Primitive resonant Rossby wavevectors at every nonzero zonal wavenumber

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Abstract

We prove Kopp's Conjecture 6 for the Rossby dispersion relation on the square torus at infinite deformation radius. For every fixed nonzero integer zonal wavenumber a, there are infinitely many positive integers b such that (a, b) belongs to a primitive resonant triad with three nonzonal modes. We can require $\gcd(a, b) = 1$. Under this requirement the admissible residue classes $c \bmod m$ for $m > 0$ are exactly those with $\gcd(c, \gcd(a, m)) = 1$, and each contains arbitrarily large such b. The construction expresses the resonance condition through a quadratic discriminant and uses Pell solutions to make that discriminant a square. The main results are formalized in Lean 4.

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1 The fixed-wavenumber problem

For the Charney–Hasegawa–Mima equation on a square periodic domain at infinite deformation radius, a mode with wavevector $(a, b) \in \mathbb{Z}^2 \setminus \{0\}$ has frequency proportional to $a/(a^2 + b^2)$. Identifying the vector with $w = a + \mathrm{i}b$ and removing the common factor gives $\omega(w) = \mathrm{Re}(1/w)$. A triad with output w and inputs $z, w - z$ is resonant if

$$\omega(z) + \omega(w - z) = \omega(w). \tag{1}$$

We require all three modes to be nonzonal. For $z = x + \mathrm{i}y$ this means $ax(a - x) \neq 0$. The triad is primitive if $\gcd(a, b, x, y) = 1$. Write Λ' for the set of integer outputs (a, b) of such triads, as in [2, (1.20)]. In particular, a nonzonal resonant triad whose output has coprime coordinates is primitive.

Bustamante and Hayat classified resonant triads using quadratic forms [1]. The pure-cube family gives outputs (s^4, st^3) on the lines $a = s^4$ [2, (1.16)–(1.18)]. Kopp also proved that each rational line through the origin with nonzero finite slope contains infinitely many points of Λ' [2, Theorem 4]. His Conjecture 6 asks whether $\{b \in \mathbb{Z} : (a, b) \in \Lambda'\}$ is infinite for every fixed nonzero a [2, p. 358]. We prove this and determine which residue classes can be reached with coprime output coordinates.

Theorem 1. *Fix a nonzero integer a. For every positive integer m, every integer c coprime to a, and every $H > 0$, there are integers b, x, y with $b > H$ such that*

$$b \equiv c \pmod{m}, \qquad \gcd(a, b) = 1,$$

and $w = a + \mathrm{i}b$, $z = x + \mathrm{i}y$, $w - z$ form a fully nonzonal resonant triad. In particular, there are infinitely many primitive resonant wavevectors on $\{a\} \times \mathbb{Z}$.

Corollary 2. *Fix $a \neq 0$ and $m > 0$. There are arbitrarily large $b \equiv c \pmod{m}$ with $\gcd(a, b) = 1$ and $(a, b) \in \Lambda'$ if and only if $\gcd(c, \gcd(a, m)) = 1$.*

Proof. A prime dividing a, m, and c divides every $b \equiv c \pmod{m}$. Conversely, assume $\gcd(c, \gcd(a, m)) = 1$. Let L be the product of the primes dividing a that do not divide c, with empty product one. Put $c' = c + mL$. If $p \mid a$ and $p \nmid c$, then $p \mid L$ and $c' \equiv c \not\equiv 0 \pmod{p}$. If $p \mid a$ and $p \mid c$, then $p \nmid L$ and the assumed gcd condition gives $p \nmid m$. Thus $p \nmid c'$ in either case. Apply Theorem 1 to c'. □

2 From the resonance equation to a Pell equation

Let a, C be positive integers and put $D = a^2 + C^2$. We will choose b of the form $C + DT$ with $T > 0$. Set $q = z(w - z)/w^2$. The frequency defect before taking real parts is $(q^{-1} - 1)/w$. To make it purely imaginary, choose $q = -T(C + \mathrm{i}a)$, which gives

$$\frac{1}{z} + \frac{1}{w - z} - \frac{1}{w} = \frac{-C + \mathrm{i}a - DT}{DTw} = \frac{\mathrm{i}}{DT}. \tag{2}$$

The last equality follows from $\mathrm{i}w = -b + \mathrm{i}a$. The normalized input z/w must solve $u(1 - u) = q$, with discriminant $1 - 4q$. Writing its square root as $W + \mathrm{i}aR$ gives

$$T = RW/2, \qquad W^2 - a^2R^2 - 1 = 2CRW.$$

The second equation is the Pell equation $P^2 - DR^2 = 1$ after setting $P = W - CR$. The next lemma records the parity and nonzonal conditions needed to turn these identities into a triad.

Lemma 3. *Suppose P, R are positive integers satisfying $P^2 - DR^2 = 1$ with R even. Put*

$$W = P + CR, \qquad T = RW/2, \qquad b = C + DT, \qquad w = a + \mathrm{i}b.$$

Then $z = w(1 - W - \mathrm{i}aR)/2$ is a Gaussian integer. These choices give a fully nonzonal resonant triad, and $\operatorname{Re} z > a$.

Proof. The Pell equation makes P odd. Thus W is odd and $z \in \mathbb{Z}[\mathrm{i}]$. Expanding the norm equation in terms of W gives $W^2 - a^2R^2 - 1 = 2CRW$. Consequently

$$z(w - z) = \frac{w^2}{4}\bigl(1 - (W + \mathrm{i}aR)^2\bigr) = -T(C + \mathrm{i}a)w^2.$$

All factors on the right are nonzero, so (2) applies. For the real part of the input we have

$$\frac{2\operatorname{Re} z}{a} = 1 - W + bR = 1 - P + \tfrac{1}{2}(P^2 - 1)W.$$

Since $R \geq 2$ and $D \geq 2$, the Pell equation gives $P \geq 3$. Also $W \geq P$, so $(P^2{-}1)W/2 \geq 4P > P{+}1$. This proves $\operatorname{Re} z > a$ and hence $\operatorname{Re}(w - z) < 0$. □

3 Infinitude in each admissible progression

Proof of Theorem 1. First take $a > 0$. Write $S = am$ and choose an integer $C > a^2$ with $C \equiv c \pmod{S}$. For $D = a^2 + C^2$ we have $C^2 < D < (C + 1)^2$, so D and DS^2 are nonsquares. Pell's theorem gives positive integers U, V with $U^2 - DS^2V^2 = 1$ [3]. In $\mathbb{Z}[\sqrt{D}]$ take

$$(U + SV\sqrt{D})^{2j} = P_j + R_j\sqrt{D}, \qquad j \geq 1.$$

Taking norms gives $P_j^2 - DR_j^2 = 1$. The coefficient of $\sqrt{D}$ in the square of $U + SV\sqrt{D}$ is $2USV$, so it stays divisible by $2S$ in every positive power of that square. The coefficients are positive and $R_{j+1} \geq P_1R_j$ with $P_1 > 1$. By Lemma 3, $T_j = R_j(P_j + CR_j)/2$ is a multiple of S and $b_j = C + DT_j \to \infty$. Thus $b_j \equiv c \pmod{am}$, which gives $b_j \equiv c \pmod{m}$ and $\gcd(a, b_j) = \gcd(a, c) = 1$.

For $a < 0$, apply the positive case to $|a|$ and reflect all three wavevectors by $v \mapsto -\bar{v}$. This fixes each meridional component, reverses the zonal components and negates every frequency, preserving (1). □

4 An explicit orbit and its verification

To obtain an infinite family on a given line $a > 0$, take $C = 1$ and $D = a^2 + 1$. A positive unit of norm one is

$$(a + \sqrt{a^2+1})^2 = (2a^2+1) + 2a\sqrt{a^2+1}.$$

Every positive power has $2a \mid R_j$, giving $b_j \equiv 1 \pmod{a}$. At $a = 2$ the first power has $(P, R) = (9, 4)$ and the construction gives

$$(512, -794) + (-510, 925) = (2, 131).$$

The companion Python script generates terms of this orbit and checks the frequency equality with exact rational arithmetic. To reproduce the example, run

```
python3 anc/check_rossby_orbit.py --zonal 2 --terms 1
```

Formalization. The Lean 4 development uses the rational dispersion relation on integer wavevectors. The declarations `kopp_conjecture_six`, `primitive_progression_unbounded`, and `admissible_progressions` establish the original conjecture, Theorem 1, and Corollary 2, respectively. For the progression result the formal proof imposes the requested height by scaling the Pell discriminant. It also verifies the $C = 1$ orbit and the example $(2, 131)$. The source and build instructions are available in the versioned GitHub repository and the Zenodo archive.

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