THE ENDPOINT CONSTANT FOR AN EXTREMAL IID SUM INEQUALITY

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Abstract. Let $S_2(c)$ be the supremum of $\mathbb{P}(X+Y<cZ)$ over all common laws of three independent nonnegative random variables. As c approaches 2 from below, we prove

$$\frac{\log(1/(2-c))}{\log c}\left(\frac{2}{3}-S_2(c)\right)\longrightarrow\frac{4}{9}.$$

Counting losses in geometric bands below a quantile gives the upper bound for arbitrary laws. A finite ladder with optimized masses gives the matching first-order lower bound. At the mth multinacci threshold c_m, splitting each ladder site into two points raises the lower bound above $2/3-4/(9m)$ by at least a positive constant times $\log(m)/m^2$ for all sufficiently large m. At the comparison level set by our upper bound at c_m, the band estimate reaches $\Theta(\sqrt{m\log m})$ consecutive thresholds, and this order is optimal within the method. The main results are formalized in Lean 4.

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1. The extremal probability

Let X,Y,Z be independent with common Borel probability law μ on $[0,\infty)$. For $1<c<2$ write

$$P_c(\mu)=\mathbb{P}(X+Y<cZ),\qquad S_2(c)=\sup_{\mu}P_c(\mu),\qquad L(c)=\frac{\log(1/(2-c))}{\log c}.$$

If a smallest observation is chosen as the target, the other two sum to at least twice its value. Exchangeability therefore gives $S_2(c)\le 2/3$. How quickly can the best law close this gap as c approaches 2?

Bellec and Fritz [1] studied this family of iid linear inequalities. For the two-summand function S_2, Zaytsev and Bellec [3, §§6.1, 7.1.3] certified eighteen discontinuities, constructed finite ladders, and asked whether the discontinuities are infinite in number. Earlier estimates [2] bounded the liminf and limsup of $L(c)(2/3-S_2(c))$ below by $1/3$ and above by $1/2$, respectively. We determine the limit.

Theorem 1.1. *For the supremum over all Borel probability laws on* $[0,\infty)$,

$$\lim_{c\uparrow 2}L(c)\big(2/3-S_2(c)\big)=\frac{4}{9}.$$

Proof idea. To bound the gap from $2/3$, we count triples for which the middle observation also loses. Grouping observations into geometric bands below a quantile gives a lower bound on this extra loss in terms of the band masses. A Cauchy–Schwarz estimate and the integral $\int_0^1\sqrt{u}\,du=2/3$ give the constant $4/9$ in the upper bound for S_2. In the opposite direction, optimizing the weights on a finite ladder gives a scalar recursion, and its first-order asymptotic yields the matching constant.

For $m\ge 2$ let c_m be the solution in $(1,2)$ of $(2-c_m)c_m^m=1$. Equivalently, $\sum_{j=1}^m c_m^{-j}=1$. This sum decreases strictly in its argument and increases in m. Consequently c_m is unique

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and $c_m \uparrow 2$. These multinacci coefficients are the ordinary ladder thresholds of [3, §6.1.7]. They satisfy $L(c_m) = m$.

Theorem 1.2. *For all sufficiently large integers m,*

$$S_2(c_m) \;\geq\; \frac{2}{3} - \frac{4}{9m} + \frac{\log(m-3)}{1000m^2}.$$

Sections 2 and 3 prove the endpoint limit. The construction in Section 4 proves Theorem 1.2 and yields a lower bound on the size of a finite block that improves on $S_2(c_m)$ (Section 5). Section 6 determines how far the band estimate extends to the right of c_m at the comparison level used for the endpoint bound.

2. A uniform upper bound from quantile bands

For a sample of three values let J count the target positions for which the other two values sum to less than c times the target. Then $2 - J$ counts the losses in excess of the one forced by the smallest observation. Exchangeability gives $\mathbb{E}(2 - J) = 3(2/3 - P_c(\mu))$.

We first recall a bound from [2]. If $1 < e < 2$, then

$$2/3 - S_2(e) \geq \frac{1}{3\lceil L(e)\rceil}. \tag{1}$$

Indeed, put $h = \lceil L(e)\rceil$, $u_0 = 0$, and $u_{i+1} = (1 + u_i)/e$. As $u_i = (1 - e^{-i})/(e - 1)$, h is the first index with $u_h \geq 1$. For an atomless law, condition on the maximum z and its index. In their original coordinate order the other two draws are independent with law truncated below z. Divide $[0, z]$ at the $h - 1$ interior cutoffs $u_i z$. If the two draws $x \leq y$ are in the same interval, then $ey \leq eu_{i+1}z = (1 + u_i)z \leq x + z$, and the middle target loses. The conditional probability of a shared interval is at least $1/h$, by Cauchy–Schwarz applied to the interval probabilities. This proves (1) for atomless laws. Adding independent uniform noise on $[0, \delta]$ to each draw makes any law atomless. Under the coupling $X \mapsto X + \delta U$ with U uniform on $[0, 1]$, every strict win persists for all sufficiently small δ. Fatou's lemma transfers the estimate back to the original law.

Proposition 2.1. *Suppose $r > 1$ is real and d, K are positive integers such that $r(1+r^{-d}) \leq c < 2$ and $r^{-(K+d)} \leq 2 - c$. Then*

$$S_2(c) \leq \frac{2}{3} - \frac{4}{9(\sqrt{K} + 4\sqrt{d})^2}. \tag{2}$$

Proof. Take μ atomless, write $H = 2/3 - P_c(\mu) > 0$ and set $s = 2 - c$. If $dH \geq 1$ the inequality follows immediately. Otherwise let $\varepsilon = (dH)^{1/3}$, and choose $z > 0$ with $\mu([0, z]) = 1 - \varepsilon$. For $i \geq 0$ put

$$t_i = 1 - r^{-i}, \qquad T_i = \mu([t_i z, \infty)), \qquad a_i = T_i - T_{i+1}.$$

Here $T_0 = 1$ and $T_i \geq \varepsilon$. Since

$$t_i + t_{i+d} - ct_{i+1} = s + r^{-i}(c/r - 1 - r^{-d}) \geq 0,$$

a triple with two values in $[t_i z, t_{i+1} z)$ and the third at least $t_{i+d} z$ loses at its middle target. The third value is outside the ith band, so the events for different i are disjoint. The event for band i has probability $3a_i^2 T_{i+d}$. Dividing the resulting bound on $\mathbb{E}(2 - J)$ by three gives, for every n,

$$H \geq \sum_{i<n} a_i^2 T_{i+d} \geq \varepsilon \sum_{i<n} a_i^2. \tag{3}$$

All unqualified band sums start at $i = 0$.

The interval $[t_K z, z]$ has mass $q = T_K - \varepsilon$. If $q > 0$, condition all three draws to lie there and rescale each as $U = (X/z - 1 + \eta)/\eta$, with $\eta = r^{-K}$. A win implies

$$U_1 + U_2 < cU_3 - s(1 - \eta)/\eta \leq (2 - s/\eta)U_3 \leq (2 - r^{-d})U_3.$$

The first weak inequality uses $U_3 \le 1$, and the second uses $r^{-(K+d)} \le s$. For $e = 2 - r^{-d}$ we have $e > r$ since $r + r^{1-d} < 2$ and $r > 1$. Thus $L(e) = d \log r / \log e \le d$. By (1) and nonnegativity of $2 - J$ off this event,

$$q^3 \le 3(d+1)H \le 6dH. \tag{4}$$

This also holds when $q = 0$.

We estimate the difference produced by the shift of d indices. The square-root difference formula, $2uv \le u^2 + v^2$, and (3) give

$$\begin{aligned}\sum_{i<K} a_i(\sqrt{T_i} - \sqrt{T_{i+d}}) &\le \frac{1}{2\sqrt{\varepsilon}} \sum_{j=0}^{d-1} \sum_{i<K} a_i a_{i+j} \\ &\le \frac{d}{2\sqrt{\varepsilon}} \sum_{i<K+d} a_i^2 \le \frac{dH}{2\varepsilon^{3/2}}.\end{aligned}$$

By Cauchy–Schwarz, $\sum_{i<K} a_i \sqrt{T_{i+d}} \le \sqrt{KH}$. Using upper sums for the integral of $\sqrt{t}$ yields

$$\tfrac{2}{3}(1 - T_K^{3/2}) = \int_{T_K}^1 \sqrt{t}\, dt \le \sum_{i<K} a_i \sqrt{T_i} \le \sqrt{KH} + \tfrac{1}{2}\sqrt{dH}.$$

By (4), $T_K \le (1 + 6^{1/3})(dH)^{1/3}$. Since $(2/3)(1 + 6^{1/3})^{3/2} + 1/2 < 4$, we get $2/3 \le (\sqrt{K} + 4\sqrt{d})\sqrt{H}$. Smoothing as above and taking the supremum over μ proves (2). □

Corollary 2.2. *For $m \ge 32$ set $D_m = 1 + \log(4m)/\log(3/2)$. Then*

$$S_2(c_m) \le C_m := \frac{2}{3} - \frac{4}{9(\sqrt{m+2} + 4\sqrt{D_m})^2}.$$

Proof. Take $r = 2 - 1/m$, $d = \lceil \log(4m)/\log(3/2) \rceil$ and $K = m + 2 - d$. Since $(3/2)^m \ge 4m$ for $m \ge 32$, we have $1 \le d \le m$, $r^{-d} \le 1/(4m)$, and $2 - c_m = c_m^{-m} < (3/2)^{-m} \le 1/(4m)$. Here $c_m \ge c_2 = (1+\sqrt{5})/2 > 3/2$. Thus $r(1+r^{-d}) \le 2 - 1/(2m) \le c_m$. Bernoulli's inequality gives $(r/2)^m \ge 1/2$. Since $r^2 \ge 2$, $r^{K+d} = r^{m+2} \ge 2^m \ge c_m^m$. Apply Proposition 2.1 using $K \le m + 2$ and $d \le D_m$. □

3. The finite ladder and the matching bound

We use the ladder pattern of Zaytsev and Bellec [3, §6.1.7]. Give N ordered sites masses $p_1, \dots, p_N$, and draw the site indices i, j, k independently. A *rank win* occurs if at least one input index is smaller than the target index, that is, $\min(i, j) < k$. Write P_N for the maximum probability of a rank win over the closed simplex of masses. With $T_k = \sum_{j=k}^N p_j$, the objective is $\sum_{k=1}^N p_k(1 - T_k^2)$. For increasing positive sites and $c < 2$, a weak win $y_i + y_j \le c y_k$ is a rank win.

Conversely, suppose $1 < e < 2$ and $(2 - e)e^N \le 1$. Take $y_k = \sum_{j=1}^k e^{-j}$ for $1 \le k \le N$. Then $y_N \le 1$, $e y_1 = 1$, and $1 + y_{k-1} = e y_k$ for $k \ge 2$. Every pair satisfies $y_i \le e y_k$, and every rank win satisfies $y_i + y_j \le e y_k$. Subtracting the same positive number less than y_1 from all sites makes every designated triple comparison strict, because its two sides decrease by 2 and e times that number, respectively. In particular,

$$S_2(c_m) \ge P_m. \tag{5}$$

Lemma 3.1. *Put $x_N = 2 - 3P_N$ and $t_N = \sqrt{1 + x_N}$. Then $x_1 = 2$, $x_N > 0$, and*

$$x_{N+1} = \frac{2x_N}{t_N(1 + t_N)}, \tag{6}$$

$$\frac{1}{x_{N+1}} - \frac{1}{x_N} = \frac{t_N + 2}{2(t_N + 1)} = \frac{3}{4} - \frac{x_N}{4(1 + t_N)^2}. \tag{7}$$

In particular, $N x_N \longrightarrow 4/3$.

Proof. Separate the smallest site of an $(N+1)$-site law, leaving mass b above it. When the target lies above the smallest site and at least one input lies at that site, the triple wins. Triples wholly above it contribute at most $b^3 P_N$, with equality at the optimal conditional law. Thus $P_1 = 0$ and $P_{N+1} = \max_{0\le b\le 1}\{b - (1-P_N)b^3\}$. Given $x_N > 0$, differentiation yields the unique maximizer $b = 1/t_N$ and $P_{N+1} = 2/(3t_N)$. This gives positivity and (6) by induction, and rearrangement gives (7). The reciprocal increments lie between $1/2$ and $3/4$, so $x_N \to 0$. Their limit is $3/4$. Taking arithmetic means proves the assertion. □

Proof of Theorem 1.1. At $c = c_m$, Corollary 2.2 and $D_m = O(\log m)$ give

$$\liminf_{m\to\infty} m(2/3 - S_2(c_m)) \ge 4/9.$$

Equation (5) and Lemma 3.1 give the matching upper limit. For $c_m \le c \le c_{m+1}$, the first m terms of $\sum_{j\ge1} c^{-j}$ sum to at most one, and the first $m+1$ terms sum to at least one. Hence $m \le L(c) \le m+1$. Monotonicity of S_2 squeezes the product between $m(2/3 - S_2(c_{m+1}))$ and $(m+1)(2/3 - S_2(c_m))$. Both tend to $4/9$. □

4. Splitting each site into two points

The rank model assigns zero probability of success to triples sampled from a single site. We use two nearby points at each site to recover part of this loss while preserving every rank win. The amount recovered depends on the cubic sum of the optimal masses.

Lemma 4.1. *There are maximizing masses $p_1, \dots, p_N$ for P_N with*

$$\sum_{i=1}^{N} p_i^3 \ge \tfrac{1}{8} x_N^2 U_N, \qquad U_N = \sum_{k=1}^{N} x_k \ge \log(N+1).$$

Proof. At stage $j \ge 2$ of the recursion in Lemma 3.1, the previous sites retain the fraction $b_j = 1/t_{j-1}$ and the newly inserted smallest site receives $a_j = 1 - b_j = x_j/2$. The latter formula also holds for the initial site, with $a_1 = 1$. Numbering sites by the stage of insertion, their masses are $a_k \prod_{j=k+1}^{N} b_j$. By (6), the inequality $b_j^3 \ge (x_j/x_{j-1})^2$ is equivalent to $(1+t_{j-1})^2 \ge 4t_{j-1}$. Multiplying these inequalities bounds the cube of the kth mass below by $x_N^2 x_k/8$. Finally (7) implies $1/x_k \le 1/2 + 3(k-1)/4 \le k$. Summing $x_k \ge 1/k$ yields the claim. □

Proposition 4.2. *With $N = m - 4$, for all sufficiently large m,*

$$S_2(c_m) \ge P_N + \tfrac{3}{64} x_N^2 U_N. \tag{8}$$

Proof. Put $c = c_m$, $s = 2 - c$, $e = 2 - 8s$, and $a = 1 - 2s$. For large m we have $0 < a < 1$, $1 < e < ca$ and $c^4 \ge 8$. Hence $(2-e)e^N \le 8sc^{m-4} = 8/c^4 \le 1$. Use the N sites realizing the weak rank comparisons at e from Section 3, with maximizing masses p_i. Conditional on site y_i, give each of ay_i and y_i probability $1/2$, independently for each draw. Every coarse rank win survives strictly, since its perturbed input sum is at most $ey_k < cay_k$. When all three sites coincide, a high target with at least one low input wins, as $1 + a = 2 - 2s < c$. The conditional success probability of these events is $3/8$. Their coarse indices coincide, so they are disjoint from the rank wins. We have $S_2(c_m) \ge P_N + (3/8)\sum_i p_i^3$. Apply Lemma 4.1. □

Proof of Theorem 1.2. By (7) and $1 + t_k \ge 2$,

$$\tfrac{1}{2} + \tfrac{3}{4}(N-1) - U_N/16 \le 1/x_N \le \tfrac{3}{4}N.$$

For $N = m - 4$, this implies $0 \le 3m/4 - 1/x_N \le 4 + U_N/16$. Because $4/(3m) \le x_N$,

$$0 \le x_N - \frac{4}{3m} = x_N \frac{4}{3m}\left(\frac{3m}{4} - \frac{1}{x_N}\right) \le x_N^2(4 + U_N/16).$$

In (8) substitute $P_N = 2/3 - x_N/3$. The result is

$$S_2(c_m) - \frac{2}{3} + \frac{4}{9m} \ge x_N^2\left(\frac{5}{192}U_N - \frac{4}{3}\right).$$

Now $U_N \geq \log(m-3)$ and $x_N \geq 1/m$. For large m the parentheses exceed $\log(m-3)/1000$, as required. □

5. A lower bound on the size of an improving block

Let ν be a probability law on ℓ positive sites. For independent X, Y, Z with law ν, define

$$B_c(\nu) = \mathbb{P}(X + Y \leq cZ), \qquad Q_c(\nu) = \mathbb{P}(X \leq cZ).$$

For $r \geq 0$ the weak block score is

$$W_c(\nu, r) = \frac{B_c(\nu) + 2rQ_c(\nu) + r^2}{1 + 3r + 3r^2}.$$

To see the role of this expression, assign a lower-scale copy relative mass r against an upper block of mass one. At a coefficient just above c, all the weak comparisons of a finite block become strict. If the target lies in the upper block, the terms in the numerator count guaranteed wins with zero, one, or two lower inputs. The denominator $(1+r)^3 - r^3$ removes triples wholly in the lower copy, where the construction repeats. We call a block improving at c when $W_c(\nu, r) > S_2(c)$.

Independently of the placement of the sites, $B_c(\nu) \leq P_\ell$ by rank order, and $Q_c(\nu) \leq 1$. Thus

$$W_c(\nu, r) \leq H_\ell, \qquad H_\ell = \sup_{v \geq 0} \frac{P_\ell + 2v + v^2}{1 + 3v + 3v^2}. \tag{9}$$

The complete weak rank pattern on m sites realizes this upper bound at c_m after taking the supremum over r, since every pair comparison holds there. The next theorem shows that $S_2(c_m)$ exceeds this benchmark even when a fixed number of sites is added to the block and m is sufficiently large.

Theorem 5.1. *For every fixed integer $k \geq 0$, $S_2(c_m) > H_{m+k}$ for all sufficiently large m. More quantitatively, for all sufficiently large m, every improving block at c_m has size*

$$\ell > m - 7 + \tfrac{9}{64}\log(m-3).$$

Proof. For large j, $x_j \leq 1/2$. The excess over P_j of the fraction in (9) has numerator $x_j v - (1-x_j)v^2$ and denominator $1 + 3v + 3v^2$. Bounding the positive part by the maximum of the numerator yields $0 \leq H_j - P_j \leq x_j^2/[4(1-x_j)] \leq x_j^2$. Equation (7) gives $x_i - x_{i+1} \leq x_i^2$. Since the x_i decrease, for $j \geq N = m - 4$ we obtain

$$H_j \leq P_N + (1 + (j-N)/3)x_N^2.$$

For $j < N$ use $H_j \leq H_N$, by monotonicity of P_j. By Proposition 4.2 the gain above P_N, divided by x_N^2, is at least $(3/64)U_N \geq (3/64)\log(m-3)$. This proves $S_2(c_m) > H_{m+k}$ for fixed k and large m. If ℓ is the size of an improving block, then $\ell > N$ for large m, and

$$(3/64)\log(m-3) < 1 + (\ell - N)/3.$$

Rearrangement proves the stated bound. □

The strict event in the definition of S_2 gives left continuity. By monotone convergence, for each law the score at c is the supremum of its scores at $d < c$. The suprema over d and over the law commute. A jump at c_m thus requires $S_2(c_m+) > S_2(c_m)$. Theorem 5.1 excludes a growing range of block sizes as witnesses of this strict inequality.

6. Reach of the band estimate

Corollary 2.2 bounds $S_2(c_m)$ by C_m. We now ask how far one application of Proposition 2.1 can keep the upper bound at or below this same comparison level. The first result gives a smaller bound on an explicit interval. We then estimate the largest offset among later multinacci thresholds that this choice of comparison level allows.

Corollary 6.1. *For $m \geq 32$ let $d_m = \lceil \log(4m)/\log(3/2) \rceil$ and set*

$$B_m = \frac{2}{3} - \frac{4}{9(\sqrt{m+2} + 4\sqrt{d_m})^2}.$$

Then $B_m < C_m$, and $S_2(x) \leq B_m$ throughout $[c_m, 2 - (2 - c_m)/(4m)]$.

Proof. Keep $r = 2 - 1/m$ from Corollary 2.2 and take $K = m + 2$. We have $r(1 + r^{-d_m}) \leq c_m$ and $r^{-(K+d_m)} \leq (2 - c_m)/(4m)$. Proposition 2.1 applies on the stated interval, and $d_m < D_m$ gives $B_m < C_m$. □

Proposition 6.2. *Let $m \geq 32$ and let $j \geq 0$ be an integer. If $4j^2 \leq m d_m$, then*

$$S_2(x) \leq B_m \quad (c_m \leq x \leq 2 - 2^{-(m+j)}), \qquad S_2(c_{m+j}+) \leq B_m.$$

Conversely, if Proposition 2.1 applies at c_{m+j} with a bound at most C_m, then

$$j < 2 + 8\sqrt{(m+2)D_m} + 16D_m.$$

The largest offset j attainable at this level has order $\sqrt{m \log m}$, even if the bound is evaluated at a larger coefficient and transferred to c_{m+j} by monotonicity.

Proof. Write $d = d_m$ and take $r = 2 - 1/m$, $e = \lceil 3d/4 \rceil$ and $K = m + 2 + j$. Since $8 \leq d \leq m$, we have $e \leq 7d/8 \leq (15/16)^2 d$. Also $r \geq 7/4$ and $r^3 \geq (3/2)^4$, whence $r^e \geq (3/2)^d \geq 4m$. It follows that $r(1 + r^{-e}) \leq 2 - 1/(2m) \leq c_m$. Using $j \leq m$ and the estimates $(r/2)^m \geq 1/2$ and $r^{m+2} \geq 2^m$ from Corollary 2.2, we obtain $r^{K+e} \geq 2^{m+j}$. The choice of j gives

$$\sqrt{K} - \sqrt{m+2} \leq \frac{j}{2\sqrt{m+2}} \leq \frac{\sqrt{d}}{4}, \qquad 4\sqrt{e} \leq \frac{15}{4}\sqrt{d}.$$

The resulting bound in Proposition 2.1 is at most B_m. Since $2 - c_{m+j} = c_{m+j}^{-(m+j)} > 2^{-(m+j)}$, the interval contains a right neighborhood of c_{m+j}.

For the converse, take any admissible parameters r, K, e at $u = c_{m+j}$. The width condition gives $r^{K+e} \geq u^{m+j}$, and the other condition gives $r < u$. Hence $m + j < K + e$. A bound at most C_m requires $\sqrt{K} + 4\sqrt{e} \leq \sqrt{m+2} + 4\sqrt{D_m}$. Squaring and using $K + e \leq (\sqrt{K} + 4\sqrt{e})^2$ proves the displayed upper bound. If the estimate is taken at $v \in [c_n, 2)$ with $n = m + j$, then $r < v < 2$ and $r^{K+e} \geq (2 - v)^{-1} \geq c_n^n$. Corollary 2.2 gives $c_n^n \geq (2 - 1/n)^n \geq 2^{n-1}$, so $n \leq K + e$ by integrality. This yields the same upper bound with a weak inequality. The order claim follows since d_m and D_m are both asymptotic to $\log m/\log(3/2)$. □

These bounds quantify how far the band argument extends from its starting threshold. The exact second-order endpoint asymptotic and the existence of infinitely many discontinuities for the unrestricted function remain open.

Formal verification and reproducibility

The Lean 4 development formalizes the endpoint limit, the logarithmic lower correction, the support-size bounds, and the estimates for the reach of the band argument. The theorem index `EndpointAsymptotics.lean` identifies the corresponding declarations. In the formal rank recursion, the index counts one fewer than the number of sites. The pinned source, build instructions, and axiom checks are available in the version 1.1.0 release and its Zenodo archive. The accompanying `anc/ladder_values.py` evaluates the rank recursion numerically using the Python standard library.

Author contributions and use of AI

Peiwei Wu and Qian Peng are responsible for the mathematical claims and the manuscript. OpenAI Codex assisted with developing the mathematical arguments, writing and checking the Lean 4 formalization, and drafting and revising the text.

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