\tau|$. Work in r variables, and grade $Y = \mathrm{Sym}^\bullet \mathbb{C}^r$ by polynomial degree. Use the cocharacter $t \mapsto t^{-1} I$, so its maximal face is the minimum-degree component. The Schur label (L, τ) fills the degree-zero sector, selects the complement of λ at degree one, saturates degrees $2, \ldots, b-1$ by rectangles of width K, and selects μ at degree b. Since $c - \lambda_1 \geq K \geq \mu_1$ and $L \geq \tau_1$, the blocks are dominant. The degree is bounded below by rT: the degree-one contribution is $rc - bd$, the intermediate degrees contribute rKu, and the last block contributes bd. If $\ell(\mu) \leq \dim \mathrm{Sym}^b \mathbb{C}^r$, the minimum-degree component given by (7) is

$$(\det \mathbb{C}^r)^T \otimes (\mathbb{S}_\lambda \mathbb{C}^r)^* \otimes \mathbb{S}_\mu(\mathrm{Sym}^b \mathbb{C}^r).$$

If $\ell(\mu) > \dim \operatorname{Sym}^b \mathbb{C}^r$, the excess rows must occupy strictly higher degrees; the component at rT is zero, as is $\mathbb{S}_\mu(\operatorname{Sym}^b \mathbb{C}^r)$. In either case the multiplicity of $(\det \mathbb{C}^r)^T$ is the original plethysm coefficient. By (29) it is $r_\kappa^{(L,\tau)}$, independently of $L \geq \tau_1$.

For stable plethysm expand the first row of Jacobi–Trudi:

$$s_{(MN-rT,\kappa)} = \sum_{j=0}^{r} (-1)^j h_{MN-rT+j} s_{\kappa/(1^j)}.$$

The adjoint of multiplication by $h_{MN-rT+j}$ extracts degree $rT-j$ after substituting $1+X$ into a homogeneous symmetric function of degree MN. Since $h_M[1+X] = \sum_{i=0}^{M} h_i[X]$, the coefficient in question is

$$\sum_{j=0}^{r} (-1)^j \langle s_{\kappa/(1^j)}, (s_{(L,\tau)}[\mathcal{H}])_{rT-j} \rangle.$$

We may evaluate in r variables, since any Schur function occurring in $s_{\kappa/(1^j)}$ has its diagram contained in κ by the Littlewood–Richardson rule. The minimum-degree computation kills every term with $j > 0$ and evaluates the remaining term. The bounds ensure $L \geq \tau_1$ and the two padded labels are partitions. Finally the displayed parameters scale homogeneously under the stated choice of K, c. □

Taking $b = 3$, $r = 3m$, $\mu = (1^{D_m^+})$, $\lambda = \Theta^+$, $K = 1$, $c = 3m^2 + 5m + 2$, we obtain the single fixed rectangle

$$\kappa_m = ((3m^2+8m+3)^{3m}), \qquad g(\alpha,\beta,\gamma) = \bar{r}_\tau^{\kappa_m} = \bar{p}(\tau, \kappa_m).$$

Here κ_m is independent of α, β, γ. Theorem A.1 instead fixes the symmetric-group representation to be trivial; this is the specialization we use next.

Appendix B. Two orbit closures

The invariant spaces in Theorem A.1 occur naturally in the multisymmetric algebra. This raises a geometric version of the preceding restriction question: can the same multiplicities be read from the coordinate ring of an explicit orbit closure? We give two constructions. The first uses the trace of multiplication operators to recover polarized power sums; the second recovers them by contraction on an open set and then clears denominators.

Let $V = \mathbb{C}^N$ and $B_N = \mathbb{C}[(V^*)^N]^{S_N}$, with the natural $\mathrm{GL}(V)$-action. By Cauchy, for $\lambda \vdash d$,

$$[\mathbb{S}_\lambda V : (B_N)_d] = \dim(\mathbb{S}_\lambda \mathbb{C}^N)^{S_N}. \tag{30}$$

The algebra B_N is generated by the entries of the polarized power sums $P_k = \sum_{i=1}^{N} v_i^{\otimes k}$, where $v_i \in V^*$. Indeed, products of power sums can be expanded by whether the indices coincide. Induction on the number of distinct indices expresses every symmetrized monomial as a linear combination of products of power sums. A homogeneous polynomial of degree d needs only generators of degrees at most d.

Theorem B.1 (Orbit closures)**.** *Let $m_N(e_i, e_j) = \delta_{ij} e_i$ and let $f_N(x) = \sum_i (x_i^2 + x_i^3)$. Put $Y_N = \overline{\mathrm{GL}(V) \cdot m_N}$ in $\operatorname{Hom}(V \otimes V, V)$ and $X_N = \overline{\mathrm{GL}(V) \cdot f_N}$ in $\operatorname{Sym}^2 V^* \oplus \operatorname{Sym}^3 V^*$. For every $\lambda \vdash d$ with $\ell(\lambda) \leq N$,*

$$[\mathbb{S}_\lambda V : \mathbb{C}[Y_N]_d] = \dim(\mathbb{S}_\lambda \mathbb{C}^N)^{S_N},$$
$$[\mathbb{S}_{\lambda + ((2d)^N)} V : \mathbb{C}[X_N]] = \dim(\mathbb{S}_\lambda \mathbb{C}^N)^{S_N}.$$

Consequently every Kronecker coefficient is an explicitly constructible orbit-closure multiplicity for each of these two families.

Proof. For coordinatewise multiplication the orbit consists of laws $m(x,y)=\sum_i u_i v_i(x)v_i(y)$ where (u_i) is a basis and (v_i) its dual. Its stabilizer is S_N, since the primitive idempotents of $\mathbb{C}^N$ must be permuted. Restriction to the orbit embeds its orbit-closure coordinate ring in $\mathbb{C}[\mathrm{GL}(V)/S_N]$. The polynomial-representation part of the latter is B_N: this follows from the Cauchy decomposition of the matrix coordinate ring, localizing at the determinant and taking right S_N-invariants. Indeed the algebraic Peter–Weyl decomposition of $\mathbb{C}[\mathrm{GL}(V)]$ contains each polynomial type with the same right multiplicity module as the matrix Cauchy decomposition. Taking right S_N-invariants preserves this equality. The scalar center acts by weight d on polynomial functions of degree d in the multiplication tensor.

To see that all of B_N occurs in the orbit closure, let $L_x(y)=m(x,y)$. On the orbit,

$$\mathrm{tr}(L_{x_1}\cdots L_{x_k})=\sum_i v_i(x_1)\cdots v_i(x_k).$$

The left side is polynomial of degree k in m, so every entry of P_k extends to the orbit closure. These generate B_N, giving equality of polynomial-type algebras and the first formula by (30).

For the second orbit, the map $(v_i)\mapsto(P_2,P_3)$ has image dense in X_N, as bases are dense in $(V^*)^N$. Hence $A=\mathbb{C}[X_N]$ identifies with the subalgebra of B_N generated by entries of P_2,P_3. Set $\Delta=\det P_2$; this is a semi-invariant of weight (2^N) on the coordinate ring. On the dense locus of bases, contraction of P_k and P_3 against P_2^{-1} in one index gives P_{k+1} for $k\geq 2$, since $v_i^{\mathsf{T}}P_2^{-1}v_j=\delta_{ij}$. Contracting two indices of P_3 with the same inverse gives P_1. Since ΔP_2^{-1} is polynomial, each entry of P_k becomes an element of A after multiplication by Δ^k (a weaker bound than necessary). It follows by homogeneous generation that

$$\Delta^d(B_N)_d\subseteq A\subseteq B_N,\qquad A[\Delta^{-1}]=B_N[\Delta^{-1}].$$

Multiplication by Δ^d twists the λ-isotypic component by $(\det V)^{2d}$. The multiplicity of the twisted type in A is at least that of $\mathbb{S}_\lambda V$ in B_N, and at most that of $\mathbb{S}_{\lambda+((2d)^N)}V$ in B_N. These bounds coincide by (30), since $(\det\mathbb{C}^N)^{2d}$ is trivial on S_N. This proves the second formula. The degree-dependent padding $f\mapsto\Delta^d f$ for $f\in(B_N)_d$ also defines an injective algebra homomorphism (the partition sizes, equivalently the B_N-degrees, become $(2N+1)d$). All partition sizes and padding exponents are polynomial in the input size of Theorem A.1. □

AI statement

These results grew out of my earlier study and work on Kostka–Foulkes polynomials and Lusztig's q-analogues, which began in August 2026. Encouraged by recent advances in AI-assisted mathematics, I returned to these ideas in November while investigating asymptotic bounds for Kronecker and plethysm coefficients.

The mathematical arguments were developed primarily using GPT-6.2 at high reasoning effort. My role consisted primarily of supplying earlier work and ideas, formulating questions, directing the investigations, and seeking stronger and more general results. I understand parts of the arguments, and several directions that I do not yet sufficiently understand have been left as conjectures, even where the models supplied purported proofs.

I am sharing these results because I find the mathematical connections they reveal interesting in their own right. Many of the proofs are relatively short, and the underlying methods appear to have considerable scope beyond their immediate applications. I hope these results stimulate discussion, offer new perspectives on existing problems, and inspire further mathematics.

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