The ordinary Chow rank of the permanent in orders three through ten

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Abstract

Over every field of characteristic zero we obtain ordinary product rank 2^{n-1} for the permanent of a generic n-by-n matrix for $3 \le n \le 10$. Each factor may involve all matrix entries. For every order we prove a lower bound with asymptotic ratio $1/\sqrt{2}$ to Glynn's upper bound. The proof chooses one quadratic annihilator for all summands and compares the kernel on each parity with the apolar algebra of the permanent. The dimensions refine according to the multiplicities of proportional factors. This yields the cubic and quartic values directly. The sextic case follows by a quadratic dimension count and a bound for cubic subproducts. Higher orders require estimates for products on a given linear support and for the joint contributions of subsets of summands.

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Let K be a field of characteristic zero. Let $P_n = \sum_{\sigma \in S_n} \prod_i x_{i,\sigma(i)}$ in the n^2 independent entries of a matrix X. Its ordinary Chow rank, also called product rank, is the least N for which an identity over K exists of the form

$$P_n = \sum_{i=1}^{N} c_i \prod_{a=1}^{n} L_{ia}, \qquad c_i \in K^\times, \quad L_{ia} \in \operatorname{span}_K\{x_{rs} : 1 \le r, s \le n\}.$$

Proportional and linearly dependent factors are allowed. The equality is a polynomial identity; each coefficient can be absorbed into one factor.

Main theorem. *For every characteristic-zero field K and every $3 \le n \le 10$, the ordinary Chow rank of P_n is 2^{n-1}.*

Order n	3	4	5	6	7	8	9	10
Exact rank	4	8	16	32	64	128	256	512
Uniform lower bound, rounded up	4	8	13	31	50	115	193	440
Proof begins on page	6	6	19	20	21	25	30	33

The second row of estimates comes from Theorem 2.3 alone. The proofs of the exact values use the sign construction over K. For a lower bound one may also work over an algebraic closure, since a decomposition over K would remain valid there. Arguments using one operator on an entire decomposition include its existence over K. The local calculations for orders nine and ten appear in Appendix A (p. 37) and Appendix B (p. 82).

What changes when factors may mix rows. In the row tensor model each summand has one linear form in each row. Glynn's formula attains 2^{n-1} terms under this restriction and therefore also gives an ordinary Chow decomposition. The reverse inequality is the issue: an ordinary product can use several factors involving the same row, and terms of higher degree in that row can cancel between summands. For $3 \le n \le 10$ the main theorem shows that admitting these products saves no terms over any field of characteristic zero. In particular, even with all one hundred entries available to each factor, 511 products cannot represent P_{10}. Liu proves the all-order row tensor rank 2^{n-1} in characteristic different from two and classifies the minimizers for $n \ge 3$ [8]. Our earlier unpublished column-separated proof gives the same rank in characteristic zero (transpose to exchange rows and columns).

How the lower bounds fit together

A derivative of the permanent indexed by a matching is a smaller permanent. A derivative of a product is a linear combination of products of some of its original factors. The space of outputs in one degree alone gives $N \geq \binom{n}{\lfloor n/2 \rfloor}$. We retain more information by using the same decomposition in several degrees and by retaining the support of each subspace of outputs.

Two uniform comparisons. Lemmas 1.2 and 1.3 bound the dimensions of outputs on both sides. On the target side the extremal count on any linear support reduces to complete square subboards of a left-justified diagram. On the product side every nonzero linear combination of indexed subproducts of a given degree has a derivative-rank lower bound determined by that degree and the original factor span. Proposition 2.2 uses an annihilator on a common image of the differential ring. Lemma 2.1 supplies one quadratic of maximal rank on all the position algebras. These comparisons give the all-order lower bound and its refinement for repeated factors.

Small orders. The cubic and quartic conclusions require only substitution in the uniform bound. For the quartic, every minimum decomposition also has thirty-two independent cubic cofactors (Proposition 2.5). The quintic proof compares polarization of the cubic outputs with cancellation between transposed derivative maps. At order six a hypothetical decomposition into at most thirty-one summands would support at most seven quadratic directions outside the target space. Every summand would then have at most twelve independent cubic subproducts, whereas the permanent has four hundred cubic outputs.

Orders seven through ten. A single parameter determinant controls the growth of cubics in orders seven and nine (Corollary 5.4). Its two applications give ratios $5/3$ and $3/2$, with an explicit loss for dependent triples. In orders seven through nine one filters the same source in several odd degrees (Lemma 3.1). Small supports in orders seven and eight admit a joint estimate. At equality in order eight the 1024 indexed cofactors are independent. In order nine the total weight in degrees one, three, five and seven is 255. The first term contributes at most $1/4$; the total excess over one from the other terms is at most $9/84$. For order ten a single physical quadratic controls the quadratic, quartic and sextic comparisons. Proposition 4.1 bounds the joint target directions visible only on a prescribed set of summands. Keeping the same quadratic in every reordering permits the final count.

Relation to earlier work. Ilten and Teitler proved the cubic product rank [5]; Ilten and Süß gave a quartic lower bound by Fano schemes [4]. Xu and Gnang considered row-homogeneous decompositions in a withdrawn preprint [11]. The row-separated proofs both use quadratics to sum binomial coefficients of one parity. Liu adds two input slots at a time: evaluation of a quadratic relation embeds the old evaluation space in the new multiplication kernel [8, Lemma 2.1].

We filter the permanent's apolar algebra by summands. On two fixed columns a separated product gives a rank-one matrix; the permanent's quadratic outputs span the symmetric zero-diagonal matrices, which contain no rank-one matrix. By duality its quadratic annihilators therefore span each summand's local degree-two space. On passing to initial ideals, the surviving subsets in each filtration quotient form an antichain in each parity. Counting maximal chains bounds each summand's contribution. Mixed factors need not give a rank-one matrix on a column pair, which is where the unrestricted proof requires the common quadratic and support estimates above.

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1 The permanent and its derivatives

Let V be the span over K of the n^2 entries of a generic square matrix. Write $S = \mathrm{Sym}(V^*)$ for the ring of constant-coefficient differential operators, $I = \mathrm{Ann}(P_n)$, and $E_j(F) = S_{\deg F - j}F$ for the space of derivatives with output degree j. We abbreviate $E_j(P_n)$ to E_j, and put $b_j = \binom{n}{j}$, with out-of-range binomial coefficients equal to zero. The perfect pairing between operators and forms of degree j is evaluation by differentiation. In particular $E_j = I_j^{\perp}$. The essential dimension of a form is the dimension of the smallest space of linear forms in whose symmetric algebra it lies. In positive degree this equals its first-derivative rank in characteristic zero.

Proposition 1.1 (Apolar presentation and the sign decomposition)**.** *The algebra S/I has $\dim(S/I)_j = b_j^2$. The ideal I is generated by products of two coordinate differentials in the same row or column (including squares) and by the differences $\partial_{ia}\partial_{jb} - \partial_{ib}\partial_{ja}$ on rectangles. Every characteristic-zero field admits the following decomposition:*

$$P_n = 2^{1-n} \sum_{\substack{\varepsilon \in \{-1,1\}^n \\ \varepsilon_1 = 1}} \left(\prod_{i=1}^{n} \varepsilon_i \right) \prod_{j=1}^{n} \left(\sum_{i=1}^{n} \varepsilon_i x_{ij} \right).$$

Proof. The stated quadrics annihilate the permanent. Modulo them, each surviving monomial is a matching. Switching two endpoints identifies any two matchings with the same row and column sets. Acting on P_n gives the complementary subpermanent. These subpermanents have pairwise disjoint monomial supports, so the upper bound for the quotient dimension is attained in every degree. In particular E_j has all j-by-j subpermanents as a basis. For the displayed identity, an entry choice from each column uses row i with multiplicity m_i. Averaging the signs kills its coefficient unless m_i is odd for every $i \geq 2$. Since $\sum_i m_i = n$, all multiplicities are one. Each such monomial survives with coefficient one. The apolar presentation is also given by Shafiei [10], and the sign identity is due to Glynn [3]. □

Lemma 1.2 (Subpermanents on a linear support)**.** *If $0 \neq f \in E_j$, then $\mathrm{ess}(f) \geq j^2$ for $1 \leq j \leq n$. For $2 \leq j < n$, the space of degree-$(j+1)$ forms whose first derivatives all belong to E_j is exactly E_{j+1}. Among all e-dimensional spaces $L \subset V$, the largest value of $\dim(E_j \cap \mathrm{Sym}^j L)$ is*

$$B_{n,j}(e) = \max_{\substack{n \geq d_1 \geq \cdots \geq d_n \geq 0 \\ \sum d_i = e}} \sum_{i=j}^{n} \binom{i-1}{j-1} \binom{d_i}{j}. \tag{1}$$

In particular for $n \geq 5$ and $0 \leq e \leq 10$ the quadratic values (in order of increasing e) are

$$0, 0, 0, 0, 1, 1, 3, 3, 6, 9, 10.$$

For smaller boards the displayed numbers remain upper bounds. We put $B_{n,0}(e) = 1$ for the constant layer.

Proof. Restrict f to a square indexing a nonzero coefficient in its subpermanent expansion. All other subpermanents disappear. For $j \geq 2$ the j^2 first partials of the remaining permanent have distinct row and column sets; for $j = 1$ the assertion is immediate. Restriction cannot increase essential dimension. The prolongation statement is dual to $I_{j+1} = S_1 I_j$.

For the last assertion use the one-parameter action scaling cell (i, a) by $t^{u_i + v_a}$, with pairwise distinct cell weights. Each subpermanent is a weight vector. A Grassmannian limit of L is therefore a coordinate space, and the dimension of the kernel of $\mathrm{Sym}^j L \to \mathrm{Sym}^j V/E_j$ can only increase. This limit can equally be constructed by row reduction over $K(t)$. On a board of coordinate cells, exactly the complete j-by-j subboards contribute, by disjoint monomial supports. Order the rows by decreasing size d_i. If row i is the last in a chosen j-set, its common neighborhood with the

others has at most d_i columns. Left-justifying every row attains this bound simultaneously for every row set, which proves (1) and its sharpness.

For the numerical values with $e \le 10$, delete isolated cells in the sense of rows or columns of degree one (they affect no rectangle). For the left-justified diagram one can instead remove all rows of length less than two, and transpose. If the smaller number of effective rows and columns is at most two, the maximum is $\binom{\lfloor e/2 \rfloor}{2}$, with the value zero for $e < 4$. Otherwise in a diagram with at most ten cells one may take at most three effective rows: four effective rows and columns would require at least $4+4+2+2 = 12$ cells after deletion. For three rows of lengths $a \ge b \ge c \ge 2$, the count is $\binom{b}{2} + 2\binom{c}{2}$. When $c = 2$ one has $b \le \lfloor (e-2)/2 \rfloor$; when $c = 3$ and $e \le 10$ one has $b = 3$. Taking the larger of these bounds and the two-row answer gives the claimed list. □

Lemma 1.3 (Derivatives of a product on a prescribed number of variables)**.** *Let T be a product of n nonzero linear forms spanning e dimensions. Let $d = n - k > 0$, and let $F \neq 0$ belong to the span of the products of d distinct indexed factors of T. This includes every nonzero derivative DT with $D \in S_k$. Set $r = \max(1, e-k)$. For $0 \le j \le d$,*

$$\dim E_j(F) \;\ge\; [t^j](1 + t + \cdots + t^{d-r+1})(1+t)^{r-1}. \tag{2}$$

The same lower bound holds for the rank of the order-j derivative map of F. Restricting the domain of that map to a subspace of codimension h decreases its rank by at most h. The coefficient bound is attained for each permissible (n, e, k).

Proof. Use e independent factors as coordinates $x_1, \ldots, x_e$, so $T = x_1 \cdots x_e H$. Deleting any k positions removes at most k of these e coordinate factors. Every remaining indexed product, and hence each of its monomials, retains at least $e-k$ of them. Leibniz's rule puts every order-k derivative in their span. Thus every monomial of F has at least r positive exponents. Choose integral coordinate weights that single out a nonzero initial monomial M of F. Diagonal changes of variables preserve derivative rank at nonzero parameter; specialization at zero can only decrease it. If the positive exponents of M are a_i, the derivative ranks are the coefficients of $\prod_i S_{a_i}(t)$, where $S_a = 1 + \cdots + t^a$: the divisors of M are a monomial basis. For $a \ge b \ge 1$,

$$S_a S_b - S_{a+1} S_{b-1} = t^b + \cdots + t^a.$$

Moving units to the largest exponent until only r positive parts remain, with all but the largest equal to one, proves (2). Complementary derivative maps are transposes. Restriction to codimension h costs at most h by rank-nullity. For sharpness take $T = x_1^{n-e+1} x_2 \cdots x_e$. Differentiate first in $\min(k, e-1)$ of the singleton variables, then in x_1 if necessary. The remaining monomial has exactly the required exponent sequence. □

At $n = 9, e = 6$, any nonzero first polar has quadratic derivative rank at least $[t^2]S_4(1+t)^4 = 11$. Any nonzero second polar gives at least $[t^2]S_4(1+t)^3 = 7$. Lemma 1.2 leaves at most three missing quadratic directions on a six-plane. The respective restricted ranks are therefore at least eight and four. This is the same estimate used below for products and first polars in orders five through eight.

2 One annihilating quadratic for the whole decomposition

For a list $L_1, \ldots, L_h$ in V, keep a separate position for each occurrence. Write

$$B_h = K[z_1, \ldots, z_h]/(z_1^2, \ldots, z_h^2), \qquad \psi(D) = \sum_{a=1}^{h} D(L_a) z_a \quad (D \in V^*),$$

and extend ψ to an algebra map on S. Let λ take the coefficient of $z_1 \cdots z_h$. In degree h, Leibniz's rule is the identity $\lambda\psi(D) = D(L_1 \cdots L_h)$. In particular the pairing on the whole position algebra

between complementary degrees is perfect. On the subalgebra $F = \psi(S)$ its radical $R_j = F_j \cap F_{h-j}^{\perp}$ is an ideal, and F/R is exactly the apolar algebra of the product. The two algebras serve different purposes when the original factors are dependent.

Lemma 2.1 (A simultaneous quadratic over the ground field)**.** *Given finitely many lists of nonzero linear forms (allowing repetitions and different lengths), there exists a single $Q \in I_2(K)$ such that multiplication by $\psi(Q)$ on each B_h is injective from degree j to $j+2$ for $0 \le j \le \lfloor (h-2)/2 \rfloor$. By complementary duality the maps in the opposite range are surjective. Any finite collection of additional rank conditions with nonempty Zariski open loci in the same parameter space I_2 may be imposed simultaneously.*

Proof. Consider first a list of even length consisting of coordinate variables; rescale the positions to absorb scalar multiples. Keep repeated occurrences of a cell together. Take each even-occupancy column as a block, then pair the odd-occupancy columns. Within each pair take the even-occupancy rows individually and pair the remaining odd-occupancy rows. Discard empty blocks. Each block has even size and lies in a row, a column or a 2-by-2 rectangle. In a row or column take the differential D with coefficient one on its occupied cells. On a rectangle take coefficients $\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ on the entire rectangle. These choices have zero coefficients on positions of the other blocks; unused physical cells of a rectangle cause no change on the list. Now $D^2 P_n = 0$: repeated-row and repeated-column pairs vanish, and the two opposite pairs in the rectangle have opposite coefficients and equal complementary permanents. Set $Q = \sum D^2$. After changing signs of position variables its position image is $q = \sum_b E_b^2$, where $E_b = \sum_{a \in b} z_a$.

Here are the ranks of this operator. In one block of size s, work over $\mathbb{Q}$ and give the squarefree monomials the orthonormal inner product. Multiplication by $E = \sum z_a$ has adjoint $F z_H = \sum_{i \in H} z_{H \setminus \{i\}}$, with $[F, E] = s - 2j$ in degree j. Starting with $Fv = 0$ in degree a gives

$$FE^i v = i(s - 2a - i + 1)E^{i-1} v.$$

The positive norm recursion gives a string from a through $s - a$; its orthogonal complement is invariant under both operators. Taking a lowest-degree vector in each remaining space decomposes the whole block into such strings. Since s is even, separating even and odd steps in a string gives strings for E^2, each centered at $s/2$. On one such string $u_0, \dots, u_\ell$ normalized by $E^2 u_i = u_{i+1}$, give the vectors squared lengths $i!\ell!/(\ell - i)!$. The adjoint lowers u_i with coefficient $i(\ell - i + 1)$. On the tensor product over the blocks, the commutator of the sum of these lowering operators with q is $h/2 - j$ in degree j. In particular $\|qv\|^2 = \|q^* v\|^2 + (h/2 - j)\|v\|^2$; for $j < h/2$ this gives injectivity. Rational nonzero minors persist in every field of characteristic zero.

For general even lists, use a row and column scaling with pairwise distinct weights on the cells. Normalize each scaled form by its lowest weight; every factor limits to a nonzero coordinate form. The construction above gives nonzero minors at zero and hence at some nonzero parameter in K. Pull the quadratic back through the torus action, which preserves I_2 since P_n is a weight vector. Normalizing the factors only rescales the separate positions. For an odd list, adjoin one nonzero form. On $z_{h+1} B_h \subset B_{h+1}$ all new terms containing z_{h+1} vanish and the degree is shifted by one. Thus $j + 1 \le (h-1)/2$ gives precisely the required odd-list range. The nonvanishing of minors defines open conditions in the vector space I_2. A finite product of their nonzero defining polynomials has a nonvanishing point over the infinite field K. □

Proposition 2.2 (A bound in each residue class)**.** *Let $F = \sum_i c_i T_i$ be homogeneous, and let D_i be finite graded quotients of S whose kernels are contained in $\operatorname{Ann}(T_i)$. For a homogeneous $q \in \operatorname{Ann}(F)_s$, $s > 0$, put $M^{[p]} = \bigoplus_{j \equiv p \bmod s} M_j$. With $A(F) = S/\operatorname{Ann}(F)$,*

$$\dim A(F)^{[p]} \le \sum_i \dim \ker\bigl(q : D_i^{[p]} \longrightarrow D_i^{[p]}\bigr). \tag{3}$$

Each D_i may also be replaced by a finite graded algebra receiving a homomorphism from S with the same kernel condition.

Proof. Let C be the common image of S in $\prod_i D_i$. Its defining kernel annihilates every T_i, hence F. Consequently C/qC surjects onto $A(F)$. On each residue class multiplication by q is an endomorphism. Its cokernel on $C^{[p]}$ has the same dimension as its kernel there, and this kernel is a subspace of the direct sum of the indicated local kernels. □

Theorem 2.3 (A lower bound for every order)**.** *Suppose $P_n = \sum_{i=1}^{N} c_i T_i$ is an ordinary decomposition over K, with zero summands omitted, $n \geq 2$. Let $m_{i1}, \ldots, m_{ih_i}$ be the multiplicities of the distinct proportionality classes among the factors of T_i. Put $S_a(t) = 1 + \cdots + t^a$. For either parity p choose a degree $j_p \equiv p \pmod 2$ closest to $n/2$. Then*

$$\sum_{k \equiv p \bmod 2} \binom{n}{k}^2 \;\leq\; \sum_{i=1}^{N} [t^{j_p}] \prod_{a=1}^{h_i} S_{m_{ia}}(t) \;\leq\; N \binom{n}{j_p}. \tag{4}$$

In particular

$$N \;\geq\; L_n := \begin{cases} \dfrac{\binom{2n}{n} - \binom{n}{m}}{2\binom{n}{m-1}}, & n = 2m, \\[2ex] \dfrac{\binom{2n}{n}}{2\binom{n}{m}}, & n = 2m+1. \end{cases} \tag{5}$$

One has $L_n / 2^{n-1} \to 1/\sqrt{2}$.

Proof. Rescale proportional forms within each term to equality, adjusting its scalar coefficient. In its position algebra let D_i be the invariant algebra for permutations within the equal classes. The original operator map lands in D_i. In each degree the squarefree subsets in one orbit have the same counts in the classes. Orbit sums give the Hilbert series $\prod_a S_{m_{ia}}(t)$.

Choose Q by Lemma 2.1. Its position image is invariant. Each injective step on the full algebra restricts injectively to D_i; a preimage in a surjective step may be averaged over the finite permutation group. Therefore every step $(D_i)_k \to (D_i)_{k+2}$ has maximal rank, increasing toward the middle and decreasing thereafter. On a fixed parity the kernel dimensions telescope: the injective steps contribute zero, and the decreasing steps contribute successive differences of dimensions. Their sum is $\dim(D_i)_{j_p}$. Apply Proposition 2.2 and $\dim A(P_n)_k = \binom{n}{k}^2$. Bounding each D_i by B_n gives the right side of (4). For $n = 2m$, take $p \equiv m - 1 \pmod 2$; the even and odd sums are separated by the coefficient of x^n in $(1-x)^n(1+x)^n$, giving the numerator in (5). For odd n, reflection exchanges parities, so either sum is $\frac{1}{2}\binom{2n}{n}$. Stirling's formula gives the limit. □

One parallel pair in a summand already decreases its available dimension by at least $\binom{n-2}{j_p-1}$, since $(1+t)^2 - S_2(t) = t$. Formula (4) counts all multiplicities in a summand simultaneously. For example at $n = 10, j_p = 4$ one pair costs at least 56, two disjoint pairs at least 97, and a triple of proportional factors at least 112.

Corollary 2.4 (The cubic and quartic values)**.** *The ordinary Chow ranks of P_3 and P_4 are four and eight.*

Proof. For the cubic, (5) gives $6N \geq 20$, so $N \geq 4$. For the quartic it gives $8N \geq 70 - 6 = 64$, so $N \geq 8$. Glynn's identity attains both bounds over K. The cubic value is the theorem of Ilten and Teitler [5]. □

Proposition 2.5 (First derivatives of all summands)**.** *Let $n \geq 4$ and let e_i be the factor-span dimension in a decomposition of P_n. Then*

$$2n^2 \;\leq\; \dim \sum_i S_1 T_i \;\leq\; \sum_i e_i. \tag{6}$$

In particular an eight-term decomposition of P_4 has independent factors in every term, and its thirty-two indexed cubic cofactors are jointly linearly independent.

Proof. On its own e variables, a product T of degree $n \geq 2$ has nonzero Hessian determinant. Take e independent factors as coordinates. Every monomial in T uses all of them. For an initial monomial with exponent vector α, the Hessian evaluated at $(1, \dots, 1)$, up to the nonzero scalar coefficient, is $\alpha\alpha^t - \operatorname{diag}(\alpha)$. Its determinant is $(-1)^e(1-n)\prod_a \alpha_a$. Diagonal scaling and specialization therefore show that the Hessian determinant of T is nonzero. At a point v with invertible Hessian, multiplication by v^{n-2} on $A(T)_1 \to A(T)_{n-1}$ is an isomorphism: its matrix in the complementary pairing is $(n-2)!$ times the Hessian evaluated at v. In particular $\dim A(T)_1 = e$.

For $n \geq 5$ one has $e \leq n < (n-2)^2$. By Lemma 1.2, $E_{n-2} \cap \operatorname{Sym}^{n-2} L = 0$ on the factor support L. Duality makes the restriction $I_{n-2} \to \operatorname{Sym}^{n-2} L^*$ surjective. Thus the intrinsic power above can be lifted to some physical annihilator. For $n = 4, e \leq 3$ the same argument applies. For $n = 4, e = 4$ the original factors are independent, hence $A(T) = B_4$ in the factor coordinates and Lemma 2.1 gives the required isomorphism. All these determinant conditions define nonempty open sets in the same K-space I_{n-2}, so choose a single q meeting them.

Take the common image C in $\prod_i A(T_i)$. The degree-one residue class modulo $n-2$ consists of degrees one and $n-1$ in the indicated range. Multiplication by q is injective on C_1, and vanishes on C_{n-1}. The argument of Proposition 2.2 gives $2n^2 \leq \dim C_{n-1}$. By transposition the latter equals the dimension of $\sum_i S_1 T_i$, proving (6). For eight quartics equality forces every $e_i = 4$ and all eight four-dimensional derivative spaces to form a direct sum. The four derivatives along the factor-dual basis in each summand are precisely its four cubic cofactors. □

Lemma 2.6 (The size of the common position image)**.** *In the common image $C \subseteq \prod_i B_n$, write $c_j = \dim C_j$. It is also the dimension of the span, over all terms, of all indexed j-factor products. For every j,*

$$c_j + c_{n-j} - b_j^2 \leq N b_j. \tag{7}$$

If $n = 2m$, then in particular $2c_{m-1} \leq N b_{m-1}$.

Proof. Transposing the common map on S_j gives the stated span. In $\prod B_n$ take the sum of the top functionals weighted by the original nonzero coefficients. The complementary pairing is perfect. Its restriction to $C_j \times C_{n-j}$ has rank b_j^2, since it is the catalecticant pairing of P_n. Thus $c_j - b_j^2 \leq N b_j - c_{n-j}$. For $d = m-1$, the form $\sum_i c_i \lambda_i(\psi_i(Q)uv)$ on $\bigoplus_i (B_n)_d$ is nondegenerate by Lemma 2.1. It vanishes on $C_d \times C_d$ because $QP_n = 0$. A totally isotropic subspace has dimension at most half the ambient dimension, giving the last claim. □

Lemma 2.7 (Cost of a dependence in even degree)**.** *Let $n = 2m \geq 4$, $d = m-1$, and retain any quadratic furnished by Lemma 2.1 for a particular product T. In its apolar algebra write $h = \dim A(T)_d$, $k = \dim \ker(Q : A(T)_d \to A(T)_{d+2})$, and define $w = (b_d - h - k)/2$. If $F = \psi(S) \subset B_n$, then $w \geq \dim F_d - h$. For any proportional pair of factors or any minimal dependent set of $g \geq 3$ factors, respectively, one has*

$$w \;\geq\; \tfrac{1}{2}\binom{n-2}{d-1} \quad \text{or} \quad w \;\geq\; \binom{n-g}{d-g+1}. \tag{8}$$

All other dependences are allowed.

Proof. The form $(a, b) \mapsto \lambda(qab)$ on $(B_n)_d$ is nondegenerate: $q = \psi(Q)$ acts between complementary, equally large degrees $d, d+2$ and is injective. Restricted to F_d, its radical has dimension $\dim F_d - h + k$. A subspace of dimension x in a space of dimension D with radical dimension v has restricted radical of dimension at most $D - x + v$. This gives the first claim, with $D = b_d, v = 0$. For a proportional pair, rescale the factors to equality and work in the invariant space for the position swap. The q-twisted pairing there is nondegenerate: the involution preserves it and its two eigenspaces are orthogonal. The anti-invariant degree-d dimension is $\binom{n-2}{d-1}$; the same restriction inequality proves the first bound in (8).

For the circuit rescale its g positions so their sum relation has unit coefficients. Inside B_g let H be the subalgebra generated by the linear forms whose coefficient sum is zero. Its graded parts are full except in degree one, which has codimension one. Indeed the annihilator of the degree-s span is the space of squarefree polynomials of degree s vanishing on that hyperplane: this follows by polarization of powers in characteristic zero. Such a polynomial is divisible by the sum of the variables. Its quotient must have degree zero in each variable, since the product has degree at most one in each. Thus only degree one has a deficiency. As $g \geq 3$, the complementary pairing has a one-dimensional radical in degree $g-1$ and zero radical elsewhere. Put $H' = H \otimes B_{n-g}$, which contains the actual image F. Its radical in degree s has dimension $r_s = \binom{n-g}{s-g+1}$. Its perfect quotient G has dimension $b_d - r_d - r_{d+2}$ in degree d. Injectivity of q on the full degree-d position space gives $\dim \ker(q : G_d \to G_{d+2}) \leq r_{d+2} - r_d$, by lifting a kernel class into the target radical modulo the image of the source radical. The actual image in G_d has dimension $x \geq h$ and twisted radical dimension $x - h + k$. Apply the restriction inequality with $D = b_d - r_d - r_{d+2}$, $v \leq r_{d+2} - r_d$. It gives $h + k \leq b_d - 2r_d$, proving (8). □

In degree ten the three costs for a pair, a dependent triple, and a four-element circuit are $28, 21, 6$. Thus a product of weight less than five has no dependence involving four or fewer factors. The statement uses the original summand and the same quadratic, including when several different circuits occur in that summand.

3 Measuring the contribution of one product

Lemma 3.1 (Successive quotients for a single ordering)**.** *Fix an ordering of $P_n = \sum_i c_i T_i$. For each i choose a graded map $\pi_i : S \to D_i$ whose kernel annihilates T_i: in particular one may take the position image or the actual apolar algebra. Put $B_i = \{i+1, \dots, N\}$, $H_B = \bigcap_{a \in B} \ker \pi_a$, $H_\varnothing = S$, and define in the i-th image*

$$J_i = \pi_i(H_{B_i}), \qquad K_i = \pi_i(H_{B_i} \cap I), \qquad U_i = \pi_i(I_2).$$

Then $U_i J_i \subseteq K_i \subseteq J_i$ and, with $\Delta_{i,j} = \dim(J_i/K_i)_j$,

$$\sum_i \Delta_{i,j} = b_j^2, \qquad \sum_i \sum_{j \text{ odd}} \frac{\Delta_{i,j}}{b_j} = 2^{n-1}.$$

Proof. The ideal containments give the first assertion. Adding index i to B_i increases $\dim(S/(I + H_{B_i}))_j$ by the dimension in degree j of

$$H_{B_i}/((H_{B_i} \cap I) + (H_{B_i} \cap \ker \pi_i)) \simeq J_i/K_i.$$

Sum along the ordering, starting with the empty set and ending with all labels. For the full set $H_B \subseteq I$ by the given decomposition. The last identity follows by summing the odd binomial coefficients. □

Lemma 3.2 (The position dimensions from the factor matroid)**.** *For one original product put $f_j = \dim \psi(S_j)$. For $B \subseteq [n]$ let $r(B)$ be the dimension spanned by the corresponding indexed factors, let $e = r([n])$, and put $r^*(A) = |A| - e + r([n] \setminus A)$. Then*

$$\sum_j f_j t^j = \sum_{B \subseteq [n]} t^{n-|B|}(1-t)^{|B|-r(B)} = \sum_{A \subseteq [n]} t^{|A|}(1-t)^{n-e-r^*(A)}. \tag{9}$$

In particular, if c original factors are coloops (deleting one lowers the span dimension), then $f_2 = \binom{e+1}{2} - c$.

Proof. The transpose of the degree-j position map spans precisely all products of j distinct indexed factors. For this product space the Hilbert-series formula of Wagner, in Berget [1, Cor. 3.1], is $t^{n-e}T_M(1+t,1/t)$ for the represented matroid M. Substitution in the defining subset expansion of the Tutte polynomial gives the first sum, and taking complements gives the second. For the quadratic specialization one can avoid the theorem: take e independent original factors as coordinates. Every distinct pair of coordinates occurs. A noncoloop basis coordinate has nonzero coefficient in another original factor, so its square occurs modulo the distinct pairs. Each coloop coordinate occurs in only its own factor, hence its square cannot occur. □

Corollary 3.3 (Uniform lists)**.** *If every e of the n factors are independent, then $f_j = \min\{\binom{n}{j}, \binom{e+j-1}{j}\}$ for all j.*

Proof. The case $j = 0$ is immediate. For $j > 0$, if $n - j \le e - 1$, choose a point vanishing on the complement of a fixed j-subset and avoiding all its factors. This is possible since no further factor lies in the span of the complement and the field is infinite. Evaluation isolates the chosen product, proving independence. If $n - j \ge e - 1$, apply the same argument to any sublist of $e + j - 1$ factors. The products thus span the entire space of degree-j forms on the factor support. □

The numbers f_j concern the span of products of indexed factors. For the ordinary apolar dimensions of the same summand one takes the complementary-pairing quotient. Orders seven through ten use this distinction when passing from an estimate in the positions to the image of actual physical differential operators.

Proposition 3.4 (Exact position dimensions for the sign decomposition)**.** *For the 2^{n-1} signed products in Proposition 1.1 the common image has*

$$c_j = \binom{n}{j} \sum_{a=0}^{\min(j,n-1)} \binom{n-1}{a} \qquad (0 \le j \le n).$$

All $n2^{n-1}$ indexed cofactors of these products are independent. For even $n = 2m$ the isotropic bound in Lemma 2.6 is attained at every order by this explicit decomposition.

Proof. Products using different j-subsets of columns have different column degrees. Fix one subset. For a choice of one entry per column the coefficient as a function of the free signs is the character $\prod_{i\in A} \varepsilon_i$, where $A \subseteq \{2, \dots, n\}$ is the set of free rows used an odd number of times. Exactly the characters with $|A| \le j$ occur: choose each row in A once and fill the other columns from row one. Character orthogonality over K gives the displayed rank. Scalar weights on whole summands do not affect these indexed spans. At $j = n - 1$ every character occurs. For $j = m - 1$ and even n, symmetry of the binomial row of odd length $n - 1$ gives $c_{m-1} = 2^{n-2}\binom{n}{m-1}$. □

4 Derivatives supported on a subset of summands

Proposition 4.1 (Joint shadows in arbitrary order)**.** *Let $P = \operatorname{perm}_n = \sum_{i\in\mathcal{I}} c_i T_i$ be an ordinary product decomposition over a characteristic-zero field, with $n \ge 2$. Fix $0 \le a \le n$, put $\ell = n - a$, and let $p \in \{0, 1\}$. For $B \subseteq \mathcal{I}$ set*

$$d = \dim E_a(P) - \dim\left(E_a(P) \cap \sum_{i\notin B} E_a(T_i)\right).$$

Put $m = \binom{n}{\ell}$. For $0 \le j \le \ell$, let $C_j(t)$ be the number of j-subsets contained in the first t members of $\binom{[n]}{\ell}$ in colex order, with $C_j(0) = 0$. Let $j_p \equiv p \pmod 2$ be closest to $n/2$, and put $\nu_{i,p} = [t^{j_p}] \prod_a (1 + t + \cdots + t^{m_{ia}})$ for the proportionality-class multiplicities of term i. Then

$$\min_{\substack{m\ge\lambda_1\ge\cdots\ge\lambda_m\ge 0\\ \sum_i \lambda_i = d}} \sum_{i=1}^{m} \sum_{\substack{0\le j\le \ell\\ j\equiv p \bmod 2}} (C_j(i) - C_j(i-1))C_j(\lambda_i) \;\le\; \sum_{i\in B} \nu_{i,p} \;\le\; |B|\binom{n}{j_p}. \tag{10}$$

Proof. The annihilator in S_a of the outside sum $X = \sum_{i\notin B} E_a(T_i)$ restricts to a d-dimensional space of functionals on $E_a(P)$. Choose $D_1, \dots, D_d$ giving a basis of these restrictions. By apolar duality $D_\alpha T_i = 0$ for each $i \notin B$, and the degree-ℓ forms $D_\alpha P$ are independent. Put $\mathcal{F} = \langle D_\alpha P \rangle$. For $d = 0$ the assertion is immediate.

Choose the single physical quadratic Q of Lemma 2.1 for the full decomposition. Its action commutes with every D_α, hence $Q\mathcal{F} = 0$. As in Theorem 2.3, rescale proportional factors to equal ones and take the invariant position algebra under permutations of the equal factors at each $i \in B$. Let C be the joint image of S in the product of these algebras. An operator in the defining kernel kills every T_i in B, hence their derivatives $D_\alpha T_i$ and every $D_\alpha P$. Thus C/QC surjects onto $S/\operatorname{Ann}(\mathcal{F})$, where the latter ideal annihilates the whole family. In parity p, multiplication by Q is a linear endomorphism of $C^{[p]}$; its cokernel dimension equals its kernel dimension. The kernel embeds in the direct sum of the corresponding invariant kernels. Each has dimension $\nu_{i,p}$ by the same maximal-rank restrictions and parity telescoping as in Theorem 2.3. This proves the upper bound on the parity-p dimension of the joint quotient.

Every element of $\mathcal{F}$ is a linear combination of ℓ-by-ℓ subpermanents. Choose row and column weights separating their row-set and column-set pairs, for example by assigning distinct powers of two to all rows and columns. In this eigenbasis a torus limit of the d-plane is spanned by d distinct subpermanents. For each j, duality identifies the degree-j piece of the joint quotient with the dual of the image of $S_{\ell-j} \otimes \mathcal{F} \to \operatorname{Sym}^j V$. The rank can only decrease at the limit. For the coordinate family the joint rank counts precisely the pairs of j-sets occurring inside some chosen pair of ℓ-sets. In particular the specialization need not preserve the chosen operator Q; its role is to lower-bound the dimensions already bounded for the original family.

In the resulting m-by-m array of chosen pairs, compress each column-set fiber over a fixed row ℓ-set to a colex initial segment of the same size. For a fixed row j-set, the original column shadows form a union at least as large as their largest individual size. By the Kruskal–Katona theorem [7, 6] each individual shadow can only shrink, and the new shadows are nested. Their new union therefore cannot be larger. Compress the row-set fibers over the columns in the same way. The columns were nested before this second compression, so their lengths are decreasing and the final diagram is Ferrers. All j-shadow inequalities hold simultaneously. With row lengths λ_i, every new row j-set first appearing at row i has column shadow of size exactly $C_j(\lambda_i)$. This yields the displayed sum and its minimum, completing the proof. □

Corollary 4.2 (Retaining a full derivative space after deletion)**.** *In the notation of Proposition 4.1, $E_a(P) \subseteq \sum_{i\notin B} E_a(T_i)$ whenever*

$$|B| < \max_{p=0,1} \frac{\displaystyle\sum_{\substack{0\le j\le n-a \\ j\equiv p \bmod 2}} \binom{n-a}{j}^2}{\displaystyle\max_{0\le h\le n,\, h\equiv p \bmod 2} \binom{n}{h}}.$$

More precisely it suffices that $\sum_{i\in B} \nu_{i,p} < \sum_{j\equiv p \bmod 2} \binom{n-a}{j}^2$ for some parity. There is also a support bound

$$d \;\le\; B_{n,n-a}(\dim L_B), \qquad L_B = \sum_{i\in B} \operatorname{span}\{L_{i1}, \dots, L_{in}\}, \quad 0 \le a < n.$$

In particular $d = 0$ if $\dim L_B < (n-a)^2$, which is automatic when $n|B| < (n-a)^2$. For quadratic outputs the two criteria together guarantee deletion of $0, 0, 1, 2, 3, 8, 13, 30$ arbitrary summands at orders 3 through 10, respectively. For linear outputs, deleting any $n-2$ summands still leaves factors spanning all n^2 matrix variables. These are sufficient bounds.

Proof. If the deficit is positive, the Ferrers diagram contains the cell $(1,1)$, whose parity-p shadow is $\sum_{j\equiv p \bmod 2} \binom{n-a}{j}^2$. Apply (10) for each parity. To evaluate the sums, Vandermonde's identity

gives $\sum_j \binom{\ell}{j}^2 = \binom{2\ell}{\ell}$, and the alternating sum is $[x^\ell](1-x^2)^\ell$. The inequality is strict, including when the ratio is an integer. All d independent forms $D_\alpha P$ also lie in $\mathrm{Sym}^{n-a} L_B$, since the terms outside B vanish. Lemma 1.2 gives the support bound. At $a = 1$ use $n(n-2) < (n-1)^2$ and $E_1(P) = V$. □

Corollary 4.3 (Deletion in the sign decomposition)**.** *For $0 \le a \le n$, the smallest number of terms whose deletion can destroy retention of $E_a(P_n)$ in the sign decomposition is $2^{n-1-\min(a,n-1)}$. Let $g_n(a)$ be the largest number of arbitrary deletions guaranteed to retain $E_a(P_n)$ in every ordinary decomposition. For fixed a, one has $g_n(a) = \Theta_a(2^n)$.*

Proof. Put $m = n-1$ and $s = \min(a, m)$. Fix a columns. Their subproduct in each signed term is isolated by differentiating once in every complementary column. Distinct column sets have different multidegrees. In the free signs $x_1, \dots, x_m$ the subproduct expands as $\sum_{|A|\le s} \chi_A(x) v_A$, where χ_A records the free rows used an odd number of times. The vectors v_A have disjoint nonempty monomial supports. Those with $|A| = a$ or $a-1$ form exactly the target subpermanents on these columns. In the latter case the remaining row is the fixed-sign row.

Failure of retention therefore gives a linear functional vanishing on all retained subproducts and nonzero on a target vector. Its evaluation on the expansion is a nonzero multilinear polynomial of degree at most s, supported on the erased signs. Such a polynomial is nonzero on at least 2^{m-s} vertices. Indeed, write it as $g + x_m h$ in a variable it uses. Wherever $h \neq 0$, one of the two values is nonzero, and induction applies to h of degree at most $s-1$. Constant functions start the induction. Conversely $\prod_{i=1}^{s}(1+x_i)$ has precisely this support size. Its coefficients prescribe a functional on the independent vectors v_A, and its top coefficient detects a target vector.

By Vandermonde's identity and Stirling's formula the cardinality threshold in Corollary 4.2 is asymptotic to $2^{n-2a-1}/\sqrt{2}$ for fixed a. Rounding gives the lower estimate for $g_n(a)$, and the signed example gives the upper estimate. □

5 Polar differentiation on a seven-dimensional support

5.1 A simultaneous polar choice on seven physical directions

Lemma 5.1. *Let E_2 be the span of the quadratic subpermanents, L a physical seven-plane, $W \subset L$ any three-plane and $Q \subset E_2 \cap \mathrm{Sym}^2 L$. For a general $v \in L^*$ the map*

$$Q \longrightarrow L/W, \qquad F \longmapsto \partial_v F \pmod{W} \tag{11}$$

is injective on a nonempty Zariski open subset of L^*.

Proof. Act on the triple (W, L, Q) by a physical row-and-column torus with distinct weights on all cells. Its limiting $W_0 \subset L_0$ is a coordinate flag, and the limiting Q_0 is a subspace of the span of the rectangles on the board of L_0. In a chart for the flag and quadratic Grassmannian, the maximal minors of (11) are polynomials in the coordinates of v with regular coefficients. At least one such polynomial being nonzero is open. It suffices to establish injectivity for all the rectangle quadrics on the limiting seven-cell board, after marking the three cells of W_0.

At most three rectangles occur. One rectangle retains a nonzero partial since it has four cells. Two distinct rectangles have at most two common cells and at least six cells in their union. At least three cells of the union are unmarked, hence at least one unmarked cell is private to one rectangle. The other rectangle also has an unmarked cell. The two-column minor from these cells is a product of nonzero coordinate linear functions.

If three rectangles occur, their union is a full two-by-three board or its transpose. Indeed, two four-cycles sharing just one cell of a seven-cell board share the corresponding graph edge. Their union has no third four-cycle: the two length-three paths joining the endpoints of that common edge are internally disjoint, and together form a six-cycle. Two four-cycles sharing two cells share

a row or column pair and supply a full two-by-three board. A seventh cell cannot add a new rectangle to that board. Disjoint four-cycles would require eight cells.

Write the three quadrics as $q_{ij} = x_i y_j + x_j y_i$ on those six cells. Of the six board cells at least three are unmarked, even when the seventh cell participates in the flag. Choose three unmarked board cells. If one in each column is available, select them. Their gradient rows at v have supports $12, 13, 23$. The two determinant terms have the same sign, since their permutations differ by a three-cycle. They are products of coordinate variables with positive integral coefficients and cannot cancel, even if they coincide. Otherwise, after relabeling, select x_1, y_1 and one cell of column 2. Row q_{23} has a single nonzero entry in the latter column. Its complementary minor on q_{12}, q_{13} equals $y_2 x_3 - x_2 y_3$, a nonzero polynomial. Thus the required open set of v's is nonempty in all cases.

□

Corollary 5.2. *Let $3 \le e \le 7$ and let a finite indexed list of nonzero forms $\ell_1, \dots, \ell_N$ span a physical e-plane L' in a matrix with at least seven cells. Let Q be any subspace of $E_2 \cap \operatorname{Sym}^2 L'$. In any prescribed nonempty open subset of $(L')^*$ one can choose v' with every $\ell_i(v') \neq 0$, and a three-plane $V \subset (L')^*$ such that $Q(v', V) = 0$ and restriction to V preserves the ranks of every indexed sublist of size at most three.*

Proof. Embed L' in a physical seven-plane L. For every independent original triple take W to be its span. The finitely many opens given by (11), the inverse image of the prescribed open, and the complements of all the factor hyperplanes meet in L^*. At such a v, $D = \langle \partial_v F : F \in Q \rangle$ meets each W trivially. Dualizing shows that every original independent triple restricts independently to $D^\perp$. A general three-plane in $D^\perp$ preserves those ranks simultaneously. Its image $V \subset (L')^*$ has dimension three. Extending each independent sublist of size one or two to an independent original triple proves that all original sublist ranks through size three are preserved. With $v' = v|_{L'}$ we have $Q(v', V) = 0$. Taking Q to be the inverse space of the actual quadratic image U gives $v'V \subset U$. □

5.2 A determinant controlling every cubic input

Theorem 5.3. *Put $B_N = k[z_1, \dots, z_N]/(z_i^2)$, $N = 2j + r$, $j, r \ge 1$. An N-by-r matrix D has rows giving linear polynomials λ_i in $R = k[t_1, \dots, t_r]$. Write $V \subset B_{N,1}$ for the column space and $\delta = \sum_{|I|=r} (r - \operatorname{rank} D_I)$. Multiplication by $\sum_i \lambda_i z_i$ gives a square map*

$$\Phi : B_{N,j+r-1} \otimes R_j \longrightarrow B_{N,j+r} \otimes R_{j+1}.$$

For $C_j = \binom{2j}{j}/(j+1)$ its determinant is

$$\det \Phi = c_{j,r} \prod_{|I|=r} (\det D_I)^{C_j}, \qquad c_{j,r} \neq 0.$$

Proof. First take a uniform matrix, so all its maximal row minors are nonzero. For $r = 1$ rescale the slots to make the Boolean operator $D_+ = \sum z_i$. Its transpose D_- removes one slot, and $[D_-, D_+] = (N - 2d)\,\mathrm{id}$ in degree d. Taking the inner product in the real subset basis at $d = j$ proves injectivity; the square matrix has integer entries and is invertible in characteristic zero. For $r \ge 2$, write a kernel vector as $\sum_T f_T z_T$. Reducing modulo λ_i in the equations for targets containing i gives this same construction on $N - 1$ positions with $r - 1$ parameters, on the coefficients with $i \in T$. The induced matrix is uniform. Thus each such f_T is divisible by λ_i. The λ_i for $i \in T$ are pairwise coprime and their number $j + r - 1$ exceeds $\deg f_T = j$. Hence $f_T = 0$ for all T, proving that the universal determinant is nonzero.

For $r \ge 2$, specialize to a general point of $\det D_I = 0$. Choose a nonzero relation $\sum_{i \in I} a_i \lambda_i = 0$. On the $2j$ outside slots take any vector $b = \sum_{|T|=j} b_T z_T$ killed by $\sum z_i$ on those slots. There are

at least $\binom{2j}{j} - \binom{2j}{j+1} = C_j$ dimensions of such vectors. Put

$$f_{(I\setminus\{i\})\cup T} = a_i b_T \prod_{t\in T} \lambda_t \quad (i \in I), \qquad f_S = 0 \text{ otherwise.}$$

For a target omitting one element of I, the kernel equation reduces to $(\sum z_i)b = 0$ on the outside slots. For a target containing I it reduces to the relation among the λ_i on I. All other targets give zero. This yields at least C_j independent kernel vectors. Local elimination at the irreducible prime $\det D_I$ gives divisibility of the universal determinant by its C_jth power: after eliminating a maximal nonsingular block, every entry of the remaining block lies in the prime. Distinct index sets yield nonassociate primes. Their total degree with multiplicity is $r\binom{N}{r}C_j = N!/(j!(j+1)!(r-1)!)$, the size of the square map. For $r = 1$ scaling the subset bases shows that the exponent of each row coefficient is $\binom{2j}{j} - \binom{2j}{j-1} = C_j$.

□

Corollary 5.4. *For every specialization of D and every subspace $W \subset B_{N,j}$ one has*

$$\dim VW \;\geq\; \frac{j+r}{j+1}\dim W - \frac{C_j}{\binom{j+r-1}{r-1}}\,\delta. \tag{12}$$

Proof. Along a general perturbation $D + tD'$ each minor has vanishing order equal to its original nullity: use rank normal form and intersect the finitely many nonempty conditions on D'. A square matrix of corank d has determinant vanishing to order at least d, by eliminating its nonsingular block. Thus $\dim\ker\Phi \leq C_j\delta$. For any $W \subset B_{N,j}$, Φ restricts on $(VW)^\perp \otimes R_j$ to a map into $W^\perp \otimes R_{j+1}$ by the Boolean top pairing. Taking dimensions, using $\dim R_{j+1}/\dim R_j = (j+r)/(j+1)$ and the equality of the dimensions of the full source and target of Φ, yields (12). □

At $j = r = 3$ the resulting inequality is $\dim VW \geq (3\dim W - \delta)/2$. The preceding polar choice makes δ the sum of the defects of the original indexed triples.

6 Quartics annihilated by products of two permanent quadrics

Let P be the generic n-by-n permanent in characteristic zero, V the space of matrix entries, and $I = \mathrm{Ann}(P)_2$. For a linear support $U \subset V$ write

$$Z(U) = \{F \in \mathrm{Sym}^4 U : AA'F = 0 \text{ for all } A, A' \in I\}.$$

On supports of dimensions at most $7, 8, 9, 10$, the respective maxima of $\dim Z(U)$ are $0, 2, 9, 10$ when $n \geq 4$. Moreover $\dim IF \geq 6$ for every nonzero member on such a support. Both assertions admit direct proofs with the same elementary switches.

The coordinate-board classification

Let B be any set of at most ten matrix cells, also denoting its linear span by B. For each complete three-by-three subboard R of B include all $x\,\mathrm{perm}\,R$ with $x \in B$ a cell variable. For any complete two-by-four subboard with rows a, b and columns S, include both row orientations of

$$D_{a,b;S} = \sum_{j\in S} b_j \prod_{k\in S\setminus\{j\}} a_k.$$

Also include the transposes of the two-by-four generators. These polynomials are a basis of $Z(B)$. In particular

$$\dim Z(B) = |B|\,\#(3\times 3) + 2\#(2\times 4) + 2\#(4\times 2).$$

The quadratic generators of I are products of differentials in a common row or column (including squares) and differences between the two diagonals of every rectangle. This follows by

grouping all second derivatives by their deleted row and column pairs. Write $f(M) = M(\partial)F$ for the value attached to a quartic monomial M. These are the ordinary coefficients multiplied by the nonzero monomial factorials. Extend f by zero to monomials outside B. Every indicated product of two generators gives a linear identity among these values. In particular, if a pair in M is in one row or column, one may exchange the other pair with the other diagonal of its rectangle without changing $f(M)$. Two common-row or common-column pairs force zero. All relations respect the row and column multiplicities, so each such component can be treated separately.

One or two occupied rows. If the row multiplicities are (4) or $(2,2)$, pair within rows to get zero. For $(3,1)$ write a monomial as $a_i a_j a_k b_l$. If l occurs among i,j,k, the same-column pair and the complementary same-row pair force zero. If i,j,k repeat a column q, switch $a_q b_l$ using the other two first-row factors as the common-row pair. The new monomial has a same-column pair at q, and again vanishes. For four distinct columns, the same switch freely changes which column carries the second-row factor. The four values must agree. A nonzero component thus uses all eight cells and is spanned by $D_{a,b;S}$. Transposition settles every case with at most two occupied columns as well.

Three rows, three columns. Label the cells $a,b,c;\ d,e,f;\ g,h,i$ in row order, with the first row and first column each repeated. The only possible monomials, in order, are

$$A = aaei, \quad B = aafh, \quad C = abdi, \quad D = abfg, \quad E = acdh, \quad F' = aceg, \quad H = bcdg.$$

In the next equalities the capital letters denote their values. The products $aa(ei - fh)$, $ab(di - fg)$, $ad(bi - ch)$, $ac(dh - eg)$ give $A = B$ and $C = D = E = F'$. The product $(bc)(dg)$ gives $H = 0$. Finally $(ae - bd)(ai - cg)$ gives $A - F' - C + H = 0$. The remaining line has coordinates $(2,2,1,1,1,1,0)$, exactly $a\,\mathrm{perm}_3$ in these monomial-pairing values. It requires all nine cells.

Three rows, four columns. Let a be the repeated row. For the two columns i,j occupied in it denote the value by $t_{ij} = t_{ji}$. The two ways of sending the complementary columns to the two single rows give the same value, by one same-row pair and one rectangle. Two rectangles interchanging respectively columns i,k and j,l between the double row and the two single rows give

$$t_{ij} + t_{kl} = t_{kj} + t_{il} \qquad (i,j,k,l \text{ distinct}).$$

The three sums on opposite edges of a four-vertex complete graph are equal. Equivalently $t_{ij} = v_i + v_j$: the equations have rank two on six coordinates, and the four-dimensional family on the right satisfies them (its map is injective in characteristic zero).

Consider the graph of nonzero t_{ij}. A nonzero edge needs both its endpoints in row a, and both complementary columns in both single rows. If the nonzero edges had no common vertex, all eight cells in the single rows would be needed and at least three in the double row, exceeding ten. They consequently have a common vertex s. All pair sums outside s vanish, which forces all three corresponding v_i to vanish. Thus all three edges at s have the same nonzero value. The resulting quartic is $a_s \operatorname{perm} R$ for the three columns complementary to s. Transposition gives the four-row, three-column case.

Four distinct rows and columns. The values are $f(\pi)$ for permutations matching the four rows to the four columns. A pair of rectangle generators using two disjoint pairs of rows gives

$$f(\pi) - f(\pi\tau) - f(\pi\nu) + f(\pi\tau\nu) = 0$$

for disjoint transpositions τ, ν of the rows. Among the four rows some row i has at most two board cells in these four columns. Choose distinct missing columns p,q there. For an assignment $\pi(i) = j$ with nonzero coefficient, take τ exchanging row i and the row currently assigned p; the two terms involving τ vanish. Thus the values are unchanged under exchanging the two columns other than j,p. The argument is valid for every permutation with i assigned j, including permutations initially having zero coefficient. The same holds with q for p. These two different transpositions generate the permutations of the three columns other than j. All six coefficients

are equal, forcing all nine cells of the complementary three-by-three board. Together with i, j this uses ten cells. Every remaining coefficient is zero since row i has only its one cell left on this board. The component is a multiple of $x_{ij} \operatorname{perm} R$.

These bounds give the claimed list. Its sufficiency is direct. For a three-by-three permanent C, applying $A \in I$ to xC gives $[A, x]C$, a first derivative of C, and each such derivative is killed by I. The polarized two-row generator is killed by same-column derivatives, by two second-row derivatives, and by rectangle differences. The only remaining nonzero quadratic derivatives in I use two distinct majority-row positions and give true two-by-two permanents. Thus it too is killed by II.

Two different complete three-by-three boards intersect in at most six cells. A three-by-three and a two-by-four (or its transpose) also intersect in at most six cells. Hence a board with a three-by-three contains only that type and only that one block. Its $|B|$ multiples are independent. Otherwise distinct two-by-four boards must have the same row pair: at different row pairs the overlap is at most four. A block and a transposed block likewise overlap in at most four cells. All the polarized generators on this board have distinct row/column multiplicities, including the two orientations. This proves independence and the dimension formula. For eight or nine cells a two-row support has at most four common columns. For ten cells it has at most five, and hence gives at most $2\binom{5}{4} = 10$ generators. Together with the three-by-three case this proves the four dimension maxima. A complete two-by-four, a complete three-by-three, and a complete three-by-three with one further cell give the three positive endpoint examples.

Linear supports and the minimum visible quadratic dimension

Choose row and column weights with distinct weights on all individual cells. The one-parameter diagonal rescaling fixes I as a space and degenerates any linear support U to a coordinate board of the same dimension, by row reduction of a basis of U. The rank of the map restricting the operators II to quartics on this support cannot increase at the limiting support. Its kernel on the dual quartic side thus has dimension at least its original dimension. This proves the bounds for arbitrary linear supports.

For a fixed nonzero $F \in Z(U)$ rescale F by a suitable scalar power of the parameter to obtain a nonzero limit on the coordinate board. The rank of $I \longrightarrow \operatorname{Sym}^2 V$, $A \mapsto AF$, cannot increase in the limit. A further generic row and column choice on the coordinate board extracts one nonzero weight component. By the classification each such component is an individual generator. For a variable x and a three-by-three permanent C, the directions $\{[A, x] : A \in I\}$ span the whole physical differential space. Indeed a functional $v \in V$ orthogonal to them would satisfy $A(xv) = 0$ for all A, so xv would be a true quadratic. Every nonzero true quadratic has rank at least four by restriction to a two-by-two subpermanent with nonzero coefficient. But xv has rank at most two, whence $v = 0$. The nine first derivatives of C are independent, so $\dim I(xC) = 9$. For a polarized generator its six majority-row second derivatives on distinct column pairs give six independent two-by-two permanents on the complementary column pairs. All other generators of I act by zero, so its image has dimension six. This proves the sharp bound $\dim IF \geq 6$.

Shared estimates for the lower bounds

In the following chapters $\mathcal{D}_j(F) = E_j(F)$ denotes the space of derivatives of output degree j. We write $q(e) = B_{n,2}(e)$ for the quadratic support bound of Lemma 1.2 in the matrix currently under consideration.

Cancellation and filtered polarization. For maps $A_i : X^* \to Y$ of ranks u_i,

$$\operatorname{rank}\Big(\sum_i A_i\Big) \;\geq\; \dim \sum_i \operatorname{im} A_i + \dim \sum_i \operatorname{im} A_i^* - \sum_i u_i.$$

Factor each A_i through an isomorphic copy of its image. In the resulting factorization through the direct sum of these copies, the second map has rank equal to the first dimension on the right and the first map has rank equal to the second dimension. The rank inequality for a composite proves the assertion. It also applies to differences by changing a sign. For self-adjoint maps on paired spaces the two image-span dimensions coincide.

Let $P_n = \sum_i T_i$. Set $F_i = \mathcal{D}_2(T_i)$, $U_i = \mathcal{D}_3(T_i)$, $F = \sum_i F_i$, $U = \sum_i U_i$. Form the cubic polarization $b : U \to V \otimes (F/E_2)$, $g \mapsto \sum_a x_a \otimes [\partial_a g]$. Then $\ker b = E_3$, since $E_3 \subset U$ by linearity, and $E_2^{(1)} = E_3$ by the quadratically generated presentation in the common target estimates. Order the original products. Let L_i be the span of the factors of T_i, $W_i = \sum_{h \leq i} L_h$, and $P_i : L_i \to W_i/W_{i-1}$ the canonical surjection. Let β_i be polarization of U_i followed by P_i on the linear component and quotient by $F_i \cap E_2$ on the quadratic component. Then

$$\dim U - \binom{n}{3}^2 \geq \sum_i \operatorname{rank} \beta_i, \qquad \sum_i \operatorname{rank} P_i = n^2.$$

Indeed, $\sum_{h<i} b(U_h)$ lies in $W_{i-1} \otimes (F/E_2)$. The contribution of $b(U_i)$ modulo that earlier image therefore has dimension at least $\operatorname{rank} \beta_i$, using the natural injection $F_i/(F_i \cap E_2) \hookrightarrow F/E_2$. Summing gives the first inequality. Differentiating E_2 spans all coordinate variables, and $E_2 \subset F \subset \sum_i \operatorname{Sym}^2 L_i$. This gives $W_N = V$ and the second equality.

Flags for the quadratic annihilator. Put $Q_L = E_2 \cap \operatorname{Sym}^2 L$. The following bounds are independent of the matrix order. For a flag $W \subset L \subset V$, put

$$\mathcal{K}_L(W) = \{F \in \operatorname{Sym}^3 L : I_2 \circ F \subset W\}.$$

Cubic flags on at most eight variables. If $\dim L \leq 8$ and $w = \dim W \leq 3$, then

$$\dim \mathcal{K}_L(W) \leq b_w, \qquad (b_0, b_1, b_2, b_3) = (0, 1, 4, 11). \tag{F1}$$

For $w \leq 2$ the space is exactly $\operatorname{Sym}^3 W$.

Proof. Degenerate the entire flag $W \subset L$ by one row-column one-parameter subgroup. Its limit is a board G of at most eight cells with a marked subset H of w cells. The condition defining $\mathcal{K}$ is closed, so this coordinate flag gives an upper bound. The coordinate condition is

$$\partial_z F \in E_2 \quad \text{for every } z \notin H. \tag{F2}$$

This follows by commuting ∂_z with each $q \in I_2$. We analyze row and column multidegrees of a cubic separately. All cubics on the marked cells already belong to $\operatorname{Sym}^3 W$.

A row or column used three times cannot give an E_2 derivative; there is no extra contribution of this kind.

For row and column multiplicities $(2, 1)$, the two possible monomials on a rectangle are $z^2 a$ and zbc, where a is opposite z. If a nonzero combination uses an unmarked cell, (F2) requires a, b, c to be marked, z unmarked, and the combination to be a multiple of

$$\tfrac{1}{2} z^2 a + zbc. \tag{F3}$$

Indeed differentiation in an unmarked a, b, c would produce a square or a same-row or same-column pair; differentiation in z equates the two rectangle coefficients. Three marked cells form at most one such L-shape.

For row multiplicities $(2, 1)$ and three distinct columns, denote the cells in the doubled row by a_1, a_2, a_3, and those in the other row by b_1, b_2, b_3. The possible monomials are

$$M_i = b_i a_j a_k, \qquad \{i, j, k\} = \{1, 2, 3\}.$$

If b_i is unmarked, its coefficient is zero. If a_i is unmarked, (F2) equates the coefficients of M_j, M_k. A surviving component with precisely two nonzero coefficients needs both corresponding b's and both corresponding a's marked, requiring four marks. A singleton is a cubic entirely on marked cells. The only possible extra component with three marks consists of all three coefficients equal, all b_i marked, and all six cells present. This is the full two-by-three board with one marked row. At most eight cells cannot contain two additional rows completing that marked triple. The transposed configuration is identical.

Finally consider three distinct rows and columns. There are six matching monomials. Place them at the vertices of $K_{3,3}$, with the two parity classes of permutations as its parts. Each cell labels the edge joining the two matchings that contain that cell. Differentiation in an unmarked cell equates the endpoint coefficients. A missing cell kills its two endpoint monomials. Since the board has at most eight cells, at most four vertices survive. A subset of two, three, or four vertices has edge boundary at least $4, 5, 4$: use $3|S| - 2e(S)$ and $e(S) \le \lfloor |S|^2/4 \rfloor$. Three marked edges cannot separate a surviving non-singleton component. A singleton can survive only when all its three matching cells are marked, and is already in $\operatorname{Sym}^3 W$.

Thus no extra dimension occurs with at most two marks. With three marks there is at most one: the marked set must be an L-shape, one row, or one column, and these possibilities are mutually exclusive. The bound is $\binom{w+2}{3}$ for $w \le 2$, and $10 + 1$ for $w = 3$. The reverse inclusion for $w \le 2$ holds without specialization, since every second derivative of a cubic on W lies in W. □

We also need the borderline four-dimensional subspace of a six-plane.

A four-dimensional subspace of a six-plane. If $\dim L = 6$ and $\dim W = 4$, then

$$\dim \mathcal{K}_L(W) \le 23. \tag{F4}$$

Proof. Write $L = W + \langle z, w \rangle$. The map

$$F \longmapsto (\partial_z F, \partial_w F) \in Q_L^2$$

has kernel $\operatorname{Sym}^3 W$, of dimension twenty. Its image consists exactly of pairs satisfying equality of mixed partials. Every such pair integrates to a homogeneous cubic: integrate the first polynomial in z, and use the second to supply the remaining z-independent part. Characteristic zero permits the required divisions. Hence

$$\dim \mathcal{K}_L(W) = 20 + 2 \dim Q_L - \dim(\partial_z Q_L + \partial_w Q_L). \tag{F5}$$

To bound this expression, degenerate the entire flag by the same torus. On a six-cell board the rectangle count is $0, 1$, or 3. Indeed two distinct rectangles with union of six cells share two edges and complete a two-by-three board, which has three rectangles. For counts $0, 1$, (F5) is at most 22. In the three-rectangle board, differentiation at a cell gives the two opposite cells in the other row. For two distinct unmarked cells the union of these sets has size three if they lie in one row, and four otherwise. Thus the last rank in (F5) is at least three, and the value is at most $20 + 6 - 3 = 23$. □

An aggregate estimate for short factor spans. For any fixed set H of k labels with $\dim L_i \le 4$, let $V_{i,j} \subset \operatorname{Sym}^j L_i$ be subspaces closed under first derivatives in the sense that

$\partial V_{i,j} \subset V_{i,j-1}$. Both the spans of the indexed subproducts and the spaces $\mathcal{D}_j(T_i)$ qualify. Write

$$d = \dim \sum_{i\in H} L_i, \quad e_j = \dim\Big(E_j \cap \sum_{i\in H} V_{i,j}\Big), \quad a_3 = \dim\ker\Big(\bigoplus_{i\in H} V_{i,3} \xrightarrow{\sum} \sum_{i\in H} V_{i,3}\Big).$$

Then

$$e_3 + a_3 \le 5\Big(\sum_{i\in H} \dim L_i - d\Big) \le 20k - 5d, \qquad e_j \le \left(\frac{\binom{n}{j}}{\binom{n}{3}}\right)^2 e_3 \quad (3 \le j \le n).$$

To prove the first bound, for $g \in V_{i,3}$ form the map $\Theta_i(g) : I_2 \to L_i,\ Q \mapsto Qg$. Every Θ_i is injective: its kernel is contained in E_3 by quadratic generation and no nonzero cubic in E_3 is supported on four variables. For a t-plane $W \subset L_i$ the cubics with obstruction image contained in W have dimension at most $5t$: the preceding flag bounds give $0, 1, 4, 11$ for $t = 0, 1, 2, 3$, and $\dim \mathrm{Sym}^3 L_i \le 20$ handles $t = 4$.

Let R be the kernel of addition on $\bigoplus_i L_i$. Filter R by support in the first i blocks, and let $W_i \subset L_i$ be the image of its i-th associated quotient in the last block. The injective blockwise map $\bigoplus_i \Theta_i$ takes all tuples with sum in E_3 into $R \otimes I_2^*$. With the same support filtration, the i-th quotient of this image embeds into the part of $\Theta_i(V_{i,3})$ with values in W_i. The preceding bound gives at most $5 \dim W_i$ dimensions. Their sum is $5 \dim R$, proving the first assertion.

For the second bound take a simultaneous row-and-column torus limit of the two spaces $E_j \cap \sum V_{i,j}$ and $E_3 \cap \sum V_{i,3}$. The inclusion of all order-$(j-3)$ derivatives of the former in the latter is a closed, equivariant condition and persists. Their limits are coordinate subpermanent spaces. Each j-board contains $\binom{j}{3}^2$ cubic boards, each cubic board belongs to $\binom{n-3}{j-3}^2$ such j-boards. Double counting and the binomial identity give the displayed ratio. This argument uses the same labels in both degrees.

Consequence for the odd-degree filtration. Let $n \ge 6$, set $b = \binom{n}{3}$, and put $s = 2^{n-1} - n - \mathbf{1}_{\{n \text{ odd}\}}$, $\alpha = s/b^2$. If $V_{i,1} = L_i$, the preceding two bounds imply

$$\sum_{\substack{j<n \\ j \text{ odd}}} \frac{e_j}{\binom{n}{j}} \;\le\; 20\alpha k - (5\alpha - 1/n)d - \alpha a_3.$$

Indeed $e_1 = d$ and the sum of the binomial coefficients of the retained odd degrees at least three equals s. For odd n the same estimate controls the full odd sum when $e_n = 0$, as happens on a proper tail of a minimum expression when its local top forms span the original products. For even n all odd degrees are already included.

The coefficient $5\alpha - 1/n$ is positive: $s \ge b + \binom{n}{5}$, and

$$1 + \frac{\binom{n}{5}}{b} - \frac{b}{5n} = \frac{n^2 - 15n + 92}{60} > 0.$$

In particular every nonempty such tail costs strictly less than its number of labels for $7 \le n \le 13$. At the endpoints $20s/b^2$ is respectively $1120/1225$ and $81640/81796$. For $8 \le n \le 12$, bound it above by $20 \cdot 2^{n-1}/\binom{n}{3}^2$. Consecutive ratios of this last sequence are $2((n-2)/(n+1))^2$, an increasing function of n. Its maximum on this interval is at one endpoint, and both $2560/3136$ and $40960/48400$ are below one.

Order 5

The quintic permanent

The rectangular catalecticant and cubic polarization.

Let $P_5 = \sum_{i=1}^{N} T_i$, with coefficients absorbed into the products. Write $F_i = \mathcal{D}_2(T_i)$, $U_i = \mathcal{D}_3(T_i)$ for the actual derivative spaces and $u_i = \dim F_i = \dim U_i$. The two catalecticants are adjoints under the differential pairing, giving the equality. The ten indexed pair products contain F_i in their span, so $u_i \le 10$. Both target dimensions are $\dim E_2 = \dim E_3 = 100$.

Lemma 0.1 (The quintic polarization estimate)**.** *Let T be any original term, L its factor span of dimension e, $F = \mathcal{D}_2(T), U = \mathcal{D}_3(T)$, and $u = \dim U$. Given a quotient $P : L \to D$ of dimension d, let $\beta : U \to D \otimes (F/R)$ be cubic polarization followed by the indicated quotients, where $R = F \cap E_2$. Then*

$$\operatorname{rank}\beta + \frac{10-u}{2} \ge 2d.$$

Proof. The assertion is immediate for $d = 0$. Henceforth $d > 0$. Put $W = \ker P$, $k = e - d$. By the target support bound, $\dim R \le 1$ and every nonzero $q \in R$ has quadratic rank at least four. A cubic in $\ker\beta$ has all its derivatives in directions annihilating W in R. If one is nonzero, choose a complementary basis so that these derivatives are $q, 0, \dots, 0$. Thus the original cubic, and therefore q, depends on at most $k+1$ variables. For $k \le 2$ this is impossible. Consequently

$$\operatorname{rank}\beta \ge u - \binom{k+2}{3} \qquad (k \le 2).$$

For $e \le 4$, the coefficient bound of Lemma 1.3 gives $u \ge [t^2](1 + \cdots + t^{6-e})(1+t)^{e-1} = \binom{e}{2} + 1$. Inserting this in the preceding inequality gives

$$\operatorname{rank}\beta + \frac{10-u}{2} - 2d \;\ge\; \frac{(e-4)(e-5)+2}{4} + 2k - \binom{k+2}{3} \;\ge 0 \qquad (k \le 2).$$

Indeed $2k - \binom{k+2}{3}$ is $0, 1, 0$ for $k = 0, 1, 2$. For $e = 5$ the five factors are independent, so $u = 10$, and the same kernel estimate gives $10 - \binom{k+2}{3} \ge 2(5-k)$ for $k = 0, 1, 2$.

If $k \ge 3$, the only possible pairs (e, d) are $(4,1), (5,1), (5,2)$. Choose $0 \ne \lambda \in D^*$ and put $v = P^*\lambda$. The polar $\partial_v T$ is nonzero: the top coefficient in the translation $T(x + tv)$ is a nonzero product, of positive t-degree since the factors span L. The λ-component of β has image $\mathcal{D}_2(\partial_v T)$ modulo R, by commutation of the three derivatives. The first-polar coefficient bound of Lemma 1.3 gives respectively

$$[t^2](1 + t + t^2)(1+t)^2 = 4, \qquad [t^2](1+t)^4 = 6$$

before quotienting. The rank after quotienting is at least three for $e = 4$, and at least five for $e = 5$, proving the required inequality. □

Set $\Delta = \sum_i (10 - u_i)$ and $\dim \sum_i F_i = 100 + h_2$, $\dim \sum_i U_i = 100 + h_3$. Both excesses are nonnegative because the target spaces lie in the corresponding sums. The image-rank inequality for the quintic rectangular catalecticants gives

$$100 \ge (100 + h_2) + (100 + h_3) - \sum_i u_i, \qquad h_2 + h_3 \le 10N - 100 - \Delta.$$

For the same terms, the filtered polarization bound, summed using Lemma 0.1, yields $h_3 \ge 2 \cdot 25 - \Delta/2$. Thus

$$h_2 + \Delta/2 \le 10N - 150.$$

If $N \le 15$, nonnegativity forces $N = 15$, $h_2 = \Delta = 0$. Then every ten-dimensional F_i is contained in $E_2 \cap \operatorname{Sym}^2 L_i$, of dimension at most one. This contradiction gives $N \ge 16$; Glynn's formula supplies all sixteen terms.

Order 6

The sextic permanent

The paired central layers force a deficit in the cubic subproducts.

For a product T let $V_j(T)$ denote the span of the products of j distinct factor positions. Repeated and proportional factors still occupy separate positions. These spaces are dual to the graded pieces of the factor-position image. The proof uses just degrees two, three, and four.

Suppose that $P_6 = \sum_{i=1}^{N} T_i$ has $N \le 31$. In the simultaneous factor-position image C write $c_j = \dim C_j = \dim \sum_i V_j(T_i)$ and $a_j = \dim E_j$. The paired-image inequality (7) on degrees two and four and the common quadratic give

$$15N \ge c_2 + c_4 - 225, \qquad c_4 \ge 225 + c_2.$$

For the second inequality, multiplication by the chosen quadratic injects C_2 into the kernel of the surjection $C_4 \to \mathrm{Ap}(P_6)_4$, whose target has dimension 225. Hence $c_2 \le \lfloor 15N/2 \rfloor \le 232$. The sum of the indexed quadratic subproduct spaces contains E_2, so each term contributes at most $232 - 225 = 7$ additional directions modulo E_2. In particular, writing e_i for its factor-span dimension,

$$\dim V_2(T_i) \le 7 + q(e_i),$$

where $q(e_i)$ is the physical rectangle bound in Lemma 1.2. For $e_i = 5, 6$, the independent factors alone give $\dim V_2(T_i) \ge \binom{e_i}{2} = 10, 15$, while $7 + q(e_i) = 8, 10$. Therefore every $e_i \le 4$.

If $e_i \le 3$, then $\dim V_3(T_i) \le \dim \mathrm{Sym}^3 K^3 = 10$. If $e_i = 4$, then $q(4) = 1$, so $\dim V_2(T_i) \le 8$. The indexed quadratic formula $\dim V_2(T_i) = \binom{e_i+1}{2} - z_i$ of Lemma 3.2 therefore gives at least two coloops, where a coloop is a factor position whose deletion lowers the span dimension. Remove two; the remaining four factors span a two-plane. Grouping the three-position products according as they use zero, one, or two of the chosen coloops gives

$$\dim V_3(T_i) \le \dim \mathrm{Sym}^3 K^2 + 2 \dim \mathrm{Sym}^2 K^2 + \dim K^2 = 12.$$

Linearity and the product rule give $E_3 \subseteq \sum_i V_3(T_i)$, contradicting

$$400 = \binom{6}{3}^2 \le 12N \le 372.$$

Thus $N \ge 32$. Glynn's thirty-two products attain the bound.

Order 7

The septic permanent

Filter the simultaneous image of the original factor-position algebras.

1. The supported filtration

Let $P_7 = \sum_{i=1}^{N} c_i T_i$ be a shortest ordinary product decomposition and $I = \operatorname{Ann}(P_7)$. The T_i are independent: a relation could eliminate a summand. For each label use its original factor-position map $S \to C_i \subset B_7$; thus $C^*_{i,j}$ is identified with the span of its indexed j-factor subproducts. The trace on the common image, weighted by the c_i, realizes the apolar algebra $A(P_7)$ as a quotient. Choose one ordering of the labels and filter that common image by support in consecutive prefixes. Lemma 3.1 gives successive target quotients $X_i = J_i/K_i$ with C_i-ideals $K_i \subset J_i \subset C_i$ and $U_i J_i \subset K_i$, where U_i is the actual image of I_2 in C_i. Define

$$\Lambda(X) = \frac{\dim X_1}{7} + \frac{\dim X_3}{35} + \frac{\dim X_5}{21} + \dim X_7.$$

The total is $\sum_i \Lambda(X_i) = 64$. The first piece has top dimension one, and each later piece has top dimension zero. Indeed the top common image is all of K^N, by independence of the T_i, and the weighted top trace is nonzero on each individual coordinate.

If a degree-j differential represents a class supported on the first label, it annihilates all the later products. Its image on P_7 is therefore in $E_{7-j} \cap \operatorname{Sym}^{7-j} L_1$, with L_1 the first factor span. This vanishes at $j = 1, 3$ by the essential-variable bound, and has dimension at most $q(\dim L_1)$ at $j = 5$. Thus

$$\Lambda(X_1) \le 1 + q(\dim L_1)/21 \le 8/7. \tag{1.1}$$

For a first label of span at most four the bound is $22/21$.

2. One middle estimate for all seven-factor products

Fix one actual list of seven factors of span $e \ge 5$. Write $C \subset B_7$ for its position image, $f_j = \dim C_j$, and $U = \psi(I_2) \subset C_2$. Let $\delta = \sum_{|S|=3}(3 - \operatorname{rank} S)$, with ranks taken on the original indexed triples. For every cubic subspace $Z \subset C_3$ we claim

$$3 \dim Z - 5 \dim UZ \;\le\; \delta \;\le\; 5(21 - f_5). \tag{2.1}$$

For the first inequality use the simultaneous polar selection of Corollary 5.2 on at most seven physical directions. Applied to $E_2 \cap \operatorname{Sym}^2 L$, it supplies a full-slot-support physical direction v and a three-plane of physical directions whose image $V \subset (B_7)_1$ preserves all original triple ranks, with $\psi(v)V \subset U$. Boolean multiplication by $\psi(v)$ is an isomorphism from degree three to four: after rescaling, its matrix is the Boolean raising operator E. With the real squarefree monomials orthonormal, $[E^*, E] = 1$ in degree three, so E is injective. The dimensions agree, and the nonzero integer determinant remains nonzero in characteristic zero. Put $Z' = \psi(v)Z$ and $H = (VZ')^{\perp} \subset (B_7)_2$, using the Boolean top pairing. Then $VH \subset (Z')^{\perp}$. The parameter shadow inequality (12), with $j = 2, r = 3$ and Catalan number two, give

$$3(35 - \dim Z) \ge 3 \dim VH \ge 5 \dim H - \delta = 5(21 - \dim VZ') - \delta.$$

The inclusion $VZ' \subset UZ$ proves the first part of (2.1), also for dependent and repeated factors.

The second inequality has the following general form. For a list of n factors write f_j for the dimension of its indexed j-fold product span. Then for $1 \le t \le n$,

$$\sum_{|S|=t} (t - \operatorname{rank} S) \;\le\; (n-t+1)\left[\binom{n}{t-1} - f_{n-t+1}\right]. \tag{2.2}$$

Index the coefficient relations among the $(n-t+1)$-factor subproducts by their omitted $(t-1)$-sets, and let R be the resulting relation space. For each S, multiply a relation among the original factors of S by the product of the complementary factors. Its coefficient vector embeds the original relation space in R, supported only on the $(t-1)$-subsets of S. Choose one echelon pivot set for R; projection onto these coordinates is injective. The dimension of vectors of R supported in a given coordinate set is therefore bounded by the number of pivots there. Each pivot is in exactly $n-t+1$ of the t-sets. Summing gives (2.2), and $n=7, t=3$ gives the second half of (2.1).

For a top-free compatible pair put $m = \dim J_1$, $r = \dim UJ_1$, $a = \dim J_3$, $s = \dim UJ_3$. Then

$$105\Lambda(J/K) \le 15m + 3(a-r) + 5(f_5 - s) \le 105 + 15m - 3r - \big[5(21 - f_5) - \delta\big]. \tag{2.3}$$

We next show this budget is at most 105 on every span of dimension at least five.

3. The linear input on six or seven directions

For a subspace $G \subset C_1 = L^*$ of dimension m put $W = G^\perp \subset L$. The rank $r = \dim UG$ is the rank of polarization on the indexed cubic subproduct space followed by projection to G^* on the linear component and quotient by E_2 on the quadratic component. This is just the transpose of the multiplication defining UG. Thus its kernel consists of cubics F in that space with $I_2F \subset W$. The cubic flag bound on any board of at most eight cells gives kernel bounds $0, 1, 4, 11$ for $\dim W = 0, 1, 2, 3$, respectively.

Also, using any e independent original factors as coordinates, the squarefree cubic space they generate is contained in the indexed source. Specializing G to a coordinate plane before quotienting shows

$$r \ge \binom{e}{3} - \binom{e-m}{3} - mq(e). \tag{3.1}$$

Indeed, every squarefree triple meeting the selected coordinates has a surviving polarized component, and distinct triples have disjoint component supports. The factor-coordinate torus preserves the squarefree cubic source; this specialization is taken before imposing any physical quadratic condition. The quadratic quotient can lose at most $mq(e)$ dimensions. When $m = e$ one may use saturation instead: for $j \ge 3$, the annihilator of the image of I_j in C_j is $E_j \cap C_j^* = 0$, since $j^2 > 7$. Quadratic generation gives $UC_d = C_{d+2}$ for $d \ge 1$. In particular, if $J_1 = C_1$, both K_3 and K_5 are full and $\Lambda(J/K) \le e/7$.

For $e = 7$ the four bounds in (3.1) for $m = 1, 2, 3, 4$ are $12, 19, 22, 22$, all at least $5m$. For $m = 5, 6$ the flag bound gives $r \ge 35 - 4, 35 - 1$, again enough. For $m = 0$, (2.3) suffices by itself.

For $e = 6$ the unique circuit has length $c \ge 2$. Its position algebra fills the Boolean core except for a single direction in degree one. This follows because a squarefree polynomial vanishing on the circuit hyperplane is a multiple of its full-support equation, whose cofactor must have degree zero in each core variable. The other $7 - c$ positions are coloops, so the tensor decomposition gives

$$f_3 = 35 - \binom{7-c}{2} \ge 25.$$

For $m = 1, 2$, (3.1) gives $7, 10$. For $c \ge 3$ one has $f_3 \ge 29$, so $m = 3, 4, 5$ give $r \ge f_3 - 11, f_3 - 4, f_3 - 1$, each at least $5m$. For $c = 2$, $f_3 = 25$ and the factors are $x^2 y_1 \cdots y_5$ in independent coordinates.

Their indexed cubic subproduct space contains no nonzero cube of a linear form, since it contains none of the pure coordinate cubes. The flag space of cubics on L with $I_2F \subset W$ contains all cubes of vectors in W. For $m = 3, 5$, $W \neq 0$, so intersecting that flag space with the indexed source subtracts at least one from its upper bound $11, 1$. This gives $r \geq 15, 25$, respectively. For $m = 4$ the ordinary flag bound gives $r \geq 25 - 4 \geq 20$. Together with (2.3) these bounds treat every six- and seven-dimensional label.

4. Five physical directions and a rank-two Gale matrix

For $e = 5, m = 1$, the rank r is at least the quadratic derivative dimension of the nonzero first polar minus $q(5)$: here the actual cubic derivative space is contained in the indexed cubic space, and differentiation commutes. Lemma 1.3 gives $[t^2]S_3(t)(1+t)^3 - 1 = 6$, with $S_a = 1 + \cdots + t^a$. Hence (2.3) gives the unit bound. For $m = 0$ it is immediate. For $m = 3, 4$, codimension of U in C_2 is at most one, and the degree-three part of the quotient by (J_1) is bounded by the cubic dimension on $5 - m$ variables. Thus

$$\dim(J/K)_3 \leq \binom{7-m}{3} + m.$$

Since $UC_2 = C_4$, one has $C_4J_1 \subset K_5$. The degree-five quotient similarly gives $\dim(J/K)_5 \leq \binom{9-m}{5}$. The budgets multiplied by 105 are at most 96 and 80 for $m = 3, 4$. Full linear input was already treated by saturation.

For $m = 2$, (2.3) requires $3r + 5(21 - f_5) - \delta \geq 30$, while the three-marked flag gives $r \geq f_3 - 11$. To compute f_3, f_5, use the following general expansion for a configuration of $n = e + 2$ factors. Let z be the number of zero columns in its rank-two Gale matrix (the number of coloops), and $s_1, \ldots, s_l$ its nonzero proportionality class sizes. Put $\sigma = \sum_i \binom{s_i}{2}$. In the matroid formula of Lemma 3.2, the subsets of dual rank zero use only the zero columns, and those of rank one use nonzero columns of exactly one class. Consequently

$$\sum_j f_j t^j = (1+t)^n - t\sum_{i=1}^{l}(1+t)^{z+s_i} + (l-2)t(1+t)^z + t^2(1+t)^z, \qquad f_3 = \binom{n}{3} - ez - \sigma. \tag{4.1}$$

For our list, $n = 7$. The defect $3-\operatorname{rank} S$ equals two minus the Gale rank of the four complementary columns: the relation space on S is the space of row combinations of the Gale matrix vanishing on its complement. Thus $\delta = \sum_i \binom{z+s_i}{4} - (l-2)\binom{z}{4}$. Comparing the degree-five coefficient in (4.1) gives

$$f_5 = 21 - \delta + \binom{z}{3}, \qquad 5(21 - f_5) - \delta = 4\delta - 5\binom{z}{3}. \tag{4.2}$$

Since there is no relation on one nonzero factor, at most five Gale columns lie together on a fixed one-dimensional linear subspace: $z + s_i \leq 5$. For $z = 0, 1, 2$, concentrating the nonzero columns into the largest permitted classes gives $\sigma \leq 11, 7, 4$, and (4.1) yields $f_3 \geq 24, 23, 21$, respectively. Thus $r \geq 10$, as required. For $z = 3$, one has $\sigma \leq 2$ and $f_3 \geq 18$. Each triple of the four noncoloop factors in the binary core is dependent, so $\delta \geq 4$. Hence $3r + 4\delta - 5\binom{z}{3} \geq 3(18 - 11) + 16 - 5 = 32$. For $z = 4$, three noncoloop factors span one line. Their own triple has defect two and each choice of two of them and one coloop has defect one, giving $\delta \geq 2 + 3 \cdot 4 = 14$. The surplus $4\delta - 5\binom{4}{3} \geq 36$ suffices alone. Larger z is impossible since the Gale matrix has rank two and $z + s_i \leq 5$. This completes all five-dimensional cases.

5. The low-span tail and the endpoint

Let M count the terms whose factor spans have dimension at most four. For any proper final group H of k such terms, the top target increment is zero. By Lemma 3.1 its residual budget is

$e_1/7 + e_3/35 + e_5/21$, with the notation of the shared tail estimate. The consequence there at $n = 7$ bounds it by $32k/35$. If $M = 0$, put any term first: (1.1) and the unit bound for subsequent terms give $64 \leq 8/7 + N - 1 = N + 1/7$. If $M \geq 1$, put one small-span term first, then every large-span term, then the other $M - 1$ small-span terms. On this single ordering,

$$64 \leq \frac{22}{21} + (N - M) + \frac{32(M-1)}{35} = N + \frac{14 - 9M}{105} \leq N + \frac{1}{21}.$$

Both inequalities give $N \geq 64$ by integrality. Glynn's identity yields equality.

Order 8

The octic permanent

Filter the simultaneous image of the original apolar algebras.

1 1. The odd layers of the octic permanent

Write $P = P_8$ and $I = \mathrm{Ann}(P)$. We prove that its ordinary Chow rank is 128. Every expression achieving equality has eight independent factors in each term, and all 1024 original degree-seven cofactors are independent. Glynn's identity gives the upper bound over the original field. For the lower bound we may work over its algebraic closure.

For a finite graded space M use

$$\Lambda(M) = \frac{\dim M_1 + \dim M_7}{8} + \frac{\dim M_3 + \dim M_5}{56}. \tag{1.2}$$

The subpermanent dimensions give $\Lambda(S/I) = 8 + 56 + 56 + 8 = 128$. Apply Lemma 3.1 to the actual apolar algebras of the original products. The individual quotients J_i/K_i of one ordering have total Λ-value 128. We first bound one such quotient on five or more directions and then estimate all smaller spans jointly.

2 2. The physical cubic flags

Set $E_j = \mathcal{D}_j(P)$. For a physical factor space L of dimension $e \le 8$, Lemma 1.2 bounds $\dim(E_2 \cap \mathrm{Sym}^2 L)$ by

$$(q(0), \dots, q(8)) = (0, 0, 0, 0, 1, 1, 3, 3, 6).$$

For $W \subset L$ put $\mathcal{K}_L(W) = \{F \in \mathrm{Sym}^3 L : I_2 F \subset W\}$. The common physical cubic flag estimate gives

$\dim W$	0	1	2	3	4
$\dim \mathcal{K}_L(W)$ at most	0	1	4	11	23

$e \le 8$ in the first four columns, $e = 6$ in the last column. (2.5)

3 3. A common inequality for both ends of an ideal pair

Let T be an original eight-factor product with factor span L of dimension e. Write $A = \mathrm{Ap}(T)$ for its ordinary apolar algebra, C for its factor-position image in B_8, and $U \subset A_2$ for the image of the physical I_2. The complementary trace pairings on A are perfect, and A is generated in degree one. Put

$$h = \dim A_3 = \dim A_5, \qquad f_3 = \dim C_3, \qquad k = \dim \ker(Q : A_3 \longrightarrow A_5). \tag{3.1}$$

Choose $Q \in I_2$ as in Lemma 2.1 and keep it fixed in these estimates. The quadratic codimension in A_2 is at most $q(e)$: the annihilator of the full intrinsic preimage of U is exactly $E_2 \cap \mathcal{D}_2(T)$. For $G \subset A_1 = L^*$, write $u(G) = \dim UG$, and $W = G^\perp \subset L$. Commuting derivatives gives

$$(UG)^\perp = \{f \in \mathcal{D}_3(T) : I_2 f \subset W\}. \tag{3.2}$$

In particular $UA_1 = A_3$: a cubic killed by I_2 belongs to E_3, whose nonzero elements need nine variables. Hence $UA_3 = A_5$ by standard generation. Some member of U acts nontrivially on A_3. This nonvanishing is open, so the same physical Q may be chosen with $k \le h - 1$ for every term. Lemma 2.7 further gives

$$h + k \le 56 - 2(f_3 - h) \le 56. \tag{3.3}$$

Thus $k \le \min(h - 1, 56 - h) \le 27$.

For each compatible pair of homogeneous ideals $K \subset J \subset A$, $UJ \subset K$, put $j_i = \dim J_i$, $k_i = \dim K_i$, $H = K_7^{\perp} \subset A_1$, and $\delta(G) = 7\dim G - u(G)$. Since $K_3 \supset UJ_1$, $K_5 \supset QJ_3$, and $UH \subset J_5^{\perp}$, we obtain

$$56\Lambda(J/K) \le h + k + \delta(J_1) + \delta(H) - 7(e - j_7 + k_1). \tag{3.4}$$

Indeed $k_3 \ge 7j_1 - \delta(J_1)$, $k_5 \ge j_3 - k$, and $7(e - k_7) - \delta(H) \le h - j_5$. Substitution in (1.2) gives (3.4). Since $UA_1 = A_3$, standard generation gives $A_{\ge 3}J \subset K$. In particular $A_6J_1 \subset K_7$, and complementary duality yields $J_1H = 0$. Each nonzero $v \in A_1$ has multiplication kernel of dimension at most one on A_1: its rank is the essential dimension of $\partial_v T$, at least $e - 1$ by Lemma 1.3. Thus nonzero J_1, H must both be lines. If $J_1 = A_1$, then K_3, K_5, K_7 are full; if $J_5 = 0$, perfect pairings and generation force $J_1 = J_3 = 0$. Either boundary case costs at most $e/8$. Otherwise J_1, H are proper: if $K_7 = 0$, then $UJ_5 = 0$ implies $A_3J_5 = 0$, contrary to $J_5 \ne 0$ and perfect pairing.

4 4. The endpoint estimates on five and six directions

For $e = 5, 6$ the following upper bounds on $\delta(G)$ hold for proper nonzero G, with $m = \dim G$:

	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
$e=5$	$\min(1, 28-h)$	$25-h$	$25-h$	$29-h$	
$e=6$	-1	$37-h$	$32-h$	$32-h$	$36-h.$

(4.1)

For $m \ge 2$, substitute $w = e - m$ in (3.2) and the flag estimates (2.5). For a line $G = Kv$, its image dimension is the quadratic derivative rank of $\partial_v T$ minus its intersection with E_2. Lemmas 1.3 and 1.2 give respectively $7 - 1 = 6$ in dimension five and $11 - 3 = 8$ in dimension six. In dimension five the physical preimage in the fifteen-dimensional intrinsic quadratic operator space has dimension at least fourteen. Multiplying by v is injective in the polynomial ring and projection to A_3 costs at most $35 - h$. Thus $u(Kv) \ge h - 21$ as well.

Except in the boundary cases, (4.1) and the endpoint relation show

$$\delta(J_1) + \delta(H) \le \begin{cases} \max(0, 29-h), & e = 5, \\ \max(0, 37-h), & e = 6. \end{cases}$$

For two lines in span five use $2\min(1, 28-h) \le \max(0, 29-h)$; in span six both deficits are negative. Hence (3.4) bounds 56Λ by $\max(h+k, 29+k)$ or $\max(h+k, 37+k)$. The first is at most 56 by (3.3) and $k \le 27$. For the second it remains to prove $k \le 19$.

5 5. The middle kernel on six directions

By Lemma 1.3, $h \ge [t^3]S_3(t)(1+t)^5 = 26$, where $S_a(t) = 1 + \cdots + t^a$. A proportional pair saves $\binom{6}{2} = 15$ in $h + k$ by Lemma 2.7, giving $k \le 41 - 26 = 15$. Suppose the eight directions are projectively distinct. The two relations form the rows of a rank-two Gale matrix. Let z be its number of zero columns, s_i the sizes of its nonzero proportionality classes, and $\sigma = \sum_i \binom{s_i}{2}$. The cubic position-dimension formula proved in the rank-two Gale calculation in Order 7 gives

$$f_3 = 56 - 6z - \sigma. \tag{5.1}$$

Indeed in that argument the polynomial cubic domain has dimension $\binom{e+2}{3}$, each zero Gale column contributes e to its kernel and each pair on the same nonzero line contributes one; here $e = 6$. Every pair of original factors is independent. Hence any six Gale columns have rank two: a relation on the other two positions would be a row combination annihilating those six columns. This gives $z + s_i \le 5$.

For $z = 0, 1, 2$, concentrating the $8 - z$ nonzero columns into the largest permissible classes bounds σ by $\binom{5}{2} + \binom{3}{2} = 13$, $\binom{4}{2} + \binom{3}{2} = 9$, and $2\binom{3}{2} = 6$, respectively. Thus $6z + \sigma \le 18$ and (3.3) yields $k \le 56 - f_3 \le 18$. For $z = 3$, all nonzero classes have size at most two, hence $f_3 \ge 36$. The three coloops split off an essential ternary quintic. Its actual Hilbert vector is $(1, 3, a, a, 3, 1)$ with $a \le 6$, by complementary duality. Tensoring with three Boolean variables gives $h = 10 + 4a \le 34$, and (3.3) yields $k \le 56 - 2f_3 + h \le 18$. For $z = 4$, a binary quartic remains; $z \ge 5$ is impossible since the Gale matrix has rank two and $z + s_i \le 5$. The binary core's position dimensions are bounded coefficientwise by $(1, 2, 3, 4, 1)$, from the polynomial dimensions in degrees one and two and the indexed dimensions in degrees three and four. Tensor the four separate coloops to obtain $\dim C_5 \le 42$. The physical Q injects the position cubic layer into the position quintic layer. Lifting an actual cubic to C_3, then projecting $C_5 \to A_5$, can therefore lose at most $\dim C_5 - h$ dimensions. In particular $k \le 42 - h \le 16$. This proves the claimed unit bound for every five- or six-dimensional factor span.

6 6. Cubic growth on independent coordinates

Here we record the elementary comparison used for larger supports. Suppose e original factors form a basis of L. The full squarefree cubic span on them, of dimension $\binom{e}{3}$, belongs to the indexed cubic space C_3^*. Its intersection with the actual space $A_3^* = \mathcal{D}_3(T)$ has codimension at most $f_3 - h$ in this squarefree span. On cubics, $u(G)$ is the rank of polarization followed by projection to G^* on the linear component and quotient by E_2 on the quadratic component; this is the transpose of physical multiplication $U \otimes G \to A_3$. Restricting the domain to this intersection thus proves

$$u(G) \ge \binom{e}{3} - \binom{e-d}{3} - q(e)d - (f_3 - h), \qquad d = \dim G. \tag{6.1}$$

To see the bound on the full squarefree source, omit the quadratic quotient first. A diagonal change of basis in the selected factor coordinates specializes G to a coordinate d-plane, preserving squarefree cubics and maintaining the dimensions. At this limit each triple meeting the chosen d variables has a surviving polarized component and different triples have disjoint component supports. The rank is at least $\binom{e}{3} - \binom{e-d}{3}$ before taking the quadratic quotient. That quotient can kill at most $dq(e)$ dimensions. Specialization is used only for the rank before quotienting, so the physical placement of E_2 remains arbitrary.

7 7. A single relation

If $e = 7$, let c be the number of positions in the unique relation, and write $f = 8 - c$. Its complement is a set of f coloops. For $c \ge 3$, the circuit computation in Lemma 2.7 shows that its position algebra fills the Boolean algebra except for one direction in degree one; its trace radical has one direction in degree $c - 1$. Tensoring with the free slots yields

$$f_3 = 56 - \binom{f}{2}, \qquad f_3 - h = \binom{f}{4-c}. \tag{7.1}$$

For $c = 4$ this gives $h = 49$ and $f_3 - h = 1$; for $c \ge 5$, $h \ge 53$ and $f_3 = h$. In either range (6.1) gives $u(G) \ge 7d$ for $1 \le d \le 3$: the three bounds before subtracting $f_3 - h$ are $12, 19, 22$. For $4 \le d \le 7$, (3.2) and the flag bound (2.5) give $u(G) \ge h - 11, h - 4, h - 1, h$, respectively, again at least $7d$. Thus $\delta(G) \le 0$ on every input. By (3.3)–(3.4), $56\Lambda \le 56 - 2(f_3 - h)$, proving the bound $27/28$ for $c = 4$, and one for $c \ge 5$.

For $c = 2, 3$, the actual Hilbert vector is $(1, 7, 22, 41, 50, 41, 22, 7, 1)$. In the repeated case this is the coefficient vector of $S_2(t)(1 + t)^6$. For $c = 3$, tensor the binary cubic's vector $(1, 2, 2, 1)$ with

the five free slots. Lemma 2.7 gives $k \le 5$ for $c = 3$ (the circuit saves $2\binom{5}{1} = 10$ in $h + k$), and $k = 0$ for $c = 2$ (the repeated pair saves fifteen).

We claim that $u(G) \ge 13, 21, 25$ when $d = 1, 2, 3$. For $c = 2$ the algebra is

$$K[t, z_1, \dots, z_6]/(t^3, z_1^2, \dots, z_6^2).$$

Specialize G to coordinates using the variable torus. Cubics outside the ideal generated by those linears number

$$\binom{6-d}{3} + \binom{6-d}{2} + 6 - d \quad (t \text{ unselected}), \qquad \binom{7-d}{3} \quad (t \text{ selected}).$$

Hence full quadratic multiplication has rank at least $16, 27, 34$; restricting to U loses at most $3d$, yielding the claim.

For $c = 3$ normalize the binary core to $x^2y + \lambda xy^2$, $\lambda \ne 0$. Its apolar algebra is

$$K[p, q]/(p^3, q^2 - \lambda pq + \lambda^2 p^2).$$

The relations annihilate the cubic and the classes $1, p, q, p^2, pq, p^2q$ form a basis: use the monic quadratic in q over $K[p]/(p^3)$; the complementary linear-quadratic trace matrix is, up to scalar, $\left(\begin{smallmatrix} 0 & 1 \\ 1 & \lambda \end{smallmatrix}\right)$, so no further annihilator occurs. This basis is free throughout the family. All nonzero fibers are related by scaling y; at zero, tensoring the five free slots gives precisely the repeated algebra just treated. Transport G in the nonzero fibers and take the Grassmannian limit; full multiplication rank can only drop. The same bounds $16, 27, 34$, and hence the stated physical estimates, follow.

For $4 \le d \le 6$, (2.5) and (3.2) give $u(G) \ge 30, 37, 40$. Therefore $\delta(G) \le 2$ on every proper subspace and $\delta(G) \le -6$ on lines. Off the boundary, at most one endpoint is nonzero unless both are lines. Thus (3.4) gives $56\Lambda \le 41 + 5 + 2 = 48$. The boundary costs at most $7/8$, so $\Lambda \le 7/8$ in both short circuits.

8 8. Eight independent directions

Here $A = C = B_8$, so $h = 56$ and $k = 0$. For $d = 1, 2, 3, 4$, the bound (6.1) is $15, 24, 28, 28$, respectively, all at least $7d$. For $d = 5, 6, 7, 8$, (3.2) and (2.5) give $45, 52, 55, 56$, also at least $7d$. Applying (3.4) gives the unit bound. We have proved:

Theorem 8.1. On any original factor span of dimension at least five a compatible ordinary apolar ideal pair has $\Lambda(J/K) \le 1$. In dimension seven the upper bound is $7/8$ for a relation on two or three positions, and $27/28$ for a relation on four positions.

9 9. The contribution of the smaller spans

Consider $P = \sum_{i=1}^{N} c_i T_i$ with nonzero scalar coefficients and products. Place all terms of factor span at least five first. Denote the remaining set by B, put $s = |B|$ and $d = \dim \sum_{i \in B} L_i$, and write a_3 for the kernel dimension of addition on $\bigoplus_{i \in B} \mathcal{D}_3(T_i)$. For this final set, Lemma 3.1 identifies its total quotient in degree j with the dimension of $S/(I + \bigcap_{i \in B} \operatorname{Ann}(T_i))$ in that degree. Its dual is

$$E_j \cap \sum_{i \in B} \mathcal{D}_j(T_i).$$

Apply the common small-span estimate to the actual derivative spaces. Since $\alpha = (128 - 8)/56^2 = 15/392$, it gives

$$\sum_{i \in B} \Lambda(J_i/K_i) \le \frac{75s}{98} - \frac{13d}{196} - \frac{15a_3}{392}. \tag{9.6}$$

This includes the empty set and the case when every term belongs to B.

10 10. The exact rank and the quantitative bound

Let b count the seven-dimensional terms whose single relation involves two or three factors, and let c count those whose relation involves four. By Theorem 8.1 the first $N - s$ terms contribute at most $N - s - b/8 - c/28$. Adding (9.6) to this bound and using the total value 128 proves

$$N \;\geq\; 128 + \frac{b}{8} + \frac{c}{28} + \frac{23s}{98} + \frac{13d}{196} + \frac{15a_3}{392}. \tag{10.9}$$

Together with the sign expression, this gives $\mathrm{CR}(P_8) = 128$.

11 11. Equality and the original cofactors

Suppose $N = 128$. Equation (10.9) forces $s = 0$, so every local quotient costs at most one in any ordering. Fix a label p, place it first, and put $H = \bigcap_{i \neq p} \mathrm{Ann}(T_i)$. The first target piece in degree j is the image of H_j in $(S/I)_j$. For $D \in H_j$ we have $DP = c_p DT_p$; under differentiation its image lies in

$$E_{8-j} \cap \mathcal{D}_{8-j}(T_p) \subseteq E_{8-j} \cap \mathrm{Sym}^{8-j} L_p.$$

By Lemma 1.2, every nonzero form in E_{8-j} has at least $(8-j)^2$ essential variables. For $j = 1, 3, 5$ this exceeds $\dim L_p \leq 8$, so these three layers vanish. The degree-seven layer has dimension at most $\dim L_p$. Since all 128 pieces have value one, this forces $\dim L_p = 8$ and the map $H_7 \to (A_p)_7$ is onto.

The image of $S_7 \to \bigoplus_i (A_i)_7$ therefore contains the full coordinate block at p: operators from H_7 vanish in the other blocks. The same conclusion holds for each choice of first label, so the map onto the direct sum is surjective. Its dual adds the spaces $\mathcal{D}_7(T_i)$ injectively. In each term, differentiation along the dual basis of its eight independent factors produces its eight original cofactors. This proves their joint independence.

Order 9

An odd-degree filtration on nine factors

The excess of an independent summand is confined to its full linear layer.

1 An odd-degree count with one exceptional linear layer

Let $P = \mathrm{perm}_9$ in its 81 physical variables, and let $P = \sum_{i=1}^{M} T_i$ be any expression as a sum of nonzero products of nine linear forms. For a fixed ordering apply the common filtration of Lemma 3.1 to the successive sets of later terms. More explicitly, put $I = \mathrm{Ann}(P)$, $B_i = \{i+1, \dots, M\}$, $H_i = \bigcap_{a \in B_i} \mathrm{Ann}(T_a)$, and take images in $A_i = S/\mathrm{Ann}(T_i)$:

$$J_i = \mathrm{im}\, H_i, \qquad K_i = \mathrm{im}(H_i \cap I), \qquad U_i = \mathrm{im}\, I_2, \qquad U_i J_i \subset K_i \subset J_i.$$

Here $H_M = S$. Set $\Delta_{i,j} = \dim(J_i/K_i)_j$, $m_i = \Delta_{i,1}$ and

$$N_i = 28\Delta_{i,1} + 3\Delta_{i,3} + 2\Delta_{i,5} + 7\Delta_{i,7}, \qquad \lambda_i = N_i/252.$$

The quotient in degree j is $H_i/((H_i \cap I) + (H_i \cap \mathrm{Ann}(T_i)))$ in that degree. Taking dimensions therefore telescopes. By Proposition 1.1 its total is $\binom{9}{j}^2$, and hence

$$\sum_i \lambda_i = 9 + 84 + 126 + 36 = 255, \qquad \sum_i m_i = 81. \tag{1}$$

Writing L_i for the span of the original nine factors and L_i' for the sum of all later supports, one has $m_i = \dim L_i - \dim(L_i \cap L_i')$, since $I_1 = 0$.

For the first term, $vP = vT_1$ whenever $v \in H_1$. Thus the degree-j increment is bounded by the dimension of the degree-$(9-j)$ target forms on at most nine variables. The essential dimension and rectangle bounds of Lemma 1.2 give zero for $j = 1, 3, 5$ and at most nine for $j = 7$. It follows that

$$\lambda_1 \le 9/36 = 1/4. \tag{2}$$

Proposition 1.1 (Local estimates for nine indexed factors)**.** *For each term of factor span $e \le 8$, $N_i \le 252$. For $e = 9$ the same inequality holds if $m_i < 9$, and $N_i \le 255$ if $m_i = 9$.*

The deductions below establish this proposition using the local calculations in the appendix. A circuit is a minimally dependent sublist of original factors. A coloop is an original factor outside the span of all the others. Both definitions refer to indexed factors, so proportional copies remain distinct positions.

1.1 Two reductions shared by the dependent cases

In one fixed source write $j_d = \dim J_d$, $u_d = \dim UJ_d$, $a = \dim A_2$, $h = \dim A_4$, and $m = j_1$. Suppose that for all $X \subset A_2$ and $Y \subset A_3$,

$$\dim UX \ge \tfrac{7}{2} \dim X - \alpha, \qquad \dim UY \ge \tfrac{3}{2} \dim Y - \beta.$$

Apply the first inequality to $X = (UJ_5)^{\perp} \subset A_2$. Since $UX \subseteq J_5^{\perp} \subseteq A_4$ and $\dim X = a - u_5$, it follows that $2j_5 + 7(a - u_5) \le 2(h + \alpha)$. Apply the second inequality to J_3; then $3j_3 - 2u_3 \le 2\beta$. The containments defining the source now give

$$N \le 28m - 3u_1 + 2(h + \alpha + \beta). \tag{3}$$

This use of the perfect pairings charges the entire kernel of every comparison with an auxiliary algebra.

If $e \le 8$ and $m \ge 2$, the transversal-derivative estimate in the low-overlap calculation already gives $N \le 246$. The only further dependent cases have $m = 0, 1$.

1.2 At most five directions

For $e \le 5$ put $b = \dim A_3$; thus $a \le 15, b \le 35, h \le 70$. On a physical five-plane U has codimension at most one by the rectangle bound, and $U^2 = A_4$ by quadratic square saturation. We show that for every x-plane $X \subset A_2$,

$$\dim UX \ge \tfrac{7}{2}x - \alpha, \qquad \alpha = \max(0, 119/2 - h).$$

In five polynomial variables the minimal quartic dimensions of full quadratic multiples of $x = 1, \dots, 7$ quadrics are bounded below by $15, 25, 31, 34, 35, 45, 51$, respectively, by counting multiples of the initial lex segment (Macaulay's theorem). Lift X to dimension x and allow dummy variables if $e < 5$. Restricting the multipliers loses at most x; projection in degree four loses at most $70 - h$. These seven entries give the assertion. For $x \ge 8$, the annihilator of UX acts faithfully by symmetric forms on $U/(U \cap X)$, by $U^2 = A_4$. Hence $\dim UX \ge h - \binom{16-x}{2}$; convexity at $x = 8, 15$ proves the same assertion. In particular $h + \alpha \le 70$. Using the first reflection alone gives $2\Delta_5 + 7\Delta_7 \le 140$. For $m = 0$ use $\Delta_3 \le 35$ to obtain $N \le 245$. For $m = 1$, lifting one linear form to the five-variable polynomial ring gives $u_1 \ge b - \binom{6}{3} - 1 = b - 21$, so $N \le 140 + 28 + 3 \cdot 21 = 231$.

1.3 Six or seven directions

For six directions first count the original coloops. With at least two, write $T = F(x_1, \dots, x_4)yz$. The actual quadratic dimension is at most 19. The two prolongation inequalities and eight scalar intervals in the two-coloop calculation give $N \le 252$. With zero or one coloop, a proportional pair is covered by the multiplicity argument. If the factors are projectively distinct, the number of dependent indexed triples is either at least two, exactly one, or zero. The corresponding arguments are the binary-core calculation, the one-triple calculation, and the triple-independent calculation. In the final case the one-coloop calculation treats the eight-position core. Each bound is for the original quadratic image and each occurrence of $m = 0, 1$ uses one background throughout.

In seven directions the dual of the factor matroid has rank two. Its zero columns record the z coloops, and its nonzero columns are points on a line with unrestricted multiplicities and cross-ratios. The rank-two calculation uses these multiplicities to compute the ordinary Hilbert function. It combines the polynomial and squarefree estimates in the seven-direction diagrams with (3). This yields $N \le 252$ for all 79 allowable multiplicity lists, with either value of m.

1.4 One relation and the independent case

For eight directions there is a unique circuit, of length r, and the remaining $9 - r$ positions are coloops. Put $q = \operatorname{codim}_{A_2} U$; the eight-cell rectangle bound gives $q \le 6$. For $r = 2, 3$ the same four covers of monomial chains give the unit bound. For $r = 4$ the core is a ternary quartic. Its six quadratic operators occur in the leading core part of U, since a physical target quadric has rank at least four and a form on the core has rank at most three. The three quartic templates give $N \le 252$. For $r = 5$, a good pair is a pair of original circuit factors whose simultaneous deletion is injective on $E_2 \cap \operatorname{Sym}^2 L$, the space of all target quadrics supported on the factor plane. The five-circuit calculation covers both the existence and the absence of such a pair, including the shear of each complementary coloop. For $r \ge 6$ a good pair exists by the rectangular polar argument. The circuit calculation retains the exact ordinary and actual layers. Its formal Boolean missing pairs lie on seven slots, with at most $q + 9 - r$ missing. The only nine-hole case is $r = 6, q = 6$, where the physical quartic estimate supplies the exceptional dimension. These arguments cover all circuit sizes.

For nine independent factors the apolar algebra is $B = k[t_1, \dots, t_9]/(t_1^2, \dots, t_9^2)$. The good-pair argument for independent factors in the rectangular polar calculation again confines all missing initial quadrics to seven slots. Boolean multiplication gives $2 \dim UW \ge 3 \dim W$ for $W \subset B_3$. The quadratic graph bounds together with the actual quartic calculation give $2 \dim UZ \ge 7 \dim Z$

for every $Z \subset B_2$. Thus (3) applies with $h = 126, \alpha = \beta = 0$. For $0 < m < 9$, the proper linear-layer calculation gives $3u_1 \geq 28m$. At $m = 0$ the inequality is immediate. At $m = 9$, $u_1 \geq 83$, as the missing cubic space is supported on nine directions and has dimension at most one by Lemma 1.2. Hence $N \leq 252 + 28 \cdot 9 - 3 \cdot 83 = 255$. This proves Proposition 1.1.

Theorem 1.2 (Order nine)**.** *Over every field of characteristic zero, the ordinary Chow rank of the generic* 9×9 *permanent is* 256.

Proof. The second identity in (1) allows at most nine terms with $m_i = 9$; each can exceed its unit budget by at most $1/84$. Combining (2) with the local result gives

$$255 \leq \tfrac{1}{4} + (M - 1) + \tfrac{9}{84} = M - \tfrac{9}{14}.$$

Integrality gives $M \geq 256$ for every decomposition. The signed product formula attains $M = 2^8 = 256$ over the original field. All dimension estimates may be proved after extending the coefficient field to its algebraic closure. □

Order 10

Joint derivative bounds on ten factors

One physical quadratic controls every reordering of the same summands.

1 The order-ten obstruction

Write $P = \mathrm{perm}_{10}$ and let K be a characteristic-zero field.

Theorem 1. *The ordinary product rank of P over K is* 512.

We exclude 511 summands. A single quadratic differential operator annihilating P controls the quartic and sextic outputs of every summand through each reordering. The upper bound follows from Proposition 1.1.

2 A dimension bound for a fixed decomposition

Let $S = \mathrm{Sym}(V^*)$ be the ring of constant-coefficient differential operators. For a form F of degree ten, define

$$E_e(F) = S_{10-e}F \subset \mathrm{Sym}^e V, \qquad A(F) = S/\operatorname{Ann}(F).$$

The two appearances of e in what follows refer to the degree of the *output* form $E_e(F)$ and the degree of the *input* operator in $A(F)_e$, respectively. The pairing of S_e with $E_e(F)$ identifies the dual of the latter space with $A(F)_e$. Write $P_e = E_e(P)$. The $e \times e$ subpermanents span P_e; each can be obtained by differentiating along a matching on its complementary rows and columns. Distinct row and column sets give disjoint monomial supports. Thus

$$\dim P_e = \binom{10}{e}^2. \tag{1}$$

Fix a representation of P, with index set $\mathcal{I} = \{1, \dots, N\}$. For a summand T_i, put $U_i = \langle L_{ia} : 1 \le a \le 10 \rangle$, $A_i = A(T_i)$ and $h_i = \dim(A_i)_4$. Choose a single quadratic operator $Q \in \operatorname{Ann}(P)_2$ as in the local results below. In particular, it can be taken over the original field. The relevant rank conditions are finitely many nonempty Zariski-open conditions on $\operatorname{Ann}(P)_2$; the assertion that each is nonempty is part of Lemma 2.1 and the local specializations of Section 7. The same choice of Q is retained through every rearrangement of the summands. Define

$$k_i = \dim\ker\bigl(Q : (A_i)_4 \longrightarrow (A_i)_6\bigr), \qquad w_i = (210 - h_i - k_i)/2.$$

The common-quadratic lemma gives $w_i \ge 0$. These weights are half-integers. For $A \subseteq \mathcal{I}$, set

$$X_e(A) = \sum_{i \in A} E_e(T_i), \qquad B_6(A) = \sum_{i \in A} \bigl(E_6(T_i) \cap \ker Q\bigr),$$
$$z(A) = \dim\bigl(X_2(A) \cap P_2\bigr), \qquad z_i = z(\{i\}).$$

In $B_6(A)$, the kernel is taken on degree-six forms under differentiation by Q. The scalar used throughout the proof is

$$\Phi(A) = \dim \frac{X_4(A) + P_4}{P_4} + \dim \frac{B_6(A) + P_6}{P_6} - z(A) + \sum_{i \in A} w_i. \tag{2}$$

It is determined by the subset itself, so two different orders of all the summands end at exactly the same value.

Lemma 2. *One has* $z(\mathcal{I}) = 2025$ *and*

$$\Phi(\mathcal{I}) \leq 105N - 46125.$$

For $N = 511$*, the upper bound equals* 7530*.*

Proof. Every derivative of $P = \sum_i c_i T_i$ is a linear combination of derivatives of the T_i with the same coefficients. In particular $P_e \subset X_e(\mathcal{I})$. Equation (1) gives $\dim P_2 = 2025$ and $\dim P_4 = 44100$.

On $L = \bigoplus_i (A_i)_4$ consider the symmetric form

$$\langle (D_i), (D_i') \rangle = \sum_i c_i (Q D_i D_i') T_i.$$

Apolar duality in each A_i gives radical dimension $t = \sum_i k_i$. Let $W \subset L$ be the diagonal image of S_4, of dimension $s = \dim X_4(\mathcal{I})$, and set $\kappa = \dim(W \cap \operatorname{rad} L)$. The restriction of the form to W is zero because $QP = 0$. The image of W in the nondegenerate quotient $L/\operatorname{rad} L$ is contained in its orthogonal complement. Consequently

$$2(s - \kappa) \leq \sum_i h_i - t.$$

The i-th radical identifies with $E_6(T_i) \cap \ker Q$ by $D_i \mapsto D_i T_i$. Summing these forms with weights c_i and reducing modulo P_6 has image of dimension $b = \dim((B_6(\mathcal{I}) + P_6)/P_6)$. Its kernel contains $W \cap \operatorname{rad} L$, since a diagonal class maps to $DP \in P_6$. Thus $b \leq t - \kappa$, and $s + b \leq \frac{1}{2} \sum_i (h_i + k_i)$. Substitution in (2) proves the bound. □

3 Estimates for adding a summand

For $i \notin A$ let $g = \Phi(A \cup \{i\}) - \Phi(A)$, and define

$$u = \dim X_2(A \cup \{i\}) - \dim X_2(A), \qquad \delta = z(A \cup \{i\}) - z(A).$$

Both are integers and $0 \leq \delta \leq u$; the quotient defining δ injects into $X_2(A \cup \{i\})/X_2(A)$. The next lemma states the specific local bounds used here. They apply at every actual subset A of the fixed decomposition.

Lemma 3 (Local derivative estimates)**.** *The following inequalities hold for the common* Q*.*

1. *At an empty prefix,* $\Phi(\{i\}) = 210 - w_i - z_i$*. For* $\dim U_i = 10$ *one has* $w_i = 0$*,* $z_i \leq 10$*. For* $\dim U_i < 10$ *one has* $z_i \leq 9$*.*
2. *For every* i *and* A*,*

$$g \geq w_i - \delta, \qquad g \geq \tfrac{10}{3} u, \qquad w_i < 5 \implies g \geq \tfrac{7}{2} u,$$
$$w_i < 4 \implies g \geq \tfrac{11}{3} \delta - \tfrac{1}{3} \mathbf{1}_{\{\dim U_i = 10,\, u = \delta = 32\}}.$$

3. *For* $u = 1, 2, 3, 4, 5$*, lower bounds for* g*, respectively, are* $8, 11, 13, 15, 17$*.*

The two positive quotients in (2) are increasing, which gives $g \geq w_i - \delta$. For a single product its factor span has dimension at most ten. A nonzero element of P_e has at least e^2 essential variables: restrict to the rows and columns of a subpermanent with nonzero coefficient, then differentiate with respect to each of its e^2 entries. The resulting first derivatives of the restricted form are linearly independent. This proves the zero intersections with P_4, P_6 used in the first identity. Hence $\Phi(\{i\}) = h_i + k_i + w_i - z_i$. When all factors are independent their apolar algebra is the squarefree algebra on ten variables. The central map of the chosen Q is an isomorphism there, giving $h_i = 210$, $k_i = w_i = 0$. The bounds on z_i follow from Lemma 1.2.

Sections 10–21 bound the quartic increment at the same prefix, distinguishing summands by the rank of their factor lists and the number of coloops (positions whose removal decreases that rank). For independent factors we allow a loss of one third at a step with $u = \delta = 32$.

4 Joint derivative bounds for a subset of summands

Proposition 4.1 and Corollary 4.2 apply to degree-two outputs of the given decomposition. Return to $n = 10$, let $a = 2, p = 0$, and put $d(B) = 2025 - z(\mathcal{I} \setminus B)$. If the summands in B are placed last, their target increments δ sum to $d(B)$.

Lemma 4. *The following upper bounds hold for every B.*

$\lvert B\rvert$ *at most*	46	95	160
$d(B)$ *at most*	1	45	171

Proof. The right side of (10) is $210|B|$. The seven explicit diagrams in Section 8 exclude areas $2, 46, 172$ at the three indicated thresholds. □

5 Ordering the summands by their weights

Assume $N = 511$, and put $H_a = \{i : w_i \geq a\}$, $h_a = |H_a|$, and $d_a = d(H_a)$. At every prefix the local estimates imply

$$w_i \geq 5 \quad \Longrightarrow \quad g \geq 5 + \tfrac{12}{5}\delta, \tag{3}$$
$$w_i \geq 4 \quad \Longrightarrow \quad g \geq 4 + 2\delta. \tag{4}$$

At $\delta = 0$ both follow from $g \geq w_i$. At $1 \leq u \leq 5$ the first uses the five absolute net bounds, in particular $g \geq 8$ at $u = 1$, independent of factor rank. For $u \geq 6$ use $10u/3 \geq 5 + 12u/5$. The second bound uses the absolute estimates at widths one and two, and $10u/3 \geq 4 + 2u$ above those widths.

The trace yields $\sum_i w_i \leq 7530 + 2025 = 9555$, because the two positive quotients in (2) are nonnegative. A minimum-weight index i therefore satisfies $w_i \leq 37/2$, by half-integrality. If $h_5 \geq 160$, put i first and choose 160 heavy indices distinct from i as a tail B. This is possible if all indices are heavy, and otherwise the minimum-weight index is light. Put $d = d(B) \leq 171$. The middle gains at least $10(2025 - z_i - d)/3$, and the tail gains at least $800 + 12d/5$. Using $z_i \leq 10$ gives

$$\Phi(\mathcal{I}) \geq 210 - \frac{37}{2} + 6750 - \frac{13}{3}\,10 + 800 - \frac{14}{15}\,171 = 7530 + \frac{257}{30}.$$

This contradicts Lemma 2, so $h_5 \leq 159$.

Now put all of H_5 last, and choose any first index $i \notin H_5$. One has $w_i + \frac{9}{2}z_i \leq 45$, since an independent summand has $w_i = 0, z_i \leq 10$, and a dependent one here has $w_i \leq 9/2, z_i \leq 9$. The rest of the complement has rate $7/2$. Inequality (3) in the tail gives

$$7530 \geq 210 - w_i - z_i + \tfrac{7}{2}(2025 - z_i - d_5) + 5h_5 + \tfrac{12}{5}d_5.$$

Thus $5h_5 \leq 555/2 + 11d_5/10$. As $h_5 \leq 159$, Lemma 4 gives $d_5 \leq 171$ and consequently $h_5 \leq \lfloor 2328/25 \rfloor = 93$.

6 Completion of the lower bound

Suppose $h_4 \geq 95$. Choose the 95 largest weights as a final subset B. It contains H_5, and $d = d(B) \leq 45$. Start with one term outside B. The remaining middle terms have rate $7/2$, and (4) holds throughout the tail. Using $w_i + \frac{9}{2}z_i \leq 45$ as above gives

$$\Phi(\mathcal{I}) \geq \tfrac{14505}{2} + 380 - \tfrac{3}{2}d \geq 7565 > 7530.$$

Thus $h_4 \leq 94$.

Finally put H_4 last. Any i in its complement satisfies $w_i + \frac{14}{3} z_i \le \frac{140}{3}$, by the same independent and dependent alternatives. The remaining summands in the complement contribute at least $11\delta/3$, apart from a possible loss of $1/3$ at a step with $u = \delta = 32$. Since the sum of the target increments is 2025, at most $\lfloor 2025/32 \rfloor = 63$ steps can lose this amount. Their total loss is at most 21. Each final summand contributes at least $w_j - \delta_j$, and the sum of those δ_j is exactly d_4. Thus

$$\Phi(\mathcal{I}) \ge 7425 + 210 - \tfrac{140}{3} + 4h_4 - \tfrac{14}{3} d_4 - 21. \tag{5}$$

For $h_4 \le 46$ we have $d_4 \le 1$, giving a margin of at least $98/3$ above 7530. If $47 \le h_4 \le 94$, use $d_4 \le 45$ and $4h_4 \ge 188$. The margin is at least $46/3$. This contradicts the trace bound in both cases.

Proof of Theorem 1. The preceding argument rules out every representation with exactly 511 nonzero summands. A shorter nonzero representation could be padded to precisely 511 terms by repeatedly replacing cT with $(c/2)T + (c/2)T$, over the same field. Consequently the ordinary product rank is at least 512. Proposition 1.1 gives the matching upper bound. □

Appendix A. Local calculations for order nine

The notation A, J, K, U and the four weights are those of Proposition 1.1. In a coefficient algebra, product dimensions carry their ordinary meaning before passage to the perfect top-trace quotient. Section labels within a calculation refer to its own formulas. All bounds concern the original nine factors.

Physical supports and cubic multiplication

A bound from the common factor support

Let a nonic product T have $e \le 8$ independent physical factor directions. If the span of the entire later background meets its support in dimension $c \le e-2$, then the source cost is at most $246/252$. This bound allows all internal factor dependencies.

Write L for the factor support, L' for the sum of the background supports, $W = L \cap L'$, and $m = e - c$. Let $D_j(T)$ be the space of degree-j derivatives, of derivative order $9-j$. By duality the increment Δ_j equals the gain in the intersection between the degree-j subpermanent space and the summed degree-j derivative spaces. Any such new target representative has the form $F - F'$, where $F \in D_j(T)$ and F' is in the background span. Their complete lists of contractions by I_2, the quadratic apolar space of the permanent, agree. The dimension Δ_j is at most the dimension of the space of compatible F: projecting these representatives onto F modulo the old intersection gives the bound. All linear polynomials are in the target, so $\Delta_1 = e - c = m$.

For $j \ge 3$ every QF, $Q \in I_2$, belongs to $\operatorname{Sym}^{j-2} W$. For any direction v annihilating W, we have $I_2(vF) = 0$. Quadratic generation of the permanent's apolar ideal puts vF in the degree-$(j-1)$ target. Every nonzero linear combination of degree-k subpermanents needs at least k^2 essential variables: row and column scaling selects a single subpermanent from their distinct weights, and first-derivative rank cannot rise in the limit. Hence $vF = 0$ for $j = 5, 7$, since $e \le 8$. All compatible quintics and septics are supported on W. For cubics the map given by the m transverse derivatives has image of dimension at most $mZ(e)$, where $Z(2), \dots, Z(8) = 0, 0, 1, 1, 3, 3, 6$. Indeed the target quadrics on an e-plane have dimension at most the maximum number $Z(e)$ of rectangles on e cells. Left packing and sorting row lengths b_i gives $\sum_i (i-1)\binom{b_i}{2}$ and the stated numbers.

When $c = \dim W \le 2$, the cubic flag lemma on eight cells gives $I_2 F \subset W \Longrightarrow F \in \operatorname{Sym}^3 W$. Its board proof applies to every physical matrix and every support of dimension at most eight.

The finite width bound Choose m original factors independent modulo W. Quotienting by their span is injective on W. At most $\binom{9-m}{j}$ indexed j-fold products survive. In coordinates including those m original factors, an order-$(9-j)$ derivative must differentiate every chosen factor at least once in its corresponding coordinate to survive the quotient, leaving $9-j-m$ free differentiations. Dimension on W and unrestricted differentiation give the other bounds. Hence the supported derivative space has dimension at most

$$R_j = \min\left\{ \binom{9-m}{j}, \binom{c+j-1}{j}, \binom{e+8-j}{9-j}, \binom{c+8-j}{9-j-m} \right\}.$$

An out-of-range binomial is zero. Put

$$D_3 = \min\left\{ R_3 + \mathbf{1}_{c\ge 3} mZ(e), \binom{e+2}{3}, \binom{e+5}{6}, 84 \right\}.$$

The source numerator is bounded by $28m + 3D_3 + 2R_5 + 7R_7$. These are all its substitutions in increasing $c = 0, \dots, e-2$.

e	Numerators with denominator 252
2	56
3	84, 68
4	112, 89, 87
5	140, 117, 106, 129
6	168, 143, 126, 153, 183
7	196, 171, 152, 180, 183, 228
8	224, 196, 171, 242, 216, 210, 246

At $e = 8, c = 6$ for example, $m = 2$, $R_3 = 35$, $D_3 \le 47$, $R_5 = 21$, $R_7 = 1$, giving $56+141+42+7 = 246$. The deduction uses a single entire background throughout.

Quadratic multiplication and its trace quotient

Write $I_2 = \operatorname{Ann}(\operatorname{perm}_9)_2$ and $Z(L) = (I_2^2)_4^{\perp} \cap \operatorname{Sym}^4 L$. The classification of quartics annihilated by products of two permanent quadrics holds for every physical matrix of size at least four in characteristic zero. On any support of dimension $e \le 7$ it gives $Z(L) = 0$; for $e = 8$ it gives $\dim Z(L) \le 2$. On a complete three-by-three board it gives

$$Z(L) = L \operatorname{perm}_3 . \tag{6}$$

This last identity is intrinsic: restricting I_2 to the board gives exactly $\operatorname{Ann}(\operatorname{perm}_3)_2$. Every same-line generator and rectangle difference on the board is the restriction of its physical counterpart. Conversely an outside generator restricts to zero, since any rectangle with a full diagonal on the board has all four corners there.

For the actual apolar algebra A of a product supported on L, write U for the image of I_2 in A_2. The dual of A_4 embeds as quartic forms supported on L; the annihilator of U^2 in this dual is contained in $Z(L)$. Consequently $\operatorname{codim}_{A_4} U^2 \le \epsilon_e$, where $\epsilon_e = 0$ for $e \le 7$ and $\epsilon_8 = 2$. For every $T \subset A_2$ this implies

$$\dim UT \;\ge\; \dim A_4 - \epsilon_e - \binom{\dim A_2 - \dim T + 1}{2}.$$

Indeed, a functional on A_4 killing UT defines a symmetric bilinear form on $U/(U \cap T)$, a space of dimension at most $\dim A_2 - \dim T$. The kernel of this construction annihilates U^2.

Products on six and seven directions

The ordinary coefficient algebra C is the image in B_9 of $\operatorname{Sym}(L^*)$ under $v \mapsto \sum_i v(\ell_i) z_i$. Top trace extracts the coefficient of $z_1 \cdots z_9$. Differentiation of $\prod \ell_i$ by the product rule identifies A with C modulo the radical of this pairing in each layer. For the ordinary position dimensions $c_j = \dim C_j$ we use the matroid formula in Lemma 3.2. This includes proportional factors and coloops. The apolar quotient may have smaller dimensions.

The cubic loss in a containing algebra. Let the original factor span have dimension $3 \le e \le 7$. Let C be its coefficient algebra and $C \subseteq D_0 \subseteq B_9$ graded subalgebras with the induced top trace. Let D be the top-trace quotient of D_0, and M the image of C in D. The nine-factor actual algebra A is the top-trace quotient of M; the top traces coincide by construction. Set

$$c = \dim C_4, \quad h = \dim A_4, \quad \nu = \dim \ker(C_3 \to A_3),$$
$$g_0 = \dim (D_0)_4, \qquad g = \dim D_4, \qquad \kappa = g - \dim M_4 \le g_0 - c, \qquad \rho = \dim M_4 - h.$$

Suppose that some operator from the original differential plane induces an isomorphism $D_4 \to D_5$. This is an open condition on the operator. Use the physical polar choice to select such an operator v and a three-plane V preserving the ranks of all original triples and satisfying $vV \subset U$. For $Y \subset A_3$, its complete preimage in C_3 has dimension $\dim Y + \nu$. The parameter determinant gives a degree-four V-image in C_4 of dimension at least $3(\dim Y + \nu)/2 - \delta/2$, where δ is the sum of the indexed triple defects. Passing to A_4 loses at most $c - h$. In D_4 the form $(f, g) \mapsto \tau(vfg)$ is nondegenerate. On its codimension-κ subspace M_4 it has nullity at most κ; the whole ρ-dimensional kernel of $M_4 \to A_4$ is in that nullspace. Hence multiplication by v on A_4 loses at most $\kappa - \rho$. Combining gives

$$\dim UY \geq \tfrac{3}{2} \dim Y - \beta, \qquad \beta = c - g + 2\kappa + \tfrac{1}{2}\delta - \tfrac{3}{2}\nu. \tag{7}$$

With $D_0 = D = B_9$ this reads $\beta = 126 - c + \delta/2 - 3\nu/2$; the full-support Boolean linear operator is invertible in central degree. For tensor-product choices below a strong linear operator is one whose complementary powers are isomorphisms. Such operators give centered strings for multiplication. On a tensor product the sum of strong operators on the factors is again strong: apply the characteristic-zero $\mathfrak{sl}_2$ tensor decomposition to the centered strings. The projection from the original linear operators to a spanning sublist is onto, and the finitely many remaining slot evaluations may be required nonzero simultaneously.

Two or more coloops in six directions

Write $a = \dim A_2, b = \dim A_3, c = \dim A_4$. In the notation $j_d = \dim J_d$, $u_d = \dim UJ_d$, suppose $m = 0, 1$. The elementary derivative estimate of Lemma 1.3 gives $a \geq 16$: after choosing six independent original factors as coordinates, each monomial of T involves all six and repeats a variable. Its quadratic divisors include all fifteen pairs and a square. The same argument for orders one and two leaves at least five or four distinct variables and a repetition. Since U has codimension at most three, every nonzero first derivative of T has at least eight independent physical quadratic outputs; every nonzero second derivative has at least four. In particular $m = 1$ implies $u_1 \geq 8$. If $u_5 < a$, there is a nonzero $\alpha \in A_2$ orthogonal to UJ_5, so $\alpha U \subset J_5^{\perp}$ and $j_5 \leq c - 4$.

Write $M(v) = v^{\langle 2 \rangle}$ for Macaulay's degree-two successor. We obtain

$$j_3 \leq M(u_3), \qquad j_5 \leq \binom{u_5 + 1}{2}, \qquad u_3 \geq c - H(b - j_3),$$

where one may take $H(d) = \binom{d+1}{2}$, or even $H(d) = \lfloor d(d+2)/4 \rfloor$ in the last inequality. Indeed $UA_1 = A_3$ since no cubic subpermanent is supported on six directions, and $U^2 = A_4$ by quartic saturation. Hence $U^3 = A_6$, since $UA_4 = UA_1A_3 = A_3^2$. For the socle trace τ, the scalar maps $f \mapsto \tau(fu^3)$ on A_3 and $g \mapsto \tau(gu^2)$ on A_5 are injective into polynomial forms on U. Differentiating once produces the corresponding traces of Uf or Ug. Macaulay's bound on cubics in terms of their quadratic derivative span, and the elementary symmetric bound on quadrics, give the first two inequalities.

For $H_0 = (UJ_3)^{\perp} \subset A_4$ consider the vector-valued quadrics $\Psi(\alpha)(v, u) = \tau(v\alpha u^2)$, $v \in A_1$, $u \in U$. This correspondence is injective because $A_1U^2 = A_5$. Its first-derivative span has dimension at most $\dim UH_0 \leq b - j_3$. For vector-valued quadrics with first-derivative span of dimension d, the dimension is at most $\binom{d+1}{2}$: in a module monomial order, every degree-one divisor of an initial quadratic belongs to the initial derivative span. Sum the bounds $\binom{d_i+1}{2}$ across the target components, where $\sum d_i = d$. This proves the first version of H.

In the present application every nonzero member of $\Psi(H_0)$ has output rank at least two, also over the algebraic closure. Its output rank is the essential dimension of αT: an element v of the linear operator space annihilates the outputs exactly when $v\alpha = 0$, because $U^2 = A_4$. An order-four derivative of a six-direction product uses at least two variables by Lemma 1.3. To obtain the

second version of H, choose two independent scalar quadratic components p, q of one member. Their Jacobian has rank at least two somewhere: vanishing of every minor and Euler's identity would give $p\,dq - q\,dp = 0$, forcing q/p constant in characteristic zero. At a point v of rank at least two, evaluation of the first-derivative space has rank at least two. All first derivatives of members in the kernel of ∂_v vanish at v, by symmetry of the second derivative. Induction gives the bound $H(d) = d + H(d-2) = \lfloor d(d+2)/4 \rfloor$, with $H(0) = H(1) = 0$.

Two original coloops write $T = F(x_1, \ldots, x_4)yz$ with F a concise septic. Thus $a = \dim A(F)_2 + 9 \leq 19$. For $a = 16, 17$, Macaulay growth gives $(b, c) \leq (36, 71), (38, 74)$. For $a = 18$ there are three independent quadrics f, g, h in the apolar ideal. No two share a linear factor: a common factor v would give two independent linear annihilators of vT, whereas the first derivative has at least five essential variables. Thus f, g form a regular sequence. The complete intersection $R = k[x_1, \ldots, x_6]/(f, g)$ is free over a linear Noether normalization in four variables. Its nonzero element h generates a module containing the free multiple of h by that normalization. Quotienting R by h therefore removes at least 4 and 10 dimensions in degrees three and four. Its Hilbert series before the quotient is $(1-t^2)^2/(1-t)^6$, giving $b \leq 40, c \leq 75$.

At $a = 19$ the septic has quadratic dimension ten. Write $t = \dim A(F)_3 \leq 20$. Symmetry and the two independent coloop factors give $b = t + 24, c = 3t + 10$. When $t = 20$ every nonzero first derivative has physical output dimension at least eleven. Indeed, if the direction has nonzero y- or z-coefficient, setting the corresponding variable to zero leaves a multiple of Fy or Fz, whose quadratic second-derivative rank is $10 + 4 = 14$. For a pure core direction v, absence of core relations through degree three implies that vF has linear and quadratic dimensions four and ten. Thus $(vF)yz$ has second-derivative dimension nineteen. Restrict the multipliers at a cost at most three in either case.

The source numerator is bounded by $7a + (28m - 3u_1) + (3j_3 - 2u_3) + (2j_5 - 7u_5)$. The first parenthesis is at most four, and at $a = 19, t = 20$ at most zero. The following eight intervals evaluate the last two parentheses.

Let $M(v)$ be the Macaulay successor in degree two: if $v = \binom{p}{2} + r$, $0 \leq r < p$, then $M(v) = \binom{p+1}{3} + \binom{r+1}{2}$. Put $L(j) = \min\{v : M(v) \geq j\}$, so L is nondecreasing, with successive increments at most one. In the foregoing bounds the cubic objective is $3j - 2u$, where $j \leq b$, $j \leq M(u)$, and $u \geq c - H(b - j)$. The two choices are $H(d) = \binom{d+1}{2}$ for the small-quadratic case and $H(d) = \lfloor d(d+2)/4 \rfloor$ for two coloops. An upper bound, even if the constraint $u \leq c$ is discarded, is

$$P(b, c) = \max_{0 \leq d \leq b} 3b - 3d - 2\max\{L(b-d), c - H(d)\}.$$

This is nondecreasing in b and nonincreasing in c by its original j-description. To evaluate it take the smaller of the two terms $3b - 3d - 2L(b-d)$ and $3b - 3d - 2c + 2H(d)$. The first decreases, the second is discretely convex. On $d \geq 1$, $H(d) + L(b-d)$ is nondecreasing. Thus one need only test $d = 0, 1, b$ and the two neighboring integers at the first $d \geq 1$ with $H(d) + L(b-d) \geq c$ (if there is no crossing, just the endpoints). The larger table values below, including those at $c = 0$, are permissible upper estimates for the originally feasible maximum.

The quintic objective is $2j - 7v$ with $0 \leq v \leq a$, $j \leq \min(c, \binom{v+1}{2})$, and $j \leq c - 4$ when $v < a$. Its upper bound $Q(a, c)$ is nondecreasing in c. On $v < a$ maximize the lesser of $v^2 - 6v$ and $2(c-4) - 7v$. It is enough to check $v = 0, a-1$, and the two integers straddling $\binom{v+1}{2} = c - 4$, as the quadratic is convex and the affine function decreases. Add the separate value $2\min(c, \binom{a+1}{2}) - 7a$.

In the first three profiles the whole-source numerator is at most $7a + 4 + P + Q$. In the two-coloop profile $a = 19, b = t + 24, c = 3t + 10$; the same upper bound holds for $t < 20$, and for $t = 20$ the constant 4 can be dropped. Use the largest b and smallest c in each interval for P, and largest c for Q.

a	Interval	Arguments for P	P upper	c for Q	Q upper	Total upper
16	$0 \le c \le 71$	$(36, 0)$	76	71	55	247
17	$0 \le c \le 67$	$(38, 0)$	80	67	49	252
17	$68 \le c \le 74$	$(38, 68)$	56	74	56	235
18	$0 \le c \le 58$	$(40, 0)$	84	58	38	252
18	$59 \le c \le 75$	$(40, 59)$	63	75	58	251
19	$0 \le t \le 15$	$(39, 10)$	81	55	32	250
19	$16 \le t \le 19$	$(43, 58)$	62	67	49	248
19	$t = 20$	$(44, 70)$	62	70	55	250

Independent triples without a coloop

Let nine indexed factors span six directions over a characteristic-zero field. Suppose every three indexed factors are independent and deleting any one leaves rank six. For the same physical quadratic multiplier space and the same later background as in the nonic filtration, their contribution in degrees 1, 3, 5, 7 is at most one. All Gale multiplicities and specializations of the coefficients are allowed.

The estimate used by the filtration In the nine-slot algebra $B = K[z_1, \dots, z_9]/(z_i^2)$ let C be generated by the differential images of the six-dimensional factor space and $A = C/N$ its full top-trace quotient. Write $c = \dim C_4$, $h = \dim A_4$, $\nu = \dim N_3$. Let U be the image of the permanent's quadratic annihilator. Then $C_2 = A_2$ consists of all 21 polynomial quadrics. Indeed no factor is a coloop, so its square is a combination of products with other factors, giving $\dim C_2 = 21$ by transposition. For each pair of original positions one can choose a differential direction killing just those two factors, by independence of all triples. Its seventh power isolates the complementary seven slots. Thus $C_7 = B_7$ and the restricted quadratic pairing has zero left radical.

We prove for every s-plane $X \subset A_2$

$$\dim UX \;\ge\; 7s/2 - (c - h). \tag{1}$$

The full ordinary-to-actual fourth kernel costs exactly $c - h$. An ordinary bound $\dim_C UX \ge 7s/2$ implies (1).

We use the physical selection and (7), with the whole nine-slot algebra as the containing algebra. The cubic allowance is $\beta = 126 - c - 3\nu/2$. At overlap six, (1) gives internal numerator at most $2(h + c - h + \beta) = 252 - 3\nu$. At overlap five, multiplication by a nonzero linear form is injective on the at least eighteen physical polynomial quadrics. Its cubic image loses at most P on passage to C_3 and then ν on passage to A_3, since the degree-three coefficient of the Gale Hilbert formula is $56 - P$. Thus $\dim UJ_1 \ge 18 - P - \nu$, which gives at most $226 + 3P$, where P counts equal-direction Gale pairs and $P \le 6$. At lower overlaps the established whole-background bounds apply. This use of (1) replaces the constituent strong-operator and central-rank calculations for this class.

Two lower bounds for the ordinary fourth dimension The dual Gale list has rank three. Its nine nonzero columns give projective points of integer weights w_i; every line has total weight at most five, by independence of original triples. If every four original factors are independent, each line has weight at most four. Write $T = \sum_i \binom{w_i}{3}$ and let L be the number of collinear triples in three distinct directions, counted with multiplicity. The ordinary coefficient-space Hilbert formula gives

$$c = 126 - 6P + T - L, \qquad P = \sum_i \binom{w_i}{2}.$$

Indeed its Hilbert series is $\sum_{S\subset[9]} t^{|S|}(1-t)^{3-r_G(S)}$, the external zonotopal Hilbert formula after matroid duality. In degree four the four-subsets contribute 126, the three-subsets subtract $7P - T + L$, and two-subsets restore P. This is the ordinary Hilbert function, before taking the trace quotient.

If every line has weight at most D, counting the three pairs in each collinear triple gives

$$3L \le \sum_{i<j} w_i w_j (D - w_i - w_j).$$

At $D = 5$ the largest weight can be four. A four must occur with five singles. A three occurs with j doubles and $6 - 2j$ singles, $0 \le j \le 3$. Otherwise take j doubles and $9 - 2j$ singles, $0 \le j \le 4$. Substitution gives the following lower bounds for c in increasing j.

Weight type	At $D = 5$	At $D = 4$
Only doubles and singles	90, 90, 90, 89, 89	102, 102, 101, 100, 100
One triple, then doubles and singles	88, 88, 88, 87	99 for $j = 0$
One quadruple and five singles	84	Impossible

Thus $c \ge 84$ always, and $c \ge 99$ if every four original factors are independent.

Physical polynomial growth and large widths For a six-dimensional factor plane, the row-and-column torus specializes the plane to six cells; specialize the same polynomial input space along it. In the limiting multiplier space all monomial quadrics except possibly the edges of a four-vertex path are available. Two distinct rectangles on at most six cells must share two cells and give a two-by-three board, or its transpose. A diagonal order there gives the three missing edges of a path. A single missing edge embeds in that path as well. This uses the restriction of the actual physical quadratic annihilator: same-row and same-column derivatives and the two rectangle diagonals with opposite signs. The limiting space contains the specialized images, even if their rank drops.

Let $P_n(s)$ be a lower bound for a polynomial four-shadow on n variables with the three path edges missing. Start with $P_4(0) = 0$, $P_4(s) = 7$ for $1 \le s \le 10$. For $n = 5, 6$ shift the quadratic inputs towards the first isolated variable, preserving the allowed multipliers. In a two-variable fiber initial intervals have smallest possible sumset and their output intervals are nested. If the first row consists of 00 and $0i$ for i in a t-set, the remaining $s - 1 - t$ inputs lie on those t variables. Count the quartics from this row and those avoiding vertex zero separately:

$$P_n(s) = \min_t \{b_n(t) + P_{n-1}(s - 1 - t)\}, \qquad 0 \le t < n, \quad 0 \le s - 1 - t \le \binom{t+1}{2}.$$

Here $P_n(0) = 0$. The row b_n, indexed from zero, is $(12, 20, 26, 31, 35)$ for $n = 5$ and $(18, 30, 39, 46, 52, 56)$ for $n = 6$. To check a row, choose t of the remaining variables, count cubic monomials containing zero or a chosen variable, then discard those for which removing any such variable leaves a path edge. Minimize over these subsets of at most five vertices. The resulting first twelve entries of P_6 are

$$18, 30, 39, 46, 52, 56, 68, 76, 79, 79, 85, 89.$$

The polynomial-to-actual map costs at most $126 - h$ quartics. Thus (1) holds when $P_6(s) \ge 126 - c + 7s/2$. For $7 \le s \le 12$ this follows by substituting $c \ge 84$. At $c \ge 99$ it also holds for $3 \le s \le 6$.

For $13 \le s \le 21$, physical seven-direction square saturation gives $U^2 = A_4$. A functional vanishing on UX is determined by its symmetric form on $U/(U \cap X)$, of dimension at most $21 - s$. Hence $\dim UX \ge h - \binom{22-s}{2}$. The convex function $7s/2 + \binom{22-s}{2}$ has endpoint values $81.5, 73.5$ on this interval, both at most 84.

For $s = 1, 2$ a single position pivot suffices. Choose six original factors as basis slots. In their slot generators $t_i + g_i$ each g_i on the outside slots is nonzero because there is no coloop. The 21 distinct quadratic leading types form the edges of K_7: pairs of basis slots, or a basis slot joined to the outside vertex via $2t_ig_i$. Order by number of basis slots and then by generic additive basis weights. Two disjoint leading edges multiply to a nonzero four-basis-slot or three-basis-slot component. For one fixed input edge the ten disjoint edges on the other five vertices yield distinct leading types. The physical multiplier space misses at most three pivots (six cells have at most three rectangles). Thus its ordinary product has dimension at least seven, sufficient for both widths.

Splitting at a four-circuit It remains to prove (1) at $3 \le s \le 6$ when an original four-circuit occurs. Its complementary five Gale labels lie on a projective line. If this five-label line has maximum multiplicity at most three, the original four-circuit is a flat: removing any one of those five Gale labels still leaves rank two. Its contraction gives a rank-three list of five nonzero factors without coloops, since the restricted five Gale columns are all nonzero. If the line has type $4 + 1$, use the five original factors complementary to the quadruple Gale class instead. They have rank three, and form a flat, since adding any one of the four others raises rank to four by duality. Every three of those five remain independent. The contraction on the other four has dual a nonzero rank-one list with four columns, and hence is a ternary four-circuit.

In either case the ordinary algebra specializes to two independent three-dimensional blocks, a full ternary four-circuit and a five-factor rank-three list without zeros or coloops. More precisely, scale the slots of the flat by a common parameter of weight -1. Split a basis of the original differential plane into lifts of its restriction and a basis of the kernel of restriction. Its limiting generator plane is the direct sum of the restriction and contraction. Their ordinary degree-two spaces have dimensions $6, 6$ by the no-coloop argument, so the limiting quadratic space has full dimension $6 + 9 + 6 = 21$. It therefore equals the limit of C_2. The limiting U and X retain their dimensions inside it. All products they give in degree four are independent lower bounds at the original parameter. Neither equality of the ordinary quartic spaces nor preservation of their radicals is required.

In the four-slot block take letters a, b, c mapping to $z_1 + z_4, z_2 + z_4, z_3 + z_4$. The four standard cubics in lex order $a > b > c$ are

$$bc^2,\ ac^2,\ abc,\ ab^2.$$

The cubes vanish, $i^2j = ij^2$ for distinct letters in this ordinary algebra, and the four displayed products are independent by their slot-support coefficients. In the five-slot block take a linear form f nonzero in every slot and extend it to a basis d, e, f. Multiplication by f injects the entire Boolean degree two into degree three. After rescaling slots, a kernel vector would satisfy $v_{ij} + v_{ik} + v_{jk} = 0$ on every triple. Subtract the equations for fixed k, l and every pair i, j outside them. The three differences $v_{ik} - v_{il}$ for the three outside indices are pairwise negatives, hence zero. Every off-diagonal entry is consequently equal, and three times that entry is zero. It follows in reverse lexicographic order with f last that the six cubics $f \cdot \{d^2, de, df, e^2, ef, f^2\}$ are standard: all come before any cubic not containing f and are independent. Use a product term order with these two restrictions.

All 36 monomials of bidegree $(2, 2)$ are standard. Also retain each displayed cubic times any one letter from the other block. These are the only quartics we use. In degree two let the s input pivots be split into pure and mixed pivots. Delete exactly three multiplier pivots; further available multipliers only help. If a_0, c_0 are the numbers of pure inputs in the two blocks and p, q, r the numbers of missing multipliers in bidegrees $(2, 0), (1, 1), (0, 2)$, their pure-input products alone

number at least

$$6(a_0 + c_0) - a_0 r - c_0 p - a_0 c_0 + (3 - q)(t_A + t_B). \tag{2}$$

The first terms count the union of two rectangles in the 6-by-6 central layer. Each cubic extension of a pure input has three opposite letters, with at most q lost. For $a_0 = 0, 1, 2, 3, 4, 5, 6$ the minimum counts t_A of distinct cubic extensions are $0, 0, 1, 2, 3, 4, 4$ by the explicit list of four cubics. For the other block $t_B \geq c_0$ by multiplying by f. Formula (2) is affine in p, q, r, so its minimum on $p + q + r = 3$ occurs at one of three endpoints. These substitutions give the following row minima, ordered by $a_0 = 0, \ldots, m$, where $m = a_0 + c_0$.

m	Lower bounds for the number of outputs from the pure inputs
2	12, 11, 9
3	18, 16, 16, 15
4	24, 21, 20, 21, 21
5	30, 26, 24, 24, 26, 27
6	36, 31, 28, 27, 28, 31, 30

Each mixed pivot has at least 15 possible products before deletion: nine mixed multipliers give central products, and at least three selected cubics in each block are divisible by the corresponding letter. A pivot involving f has at least 18. At most three are lost per pivot. Two distinct pivots have at most six common outputs (multiples of their cubic least common multiple), or at most one if relatively prime.

With no mixed pivot the table proves the result. With exactly one, its twelve surviving products suffice through $s = 3$ and the pure row $m = s - 1$ suffices thereafter. Two mixed pivots already give at least $15 + 15 - 6 - 6 = 18$, sufficient through $s = 5$. A relatively prime mixed pair gives at least 23, sufficient for $s = 6$. If exactly two mixed pivots occur at $s = 6$, four pure pivots suffice except possibly the central entry 20 in row four. For $a_0 = c_0 = 2$, (2) gives $23 - q$, so only $q = 3$ matters. Then all pure multipliers survive. The pure rows and columns give 20 central products, and any mixed pivot gives three further products in bidegree $(3, 1)$.

Finally, three or more pairwise intersecting mixed pivots form a three-edge star in $K_{3,3}$. If the common variable is in the four-slot block, one pivot involves f; that pivot and another already give $18 + 15 - 6 - 6 = 21$. If the common variable is in the five-slot block, each pair of pivots has at most five common outputs: three from an extra five-slot letter and at most two selected four-slot cubics containing its two distinct four-slot letters. The three sets have the common product abc times the star center, so their union before deletion has at least $45 - 15 + 1 = 31$ outputs. Deletion costs at most nine, leaving 22. This proves ordinary growth $7s/2$ at every required width and finishes (1).

Independent triples with one coloop

Suppose the nine original nonzero factors span six dimensions, every three are independent, and precisely one is a coloop. Their ordinary coefficient algebra is

$$C = C^G \otimes K[t]/(t^2),$$

where G is the coloop-free product of the eight other factors on five directions. Write $A = C/N$ for the actual top-trace quotient, $c = \dim C_4$, $h = \dim A_4$, and $\nu = \dim N_3$. We prove the unit source bound without separate parameter calculations on rank-four sextic cores. We use the physical degree-two square saturation $U^2 = A_4$, six-direction cubic selection, the polynomial bound P_6 from the accompanying rank-six proof, and the same original lower-overlap estimates.

Ordinary dimensions The rank-three Gale list of G consists of eight weighted nonzero projective points, with every line of total weight at most four. A line of weight five would be complementary to a dependent original triple. Write $P = \sum \binom{w_i}{2}$, $T = \sum \binom{w_i}{3}$, and L for the number of collinear triples on distinct directions counted with weights. The external coefficient Hilbert formula gives

$$\dim C_2^G = 15, \quad \dim C_3^G = 35 - P, \quad \dim C_4^G = 70 - 5P + T - L, \quad c = 105 - 6P + T - L.$$

The third-degree identity follows by taking the coefficient of degree three in $\sum_{S\subset[8]} z^{|S|}(1-z)^{3-r(S)}$. In degree four the three-subsets subtract $6P - T + L$ and the two-subsets restore P. Counting pairs in collinear triples gives $3L \le \sum_{i<j} w_i w_j (4 - w_i - w_j)$. One may have one triple with five singletons, or $j = 0, 1, 2, 3, 4$ doubles and $8 - 2j$ singles. Substitution gives the table.

multiplicities	$3, 1^5$	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$
P	3	0	1	2	3	4
Lower bound for $\dim C_4^G$	50	52	51	51	51	50
Lower bound for c	82	87	85	84	83	81

There are 20 effective quadrics C_2, the only polynomial relation in degree two being t^2. Every pair of original factors can be killed by one differential which kills no others, since every triple is independent over an infinite field. Its seventh power isolates the complementary seven slots. Hence $C_7 = B_{9,7}$, the left radical in degree two is zero, and $A_2 = C_2$.

Quadratic estimate We prove $\dim UX \ge 7s/2 - (c - h)$ for every s-dimensional $X \subseteq A_2$. For $6 \le s \le 11$ lift X to its full $(s+1)$-dimensional preimage in the polynomial ring on all six directions. Take the physical polynomial restriction of I_2 in that ring; its actual projection is U. The row-column path bound $P_6(s+1)$ from the accompanying rank-six argument gives respectively $68, 76, 79, 79, 85, 89$ polynomial products. These exceed or equal $126 - 81 + 7s/2$. Quotienting to A_4 costs $126 - h$. For $12 \le s \le 20$, a functional annihilating UX is determined, by $U^2 = A_4$, by its symmetric form on $U/(U \cap X)$, which has dimension at most $20 - s$. On this interval $7s/2 + \binom{21-s}{2} \le 81$ by convexity at the endpoints. This proves the claim in these ranges.

For $1 \le s \le 5$, we prove the stronger ordinary estimate $\dim_C UX \ge 7s/2$. Take highest core degree in the displayed coloop splitting. The leading multiplier components are a subspace $U_2 \subset C_2^G$ of codimension $p \le 1$ and tV, where $V \subset C_1^G$ has dimension at least $2 + p$. Indeed the projection to the five-dimensional physical core is the physical quadratic restriction there and its dimension loss is at most one (five cells support at most one rectangle). The whole physical dimension loss in the six-space is at most three. The leading test space consists of $X_a \subset C_2^G$ and tY_b with $a + b = s$ and dimensions a, b. Associated products can only lose rank. The two disjoint output layers give

$$\dim_C UX \ge \dim U_2 X_a + \max\{\dim U_2 Y_b,\ \dim VX_a\}. \tag{1}$$

For $a = 1, 2$, $\dim U_2 X_a$ is at least 5 and 7 respectively. To see this, write a basis for the five-position projection of the eight-slot generator plane as $z_i + g_i$, $i = 1, \dots, 5$, with each g_i nonzero on the other three slots. This holds because the core has no coloop. Order by number of basis slots and then generic additive basis weights. The 15 distinct leading quadratic types are edges of a virtual K_6; its last vertex represents the outside component in $2z_i g_i$. Disjoint edges produce a nonzero leading four-set type (four basis slots or three and the outside vertex). For each input edge there are six disjoint multipliers, at least five retained. Two input edges have at most three four-sets in common, since their union has at least three vertices. These counts give 5 and $5 + 5 - 3 = 7$ independent leading types. The virtual vertex is used only in this lower count; no equality of algebras is asserted.

For $a = 3, 4, 5$ count first in the five-variable polynomial ring. Simultaneous compression preserves the sizes of the two quadratic input families and can only decrease their product shadow: in

a two-variable fiber the sumset of two nonempty exponent sets has at least the sum of their cardinalities minus one, and the compressed sumsets are nested intervals at the same end of the output fiber. For $p = 1$ all multipliers except the last square are retained after shifting. Shifted input diagrams are strict partitions, with row i starting at column i. The table records all polynomial counts. Each entry is obtained by partitioning a sorted quartet into two pairs, one in each family.

a	Shape	All multipliers	All except the last square
3	3, 21	31, 35	28, 32
4	4, 31	34, 41	30, 37
5	5, 41, 32	35, 44, 47	34, 39, 42

Passing to C_4^G costs at most $70 - 50 = 20$. At $a = 3, 4, 5$ this gives respectively $11, 14, 15$ for $p = 0$ and $8, 10, 14$ for $p = 1$. Also

$$\dim U_2 Y_b \geq 35 - \binom{7-b}{3} - pb - P.$$

The full cubic polynomial space containing a factor from a b-plane has codimension $\binom{7-b}{3}$; removing p multipliers costs at most pb, then the ordinary cubic kernel costs P. At $b = 1, 2, 3, 4, 5$ this bounds the mixed layer below by $10, 19, 24, 26, 26$. These numbers and the preceding pure bounds in (1) imply $7s/2$ whenever $b > 0$, including $a = 0$.

It remains to take $b = 0$. For $p = 0$ the pure bounds already suffice through $a = 4$. At $a = 5$ choose two independent letters in V. A five-element family of quadratic leading monomials has at least seven cubic products with the two letters: group by the outside monomial; each nonempty group gains at least one in its two-variable shadow, and each has at most three quadratic members. The cubic quotient loses at most $P \leq 4$, giving the missing three products in (1).

For $p = 1$ choose three independent letters x, y, z in V. For $a \geq 2$ their product with X_a spans at least $a + 3$ polynomial cubics. Group its input pivots by outside monomial. In each nonempty group use the exponents of x, y as plane lattice coordinates. Its shadow is the union of the original points and their unit translates up and to the right. Each occupied row supplies a new right-hand endpoint; every point in the highest row supplies a new point above. A group of size at least two gains at least three points, and each singleton gains two. This proves the bound across all groups.

The loss in these particular cubic products is at most three even if $P = 4$. Only that last case needs proof. The four parallel pairs of the eight Gale columns have disjoint supports. For each pair the core generator plane contains a nonzero direction d_i supported precisely on its two slots, by Gale duality. The four d_i are linearly independent. The formal cubes d_i^3 are independent, vanish in the Boolean ordinary algebra, and span the entire polynomial cubic kernel of dimension four. If all four cubes lay in the degree-three ideal generated by x, y, z, all four d_i would lie in their three-space, since this linear ideal is prime. Thus the intersection of their span with the particular product space has dimension at most three. For $a = 3, 4, 5$ we now add at least $3, 4, 5$ cubic products to the pure $8, 10, 14$. This proves (1)'s required ordinary bound also in this last case; $a = 1, 2$ were already paid.

First-layer and source accounting Every nonzero linear input has ordinary product with the physical quadrics of dimension at least ten. In the coloop splitting, if it has nonzero core leading component, the $(15 - p)$-dimensional pure multiplier space acts injectively in the polynomial ring and the ordinary cubic kernel costs at most $P \leq 4$. For a pure coloop input it gives $15 - p \geq 14$ independent products directly. The actual cubic loss is at most ν. For the full cubic shadow use the same Boolean container B_9, selecting a nonzero coordinatewise linear operator and the

same physical three-plane preserving all original triple ranks. Triple independence gives the full normalized Boolean shadow, and the complete radicals give $\beta = 126 - c - 3\nu/2$ exactly as in the preceding no-coloop proof. This selection is stated for arbitrary supports of ranks three through seven and the ranks of their original triples; its theorem has no no-coloop hypothesis.

The actual perfect degree-seven pairing has dimension 20 here. At $X = (UJ_5)^{\perp} \subset A_2$ the quadratic inequality yields $2\dim J_5 + 7(20 - \dim UJ_5) \le 2(h + c - h)$. Together with the cubic estimate and $UJ_d \subseteq K_{d+2}$, this gives the numerator bound $252 - 3\nu + 28(m - k_1) - 3\dim UJ_1$. At overlap six $m = 0$. At overlap five $m = 1$ and the first-layer bound gives at most 250. When $m \ge 2$, apply the low-overlap calculation. This completes the triple-independent, exactly-one-coloop case.

One dependent triple

Let nine distinct projective physical factors span six directions. Suppose exactly one of the indexed triples is dependent and at most one original factor is a coloop. For either coloop count and every fixed set of later terms, we prove $N \le 252$.

One binary cubic container The three factors of the distinguished triple form a distinct binary cubic G. In the nine-slot algebra put the ordinary binary coefficient algebra on these three slots and the Boolean algebra on the six other slots:

$$D_0 = C(G) \otimes B_6, \qquad D = A(G) \otimes B_6.$$

The ordinary/actual binary rows are $(1, 2, 3, 1)$ and $(1, 2, 2, 1)$, as follows directly from a split squarefree cubic in two variables. Therefore $g_0 = \dim(D_0)_4 = 106$, $g = \dim D_4 = 91$. The original ordinary algebra C is a subalgebra of D_0; its map to D has image M, whose top-trace quotient is the actual algebra A. Put

$$c = \dim C_4, \qquad h = \dim A_4, \qquad \nu = \dim\ker(C_3 \to A_3),$$
$$\kappa = 91 - \dim M_4, \qquad \rho = \dim\ker(M_4 \to A_4).$$

Then $0 \le \kappa \le 106 - c$ and $h = 91 - \kappa - \rho$.

On D a strong linear operator can be selected from the original differential plane. Its projection to the binary differential plane is onto. The binary strong open is nonempty by the cubic and its nonzero binary Hessian. The additional six linear values can all be required nonzero. The characteristic-zero centered-string tensor rule then gives a central isomorphism on D. In (7) the only dependent triple contributes $\delta = 1$, yielding

$$\dim UW \ge \tfrac{3}{2}\dim W - \beta, \qquad \beta = c - 91 + 2\kappa + \tfrac{1}{2} - \tfrac{3}{2}\nu \qquad (W \subset A_3). \tag{2}$$

The six Gale counts The rank-three Gale dual has one zero for each coloop. The nonzero labels consist of one projective line of weight $b = 6$ or 5 and three outside labels. This follows from the rank formula on the complement of the one dependent triple. Every other line has weight at most $b - 1$. Two such lines of weight at least b would give two dependent triples, and more than b labels on one line would do the same. Denote inside and outside sums of $\binom{w}{2}, \binom{w}{3}$, taken over the distinct projective directions with their multiplicities, by P_i, T_i, P_o, T_o. Put $P = P_i + P_o$. Let J_o count triples from three distinct collinear outside directions, with multiplicity, and $J_\times$ the collinear triples with two distinct outside and one inside direction. The number of triples using distinct inside directions equals $\binom{b}{3} - (b-2)P_i + 2T_i$.

For zero coloops the ordinary external coefficient formula gives $c = 126 - 6P + (T_i + T_o) - J$, where J counts collinear triples on three distinct directions. For one coloop the eight-factor core has ordinary layers $35 - P$ and $70 - 5P + (T_i + T_o) - J$ in degrees three and four. Tensoring with the square-zero coloop gives $c = 105 - 6P + (T_i + T_o) - J$. These are the degree-four

coefficients of $\sum_S t^{|S|}(1-t)^{3-r_G(S)}$ for the nonzero Gale list and the extra coloop factor when present. Substituting the inside identity gives

$$c = \begin{cases} 106 - 2P_i - T_i - 6P_o + T_o - J_o - J_\times, & b = 6, \\ 95 - 3P_i - T_i - 6P_o + T_o - J_o - J_\times, & b = 5. \end{cases}$$

Coloop count	Three outside labels all equal	Two equal	All distinct	Maximum P
Zero	83	82	78	7
One	78	81	78	4

Here are the elementary bounds behind the table. An inside class and an outside class together have weight at most $b-1$. For $b=6$, the possible maxima of $2P_i + T_i$ in the three columns are $6, 14, 18$ (the extreme inside partitions are $222, 33, 42$). The bounds on $J_\times$ are $0, 4, 9$, since each distinct outside pair meets the inside line in at most one direction, of weight at most $5 - w - w'$. Only in the third column can J_o be nonzero, and then it is at most one. For $b=5$ the first two columns have all inner classes single, or of size at most two respectively, giving $J_\times \le 0, 2$. In the last column all inside weights are at most three and $J_\times \le 6$. Except for inside partition 32, one has $3P_i + T_i \le 10$, so $c \ge 95 - 10 - 1 - 6 = 78$. For 32, no outer pair line can pass through the weight-three point. At most one outer pair line can pass through the weight-two point, since two such pairs share an outside point and would put all three on the same line of total weight five. Thus $J_\times \le 2$ in this exceptional partition and $c \ge 95 - 13 - 1 - 2 = 79$. The pair-count column follows from the same inside multiplicity caps. In every case $c \ge 78$.

One quadratic estimate Set $\alpha = 251/2 - c - \kappa + \rho$. Then $\alpha \ge 19.5$, $h + \alpha \ge c + 4.5 \ge 82.5$, and $h + \alpha + \beta = 126 - 3\nu/2$. We prove $\dim UX \ge 7s/2 - \alpha$ for each s-plane $X \subset A_2$. Through $s = 5$ the right side is nonpositive. At $s = 6$ it is at most 1.5, and any one nonzero actual septic in XT has at least four independent physical quadratic outputs by the derivative estimate at the start of the two-coloop calculation. For $7 \le s \le 12$, lift X to the six-variable polynomial ring and use $P_6(s) = 68, 76, 79, 79, 85, 89$ from the polynomial path calculation. Quotienting degree four loses at most $126 - h$. The required bounds $126 - 82.5 + 7s/2$ are $68, 71.5, 75, 78.5, 82, 85.5$. For $13 \le s \le \dim A_2 \le 21$, square saturation gives $\dim UX \ge h - \binom{22-s}{2}$; the convex endpoint values for $7s/2 + \binom{22-s}{2}$ are $81.5, 73.5$.

With no coloop every nonzero first input produces at least $18 - P - \nu$ actual cubics. The six-dimensional physical polynomial restrictions have dimension at least eighteen, multiplication by a fixed letter is injective, and the ordinary and actual cubic quotients cost P and ν. With one coloop use the highest-core-degree filtration. The pure quadratic restrictions on the remaining coloop-free five-space have dimension at least fourteen, since its ordinary quadratic dimension is fifteen and its physical deficiency at most one. Its ordinary cubic dimension is $35 - P$, giving at least $14 - P - \nu$ actual cubic products when the input has nonzero core leading part. For a pure coloop the fourteen pure multipliers remain independent before the final actual cubic quotient.

The common actual-source reflection now gives $252 - 3\nu + 28(m - k_1) - 3\dim UJ_1$. At $m = 0$ this is at most 252. At $m = 1$ use $P \le 7, 4$ in the two coloop counts to obtain at most $247, 250$. At $m \ge 2$ use the whole-background low-overlap theorem on six directions.

Repeated factors

Let nine nonzero original factors span six directions, with zero or one original coloop and at least one proportional pair. Every whole-background source has numerator at most 252. The three

possible multiplicity lists have at most one tripled direction. Indeed six distinct directions would form a basis, and distributing three additional copies leaves at least three independent singleton coloops; five distinct directions cannot span six. Thus the possibilities are one double and seven further distinct factors, two doubles and five further distinct factors, or one triple and six further distinct factors.

Use the actual A, ordinary C, physical U, cubic radical ν, and $c = \dim C_4,\ h = \dim A_4$ as in the unique-dependent-triple proof. Treat the repeated directions as an independent monomial core and all other slots as Boolean. In all three containers the ordinary-to-actual map is an isomorphism. Their linear strong opens are nonempty by centered tensor strings; each is attained on the original generator plane, since the original core directions are independent and the remaining linear evaluations can simultaneously be made nonzero. The crucial numeric inputs are

core and outside slots	$g_0 = g$	c at least	δ at most
$h^2,\ B_7$	91	64	9
$h^2k^2,\ B_5$	66	48	18
$h^3,\ B_6$	56	41	23

Here $\delta = \sum_{|I|=3}(3 - r(I))$ uses the original indexed triples with all copies retained.

The first row Let z be the number of coloops. In the rank-three Gale dual there are z zero labels. The complementary seven original slots to the one doubled pair have nonzero Gale labels on one line, of weight $b = 7 - z$; precisely two outside labels remain. Other lines have weight at most $b - 1$, since two different seven-slot complements of rank at most two would give a second proportional pair. A line cannot carry more than seven original indices, since an eight-slot complement would give a zero original factor. Use P_i, T_i, P_o, T_o for the inner/outer sums of $\binom{w}{2}$, $\binom{w}{3}$, and $J_\times$ for weighted triples containing two distinct outside directions and one inside direction on their line. The same external Hilbert formula as in the unique-triple proof simplifies to

$$c = \begin{cases} 91 - P_i - T_i - 6P_o - J_\times, & b = 7, \\ 85 - 2P_i - T_i - 6P_o - J_\times, & b = 6. \end{cases}$$

When the outside labels coincide the maximum inner weight is $b - 3$ and $J_\times = 0$; when they differ, the maximum inner weight is $b - 2$, with $J_\times \le b - 3$. Thus the lower bounds for equal/distinct outside labels are $71, 66$ at $b = 7$ and $65, 64$ at $b = 6$. These follow by concentrating the inner weights, with respective extreme partitions $43, 52, 33, 42$.

A dependent indexed triple corresponds exactly to a six-slot complement of rank two. The seven ways of choosing six of the original slots on the main line give seven triples. Each other line of weight $b - 1$ gives one more, with every zero included. There are at most two such lines. With the two outer labels equal, each requires an inside point of weight $b - 3$, and at most two such points fit. If the outer labels differ, a line through just one of them must have an inner point of weight $b - 2$; there is at most one such inner point. If it exists there are at most two lines through it, and the remaining inner weight is two, too small for a third line through both outside points. If it does not exist only the line joining both outside points can qualify. There are no rank-one triple complements, as no original triple is proportional. This proves $\delta \le 9$.

The last two rows Take a basis of six original factors including the two doubled directions, or the tripled direction, respectively. Beyond the basis there are the two dictated copies and one additional factor p. In the two-double case p has a nonzero coefficient at some basis factor outside h, k, since otherwise the other four basis factors would be coloops. In the triple case it has a nonzero coefficient outside h, by distinctness. Give that coordinate the first weight. The product specializes to exponent lists $(2, 2, 2, 1, 1, 1)$ or $(3, 2, 1, 1, 1, 1)$. The dimensions of spans of indexed

fourfold subproducts cannot go up in this limit. Their limiting dimensions are the coefficients of t^4 in $(1+t+t^2)^3(1+t)^3$ or $(1+t+t^2+t^3)(1+t+t^2)(1+t)^4$, namely 48 and 41.

Triples involving exactly one extra and two basis factors have total defect at most $5+5+1=11$: the two extra copies each lie on the five basis pair planes through their axis, and the distinct p lies on at most one basis pair plane. Two extras and one basis factor contribute at most six in the two-double row, because each independent extra pair plane contains at most two basis elements. In the triple row the equal extra pair contributes at most seven and the other two pairs at most four. All three extras contribute at most one in either row. These give $18, 23$.

Complete source accounting Write M for the original image in the strong container, $\kappa = g - \dim M_4 \le g_0 - c$, and $\rho = \dim\ker(M_4 \to A_4)$. As in the unique-triple proof $h = g - \kappa - \rho$, and the prescribed-open physical three-plane and original Boolean cubic-shadow theorem for the full indexed defect give

$$\dim UW \ge \tfrac{3}{2}\dim W - \beta, \qquad \beta = c - g + 2\kappa + \delta/2 - 3\nu/2, \qquad W \subset A_3.$$

Take $\alpha = 116 - c - \kappa + \rho - \delta/2$. The table implies

$$\alpha \ge \min(20.5, 41, 48.5) = 20.5, \qquad h + \alpha \ge \min(84.5, 89, 89.5) = 84.5.$$

Every $X \subset A_2$ of dimension s has $\dim UX \ge 7s/2 - \alpha$. For $s \le 5$ the right side is nonpositive. For $s = 6$ it is at most $1/2$, and a nonzero septic on six variables has a nonzero physical square output by the squarefree argument in the unique-triple proof. For $7 \le s \le 12$ use the six-variable path bounds $P_6(s) = 68, 76, 79, 79, 85, 89$ and the entire polynomial-to-actual loss $126 - h$. For $13 \le s \le 21$ use $U^2 = A_4$ and the symmetric quotient bound as in that proof. Both tests already work at $h + \alpha \ge 82.5$.

For the first layer let P be the pair count on the nonzero Gale classes. With no coloop their total weight is nine, every line has weight at most seven, and the Gale rank is three. A class of weight six forces the other weights one; with maximum at most five and at least three directions one has $P \le 15$ as well. With one coloop the total weight is eight, every line has weight at most six, and a class of weight five forces all others single. With maximum at most four and at least three classes one again has $P \le 10$. The cubic dimension is $56 - P$ without a coloop and $35 - P$ on the five-dimensional coloop-free core with one coloop. The same polynomial and split first-line arguments as in the unique-triple proof therefore give at least $18 - P - \nu$ and $14 - P - \nu$ actual cubic outputs, respectively.

The source reflection gives $232 - 3\nu + 28m - 3\dim UJ_1$. It is at most 232 when $m = 0$, and at most 236 when $m = 1$, by the six-direction derivative bound $\dim UJ_1 \ge 8$. The low-overlap calculation covers $m \ge 2$.

Several dependent triples

Let the nine original factors be pairwise nonproportional, span six directions and contain at least two dependent indexed triples. Suppose zero or one of the original factors is a coloop. The same whole-background source bound holds in all these cases using three possible small cores. A characteristic-zero field is understood.

Core and contraction Take two dependent triples. If they share two factors, their union is a squarefree binary quartic Q_4. If they share just one factor and their binary planes coincide, there would be five factors in a binary plane, forcing the other four directions to be original coloops. Otherwise their union is the rank-three quintic

$$H_5 = xyz(x+y)(x+z)$$

after nonzero rescalings. If the triples are disjoint, their two binary planes must be independent, since otherwise the other three factors would all be coloops. The third possible core is then a product G_3G_3' of binary cubics on independent planes.

Each displayed core has rank r and $r+2$ factors, and is a flat. A further factor on its span would leave only $6-r$ factors outside a rank-r space; each would then be a coloop, whereas $6-r \geq 2$. Upon contraction of this flat the remaining $7-r$ labels have rank $6-r$, no zeros, and exactly the original number z of coloops. Indeed contraction preserves the coloop status of each other label by the deletion rank formula. A rank-one deficiency on a nonzero list is one circuit on all noncoloop positions. Denote a full circuit on $k+1$ factors of rank k by U_k.

A linear coordinate change scaling the core support to zero on the outside labels and preserving each rescaled core factor gives the product of the restriction and contraction in the limit. Explicitly extend a basis of the flat's span to a basis of the six-space, multiply its r coordinates by λ, and divide each original core factor by λ. At $\lambda = 0$ no outside factor vanishes, since this is a flat. The ranks of matrices of indexed subproducts are lower semicontinuous. The resulting tensor rows thus give lower bounds on the original ordinary dimensions C_3 and $c = C_4$.

Core	$z=0$, lower (C_3, c)	$z=1$, lower (C_3, c)	Maximum triple defect δ
Binary quartic	$(46, 72)$	$(44, 69)$	4
H_5	$(48, 78)$	$(45, 72)$	3
Two binary cubics	$(47, 75)$	$(45, 72)$	3

The ordinary rows of a squarefree binary cubic or quartic are $(1, 2, 3, 1)$ and $(1, 2, 3, 4, 1)$; the subproducts at one below the top are independent by evaluating at their distinct roots. A full circuit U_k has row $1, k, \binom{k+1}{2}, \ldots, \binom{k+1}{k}, 1$. Indeed its ordinary slot generators can be normalized to $s_i + t$; their squares and pair products span the whole squarefree quadratic layer, and at higher degrees pure or mixed squarefree position monomials are then generated inductively. The ternary H_5 has ordinary row $(1, 3, 6, 8, 5, 1)$ by the rank-two Gale formula (two parallel pairs and one singleton). Multiplying these rows by the contraction rows just described gives all displayed entries except the improved last $z = 1$ entry.

For this last entry split off the original coloop. The remaining pair outside the two cubics can be written $y, y + h$, where $0 \neq h = h_1 + h_2$ in the two binary planes. Indeed their quotient projections are proportional and nonzero, and the original factors are distinct. Rescale so $h_1 \neq 0$ after interchange. Independently scaling the second binary plane leaves both cubics fixed up to scalar and specializes h to h_1. This leaves an independent cubic and coloop alongside the ternary quintic $G_3 y(y + h_1)$. Its ordinary quadratic and fourth dimensions are six and five: deleting any single factor preserves its rank, and the original five rays are distinct, giving independent fourfold subproducts by restriction to a generic point of each factor hyperplane. Its third dimension is at least eight. Indeed the only dependent triples are the binary core triple and possibly the two y-factors with the ray h_1; each gives a parallel Gale pair complementary to that triple. For a five-slot rank-three list without a coloop the rank-two Gale formula gives ordinary third dimension ten minus the number of these pairs. Multiplication by the cubic and coloop rows now gives $(45, 72)$.

To bound δ, a triple with two core factors and one outside is independent. A triple with one core factor and two outside is independent unless the two contract to parallel rays. The latter occurs only for the two-cubic core at $z = 1$; their original plane meets the core along the single line Kh, so at most one core factor completes a dependent triple. For outside triples their dependence requires dependent contraction; the only possible circuit of size three contributes at most one. At a two-cubic core with $z = 1$ an outside triple contains the original coloop and the two independent original y-rays, so is independent. These counts add to the four, two and two internal dependent

triples, respectively. In the quartic case the contraction circuit has length at least four and supplies no further dependent triple. This proves the last column.

Strong containers and quadratic payment Put the actual or ordinary core in the indicated slots, adjoining separate Boolean slots on the complement as in the unique-triple proof. Their fourth dimensions g_0, g are as follows:

$$(g_0, g) = (76,\ 36 + 10\sigma), \quad (86,\ 76), \quad (89,\ 66).$$

Here $\sigma \geq 2$ is the actual binary quartic second dimension. It is at least two because a one-dimensional space of second derivatives would force the concise binary quartic to be a univariate fourth power (or by the binary Hilbert function). Binary cubics and quartics have a strong linear open in characteristic zero. For a binary quartic this follows from nonvanishing of the top evaluation and first Hessian; only the complementary maps require testing. For H_5, the direct Hessian calculation below gives the actual Hilbert row $(1, 3, 6, 6, 3, 1)$ and the strong open. For two cubics take the tensor strong open. The projection of the original plane onto the core differential plane is onto, and nonvanishing of the remaining slot evaluations imposes only nonempty opens. All comparisons thus use an available fixed physical quadratic via the prescribed-open three-plane theorem.

Use the $\kappa, \rho, \nu, h, \beta$ as in the one-triple calculation:

$$\kappa \leq g_0 - c, \quad h = g - \kappa - \rho, \quad \beta = c - g + 2\kappa + \delta/2 - 3\nu/2.$$

Take $\Lambda = 245/2$ and $\alpha = \Lambda - c - \kappa + \rho - \delta/2$. The two tables give $\alpha \geq 32$ universally, and

$$h + \alpha \geq \Lambda + g - 2g_0 + c - \delta/2 \ \geq \ \begin{cases} 84, & z = 0, \\ 81, & z = 1. \end{cases}$$

Thus $\dim UX \geq 7s/2 - \alpha$ is immediate through $s = 9$. At $z = 0$ and $s = 10, 11, 12$, use the six-variable polynomial bound $P_6(s) = 79, 85, 89$, paying the full loss $126 - h$; at $s \geq 13$ use square saturation and the convex maximum 81.5 from the unique-triple proof. At $z = 1$ there is a coloop square relation, so the full polynomial preimage of any s-plane $X \subset A_2$ has dimension at least $s + 1$. The path bounds $P_6(11), P_6(12) = 85, 89$ suffice at $s = 10, 11$; at $s \geq 12$, $\dim A_2 \leq 20$ and $7s/2 + \binom{21-s}{2}$ has endpoint values $78, 70$.

By the six-direction derivative bound every nonzero first derivative has at least eight physical quadratic outputs. Equation (3) now gives $245 - 3\nu$ at $m = 0$ and at most $245 - 3\nu + 28 - 3(8 - \nu) = 249$ at $m = 1$. Use the low-overlap calculation for $m \geq 2$. The unit bound follows on all original coefficients, including the direct product of three binary cubics. No classification of the coupling ratios or catalecticant determinants for the ternary quintic remains.

The fixed ternary quintic. Take $H_5 = xyz(x + y)(x + z)$. At $(1, 1, 1)$ its value is four and its ordinary Hessian has rows $(10, 11, 11), (11, 4, 9), (11, 9, 4)$, with determinant 560. At $(1, 0, 0)$ the second Hessian, indexed by the six quadratic differential monomials, has determinant -2304. The x^2 row pairs only with yz, with entry six. On deleting these two indices, the remaining array in the block order $xy, y^2 \mid xz, z^2$ is $\left(\begin{smallmatrix} 0 & B \\ B^\top & 0 \end{smallmatrix}\right)$ with $B = \left(\begin{smallmatrix} 6 & 4 \\ 4 & 4 \end{smallmatrix}\right)$, whose determinant is eight. The resulting three nonzero polynomial conditions have a common point over every infinite characteristic-zero field. By the perfect graded pairings they give isomorphisms $A_i \xrightarrow{\ell^{5-2i}} A_{5-i}$, $i = 0, 1, 2$. This also gives $\dim A_2 = 6$.

Two diagrams on a seven-dimensional support

In the seven-variable polynomial operator ring specialize the physical support and the quadratic test space by the same row-and-column torus. The limit is a seven-cell plane, and product dimension can only drop. A seven-cell board has at most three rectangles. Indeed two rectangles sharing a single cell already use all seven cells. They have just one common row and one common column, and their union has no third rectangle. Two rectangles sharing two cells form a complete two-by-three board, up to transposition, with three rectangles. An additional single cell cannot belong to another rectangle. Choosing a diagonal monomial order on this two-by-three board makes the three missing quadratic monomials the edges of a four-vertex path. With at most two rectangles, the missing pairs, which involve distinct variables, embed in the same path. All squares, same-line pairs and pairs lacking a complete rectangle belong to the physical annihilator on the board. The rectangle differences provide one leading diagonal for each rectangle. Thus the polynomial leading multipliers contain *all* quadratic monomials except the three path edges. A quadratic input of dimension s gives s leading monomials. Count all quartics reached by multiplying them by the retained quadrics.

Use $n = 4, 5, 6, 7$ variables with a path on the last four vertices and $n-4$ isolated vertices. All pairs with repetition except the three path edges are available multipliers. Let $P_n(s)$ be a lower bound for the number of quartics in their product with any s quadratic monomials. Put $P_4(0) = 0$, $P_4(s) = 7$ for $s > 0$; one polynomial monomial already has seven distinct products. With $n > 4$, shift the input toward the first, isolated, vertex 0. This preserves the multiplier family. To verify the compression, fix the outside exponents for a two-variable fiber. Both factors now occupy consecutive intervals at the same end, whose sum has minimum possible size and whose outputs for different factorizations are nested. Repeated moves give stability. For a nonempty input its first-vertex part is exactly x_0^2 and x_0x_i for $i \in J$, $|J| = t$. The remaining inputs use only J, because replacing either endpoint by 0 is an allowed shift. Their count is $s-1-t \le \binom{t+1}{2}$.

The first-vertex inputs contribute at least $b_n(t)$ different quartics containing that vertex. The others give at least $P_{n-1}(s-1-t)$ quartics without it. Here $b_n(t)$ is the minimum, over t-subsets J of the other vertices, of the number of distinct products x_0x_im for $i \in \{0\} \cup J$ and any allowed quadratic m. Each row below can also be checked by splitting off the common factor x_0. Among all cubic monomials containing a variable from $J \cup \{0\}$, remove exactly those for which deletion of every such variable leaves one of the three path edges. The index t starts at zero.

n	$b_n(t)$
5	12, 20, 26, 31, 35
6	18, 30, 39, 46, 52, 56
7	25, 43, 55, 65, 73, 80, 84

Set $P_n(0) = 0$ and take the minimum of $b_n(t) + P_{n-1}(s-1-t)$ for $0 \le t < n$, $0 \le s-1-t \le \binom{t+1}{2}$. The nonzero entries $P_7(1), \dots, P_7(28)$ are

Inputs	Consecutive values of P_7
1–7	25, 43, 55, 65, 73, 80, 84
8–14	102, 114, 121, 129, 136, 140, 152
15–21	152, 159, 163, 169, 170, 170, 170
22–28	174, 178, 182, 182, 182, 182, 182

After projection to a quartic quotient of dimension h, the resulting product bound is $P_7(s) - 210 + h$.

The small position widths Write c for the number of coloops and $a = 28 - c$. For u quadratic inputs write their lifts in the ordinary coefficient algebra. Choose seven independent original factors as basis slots and represent each square with one basis position and its outside component, as detailed in the rank-two calculation. The position types form a complete graph on seven vertices and an eighth vertex joined to the first $7 - c$. This graph has a edges. Multipliers have at least $a - 3$ independent pivots, and the source has at least u. Counting disjoint unions of one pivot of each family gives a lower bound on the ordinary quartic product dimension. The diagram is shifted towards earlier vertices. Simultaneously shifting the two families cannot increase the union shadow. In a two-vertex degree-one fiber compress both towards the same vertex. The only possible nonzero degree-two product pairs distinct vertices, and was already present unless both original singleton fibers were equal; in that case the compressed product is zero. Degree-one output fibers become nested. Thus the disjoint product shadow does not grow under the complete simultaneous shift.

A shifted edge graph is a strict partition with largest part at most seven. A row of length p_i starting at vertex i, indexed from zero, consists of edges (i, j) for $i < j \le i + p_i$. The following lists include *all* multiplier partitions of size $25 - c$ in the available diagram. Each string lists its nonzero parts. The last column gives the minima $B_c(1), B_c(2), \ldots$ at precisely the widths needed. To obtain an entry, take each strict partition of the required small input size (at most eight), discard any partition extending outside the available diagram, and count the four-subsets $i < j < k < l$ for which one of the three complementary pairings has its two edges in the indicated families in either order. This gives a finite sum of at most seventy zero-one terms per pair of partitions.

c	Multiplier partitions	Small-width bounds B_c
0	765421, 76543	12, 19, 22, 22, 29, 32, 32, 35
1	765321, 76542	11, 17, 19, 22, 27, 29, 32
2	764321, 76532, 76541	10, 15, 16, 21, 22

Use $B_c(u) = 0$ outside the printed widths. For $c = 2, u = 5$, each of the two independent coloop operator squares vanishes in the coefficient algebra. The polynomial ring modulo these squares has 26 quadratic and 155 quartic monomials. The physical multiplier space in this ring has codimension at most three. Take a five-dimensional space of quadratic inputs. Shift both families toward variables $0, 1, 2, 3, 4, 5, 6$, the first two being square-zero. For two square-zero or two free variables, the fiber arguments above apply. If the earlier variable has square zero and the later is free, both exponents in a positive-degree fiber are zero or one in the earlier variable. Compress toward exponent one; two singletons can then contribute zero, whereas a full fiber times a nonempty one yields a terminal interval no larger than the original output. Thus simultaneous compression still works. The multiplier holes after compression are $\{55, 56, 66\}$ or $\{46, 56, 66\}$, and the complete list of size-five inputs and resulting counts follows. Here digits name variables and monomials are unordered.

Input	First holes	Second holes
01, 02, 03, 04, 05	42	41
01, 02, 03, 04, 12	46	49
01, 02, 03, 12, 13	50	50
01, 02, 03, 12, 22	55	55

Thus these products in the 155-dimensional quartic layer span at least 41 dimensions. With three coloops the corresponding ring has 129 quartic monomials. At widths one through three its complete shifted inputs are respectively 01, 01, 02, and 01, 02, 03 or 01, 02, 12. The same two multiplier-hole

patterns give, respectively, 11, 18, 25, and 25 quartics for either pattern. Thus at $c = 3$ the 129-dimensional quartic layer contains at least $v(u)$ products, where $v(1) = 11,\ v(2) = 18,\ v(u) = 25$ for $u \geq 3$. In any actual quartic quotient of dimension h one may subtract the full loss $129 - h$.

The rank-two Gale calculation

Rank-two Gale data and a true containing circuit Let z be the number of original coloops, equal to the number of zero Gale columns. Other Gale columns are indexed points on a line with multiplicity partition $p_1 \geq \cdots \geq p_l \geq 1$, summing to $9 - z$. Their rank is two, so $l \geq 2$. An original factor is nonzero precisely when it is not a loop. Consequently $r = 9 - z - p_1 \geq 2$. These conditions imply $0 \leq z \leq 6$, and give exactly 79 partitions over all z. Special cross ratios of the distinct points require no exclusion.

With a binomial coefficient zero outside its range, the ordinary Hilbert row is

$$C_j = \binom{9}{j} + (l-2)\binom{z}{j-1} - \sum_i \binom{z+p_i}{j-1} + \binom{z}{j-2}. \tag{1}$$

Indeed a Gale subset has rank zero if contained in the zero columns and rank one if its nonzero columns have exactly one direction. Expanding the subset formula gives (1). We use $a = C_2 = 28 - z$, $c = C_4$, and C_3. Matroid duality gives the indexed triple loss

$$s = \sum_i \binom{z+p_i}{6} - (l-2)\binom{z}{6}. \tag{2}$$

This follows by applying the dual rank formula to the six-point complement of a triple. An all-zero complement contributes two units.

The r original positions complementary to a maximal Gale flat (including the zero columns) form a circuit. Its intrinsic ordinary algebra, on r squarefree slots, has Hilbert series $d_r(t) = (1+t)^r - t$. In fact the generating hyperplane of the squarefree slot space misses all the coordinate lines, and its pair products fill the entire squarefree degree-two space. To see this, transpose to products of distinct linear factors: each factor is a linear combination of the remaining ones, so its square is a combination of distinct products. There are exactly $\binom{r}{2}$ independent ordinary quadrics on $r-1$ variables. For degree $j > 2$ extend any pairwise squarefree coordinate to a j-subset by a linear generator whose coefficient on the extra slot is nonzero and whose other coefficients vanish away from that slot and a previously used slot. This proves fullness in every higher degree. The actual circuit algebra has Hilbert polynomial $b_r = d_r - t^{r-1}$ for $r \geq 3$ and $b_2 = d_2$, by the perfect complement pairings in the squarefree algebra.

A circuit has a strong linear operator. In degrees at least two with complementary degree at least two the map is the ordinary Boolean Lefschetz map. For degree one, after diagonal rescaling the circuit hyperplane is $H_r = \ker v$, with $v = (-1, \ldots, -1, 1)$; take a full-support $\ell = (1, \ldots, 1, r-1) \in H_r$. The ambient Hessian, up to a nonzero scalar, is $\operatorname{diag}(\ell)^{-1}(\mathbf{1}\mathbf{1}^T - I)\operatorname{diag}(\ell)^{-1}$. With respect to its inverse the squared norm of v is $(\sum v_i\ell_i)^2/(r-1) - \sum(v_i\ell_i)^2 \neq 0$, so its restriction to H_r is nondegenerate. The top evaluation is nonzero. For a binary degree-two circuit the claim is immediate. The Boolean strong Lefschetz theorem in characteristic zero follows from its elementary tensor decomposition into centered $\mathfrak{sl}_2$ strings.

Use this circuit on its true positions and all other positions free to form ordinary and actual containing algebras. Their Hilbert rows are respectively

$$D^0(t) = d_r(t)(1+t)^{9-r}, \qquad D^*(t) = b_r(t)(1+t)^{9-r}.$$

Write $g_0 = D^0_4,\ g = D^*_4$, and define $\Gamma = 126 - 2g_0 + g$. The true original operator space projects onto the circuit operator space, and every remaining evaluation is nonzero as a linear functional.

Therefore the open condition of being strong in the circuit and nonzero on the remaining positions meets the original operator space. The tensor has a strong central map at that choice. The same holds with a stronger genuine direct-product container where specified below.

To bound ν, first A_3 is a quotient of the image in the actual container, so $\nu \geq C_3 - D_3^*$. For a second bound, let C_j^{core} be the ordinary Hilbert row of the noncoloop list, obtained by replacing $9, z$ in (1) by $9-z, 0$. Put $u_j = \min(C_j^{\mathrm{core}}, C_{9-z-j}^{\mathrm{core}})$. True apolar duality and the genuine independent coloop splitting give

$$\nu \;\geq\; \nu_0 := \max\bigl(0, C_3 - D_3^*,\; C_3 - [t^3](1+t)^z \sum_j u_j t^j\bigr). \tag{3}$$

One may use a stronger valid lower bound in the direct products below.

One source allowance, including the radical The needed quantitative claim is that every $X \subset A_2$, $x = \dim X$, satisfies

$$\dim UX \;\geq\; \tfrac{7}{2}x - (c-h) - L, \qquad 2L = 2\Gamma - s + \min(3\nu_0, 26 - 3z). \tag{4}$$

It suffices to establish this with any of the listed genuine containers. Here is why (4) closes the source numerator.

Use (7) for the chosen containing algebra. Since $\kappa \leq g_0 - c$ and $\Gamma = 126 - 2g_0 + g$, it gives

$$\dim UW \geq \tfrac{3}{2}\dim W - \beta, \quad \beta = \tfrac{s}{2} + c - g + 2\kappa - \tfrac{3}{2}\nu \leq \tfrac{s}{2} + 126 - \Gamma - c - \tfrac{3}{2}\nu.$$

Combining (4) with the reflection (3) gives the numerator bound

$$252 - 2\Gamma + 2L + s - 3\nu + 28m - 3\dim UJ_1. \tag{5}$$

For $m = 0$ this is at most 252 by (4).

For $m = 1$, let $y \neq 0$ span J_1. Choose one slot where its coefficient is nonzero. Operators vanishing on that slot form a six-dimensional space. On the remaining positions their arrangement is contraction of that original nonloop; its Gale dual is deletion of the selected Gale column. The remaining dual columns still span a two-space since the original column was not a loop. Hence the contracted primal configuration has at most z coloops (zero original factors after contraction cause no problem). For a rank-e factor list the ordinary quadratic dimension is $\binom{e+1}{2}$ − #coloops: split coloops, and express every square of a noncoloop linear form using products of distinct factors. Thus the degree-two image of the vanishing operators has dimension at least $21 - z$. Multiplication by y, followed by projection to Boolean cubics containing the chosen slot, identifies this space with itself times that one nonzero coefficient. Lifting U to its complete preimage in C_2 costs at most three dimensions, and passing to A_3 costs at most ν. We have $\dim Uy + \nu \geq 18 - z$. In (5) the remaining allowance is precisely $26 - 3z$ in (4), as required. This argument does not assume $C_2 = A_2$.

Three smaller containing algebras When the Gale partition has exactly two directions, the noncoloop factors already are the direct sum of two circuits. Use their actual tensor with the coloops as the container, with $g_0 = c$ and $g = [t^4](1+t)^z b_{p_1}(t) b_{p_2}(t)$. Its operator is strong, as above. Improve (3) using its actual third dimension as well.

For $z = 4,\ p = (2,2,1)$, the five-factor core is, after linear coordinates and slot scalings, $def(d+f)(e+f)$: three distinct binary Gale directions have no cross ratio. It is the fixed quintic H_5 in the multiple-triple calculation, after permuting its three variables. The Hilbert row is therefore $(1,3,6,6,3,1)$ and a strong operator exists. The total actual fourth dimension is 76; hence we may take $g_0 = c = 86,\ g = 76$, giving $\Gamma = 30$.

For $z = 5$, $p = (1, 1, 1, 1)$ the core is a squarefree binary quartic. Its first Hessian does not vanish identically: a binary form in characteristic zero with zero Hessian is a power of a linear form, as follows by writing $F = x^4 f(y/x)$ and solving $4ff'' = 3(f')^2$, or inspecting the lowest order at any simple root. Choose a common nonzero Hessian and top evaluation. These are all complementary Lefschetz tests other than the identity in degree two. The actual core row is $(1, 2, h_2, 2, 1)$ with $h_2 \geq 2$; for example the chosen strong operator injects the first layer into the second. Therefore the genuine tensor with five coloops has actual fourth dimension at least 56, and $\Gamma \geq 126 - 2 \cdot 76 + 56 = 30$, uniformly even if the binary middle rank changes.

The common polynomial diagram. Lift the actual physical quadratic operator space to the image of the unrestricted physical I_2 in the polynomial ring on the term's seven-dimensional factor support. Lift X to its complete preimage in the polynomial quadrics. This preimage has dimension $28 - \dim A_2 + x \geq z + x$. The product of these two polynomial spaces has dimension at least $P_7(z + x)$ by the simultaneous physical torus specialization and four-vertex-path argument above. This argument only uses the physical quadratic generators of the permanent and the support dimension. After passing to the original actual fourth layer it gives $\dim UX \geq P_7(z+x) - 210 + h$.

The common position diagram. Use seven original factors as a basis. In formal squarefree differential coordinates their associated generators are $t_i + g_i$, with g_i supported only on the outside positions, nonzero precisely for the noncoloops. The initial terms of their cross products are the distinct $t_i t_j$, while the initial components of their squares are $2t_i g_i$. Give priority to the number of basis slots and then refine by distinct additive basis weights. Encode each $t_i g_i$ as an edge to a single eighth vertex. Products involving two such edges are discarded. An ordinary edge disjoint from this virtual edge gives a nonzero component $t_i t_j t_k g_i$; different three-basis-slot sets give independent components. Four ordinary endpoints similarly give distinct nonzero Boolean monomials. Thus the same shifted graph on eight vertices and its simultaneous disjoint-union bound $B_z(x)$ apply to the complete preimage of the multipliers in C_2 and an equal-dimensional lift of X. This is the positional basis template above and works with any number of outside positions. Passing to the actual fourth layer gives $\dim UX \geq B_z(x) - (c - h)$. As above, the physical square gives $\dim UX \geq h - \binom{a-x+1}{2}$.

Write $A = 2L$ in the following scalar inequality. After accounting for the quartic loss, it suffices to verify

$$7x - 2 \max \left\{ B_z(x),\ P_7(x + z) - 210 + c,\ c - \binom{a - x + 1}{2} \right\} \leq A.$$

For $z = 3$, the uniform noncoloop Gale partition (1^6) at $x = 6$ also uses the polynomial ring with three square-zero coloop variables above. It has 129 quartic monomials, and the extra bound inside the maximum is $40 - 129 + c$. As before the only multiplier-hole patterns are $55, 56, 66$ and $46, 56, 66$. The entire list of shifted six-element inputs and their quartic products in these two cases is as follows.

Input	First holes	Second holes
01,02,03,04,05,06	40	41
01,02,03,04,05,12	43	42
01,02,03,04,12,13	42	45
01,02,03,12,13,23	46	46

Each width curve below is calculated by the same strict-partition formula above: a multiplier partition of size $25 - z$ and an input partition of size x fit in the rank-eight complete graph minus its last z star edges. Take the minimum number of four-sets admitting a complementary pair. Beyond the displayed widths use zero. The last column gives the minimum margin over the allowable Gale partitions at that coloop count and all positive input dimensions.

z	$B_z(1), B_z(2), \ldots$	Minimum margin
0	12,19,22,22,29,32,32,35,35,35,43	2
1	11,17,19,22,27,29,32,34,35,35	3
2	10,15,16,21,22,23,29,32,32,35,35	2
3	9,13,13,17,17,17,26,30,32,34	1
4	8,11,13,16,17,17,25,29,29	5
5	7,11,12,15,15	10
6	7,10,10,14,15,16,22,24,25,25,28,29,31,31	1

The final comparison can also be performed on just the following pairs. For fixed z it improves if either c or $A = 2L$ increases, so delete any pair dominated coordinatewise by a listed pair. In each row the pairs are ordered by increasing c; the values of A appear in the corresponding order. These are obtained by the two binomial-coefficient formulas for C_j and the triple loss, and the stated circuit and coloop bounds for the trace loss.

z	c	A
0	91,105,106,112,115,118,123,126	63,26,24,21,10,6,2,0
1	90,91,102,105,112,117,122,126	65,63,32,18,10,6,2,0
2	88,90,97,102,114,120	67,63,41,9,6,2
3	84,87,89,96,109	72,63,43,9,6
4	77,81,86,90	82,66,64,18
5	66,71,76	106,64,62
6	56	50

In the $z = 3$ row the extra width-six bound belongs only to the last pair $(109, 6)$, from (1^6). The four omitted lists give pairs $(99, 9), (100, 10), (103, 10), (106, 10)$, all dominated by $(96, 9)$ without the extra bound.

A single circuit and a basis

Rank-four quadrics and good original pairs

Work over a field of characteristic zero. Write the space of physical matrix linear forms as $M = X \otimes Y$, using its fixed physical row and column bases. Let $P_2 \subset \operatorname{Sym}^2 M$ be the span of the 2×2 subpermanents. A quadric is supported on its smallest linear space of variables, the image of its polar. An m-factor circuit means m pairwise nonproportional linear forms of rank $m - 1$, admitting one relation and no relation on a proper subset.

Rank four Any nonzero $F \in P_2$ has rank at least four. Pick a physical 2×2 coefficient that is nonzero. The restriction to the four derivatives in that rectangle is its nonzero two-by-two permanent and has polar rank four. Suppose the entire F has rank four. After permutation and rescaling assume the selected rectangle has rows and columns 1 and 2 and the polar entries on its two diagonals equal one. Let $W = M^*/\operatorname{rad} F$ with polar B, and denote by $v_{rc} \in W$ the class of the derivative at (r, c). The four anchor classes $e_{ij} = v_{ij}$ form a basis of W. Rows and columns are separately totally isotropic. For all indices one has

$$B(v_{rc}, v_{sd}) = B(v_{rd}, v_{sc}).$$

Each column plane in the anchor is its own perpendicular. Therefore

$$v_{r1} = \alpha_r e_{11} + \beta_r e_{21}, \qquad v_{r2} = \gamma_r e_{12} + \delta_r e_{22}.$$

The rectangle identities against the first and second anchor rows give $\delta_r = \beta_r,\ \gamma_r = \alpha_r$. Same-row isotropy gives $2\alpha_r\beta_r = 0$. Likewise

$$v_{1c} = p_c e_{11} + q_c e_{12}, \qquad v_{2c} = p_c e_{21} + q_c e_{22}, \qquad p_c q_c = 0.$$

Pairing v_{rc} with each e_{ij} is equivalent to pairing v_{rj} with v_{ic}. Nondegeneracy of B now gives

$$v_{rc} = \alpha_r p_c e_{11} + \alpha_r q_c e_{12} + \beta_r p_c e_{21} + \beta_r q_c e_{22}.$$

Regard α, β as physical row vectors and p, q as column vectors. The first pair have disjoint supports, as do the second pair. The support of F is

$$H = \langle \alpha, \beta \rangle \otimes \langle p, q \rangle,$$

and F equals a nonzero multiple of

$$(\alpha \otimes p)(\beta \otimes q) + (\alpha \otimes q)(\beta \otimes p).$$

All four vectors are nonzero because of the chosen anchor. In particular this description is valid over the original coefficient field.

Pairs inside a quadratic source Let $L \subset M$ be a given physical support, and call two independent factors $\ell, m \in L$ bad if there is a nonzero $F \in P_2 \cap \operatorname{Sym}^2 L$ in the ideal (ℓ, m). For a bad pair write $F = \ell f + m g$ with $f, g \in L$. Rank at least four forces ℓ, m, f, g to be independent, and they span H. The inverse polar on H makes $\langle \ell, m \rangle$ totally isotropic: in this basis the polar pairs its first two coordinates only with the last two, and the same holds for its inverse.

If $\ell = (a\alpha + b\beta) \otimes (cp + dq)$ is a simple tensor in this plane, its inverse norm is a nonzero constant times $abcd$. For another simple tensor $m = (a'\alpha + b'\beta) \otimes (c'p + d'q)$, the mixed inverse pairing is a nonzero constant times

$$(ab' + ba')(cd' + dc').$$

Suppose the two row directions and the two column directions are each independent. If the row factor $ab' + ba'$ vanishes, then a, b, a', b' are all nonzero, since otherwise vanishing forces proportional row vectors. The two row supports coincide, and separate isotropy forces both column vectors to be axes in the basis p, q. Their independence makes these axes distinct, so their physical supports are disjoint. Interchanging modes gives the corresponding assertion if $cd' + dc' = 0$. If row or column directions are proportional then their physical supports coincide in that mode. Thus any simple bad pair has equal support in at least one mode; if both directions in each mode differ and the row supports coincide, its column supports are disjoint.

A good pair in a circuit Let the above circuit span $d = m - 1 \geq 5$ dimensions and be contained in L. If every pair were bad, a circuit factor of matrix rank at least three would be impossible by the rectangular support theorem. A rank-two circuit factor determines its two-dimensional column and row spaces uniquely. Every rectangular support of a bad pair containing it must be exactly their tensor product. This would put all factors in dimension four. Hence all factors are simple.

Write each factor as $u_i \otimes v_i$. Every two factors have equal row support or equal column support. If two row supports differ, their column supports coincide and any third factor whose column support differs must have its row support equal to both. This contradiction shows that all column supports coincide in this case. Transpose if necessary so that all row supports coincide.

Group the factors by their projective column direction. Join two distinct groups when their column supports intersect. Across an edge the column rays are independent. Two unequal row rays would force the column supports to be disjoint by the preceding simple-pair computation. Hence *all* row rays in both incident groups coincide. Distinct connected components have disjoint

physical column supports. The unique circuit relation has every coefficient nonzero, so if there were two such components, projecting to either column support would yield a relation on a proper circuit subset. The graph is connected. With one group the columns are all proportional. With more than one, the row rays are all proportional. After transposition we may assume that the entire circuit is contained in the ruling $u \otimes V$, where $\dim V = d$.

Select d independent factors $u \otimes v_i$. For each i choose a different circuit partner and a bad witness with separated axes α, β, p, q. These two column rays are independent. If the common row $u = a\alpha + b\beta$ had $ab \neq 0$, separate isotropy would force the two columns onto the distinct axes p, q, while their mixed pairing would be nonzero. Thus u is one of the two row axes. Write w_i for its opposite row axis, supported outside $\operatorname{supp}(u)$. The whole rectangle belongs to L, hence $w_i \otimes v_i \in L$. Projection onto the physical rows outside $\operatorname{supp}(u)$ keeps these d tensors linearly independent, because the v_i are independent and all $w_i \neq 0$. Its kernel contains the given d-dimensional ruling. Therefore

$$\dim L \geq 2d.$$

Consequently every such circuit inside an e-dimensional physical support with $e < 2d$ has at least one good pair. The independent nine-frame uses a different application, since independent components need not form a single circuit.

A good pair for an independent nine-frame Suppose $n > 4$ physical factors form a basis of their support L, and every pair is bad. The preceding argument again makes all factors simple, and after transposition they have the same physical row support S. Since they span L, every row direction occurring in L is supported on S. Two factors with the same row direction would have independent column directions. The common-row computation above would then supply a second nonzero row axis supported outside S, with its rectangle in L, a contradiction. Hence all n row rays are distinct.

There are at least two column rays, since L contains a full rectangle. Between distinct column classes, both mode pairs are independent and the row supports agree, so the column supports are disjoint. In the disjoint physical row blocks the two vectors have coordinate ratios $\lambda, -\lambda$, by $ab' + ba' = 0$. All row rays therefore belong to one projective coordinate-sign orbit: use a second column class to compare any two within the first. Write the distinct column representatives as $v_1, \ldots, v_s$ and their respective row spans as $A_1, \ldots, A_s$. The decomposition $L = \bigoplus_j A_j \otimes v_j$ is direct. For u and w from distinct classes j, k, their bad rectangle is precisely $\langle u, w \rangle \otimes \langle v_j, v_k \rangle$. Its containment in L forces $u, w \in A_j \cap A_k$. Thus all A_j equal a single r-plane A, each class is a basis, and $n = rs$, $r, s \geq 2$.

Select r physical coordinates whose restrictions to A are independent. After normalizing row rays to the common sign orbit, these coordinates take just two values each, and a vector is determined by them. Opposite signs represent the same line, so

$$n \leq 2^{r-1}.$$

For $n = 9$ the only factorization is $r = s = 3$, inconsistent with this inequality. An independent nine-frame has a good pair.

Circuits of length two and three

Suppose nine nonzero factors span eight directions. If their unique relation has length two or three, the global proof needs the unit inequality for every actual source. The four chain covers below apply to every quadratic multiplier subspace of codimension at most six.

Transfer and simultaneous compression For a repeated factor the actual algebra is the tensor product of seven square-zero variables and one variable of exponent at most two. For a three-term circuit, normalize the binary cubic to $XY(X+Y)$. The degree-two and degree-three operators x^2-xy+y^2, y^3 annihilate it. With $x>y$ the initial quotient has basis $1, x, y, xy, y^2, xy^2$; its dimensions agree with those of the actual concise binary cubic in every degree. The other six factors are coloops. Thus in both cases the same initial algebra is

$$R = K[z_1, \ldots, z_7, y]/(z_1^2, \ldots, z_7^2, y^3).$$

Apply this same order to the full preimages of the original ideals $UJ \subseteq K' \subseteq J$ and subspace U. Take the leading standard monomials. Graded dimensions of the ideals are preserved and every nonzero initial product belongs to the initial space of K'. It suffices to estimate the weighted dimensions of J/UJ in the monomial model. Delete additional quadratic multipliers to leave exactly six missing among the 29 effective monomials.

For two of the seven square-zero coordinates shift both the multiplier family and each graded piece of the ideal towards the earlier variable. The product of the shifted families is contained in the shift of their original product. To see this, fix the other exponents in each factor. A fiber has either one monomial (special degree zero or two), or one or two singleton choices (degree one). In output degree one the compressed fibers are nested at the chosen end. In output degree two the only potentially new product would pair two singleton fibers; if they had equal unique choices their compressed product is still zero. This proves the inclusion after taking the union of all factorizations of the outside monomial. Applying the same fact with the full linear layer shows the shifted ideal stays an ideal. Thus each dimension of UJ can only decrease; graded dimensions of J stay fixed. Repeat to stability, then reverse the labels so that the *holes* in the pure Boolean pairs form a shifted graph. Its row lengths form a strict partition p, with at most six as largest part. The holes $z_i y$ have $i = 1, \ldots, c$; write $b = 1$ if y^2 is missing and zero otherwise. The rule $|p| + c + b = 6$ gives the 24 rows below.

Four larger forbidden sets Give the Boolean variables class labels A, B, C and let D stand for the double variable y. In each template delete *all* quadratic multipliers with the class pairs indicated below. This can only enlarge the upper bound when its forbidden set contains the given six holes.

Template	Sizes of A, B, C	Forbidden class pairs
1	4,2,1	AB, AC, AD, CD, DD
2	4,2,1	AB, AC, BB, BD, DD
3	3,3,1	AA, AC, BB, BC, DD
4	3,3,1	AA, AD, BC, BD

The following embeddings can be checked directly. For each word assign its seven consecutive letters to the Boolean variables in their numbered order. A graph with row length p_i has the edges (i, j) for $i < j \leq i + p_i$. Each table half is independent. A dash denotes the empty partition.

p	c	b	Template	Word	p	c	b	Template	Word
6	0	0	3	CAAABBB	1	5	0	1	ACAAABB
51	0	0	2	BABAAAC	—	6	0	4	AAABBBC
42	0	0	2	BBAAAAC	5	0	1	2	BAAAABC
321	0	0	3	AAACBBB	41	0	1	2	BABAAAC
5	1	0	2	BAAAABC	32	0	1	2	BBAAAAC
41	1	0	2	BABAAAC	4	1	1	1	CAAAABB
32	1	0	2	BBAAAAC	31	1	1	2	BABAAAC

p	c	b	Template	Word	p	c	b	Template	Word
4	2	0	1	CAAAABB	3	2	1	1	ACBBAAA
31	2	0	2	BBAAAAC	21	2	1	2	BBAAAAC
3	3	0	1	CAAAABB	2	3	1	1	CAAAABB
21	3	0	4	AAABBBC	1	4	1	1	ACAAABB
2	4	0	1	CAAAABB	—	5	1	1	AAAACBB

Checking a cover on class counts For an exponent type $v = (v_A, v_B, v_C, v_D)$, let $0 \le v_i \le n_i$ with $n_D = 2$. Use the types of total degrees $d = 1, 3, 5, 7$, in this order; within a degree sort the four entries lexicographically in ascending order. Starting at zero, label the types consecutively by the characters in this list:

`0123456789ABCDEFGHIJKLMNOPQRSTUVWXYZabcdefghijk`.

The required weight at a type is

$$m(v) = w_d \binom{n_A}{v_A}\binom{n_B}{v_B}\binom{n_C}{v_C}, \qquad (w_1, w_3, w_5, w_7) = (28, 3, 2, 7).$$

For types $v < v'$ coordinatewise (with weak inequalities in every coordinate, and at least one strict one), a step is legal if the multiset $v' - v$ contains a pair allowed by the template. Such a step lifts to comparable standard monomials. Along a chain of types choose nested subsets of each Boolean class of the appropriate cardinalities, for instance by taking initial segments of one random permutation in each class. At each type this gives the uniform distribution on its monomials. The monomials of $J \setminus UJ$ meet any such lifted chain at most once: a later monomial is divisible by an earlier one times an allowed quadratic, with a remaining monomial factor. Therefore a list of type chains with total multiplicity at most 252 and coverage at least $m(v)$ at each type proves the claimed weighted bound, by taking expectations and summing chain multiplicities.

Each table entry gives the positive multiplicity followed by the chain of consecutive labels; case matters. For each template the multiplicities incident to a label sum exactly to $m(v)$. Thus the tables permit direct checking without finding a matching or proving it optimal. Every listed edge is checked by subtracting the two four-tuples and reading the forbidden pairs above.

Weight	Template 1 chain	Weight	Template 2 chain
12	`3HUc`	14	`2GTe`
4	`3CUc`	2	`2GTa`
8	`3CLc`	12	`3HUf`
4	`3BTc`	24	`3COa`
12	`3BTb`	8	`3ERa`
8	`3BKb`	7	`3AMd`
4	`2GYh`	16	`2BJa`
3	`2GQh`	8	`2BNZ`
7	`2GPg`	7	`2GQh`
2	`1FXf`	2	`3EWf`
12	`1FOf`	5	`2GQc`
4	`1FOZ`	3	`28LZ`
6	`16OZ`	5	`3AZ`
2	`0EWe`	2	`7IZ`
2	`0EOZ`	8	`3EZ`
12	`0ENe`	7	`3Vg`
2	`0ENZ`	16	`3NZ`
6	`05NZ`	1	`2Ga`

Weight	Template 1 chain	Weight	Template 2 chain
8	ASa	3	04a
4	AJa	1	3Vc
8	9Rb	1	7Kb
4	9Ja	2	1Fa
3	4Md	2	1FX
4	3NZ	8	1FS
8	3Ja	7	0Kb
8	3DV	6	1Fc
12	3Ca	5	3Lc
2	2IZ	12	3Pb
16	2GZ	8	09b
6	2GQ	6	06c
3	28Q	5	3M
3	27P	4	1U
4	1Md	4	GY
20	3a	3	Gc
2	3P	2	1c
4	3D	4	09
4	0M	4	1
1	M	12	D
6	3	6	5
12	2		

Weight	Template 3 chain	Weight	Template 4 chain
18	3CUd	18	3EVd
9	3CMd	18	2DMd
3	04Md	18	3HTi
18	2EOd	9	2ENd
6	27Jc	18	0CUd
6	3BMd	9	26Of
6	3FRg	9	3HSh
9	05Ne	9	25Ne
3	3FZg	9	3ASg
6	2EPe	6	2DQc
9	3HTd	9	3BOf
3	27Ne	6	18Jc
6	3Hbi	6	3Gag
3	3HZg	6	0CPc
7	1DVk	3	3GRg
11	1DVf	2	29Kc
3	2EWf	6	2bh
9	0ASh	6	2We
9	3HTg	3	1Dk
3	3Ne	6	3Zj
2	8Lc	3	18k
3	3Bc	4	07e
9	1Dc	3	1Rg
3	3Wf	1	29f

Weight	Template 3 chain	Weight	Template 4 chain
1	`26c`	2	`3Xh`
6	`3Sh`	2	`FYi`
4	`8Qf`	1	`14c`
3	`2Sh`	4	`2h`
1	`1ai`	2	`2f`
8	`26i`	2	`2e`
5	`Gai`	2	`1L`
2	`0X`	2	`14`
2	`2K`	1	`2k`
1	`8i`	1	`2j`
3	`2h`	1	`2i`
2	`8Q`	8	`1`
3	`2N`	7	`F`
7	`2j`	7	`2`
2	`IY`	5	`7`
21	`2`	4	`3`
3	`9`	3	`I`
5	`0`	3	`C`
1	`I`		
4	`G`		

The four totals are respectively 252, 252, 248, 244. Dividing by 252 proves the unit bound at relation lengths two and three.

A ternary quartic with five coloops

Let D be a standard graded Artinian Gorenstein algebra of socle degree four over a characteristic-zero field, with Hilbert vector $h = (1, 3, 6, 3, 1)$. Put $B = K[t_1, \dots, t_5]/(t_1^2, \dots, t_5^2)$ and $R = D \otimes B$. Give all generators degree one. Suppose $U \subset R_2$ has codimension at most six and its projection on $D_2 \otimes 1$ is onto. Then every homogeneous ideal J of R satisfies

$$28 \dim(J/UJ)_1 + 3 \dim(J/UJ)_3 + 2 \dim(J/UJ)_5 + 7 \dim(J/UJ)_7 \leq 252. \tag{1}$$

This allows six deficiencies incident with a single independent factor. The three integer certificates below use additions, comparisons and binomial coefficients on at most three elements.

Multiplication in the quartic algebra Write $V = D_1$. Dimension gives $D_2 = \operatorname{Sym}^2 V$. If $L \subset V$ has dimension u and $W \subset D_i$ has dimension k, the following numbers are lower bounds for $\dim LW$. Set $b_i(0, k) = b_i(u, 0) = 0$, and otherwise set

$$\begin{aligned} b_0(u, 1) &= u, \\ b_1(u, k) &= uk - \binom{\min(u, k)}{2}, \\ b_2(u, k) &= \min\{e \in \{0, 1, 2, 3\} : \ (3 - e)u - \binom{\min(3 - e, u)}{2} \leq 6 - k\}, \\ b_3(u, k) &= \mathbf{1}_{u+k>3}. \end{aligned} \tag{2}$$

In degree one the kernel of symmetrization on $L \otimes W$ is $\bigwedge^2(L \cap W)$; extend a basis of the intersection separately in the two subspaces. In degree two the annihilator of LW in V under the

perfect D_1, D_3 pairing has dimension $3 - \dim LW$. Its product with L is orthogonal to W under the perfect D_2, D_2 pairing. The degree-one bound gives (2). The last formula follows from perfect pairing of two three-spaces.

For $e \geq 2$, $i + e \leq 4$, the full D_e takes every nonzero subspace of D_i onto D_{i+e}. For $i = 0$ this is immediate. For $i = 1, e = 2$ a nonzero annihilator in D_1 would give a product of two nonzero linear forms vanishing in $\operatorname{Sym}^2 V$. All other cases are top-degree perfect pairings. Keeping the actual quartic algebra gives these surjections simultaneously.

The five independent directions and simultaneous shifting Take leading components by largest core degree, breaking ties by additive weights on the five t's distinguishing squarefree subsets. The ring and multiplication are homogeneous for these weights. Each core degree on a fixed subset retains its actual vector space and multiplication. Leading subspaces of $U, J, K = UJ$ preserve all ordinary graded dimensions and satisfy $(\operatorname{in} U)(\operatorname{in} J) \subseteq \operatorname{in} K$; a nonzero product of leading components is the leading component of the product. Both leading ideals remain ideals and $\operatorname{in} K \subseteq \operatorname{in} J$.

All six core quadrics belong to $\operatorname{in} U$ by the projection hypothesis. Let E denote the missing pure coloop pairs and v_a the dimension of the retained coefficient subspace of $t_a D_1$. Discard additional pure or mixed multipliers until

$$|E| + \sum_{a=1}^{5} (3 - v_a) = 6. \tag{3}$$

Every retained product still belongs to $\operatorname{in} K$.

Encode the dimension k of a slice $J_{i,S} \subseteq D_i t_S$ by the thresholds $1, \ldots, k$. Encode K and each mixed coefficient space in the same manner. For every fixed pair of input thresholds u, k, a mixed multiplication forces at least the first $b_i(u, k)$ output thresholds. Use a single multiplier label for the entire full D_2, since it occurs at the empty coloop subset. A nonzero input multiplied by full D_2 forces all output thresholds in that degree. A pure coloop product retains the same thresholds. Include also the inclusions expressing the ideal property under each linear coloop generator and full D_1, where (2) with $u = 3$ applies. All these implications hold for the original leading slices. Their rules on coloop subsets use disjoint union.

Shift the threshold families of the three spaces to the same end in a pair of coloop coordinates. In a fiber containing singleton choices on those coordinates, replace the choices by the initial interval of equal cardinality. The full multiplier spaces are invariant; the threshold families stay nested. If $F * G$ denotes the family of disjoint unions from two families, then

$$(\operatorname{sh} F) * (\operatorname{sh} G) \subseteq \operatorname{sh}(F * G). \tag{4}$$

Fix the outside portions from each input. Zero distinguished positions is unchanged. With one distinguished position the shifted output choices form nested initial intervals; their union cannot exceed the previous union's cardinality. With two positions, either a full pair and an empty fiber contribute, or two singleton fibers do. In the latter case a disjoint choice after the shift requires at least one input fiber with both singleton choices, so a disjoint choice existed before. This proves (4) after union over outside factorizations. Apply it separately for each fixed pair of input thresholds and every forced output threshold. All product and ideal implications persist. So do $K \subseteq J$ and every total threshold count by degree. Shifting terminates, since each nontrivial move decreases the sum of the position indices over the finitely many families. This operates on integer dimensions and makes no common choice of vectors in different slices.

Label the opposite end first, with vertices $0, \ldots, 4$. The missing pure pairs form a shifted graph with strictly decreasing positive row lengths p drawn from $4, 3, 2, 1$. Row i of length p_i has edges $i\,j$ for $i < j \leq i + p_i$, with indices starting at zero. The mixed dimensions satisfy $v_0 \leq \cdots \leq v_4$.

Three templates Group the five positions into classes A, B, C. Each template removes all pure pairs in the indicated class types, keeps every other pure pair, and uses the indicated lower bounds for the retained mixed dimensions. The full core D_2 is available throughout.

Template	Sizes	Missing pure pair types	Retained mixed dimensions
I	$3,1,1$	AB	$1,0,0$
II	$3,1,1$	AB, AC, BC	$1,1,3$
III	$2,2,1$	AA, AB, AC, BB	$2,3,0$

The following list exhausts the strict partitions with largest part at most four and size at most six. A word gives the classes of the consecutive vertices. In each row the template removes every missing pair and retains at most each available mixed dimension.

Shifted pure graph	Embedding and reason
Empty	I. Put all vertices of mixed dimension zero in B, C; by (3) there are at most two.
$1, 2, 3$ (single rows)	I, $BAAAC$. Only vertex zero could have mixed dimension zero.
21	III, $CAABB$. There are three mixed deficiencies, hence $v_1, v_2 \geq 2$ and $v_3 = v_4 = 3$.
4	II, $BAAAC$. All $v_i \geq 1$ and $v_4 = 3$.
31	II, $BACAA$. All $v_i \geq 1$ and $v_2 = 3$.
$41, 32, 42$	II, $BCAAA$. At most one mixed deficiency remains, and vertices zero and one meet every missing pair.
321	III, $BABAC$. Vertex four is isolated and all mixed dimensions equal three.

Checking the three chain tables For class sizes $s = (s_A, s_B, s_C)$, use types (i, k, a, b, c) with $0 \leq i \leq 4$, $1 \leq k \leq h_i$, and $u = (a, b, c)$ between 0 and s coordinatewise. The total degree $d = i + a + b + c$ is $1, 3, 5$, or 7. The weight of a type is

$$m = w_d \binom{s_A}{a} \binom{s_B}{b} \binom{s_C}{c}, \qquad (w_1, w_3, w_5, w_7) = (28, 3, 2, 7).$$

Sort by total degree then lexicographically by (i, k, a, b, c), all ascending. Label from zero in two-character base 36, using digits then capital letters. Templates I and II have 111 types each; III has 125.

For a later type (j, l, u') on a chain, require strictly larger total degree, $j \geq i$ and $u' \geq u$. Put $\Delta = u' - u$. The following steps are legal.

- If $j - i \geq 2$, every $l \leq h_j$ is legal.
- If $j = i$, Δ must contain an allowed pure coloop pair, and $l \leq k$.
- If $j = i + 1$ and Δ contains an allowed pure pair, require $l \leq b_i(3, k)$. Otherwise require $l \leq b_i(v, k)$, where v is the largest retained mixed dimension among classes used in Δ, or zero if $\Delta = 0$.

For nested subsets of the given class counts, whenever the earlier threshold belongs to J, a legal step forces the later threshold into K. Use the appropriate quadratic multiplication above, the full linear ideal inclusions for any additional core degree, and coloop ideal inclusions for the remaining subset difference. Since $K \subseteq J$, $J \setminus K$ meets each lifted chain at most once.

In each class choose a uniformly random permutation; take its initial segments to realize the counts along a chain. Each type is uniform on its subsets. If the chain multiplicities cover each type at least m times, their sum bounds the left side of (1) by summing these expectations. The three columns below have respective lengths 68, 67, and 77, and each total multiplicity is 252. Reading two characters at a time gives the labels of a chain. The incident multiplicities sum to at least m at each type. For example the degree-one labels in template I are $00 = (0,1,0,0,1)$, $01 = (0,1,0,1,0)$, $02 = (0,1,1,0,0)$, $03 = (1,1,0,0,0)$, $04 = (1,2,0,0,0)$, $05 = (1,3,0,0,0)$.

Multiplicity	I	Multiplicity	II	Multiplicity	III
2	000B192R	3	0007182Q	1	000D1O3F
7	000B2E2X	7	00071U2Q	2	000D1T34
3	000M1I30	3	000A182W	3	000D1W36
3	000P1I30	9	000B1I30	2	000I1Y37
4	000S1M30	1	000B2E2X	2	000I2639
3	000V2H30	3	00112D2W	2	000I2M3F
3	000Y1Y	2	001E2W	3	000Q2F
3	00111Y2R	3	01081V2W	2	000T2U
3	01081Z31	7	0108232W	3	000W2H3D
3	0108232Y	3	010E1B2W	3	000Z2P3D
7	010C272S	4	010I1E2M	3	00122R3D
3	010N1J31	2	010Q292T	2	00152M3D
3	010Q1N31	3	010W1V2Q	1	00152R3F
2	010T2I2Z	3	010Z1Z2T	3	01
3	010W1V2N	3	01121Z2T	2	01062036
3	010Z1Z2O	3	02	2	01062A36
3	01122320	2	02091W2S	3	010C1N
2	0206172N	2	0209242Y	4	010E1C2Y
7	0206212P	2	020C1A2L	4	010E1U2Y
9	0207222U	6	020C1L2K	4	010H1X3E
2	02091W2K	6	020C1P2M	4	010J1Z2X
1	02092432	3	020D2G31	7	010J2235
11	020O1H2Z	2	020F192X	4	010J273B
7	020R1L2L	7	020F2E2Z	1	010M1N3E
2	020U1P2Z	2	020J1C2X	2	010M2G
2	020U1S32	2	020J1F2M	2	01102L3E
6	020U2J2Z	10	020J1H2Z	2	01102Q
6	020X1Q30	3	020R2A	5	01162S3G
2	020X1T2N	2	020R2Z	2	011K38
6	020X1U30	2	020U1S2U	5	012E32
4	02101P2M	6	020X272S	4	02071D2Z
6	02101T2Q	3	020X2A2U	4	02071G30
2	0210202O	2	0210202X	4	02071I30
2	02131O32	7	02102B2V	4	020E1P3G
6	02131X2T	6	02132C	3	020F2T3A
1	0213242P	6	02132F2Y	3	020K242X
6	030D2G2T	6	030D1Q30	4	020O1F2Z
3	031426	3	030M2H	7	020O2931
3	03152C2T	3	030N2I31	4	020O2C3B
1	0316252Q	2	030O2Z	6	020S1M33
3	031J31	3	030S1M	6	02112137
4	032F2Y	3	03142830	7	02172B3A

Multiplicity	I	Multiplicity	II	Multiplicity	III
1	032Q	3	03152830	2	030R1L3C
7	032W	3	03162G31	4	030R2V3C
1	040A252Q	2	032Q	2	030V2J3G
2	040A292T	8	040H1J31	6	030V2N36
3	040D2C2T	3	040H1Y2T	2	030X1V3C
6	040E182K	3	040P1M	3	03182F
5	040F262R	1	040Q1N2T	4	031A2W3D
9	040H1B2W	2	040R1K32	5	033D
2	04162D2W	2	040R1O2U	7	040B1R34
2	050F1C2X	3	040V252Q	5	040G1W34
2	050F1M	3	040Y1Y2Q	2	040U1Q33
3	050I1E2T	3	041B2L	4	040U2I33
2	050J1F	3	050L1R2T	1	040Y1R36
7	050J2A2U	6	050L222T	6	04132S39
3	050L1E2W	7	050O262R	4	04192K3D
6	050L282Q	3	050T1R31	1	050G2P3D
2	050R1K32	7	050U2J32	6	050L2139
2	051G2V	2	051D2O	4	050X2I3E
3	081V2M	2	06172L	5	050Y1M3D
2	0C1A2S	6	061T2N	2	0510 2N3G
3	0G1N31	1	061X2N	6	05142637
6	0G2B2Q	5	0G1N31	2	05142O37
2	0K1D2V	5	0G1X2O	2	05162533
5	0K1R31	2	0K1G2K	2	081E38
3	0K2F2V	7	0K212P	4	081S3E
1	0T1R31			1	091O3F
				2	092D34
				2	0A1B32
				3	0A1H38
				1	0A243G
				2	0N233A
				2	0N283F
				2	0N2B3A
				4	0P1J37
				1	0T2H

Transfer to the permanent The nonic circuit $XYZ(X+Y+Z)$ has the required middle dimension six. The square second derivatives give the three mixed quadrics, and the three mixed second derivatives then supply independent square terms. Conciseness and apolar duality give the remaining dimensions. Each of the five independent factors contributes one square-zero tensor variable.

Write $Q = U^{\perp}$ for the physical quadratic inverse-system space. Every nonzero linear combination of matrix 2×2 subpermanents has quadratic rank at least four: restrict to the four entries of a subpermanent with nonzero coefficient and only that multiple of $ad + bc$ remains. A quadric supported on the three circuit variables has rank at most three. Thus Q has zero intersection with quadrics supported on the core, precisely the hypothesis that $U \to D_2$ is onto. The eight-plane quadratic capacity theorem supplies codimension at most six. Dividing (1) by 252 proves the internal unit source bound for every permissible ideal and physical background in this four-circuit case.

Factor-critical graphs and Boolean products

Let G be a graph on $n = 2h + 1$ vertices such that deletion of every single vertex leaves a perfect matching. Write $U_G \subset B_{n,2}$ for the span of its edge monomials. For every $W \subset B_{n,h-1}$,

$$h \dim U_G W \geq (h + 2) \dim W.$$

Choose a weight w_{ab} on each edge so that for every i the quadratic $q_i = \sum_{a,b \neq i} w_{ab} z_a z_b$ induces an isomorphism from degree $h - 1$ to $h + 1$ on the remaining $2h$ slots (each unordered edge is counted once). Such weights form a nonempty open: for any fixed i specialize to its perfect matching. Decompose according to matching pairs containing a prescribed single variable. On a component with s single pairs the other $h - s$ pairs are empty or full, and multiplication by their sum of pair products is the central Boolean inclusion on that many pair variables. It is invertible by the commutator identity in the parameter calculation. The finitely many determinant opens therefore meet.

Take distinct binary linear forms $l_i = a_i s + b_i t$ up to scale. Write $\sum w_{ij} l_i l_j z_i z_j = q_0 s^2 + q_1 st + q_2 t^2$. Top pairing and binary differential pairing transpose the map

$$q_0 \partial_s^2 + q_1 \partial_s \partial_t + q_2 \partial_t^2 : B_{n,h-1} \otimes k[s,t]_{h+1} \longrightarrow B_{n,h+1} \otimes k[s,t]_{h-1}$$

to a map sending coefficients H_T, $|T| = h$, of binary degree $h - 1$ to $\sum_{T \subset C,\ |T|=h} w_{C \setminus T} l_{C \setminus T} H_T$, for $|C| = h + 2$. Here $l_X = \prod_{i \in X} l_i$ and forbidden weights are zero. The polynomials $l_{T \setminus i}$, $i \in T$, are a basis by evaluation at the distinct roots of the l_i. In those bases the transposed map splits, indexed by i, as the direct sum of the invertible multiplication maps for q_i. The displayed map is consequently injective on $W \otimes k[s,t]_{h+1}$; its range lies in $U_G W \otimes k[s,t]_{h-1}$. Dimensions prove the claim, in particular $2 \dim U_G W \geq 3 \dim W$ at $h = 4$.

A graph on nine slots with two universal vertices and at most nine missing edges is factor-critical. More generally, at ten missing edges the only failure has all ten missing pairs on a five-set. Delete any vertex. The eight-vertex graph has at least one universal vertex. If it has no perfect matching, Tutte's theorem provides a set of s vertices whose removal leaves at least $s + 2$ odd components, by parity. The universal vertex forces $s \geq 1$; the component count forces $s \leq 3$. At $s = 1, 2$ the fewest missing cross edges occur at component sizes $5, 1, 1$ or $3, 1, 1, 1$ and give 11 or 12. At $s = 3$ all five remaining vertices are isolated, giving all ten missing pairs on one five-set.

Nested missing diagrams for quadratic and linear tests

Keep the original factor variables. When the two selected slots meet no quadratic hole, give precedence to these slots in the initial space and place them last, numbered 7 and 8. For disjoint leading monomials the product is again the leading term. Thus a lower bound for the disjoint products of the initial families applies to the original vector spaces. Simultaneously shift the two input families towards larger labels. In a two-slot fibre, inputs of degree one each have one or two choices. Their shifted choices are nested at one end. The only possible nonzero output of special degree two pairs distinct options, and forcing singleton fibres towards the same end can only remove such a choice. Taking the union over the outside slots proves that product size cannot increase. The family of holes stays on the first seven slots. Its terminal rows are therefore a strict partition with largest part at most six.

At exactly nine holes these are $63, 621, 54, 531, 432$. For a smaller diagram add holes within the first seven positions to reach size eight. The only size-eight shapes are $62, 53, 521, 431$. They embed, respectively, in $63, 63, 621, 531$. The positive allocations below cover every proper quadratic test for the three latter nine-hole shapes (for 63 they even cover the full test). Shrinking the set of holes only increases products. Thus the same nine-hole calculation establishes every proper quadratic test with at most eight holes. At nine holes the only failed abstract test widths are $27, 35, 36$; their physical treatments depend on the original factor rank.

For the following allocations, E is the set of all 36 pairs, H the shifted missing diagram on slots $0, \dots, 6$, and $F \subset E$ an upper set of input pairs. The available multipliers are $A = E \setminus H$. Count each four-set once if it is the disjoint union of a pair in F and a pair in $E \setminus H$. A row of length p_i, indexed from zero, contains pairs ij, $i < j \le i + p_i$. To prove normalized growth we send seven units from each input edge to these four-sets with capacity two per four-set. Restricting a valid allocation to a smaller input family preserves the bounds.

For $H = (6, 3)$ use all E. For $H = (6, 2, 1), (5, 3, 1)$ every proper upper set misses 01, and the two full families each have shadow 125. For $H = (5, 4)$, every upper set of size at most 34 misses 01 and 02. The full family has shadow 126; deleting 01 alone leaves shadow 120. For $H = (4, 3, 2)$ the minimal edges of A are 05 and 34. An upper set missing either is covered by omitting, respectively, the five edges $01, \dots, 05$, or all ten edges on $0, \dots, 4$.

The upper sets containing A in the last case have the following fifteen possible missing diagrams. Listed shadow sizes are obtained by the literal four-subset test. One elementary way to count is to fix the three smallest vertices of a four-set and inspect its fourth vertex in increasing order; once one partition into an allowed multiplier and input exists, that property persists as the fourth vertex increases.

Diagram missing from F	Shadow size
432	93
431	101
421	105
43	109
321	105
42	113
32	113
41	117
31	117
4	122
21	117
3	122
2	122
1	123
empty	124

It remains to check six positive certificates. The following tables contain all their nonzero entries. In each, use the full permutation group preserving the displayed H and the indicated maximal F. The group can be read directly from the vertex degree classes of the two graphs, retaining only permutations preserving both. A pair or quartet written without punctuation denotes the lexicographically first representative of its orbit. The listed letter denotes an input-edge orbit. On a line $q : Au, Bv$, place total mass u from orbit A and total mass v from orbit B into the orbit of q. Distribute each mass uniformly on incidences $e \subset q,\ q \setminus e \in E \setminus H$ between those two orbits. Whenever the incidence exists, it is biregular because the group is transitive on each orbit. Thus it suffices to sum to seven times each input orbit size and at most twice each output orbit size. The tables display input symbols with their orbit sizes; the number after each output representative is twice its orbit size. Zero-only output orbits are omitted.

Missing multipliers 6,3 with all edges Input orbits are A=01 (1), B=02 (3), C=05 (2), D=07 (2), E=12 (3), F=15 (2), G=17 (2), H=23 (3), I=25 (6), J=27 (6), K=56 (1), L=57 (4), M=78 (1).

Quartet	Capacity	Assigned input masses
0123	6	A 6
0125	12	A 1, B 11
0127	12	B 2, E 10
0156	2	C 2
0157	8	D 7, F 1
0178	2	D 2
0234	2	B 2
0235	12	C 12
0237	12	D 5, H 7
0256	6	B 6
0257	24	I 24
0278	6	J 6
0567	4	K 4
0578	4	L 4
1234	2	E 2
1235	12	E 4, F 7, H 1
1237	12	E 5, G 7
1256	6	F 5, I 1
1257	24	F 1, G 7, I 9, J 7
1278	6	J 6
1567	4	K 2, L 2
1578	4	L 4
2345	4	H 4
2347	4	H 4
2356	6	H 5, K 1
2357	24	I 8, J 8, L 8
2378	6	M 6
2567	12	J 12
2578	12	J 3, L 8, M 1
5678	2	L 2

Missing multipliers 6,2,1 with all edges except 01 Input orbits are A=02 (2), B=04 (3), C=07 (2), D=12 (2), E=14 (3), F=17 (2), G=23 (1), H=24 (6), I=27 (4), J=45 (3), K=47 (6), L=78 (1).

Quartet	Capacity	Assigned input masses
0124	12	A 12
0127	8	A 1, D 7
0145	6	B 6
0147	12	B 1, C 8, E 3
0178	2	C 2
0234	6	A 1
0237	4	G 4
0245	12	B 12
0247	24	C 4, H 20
0278	4	I 4
0456	2	B 2
0457	12	J 12
0478	6	K 6

Quartet	Capacity	Assigned input masses
1234	6	D 5, G 1
1237	4	D 2, G 2
1245	12	E 12
1247	24	E 6, F 14, H 1, I 3
1278	4	I 4
1456	2	J 2
1457	12	J 6, K 6
1478	6	K 1, L 5
2345	6	H 6
2347	12	H 12
2378	2	I 2
2456	4	H 3, J 1
2457	24	I 15, K 9
2478	12	K 10, L 2
4567	4	K 4
4578	6	K 6

Missing multipliers 5,3,1 with all edges except 01 Input orbits are A=02 (2), B=04 (1), C=05 (1), D=06 (3), E=12 (2), F=14 (1), G=15 (1), H=16 (3), I=23 (1), J=24 (2), K=25 (2), L=26 (6), M=45 (1), N=46 (3), O=56 (3), P=67 (3).

Quartet	Capacity	Assigned input masses
0125	4	A 4
0126	12	A 7, E 5
0145	2	B 2
0146	6	B 5, F 1
0156	6	C 3, G 3
0167	6	D 6
0234	2	A 2
0235	2	A 1
0236	6	I 6
0245	4	C 4
0246	12	D 12
0256	12	D 3, K 9
0267	12	L 12
0456	6	M 6
0467	6	N 6
0567	6	O 6
0678	2	P 2
1234	2	E 2
1235	2	E 1, I 1
1236	6	E 6
1245	4	F 2, J 2
1246	12	F 4, H 8
1256	12	G 4, H 5, L 3
1267	12	H 8, L 4
1456	6	M 1, N 5
1467	6	N 6
1567	6	O 3, P 3

Quartet	Capacity	Assigned input masses
1678	2	P 2
2345	2	J 2
2346	6	J 6
2356	6	K 5, L 1
2367	6	L 6
2456	12	J 4, L 5, O 3
2467	12	L 9, P 3
2567	12	L 2, O 9, P 1
2678	4	P 4
4567	6	N 4, P 2
4678	2	P 2
5678	2	P 2

Missing multipliers 5,4 with all edges except 01 and 02 Input orbits are A=03 (3), B=06 (3), C=12 (1), D=13 (3), E=16 (3), F=23 (3), G=26 (3), H=34 (3), I=36 (9), J=67 (3).

Quartet	Capacity	Assigned input masses
0126	6	C 6
0136	18	A 15, D 3
0167	6	B 6
0234	6	A 6
0236	18	B 15, F 3
0267	6	G 6
0346	18	H 18
0367	18	I 18
0678	2	J 2
1234	6	C 1, D 5
1236	18	D 11, E 6, F 1
1267	6	G 6
1345	2	D 2
1346	18	E 15, H 3
1367	18	I 18
1678	2	J 2
2345	2	F 2
2346	18	F 15, G 3
2367	18	G 6, I 5, J 7
2678	2	J 2
3456	6	I 6
3467	18	I 16, J 2
3678	6	J 6

Missing multipliers 4,3,2 with all edges except 01 through 05 Input orbits are A=06 (3), B=12 (1), C=13 (4), D=15 (2), E=16 (6), F=34 (1), G=35 (2), H=36 (6), I=56 (3), J=67 (3).

Quartet	Capacity	Assigned input masses
0125	2	B 2
0126	6	B 5

Quartet	Capacity	Assigned input masses
0135	8	C 8
0136	24	C 20
0156	12	A 12
0167	12	A 9, E 3
0345	2	F 2
0346	6	F 5
0356	12	G 12
0367	12	H 12
0567	6	I 6
0678	2	J 2
1256	6	D 6
1267	6	E 6
1345	4	D 4
1346	12	E 12
1356	24	D 4, E 20
1367	24	E 1, H 23
1567	12	I 12
1678	4	J 4
3456	6	G 2, H 4
3467	6	H 3, J 3
3567	12	I 3, J 9
3678	4	J 3

Missing multipliers 4,3,2 with all edges except those on vertices 0 through 4 Input orbits are A=05 (12), B=35 (8), C=56 (6).

Quartet	Capacity	Assigned input masses
0156	36	A 36
0345	24	A 24
0356	72	A 24, B 48
0567	24	C 24
3456	12	B 8, C 4
3567	16	C 14

The complete proper linear-layer bound

For a linear m-plane $Y \subset B_{9,1}$, put $u = \dim UY$. There are at most nine missing quadrics, supported on seven slots. For $1 \le m \le 4$ take initial linear forms, multiply by all Boolean quadrics and then remove a complement of U, of dimension at most nine. This gives

$$u \ge 84 - \binom{9-m}{3} - 9m = 19,\ 31,\ 37,\ 38$$

in the four respective dimensions. All four values are at least $28m/3$.

For $m = 5, 6$, shift the allowed pairs and the m vertices simultaneously as in the quadratic calculation. The marked vertices can be taken to be $S_m = \{9-m, \dots, 8\}$; enlarge the missing shifted diagram to nine pairs on the first seven slots. Each entry below is the number of triples $a < b < c$ for which deletion of some marked vertex leaves an allowed pair. Equivalently, test the

smallest marked vertex of the triple, since the allowed pair family increases in each coordinate. This gives the explicit count

$$\sum_{a<b<c} [\, a \in S_m,\ bc \notin H \,] + [\, a \notin S_m,\ b \in S_m,\ ac \notin H \,] + [\, a, b \notin S_m,\ c \in S_m,\ ab \notin H \,],$$

where the summation covers all three brackets. Here H is the indicated missing diagram, and slots range from 0 to 8.

H	63	621	54	531	432
$m = 5$	52	47	53	49	50
$m = 6$	58	59	59	61	62

The smallest entries, 47 and 58, both exceed $28m/3$.

For $m = 7, 8$ put $W = Y^{\perp} \subset L$, so $w = \dim W = 2, 1$. The annihilator of UY consists of actual cubic inverse forms F with $I_2 F \subset W$. The two-marked-cell argument in the low-overlap calculation applies on nine cells with one addition. If the three distinct rows and columns make a complete three-by-three board, their matching graph is $K_{3,3}$; removal of at most two marked edges leaves it connected. This admits one common matching coefficient, hence one cubic permanent. At most one complete three-by-three board fits. All other cases give only cubics on the marked cells. Consequently

$$u \geq 84 - \binom{w+2}{3} - 1 = \begin{cases} 79 & m = 7, \\ 82 & m = 8. \end{cases}$$

These give $3u \geq 28m$ for every proper linear input.

The maximal physical nine-plane

In an independent nine-frame write $B = B_9$ and $q = \operatorname{codim}_{B_2} U$. For $q \leq 8$ every proper test is covered by the graph argument, and the full test gives $UB_2 = B_4$. Indeed, an inverse quartic killed by I_2 is a combination of four-by-four subpermanents, requiring sixteen directions. There remain the three dimensions $27, 35, 36$ at $q = 9$.

First $q = 9$ forces the support L to be a separated three-by-three tensor plane. The proof is short at this maximum. The full physical intersection $E_2 \cap \operatorname{Sym}^2 L$ has dimension nine by the rectangle bound. A coordinate limit therefore has nine rectangles on nine cells, hence is a full three-by-three board. For the last assertion compress row lengths: if the second longest has length at least four, at most six rectangles occur; if it has length at most two, at most four nontrivial rows contribute, again giving at most six. With second length three, nine rectangles require row lengths $3, 3, 3$. Equality in the pairwise row-intersection count forces the three original neighborhoods to coincide. Projection to these anchor coordinates is an isomorphism on L, since the corresponding Plücker coordinate is nonzero at the limit. The restriction of every physical quadric $F \in \operatorname{Sym}^2 L$ to the anchor determines it, and the nine target quadrics thus restrict to the full nine-dimensional span Q_3 of the anchor subpermanents.

Express the derivative in physical cell rc, restricted to L, as a vector v_{rc} in the nine-dimensional anchor dual, with basis $e_{ij} = v_{ij}$ for $1 \leq i, j \leq 3$. For the polar form B_F of *each* intrinsic $F \in Q_3$ the equations

$$B_F(v_{rc}, v_{sd}) = B_F(v_{rd}, v_{sc})$$

hold, and same-row and same-column pairings are zero. In particular v_{rj} for an anchor column j lies in that column span. To test a putative component e_{ab} with $b \neq j$, pair with e_{ij}, $i \neq a$, against the single rectangle on rows i, a and columns j, b: it extracts just that component. Put $v_{rj} = \sum_a \alpha_{rj,a} e_{aj}$. The rectangle identity against e_{ik}, $k \neq j$, implies $\alpha_{rj,a} = \alpha_{rk,a}$ by taking

$i \neq a$ and that same row pair. Thus the coefficients are α_{ra}, independent of j. Pairing v_{rj}, v_{rk} for the rectangle with row axes a, b gives $2\alpha_{ra}\alpha_{rb} = 0$ for $a \neq b$. Transposition likewise gives $v_{ic} = \sum_b \gamma_{cb} e_{ib}$ with at most one nonzero coefficient γ_{cb} per physical column c. Finally

$$v_{rc} = \sum_{a,b} \alpha_{ra}\gamma_{cb} e_{ab}.$$

Indeed both sides have the same pairing with every e_{ij} for every intrinsic rectangle, by the rectangle identity. The joint radical of these polar forms is zero since every coordinate can be extracted by pairing against an opposite corner. The columns of α have pairwise disjoint physical row supports and those of γ have pairwise disjoint physical column supports. The formula thus identifies the original L with their three-by-three tensor product. In intrinsic coordinates its target cubic is $P_3 = \mathrm{perm}_3$, and its physical quadratic ideal is the annihilator of the nine rectangle quadrics Q_3.

At $q = 9$ all of Q_3 is squarefree in the given nine independent factor coordinates. The square of each dual factor derivative D_i therefore kills every first derivative of P_3, so $D_i^2 P_3 = 0$. Consequently P_3 itself belongs to the actual cubic inverse space. The only intrinsic cubic killed by all physical quadratic operators is a multiple of P_3, by the matching monomial argument. Hence $\dim UB_1 = 83$. The intrinsic square calculation, with I_2 now denoting the restricted quadratic operators on the three-by-three model, gives $(I_2^2)_4^\perp = LP_3$. If an actual quartic $F = fP_3$ annihilates U^2, its squarefreeness yields $0 = D_i^2(fP_3) = 2(D_i f)(D_i P_3)$. The cubic is concise, so $D_i P_3 \neq 0$ for every i. Thus $f = 0$, giving $U^2 = B_4$.

The hyperplane test. For Z a hyperplane of B_2, write $Z^\perp = kr$. A quartic $F \in (UZ)^\perp$ has all its intrinsic I_2 outputs in kr. If a coordinate square $d_a^2 F = cr \neq 0$, every b in the row or column of a satisfies $c\, d_b r = \lambda_{ab} d_a r$, by commuting three derivatives. For an opposite cell b, let p, q be the cross corners. The same commutation with the rectangle difference gives $c\, d_b r = \lambda_{\square} d_a r + \lambda_{ap} d_q r$. Thus all first derivatives of r are proportional. This makes r a rank-one quadric, impossible for a nonzero quadric squarefree in nine independent coordinates.

It follows that F is squarefree in the intrinsic physical cells. If $F \neq 0$, choose a four-cell monomial S with nonzero coefficient. The pigeonhole principle supplies a same-row pair e and a distinct same-column pair f in S. The two outputs $d_e F$ and $d_f F$ on columns $S \setminus e, S \setminus f$ have equal nonzero diagonal coefficients and zero off-diagonal coefficients: an off-diagonal term would repeat a cell. This contradicts their containment in a line. Hence $UZ = B_4$ for all such hyperplanes. The full test $Z = B_2$ follows as well.

The 27-dimensional test. Take $\dim Z = 27$. Write $K' = Z^\perp$, $Q = U^\perp$, $k = \dim(K' \cap Q)$, $S = U \cap Z$ and $\mathcal{H} = (UZ)^\perp$. Then $\dim S = 18 + k$. The target is $\dim \mathcal{H} \leq 31$. At $k \geq 2$ the restriction to symmetric forms on U/S is injective since $U^2 = B_4$, giving $\dim \mathcal{H} \leq \binom{10-k}{2} \leq 28$. At $\dim S = 18, 19$, the quotients B_4/SB_2 have dimension at most $16, 15$, respectively. Indeed their standard four-sets must be cliques on the remaining $18, 17$ graph edges. By the Kruskal–Katona theorem the maximum clique count is attained by the initial colex segments; they consist of a clique on six vertices and a seventh vertex joined to $3, 2$ of them, giving $\binom{6}{4} + \binom{3}{3} = 16$ and $\binom{6}{4} = 15$. At $k = 0$, $SB_2 = S(U + Z) \subset UZ$, proving the result.

If $k = 1$, let $K' \cap Q = kr_0$ and $\rho = \dim\langle dr_0 : d \in B_1\rangle \geq 4$, by the four-cell rank bound. For $F \in \mathcal{H}$, $s \in S$, write $sF = \lambda_F(s) r_0$. Set $E = \{\lambda_F : F \in \mathcal{H}\}$, $t = \dim E$. The kernel has dimension at most 15 by the preceding quotient bound. Every relation $\sum_i d_i s_i = 0$ gives $\sum_i \lambda_F(s_i) d_i r_0 = 0$. Thus a surjection

$$SB_1 \twoheadrightarrow (B_1 r_0) \otimes E^*, \qquad ds \longmapsto (dr_0) \otimes \mathrm{ev}_s$$

is well defined. Its dimension gives $\rho t \leq 83$. For $\rho \geq 5$, $t \leq 16$. For $\rho = 4$ let $W = \mathrm{Ann}_{B_1} r_0$, of dimension five. Then WS is in its kernel and has dimension at least sixteen: the leading monomials of W, S are five vertices and nineteen edges. Two of the five vertices have total degree

at most 15 in that edge set, by averaging the 38 incidences. Their disjoint products with the nineteen edges overlap in at most seven triples and therefore give at least $38-15-7=16$ distinct triples. It follows that $4t+16\le 83$, again $t\le 16$. Consequently $\dim \mathcal{H}\le 15+16=31$ in every case, as required.

Long circuits and the exact trace correction

Let the sole circuit have $r\ge 5$ slots and let $s=9-r$. The ordinary circuit algebra on those positions is the hyperplane algebra of the rank-two calculation, with ordinary layers full from degree two onwards and one trace-radical line in degree $r-1$. Hence for the nine-factor product

$$A=C/R,\quad C=C^{(r)}\otimes B_s\subset B_9,\qquad \dim C_j=\binom{9}{j}-\binom{s}{j-1},\qquad r_j:=\dim R_j=\binom{s}{j-r+1}.$$

In particular $A_2=C_2$, $r_3=0$, and $C_7=B_{9,7}$. The number of Boolean missing quadrics is $b=q+s$, where $q=\operatorname{codim}_{A_2}U\le 6$.

On the dual squarefree quadrics in formal positions X_i, let σ substitute the original factors. For $\lambda=\sum_{i\text{ in circuit}}\lambda_iX_i$ their relation,

$$\ker\sigma=\lambda\cdot\langle X_j : j\text{ a coloop}\rangle.$$

Indeed these s elements are independent and span by the formula for C_2; transposition identifies the rank of substitution with $\dim C_2$. Its image is the actual quadratic inverse space, since $A_2=C_2$. The inverse of the lifted U is $\sigma^{-1}(E_2\cap D_2(T))$. Suppose a,b are a good pair of original circuit slots. If a formal inverse quadric vanishes on deleting them, its substitution is in the quadratic part of (ℓ_a,ℓ_b) and in E_2, so is zero. It must then be $\lambda\eta$ with η a coloop form. Deleting the two slots leaves a nonzero circuit form times η, forcing $\eta=0$. By duality the projection of U onto the fifteen pairs incident to the good slots is onto. Giving precedence to those two slots produces the seven-supported diagram of the graph calculation.

For $r\ge 6$, if $b\le 8$, the allowed graph is factor-critical and all proper quadratic tests satisfy $2\dim_B UZ\ge 7\dim Z$. For the full preimage $\widetilde{J}_d$ of J_d in C_d, the Boolean cubic estimate gives $u_3\ge\frac{3}{2}(j_3+r_3)-r_5$. If $J_5\ne 0$, then $U\widetilde{J}_5\ne 0$: intrinsically $UA_2=A_4$ by the sixteen-direction lower bound for quartic targets, and the perfect pairing rules out any nonzero element of A_5 annihilated by U. Thus $Z=(U\widetilde{J}_5)^{\perp}\subset B_2$ is proper. Boolean reflection gives $u_5\ge\frac{2}{7}(j_5+r_5)-r_7$. For $J_5=0$ one has $J_3=J_1=0$ by the ideal property and the perfect pairing. Otherwise cancellation of r_5 gives

$$\lambda\le\frac{j_7+r_7}{36}+\frac{m}{9}-\frac{u_1+r_3}{84}\le 1+\frac{m}{9}-\frac{u_1+r_3}{84}.\tag{8}$$

For $m=0,1$ this is at most one. Indeed when $m=1$ multiplication by all Boolean quadrics produces at least $\binom{8}{2}=28$ cubics, and restricting to U loses at most $b\le 10$. The low-overlap calculation treats $m\ge 2$.

For $r\ge 6$ the sole remaining possibility is $r=6,q=6$, hence $b=9,\dim A_2=33,\dim A_4=125,r_5=1,r_7=3,r_4=0$. Every actual $Z\subset A_2$ has dimension at most 33, and at width 27 the physical square calculation gives $\dim UZ\ge 125-2-\binom{7}{2}=102$, since $\dim U/(U\cap Z)\le 6$. As $r_4=0$ there is no quartic loss, so the nine-hole bounds at the other widths give $2\dim_A UZ\ge 7\dim Z$ for every actual test. On $Z=(UJ_5)^{\perp}\subset A_2$ this implies $u_5\ge\frac{2}{7}(j_5+1)-3$, exactly the formula above. Thus (8) again gives the unit inequality.

A five-circuit with four coloops

Here $A=A(F_5)\otimes B_4$, where F_5 is the full five-circuit. The ordinary algebra C is contained in B_9, with radical $R=kr\otimes B_4$ generated in core degree four. Consequently $\dim A_2=32,\dim A_4=121,r_3=0,r_4=1,r_5=r_7=4$. We first consider a good pair, as defined in the circuit calculation.

The projection of U to the coloop-pair space $(B_4)_2$ has deficiency at most one, since the inverse quadrics there are physical target quadrics supported on a four-plane. A two-dimensional pencil of quadrics on four variables has a singular member over the algebraic closure, whereas every nonzero physical target quadric has rank at least four. Such a codimension-at-most-one subspace of $(B_4)_2$ multiplies $(B_4)_1$ onto $(B_4)_3$: any inverse cubic in its cokernel would have all its first derivatives on a line, impossible for a nonzero squarefree cubic. Since every positive core operator kills the radical generator, $UR_5 = R_7$. The reflected test $Z = (U\widetilde{J}_5)^\perp \subset B_{9,2}$, with $\widetilde{J}_5$ the full preimage in C_5, therefore satisfies $Z \subset R_7^\perp = C_2 = A_2$, and $z = \dim Z \le 32$.

A good pair confines the $q+4$ holes to seven original slots. For $q \le 5$ the allowed graph is factor-critical. The only exceptional quadratic test width below 33 is 27, where square saturation gives $\dim_A UZ \ge 121 - 2 - \binom{6}{2} = 104 \ge 95$. The Boolean product dimension dominates the actual one. Consequently $2\dim_B UZ \ge 7z$ and the same cancellation of r_5 in (8) proves the unit bound.

For $q = 6$ there are ten holes. If the original allowed graph is factor-critical, the cubic growth still holds. For $z \le 23$ apply the ten-interval quadratic bound below; for $25 \le z \le 32$ the eight-direction square bound gives $\dim_A UZ \ge 119 - \binom{33-z}{2} \ge 7z/2$ (check the two endpoints by convexity). At $z = 24$ put $S = U \cap Z$ and $k = \operatorname{codim}_{A_2}(U+Z)$; then $\dim S = 18 + k$. If $k \ge 1$, $\dim U/S \le 7$, so the same square bound gives at least $119 - \binom{8}{2} = 91$.

If $k = 0$, then $SA_2 = S(U+Z) \subset UZ$. We show directly that any eighteen-plane $S \subset A_2$ has $\dim_A SA_2 \ge 85$. Choose an initial order for this isolated multiplication test. The ordinary degree-two space includes all circuit pairs, all coloop pairs and, for each of the four coloop positions, the same hyperplane of five mixed coefficients. Thus its four missing pairs form a star with centre a specified circuit slot v (the last core pivot), and the other four core slots are its allowed neighbors. Let t of the 18 leading pairs of S meet v. Four-sets avoiding v are uncovered only if all six pairs belong to the complement of the remaining $18 - t$ pairs on eight vertices. This complement has $10 + t$ edges. Writing $10 + t = \binom{a}{2} + b$, $0 \le b < a$, the clique bound of Kruskal–Katona gives at most $\binom{a}{4} + \binom{b}{3}$ missed four-sets. The respective lower counts for $t = 0, \dots, 8$ are

$$65,\ 65,\ 65,\ 64,\ 61,\ 55,\ 55,\ 55,\ 54.$$

For $t > 0$, four-sets containing v add at least $56 - \binom{8-t}{3}$, by choosing a triple containing one of the t neighbors of v in the test. Every complementary pair avoids v and is available. The sum is at least 86.

For $t = 0$ divide the eight other vertices into four allowed neighbors D of v and four forbidden neighbors E. There are 52 triples meeting D. A triple is missed only if deletion of every member of D leaves one of the ten complement pairs of the test. Let b_0, b_1, b_2 be the numbers of complement pairs of types EE, DE, DD. At most $4b_0$ missed triples have one D-vertex. Those with two D-vertices number at most $\sum_{x \in E} \binom{\deg_D(x)}{2} \le 3b_1/2$. Those with three are triangles on D, at most $2b_2/3$. Since $b_0 \le 6$ and $b_0 + b_1 + b_2 = 10$, the total is at most 30. Thus at least $65 + 52 - 30 = 87$ four-sets are covered in this case. The passage from $C_4 \subset B_{9,4}$ to A_4 costs one dimension throughout, proving the claim and treating the 24-dimensional test. The reflected quadratic inequality and (8) now apply in every width.

The remaining ten-hole possibility from the matching argument is that the raw hole graph is a complete five-set D. The formal inverse quadratic space has dimension ten, projecting isomorphically onto the ten coordinates on D by echelon duality. Every coloop must lie in D: the kernel element λX_b for a coloop $b \notin D$ would project to zero. Thus D consists of all four coloops and one circuit slot a. The four kernel elements project to the four star coordinates $\lambda_a X_a X_b$. Quotienting shows that the actual $Q = U^\perp$ projects injectively onto the six pure-coloop coordinates. Scaling the coloop operators by a parameter and taking the limit therefore puts all of Q in the pure-coloop dual component. The limit of U contains all pure circuit quadrics and all mixed circuit–coloop quadrics. The mixed column of the grid bound below proves a numerator at most 242.

Finally suppose every pair of original circuit factors is bad. Each witness is a rank-four rectangle and is supported on a separated two-by-two tensor plane inside L. A factor of matrix rank at least three is excluded. If one factor f has rank two, its two intrinsic row and column planes determine the same four-plane for every bad pair containing f. That plane must be the circuit span, since both have dimension four. The space of physical quadrics on this separated four-plane is a single nondegenerate line kQ. Inverse isotropy gives $Q^{-1}(f, \ell_j) = 0$ for every other circuit factor. They span the circuit, forcing $f = 0$. All factors are therefore simple tensors. The connected-component argument in the rectangle calculation places the full circuit in a ruling $u \otimes C$, $\dim C = 4$, after transposition if needed. Take four independent circuit columns v_i. The bad-pair construction puts $w_i \otimes v_i$ in L with w_i supported outside $\operatorname{supp}(u)$. The projection outside these rows has dimension at most four and contains all four independent tensors. For each $i \neq j$, a bad witness has a single opposite row w_{ij} with both $w_{ij} \otimes v_i, w_{ij} \otimes v_j \in L$. Expanding their projections in the four-vector basis and applying the dual column functionals forces w_i, w_{ij}, w_j proportional. Thus $L = (ku \oplus kw) \otimes C$, with separated row supports.

The ruling and its grid. With four actual circuit coordinates a and four actual coloop coordinates, the product has form $F_5(a) \prod_i (b_i + M_i(a))$, where the b_i form a basis of the second ruling. First take $M = 0$. The physical target has only mixed-row components. Indeed the same-row equations on one row in the support of each axis kill the two pure parts. The cross-column equations make the remaining matrix symmetric with zero physical diagonal. Four independent physical column evaluations form a dual column basis on C; in these coordinates the entire inverse space is contained in the symmetric zero-diagonal four-by-four matrices. Thus U includes the two full pure quadratic spaces and a ten-dimensional mixed subspace Γ, generated by the four mixed diagonal operators and six antisymmetric mixed differences. Separate coordinate changes for the two factor blocks preserve the dimension arguments.

For $0 \neq w' \in (B_4)_3$, $\Gamma w' = A_{1,4}$, where A_{ij} denotes the (i, j)-piece. Otherwise a nonzero decomposable tensor would belong to $\Gamma^\perp$; a nonzero symmetric rank-one matrix cannot have vanishing diagonal. Also $\Gamma A_{2,1} = A_{3,2}$: if an inverse form of bidegree $(3, 2)$ is killed by Γ, its polarized coefficient tensor must be fully symmetric in the five column indices. The antisymmetric equations permit every cross-block transposition, and the pure-block symmetries are built in. The four mixed diagonal equations then force the tensor to vanish when any two indices coincide. Five indices on four columns necessarily repeat. Hence for $V \subset A_{2,1}$, $\dim \Gamma V \geq 60 - 10(40 - \dim V)$.

By block scaling we may take a bihomogeneous limit of any ideal. The normalized densities $x_{ij} = \dim J_{ij}/d_{ij}$ and $u_{ij} = \dim(UJ)_{ij}/d_{ij}$ use $d_{ij} = H_i\binom{4}{j}$, $H = (1, 4, 10, 10, 4, 1)$. The sequence x is nondecreasing in both coordinates. To see this, in the circuit algebra the first multiplication into degree two of a w-plane has dimension $10 - \binom{5-w}{2} \geq 10w/4$. The central step from degree two to degree three is an isomorphism for a suitable linear form by the circuit calculation. Reflecting in the perfect trace pairing proves the opposite normalized inequalities in higher degrees, and the endpoint edges are automatic. On the Boolean factor use leading subsets and count edges between consecutive ranks in the inclusion graph, which is regular on each part. Each normalized inequality extends to graph subspaces in a tensor slice: filter by a basis of the inactive factor and sum the bounds on its initial coefficients. Consequently full pure quadratic multipliers imply $u_{ij} \geq x_{i-2,j}, x_{i,j-2}$ whenever the subscripts exist. For a limiting multiplier containing pure circuit and all mixed quadrics one instead has $u_{ij} \geq x_{i-2,j}, x_{i-1,j-1}$. These latter inequalities apply to the raw five-clique above. In the ruling itself, either $x_{03} = 0$, or the two facts about Γ give $u_{14} = 1$ and $(20/3)x_{21} - u_{32} \leq 17/3$. The three allocations below apply.

For a shear M, use the new variables $c_i = b_i + M_i(a)$ to identify all the actual algebras with $A(F_5) \otimes B_4$. Scaling the coloop differential operators by t conjugates the quadratic image for M to the image for tM, for $t \neq 0$. Its dimension is constant even at zero. In fact every physical inverse quadric is a mixed form $a(c - tMa)$ and its pure circuit and mixed components both belong to full actual inverse layers. Taking the ideal and multiplier limits simultaneously preserves the ideal

dimensions and can only decrease the product dimensions. Thus the unsheared upper bound holds for the original ideal and shear.

Ten-hole tests through width 23. Use indices 0 through 8. Order pairs componentwise, and write E for all 36 pairs. After the stated shifting, the missing pairs form an order ideal on the first seven vertices and the input F is an upper set. We may enlarge the missing ideal to cardinality ten. Its strictly decreasing positive row lengths are one of 64, 631, 541, 532, 4321, in that order below. For row lengths p, the missing pairs are ij with i<j≤i+p_i, starting with row 0. Write A for the remaining pairs and N for the set of admitted four-sets.

Take any integer interval [d,h] in the table. Every upper set of size between d and h contains an upper set of size exactly d, by successively removing minimal elements. Consider the listed pivots in this latter upper set. For any yes/no answers define G as the union of the principal upper sets of yes pivots, and B as E minus the union of the principal lower sets of no pivots. Skip the answer if G is not contained in B or if d lies outside [|G|,|B|]. In the other cases the tested upper set necessarily contains

$D = G \cup \{e \in B : |B \setminus {\downarrow e}| < d\}$.

Indeed, omitting e excludes every pair below it. This proves the completeness of these tests for all original input sets. The numbers in the five right columns are |N(D)| minimized over the retained pivot answers, with the five possible missing diagrams as headers.

Input size	Pivots	Feasible answers	64	631	541	532	4321
1–3	none	1	11	11	11	11	11
4–6	48	2	28	25	26	24	21
7	none	1	28	25	26	27	28
8–11	18,38	3	41	39	39	40	41
12–14	18,28	3	51	50	51	54	56
15	18	2	54	55	54	57	63
16–17	45	2	60	61	62	64	66
18–19	08,17	4	69	71	73	72	72
20–22	08,17	4	79	79	78	77	78
23	16	2	84	83	85	81	81

Here is a direct arithmetic formula for each entry. For any upper set S let $r_S(v)$ be the least w>v with vw in S, taking 9 if none exists. For each triple T={i,j,k}, i<j<k, take m to be the minimum of 9 and all the following entries indexed by v∈T.

- Include $r_D(v)$ when the complementary pair T\{v} belongs to A.
- Include $r_A(v)$ when the complementary pair T\{v} belongs to D.

Add 9−max(k+1,m) over the triples. This counts the admitted choices of the fourth vertex separately for each three smallest vertices. For example, in the first row D consists of 78, and exactly 21−10=11 pairs among the seven remaining vertices are available. In the final row both pivot answers force a 20-edge D and the minima over the five hole shapes for the no and yes answers are 84 and 81. Twice every entry in every row is at least seven times that row's upper width. This proves the asserted inequality.

Three grid allocations. Write x for the ideal and u for its quadratic image. Always 0≤u≤x≤1, and x increases in each coordinate. Let H=(1,4,10,10,4,1). In units of 1/252, the required cost is

$$C = \sum_{i,j} w_{ij}(x_{ij} - u_{ij}), \qquad w_{ij} = \frac{252 H_i \binom{4}{j}}{\binom{9}{i+j}}.$$

for i+j=1,3,5,7. All other weights are zero. This argument proves C≤251, 250, 242 in the three geometric cases.

In the two axis cases, all moves increasing either coordinate by two satisfy x at the source ≤u at the destination. In the mixed case the same holds for the increments (2,0) and (1,1). In the positive axis case also u_{14}=1 and $(20/3)x_{21}-u_{32}\leq 17/3$. In the zero axis case x_{03}=0.

Each entry "d (v)" in the following table multiplies the inequality x at the row source minus u at destination d ≤0 by v. Form the sum of all indicated terms in the applicable column.

Source	Axis positive	Axis zero	Mixed
01	21 (112)		21 (38), 12 (72)
03	23 (12)		23 (12), 14 (2)
10	12 (82), 30 (30)	12 (82), 30 (30)	30 (30), 21 (82)
12	32 (82)	14 (8), 32 (74)	32 (4), 23 (68)
14	34 (70)	34 (8)	34 (8)
21	23 (68), 41 (32)	23 (80), 41 (32)	41 (4), 32 (116)
23	43 (80)	43 (80)	43 (2), 34 (62)
30	32 (30)	32 (30)	50 (2), 41 (28)
32	52 (42)	34 (62), 52 (42)	52 (10), 43 (110)
41	43 (32)	43 (32)	52 (32)

In the first column add 70 times $-u_{14}\leq -1$ and 3 times the special inequality at cells 21 and 32. Their right sides sum to −53. In the second column add 112 times $x_{01}-x_{03}\leq 0$ and 124 times $x_{03}\leq 0$. In the third column add 2 times $x_{01}-x_{03}\leq 0$. The right sides in the last two columns are zero.

For a compact coefficient check subtract the combined left side from C. In each cell write the remainder as ax+bu. The upper bound from this one cell is max(0,a,a+b), by the three vertices of $0\leq u\leq x\leq 1$. The nonzero cell contributions in the three cases are as follows.

Case	21	23	32	34	43	50	52	Added constant	Bound
Axis positive	0	0	78	70	112	2	42	−53	251
Axis zero	8	0	16	70	112	2	42	0	250
Mixed	0	16	0	70	112	2	42	0	242

For example in the first column the weight of cell 32 is 120. Its outgoing coefficient is 42, leaving a=78. The incoming coefficients are 82 and 30 and the special inequality contributes another 3 to the coefficient of −u, giving b=−5. The maximum on this cell is 78. All other entries follow from the same subtraction.

Appendix B. Local calculations for order ten

The local estimates used in the order-ten argument are proved here. The section numbers continue those of the main argument.

7 Position algebras and physical increments

Write $I_2 = \mathrm{Ann}(P)_2$. For one original product T, let $B = B_{10}$, $F = \psi(S)$, $f_j = \dim F_j$, $h_j = \dim A(T)_j$, and $r_j = f_j - h_j$, as in the construction of the common quadratic. The linear map $\sigma_j : B_j \to \mathrm{Sym}^j V$ sends each squarefree position monomial to the product of the corresponding physical factors. Its transpose under the degree-j coefficient pairing is $\psi : S_j \to B_j$.

Apply Lemma 2.1 to all the original lists. The extra central ranks constructed below each hold on nonempty open subsets of the same physical space I_2. Their defining nonzero minors can all be imposed on this same Q.

7.1 Two bounds for the increment at the same prefix

Fix a prefix A and a new term T, with weight w, factor span U, factor rank ρ, and c coloops. Put

$$K_0 = E_2(T) \cap X_2(A), \qquad L = E_4(T) \cap (X_4(A) + P_4), \qquad k = \dim K_0,$$
$$z = \dim(\mathrm{Sym}^2 U \cap P_2), \qquad R = \mathrm{rank}\big(Q : A(T)_4 \longrightarrow A(T)_6\big).$$

On the 210-dimensional space B_4 the symmetric form $(a, b) \mapsto \lambda(\psi(Q)ab)$ is nondegenerate. Restricted to F_4, its kernel contains the complementary-degree pairing radical of dimension r_4. Modulo that radical the kernel is precisely the kernel of the apolar map displayed above, of dimension $h_4 - R$. Thus the restricted form has rank $R \geq 2f_4 - 210$, and

$$w = 105 - h_4 + R/2, \qquad w - r_4 = 105 - f_4 + R/2 \geq 0.$$

The new quartic quotient dimension and the weight together give

$$G := h_4 - \dim L + w = 105 + R/2 - \dim L, \qquad g \geq G - \delta, \tag{9}$$

since the sextic quotient increment is nonnegative.

Let $H = \psi(I_2)$ and $W = \psi(\mathrm{Ann}_2 K_0)$. The image of W in $A(T)_2$ has dimension $h_2 - k = u$, and it contains the full positional pairing radical before passage to this quotient. Hence $\dim W = u + r_2$. For $f \in L$, the condition $I_2 P_4 = 0$ gives $I_2 f \subseteq K_0$. Consequently the operators represented by HW vanish on L. Passage from F_4 to $A(T)_4$ costs at most r_4 dimensions, and therefore

$$G \;\geq\; M + \max(0, 105 - f_4 + R/2) \qquad \text{whenever } M \leq \dim(HW). \tag{10}$$

Both inequalities concern this same prefix and this same Q.

For $1 \leq \rho < 10$ one has $0 \leq c < \rho$, and for $\rho = 10$ one has $c = 10$. A basis of ρ original factors exhibits every cross product of distinct basis factors. If a remaining factor has a nonzero coefficient at the i-th basis vector, multiplying it by that vector produces its square modulo the cross products. The omitted squares are exactly the coloop squares. Thus $f_2 = \binom{\rho+1}{2} - c$. The support rectangle bound gives

$$Z(1), \ldots, Z(10) = 0, 0, 0, 1, 1, 3, 3, 6, 9, 10, \qquad 0 \leq z \leq Z(\rho).$$

In evaluating (10), a basis of the factor span gives a template of $\binom{\rho}{2}$ distinct basis pairs and $\rho - c$ additional edges incident to an outside vertex. With independent factors the template is just the graph on the ten positions. To see why the outside construction yields a lower bound, write the coordinate images of the dual basis as $y_i + g_i$, with g_i supported on the outside positions. Each noncoloop square has initial component $2y_i g_i$; each cross product has initial component $y_i y_j$.

A disjoint union of two initial edges gives a nonzero degree-four initial component, even when one edge uses the outside vertex. Different four-sets give independent components. Counting only one dimension for each such four-set is valid for any number of outside directions.

The two input edge families can be taken to have sizes $f_2 - z$ and $u + \underline{r}_2$, since restricting $\mathrm{Ann}(P)_2$ to the support gives codimension z.

The source family may be trimmed to any claimed lower bound. A simultaneous shift $j \to i$ of the two edge families towards a smaller vertex has its disjoint-union shadow contained in the shifted original shadow. If both special vertices occur in different edges, creating the earlier edge while retaining the later vertex forces the alternative earlier edge in the second original family. Interchanging assignments gives an original witness. If only one special vertex occurs, the assertion follows on a two-point fiber. The remaining cases are unchanged. The template is shifted after its noncoloop neighbors are placed first.

The upper bound for f_4 follows by selecting j of the c coloop factors and the remaining $4 - j$ positions from the core:

$$f_4 \leq \sum_{j=0}^{\min(4,c)} \binom{c}{j} \min\left\{ \binom{10-c}{4-j},\ \dim \mathrm{Sym}^{4-j}(K^{\rho-c}) \right\}.$$

By Lemma 3.2, the external zonotopal Hilbert formula for a list of length n, rank ρ, Gale corank $d = n - \rho$ and Gale rank function r^* is

$$\sum_j f_j t^j = \sum_{S \subseteq [n]} t^{|S|}(1-t)^{d-r^*(S)}.$$

Its coefficient of t^4 gives the Gale-rank formulas below. Binomial coefficients outside their usual range are zero.

8 Three joint shadow bounds by seven diagrams

We specialize Proposition 4.1 to $\ell = 8, n = 10, p = 0$. The exact even shadow of a Ferrers diagram λ is $\mathcal{E}(\lambda) = \sum_{j=0,2,4,6,8} \sum_i (C_j(i) - C_j(i-1)) C_j(\lambda_i)$. The increment for the i-th eight-set in colex order is $\binom{8-t_i}{j-t_i}$, where the list t_i consists of $0, \ldots, 8$, then $1, \ldots, 8$, and so on down to 8. Indeed, a subset of the current eight-set is new precisely if it contains every occupied element larger than the smallest omission. There are t_i such elements. This gives all entries in the following table. The five inner columns are the individual degree contributions. Transposition preserves the sum.

row lengths	0	2	4	6	8	$\mathcal{E}$
(2)	1	980	7350	1372	2	9705
$(2, 1^{10})$	1	1428	14770	3948	12	20159
$(4^2, 1^5)$	1	1288	14595	4606	13	20503
$(19, 1^{18})$	1	1736	23380	8456	37	33610
$(10^3, 1^{11})$	1	1772	23660	8820	41	34294
$(4^{12}, 1^2)$	1	1584	22695	9872	50	34202
$(4, 2^{21})$	1	1603	22715	9275	46	33640

Orient the diagram so that its height is at least its width. At area at least two it contains (2) or its transpose, and $9705 > 210 \cdot 46$.

At area at least 46 its height is at least seven and its width is at least two, as it lies inside a 45-by-45 board. If its even shadow is at most $210 \cdot 95 = 19950$, the second row of the table rules out height eleven, and the third rules out a second row of length four. Its area is at most $10 + 9 \cdot 3 = 37$, a contradiction.

At area at least 172 the height is at least fourteen and the width at least four. If the even shadow is at most $210 \cdot 160 = 33600$, the last four rows forbid cells $(1, 19), (3, 10), (12, 4), (22, 2)$,

respectively, with indices starting at one. For example, cell $(1, 19)$ forces height at least nineteen by the orientation, while $(22, 2)$ forces twenty-two rows of length at least two and the first row has length at least four. Thus the first two rows have length at most eighteen, the next nine at most nine, the next ten at most three, and the last twenty-four at most one. Their total is at most $2 \cdot 18 + 9 \cdot 9 + 10 \cdot 3 + 24 = 171$. This proves Lemma 4.

9 Quartics with at most two independent quadratic responses

Let P be a generic matrix permanent over a characteristic-zero field, $I = \mathrm{Ann}(P)_2$, and $P_j = E_j(P)$. A useful consequence of the two-coefficient prolongation argument is the following.

Proposition. After extending to an algebraic closure, every quartic F for which $\dim(IF) \le 2$ can be written

$$F = F_0 + B, \qquad F_0 \in P_4,$$

where the quartic B uses at most two essential variables. If $IF \neq 0$, these variables can be chosen in the common linear support of the quadrics in IF. In the two-dimensional case B is either a linear combination of two fourth powers or belongs to $\langle x^4, x^3y \rangle$.

Put $R(xy)(A) = A(xy)$ and $e_x(A) = R(x^2)(A)/2$ for $A \in I$; write

$$T = (A \longmapsto AF) \in \mathrm{Sym}^2 V \otimes I^*.$$

Its coefficient span is a subspace $E \subseteq I^*$ of dimension $\dim IF$. Let $U \subset V \otimes E$ be the span of all first derivatives of T, so that U is active, meaning it is generated by the derivatives of the very quadratic tensor in question. For a linear syzygy $\sum_j d_j A_j = 0$ of I, acting on any derivative of F shows that every element of U obeys this syzygy. We use the following consequences of the grid relations.

- A nonzero simple tensor $x \otimes \xi \in U$ has $\xi \in Ke_x$. One can see this directly from the grid syzygies: choose a cell differential a with $a(x) \neq 0$, subtract a multiple of e_x to set $\xi(a^2) = 0$, then successively use $a(au) = ua^2$ on the pivot row and column and $a(ay_{ij} - u_j v_i) = y_{ij}a^2 - v_i(au_j)$ on the pivot rectangles. These kill the corresponding evaluations. Same-row or same-column products involving a pivot row or column reduce using $a(u_j u_k) = u_j(au_k)$ and $a(v_i y_{ij}) = y_{ij}(av_i)$, and their transposes. Put $R_{ij} = ay_{ij} - u_j v_i$. Pivot-row rectangles satisfy $a(u_j y_{ik} - u_k y_{ij}) = u_j R_{ik} - u_k R_{ij}$. Remote same-row products satisfy $a(y_{ij}y_{ik}) = y_{ij}R_{ik} + u_k(v_i y_{ij})$; transposition covers the column versions. Finally,

  $$a(y_{ij}y_{hk} - y_{ik}y_{hj}) = y_{hk}R_{ij} - y_{hj}R_{ik} + v_i(u_j y_{hk} - u_k y_{hj}).$$

  These exhaust the generators and make every residual evaluation zero. This argument also works coefficientwise over $K[t]/t^m$ if the pivot value at $t = 0$ is nonzero: the only denominators are powers of this unit.
- If $0 \neq C \in P_2$, projection to any two-by-two board with nonzero coefficient gives a quadratic of rank four. Therefore $R : \mathrm{Sym}^2 V \to I^*$ is injective on the quadrics supported by any at most three variables. With $\dim E = 2$ at most two lines $K\xi \subset E$ can occur in simple tensors in U: three distinct such lines would give three independent squares of nonproportional vectors in at most three variables whose images all lie in E. Each fixed $K\xi$ has fiber of dimension at most one by the same argument for two squares.
- Choose a scalar projection on E whose kernel avoids those lines. It is injective on U; call its image $D \subset V$. The projected quadratic part of T has all gradients in D, so it depends only on D. Any derivative annihilating D annihilates all of T by projection-injectivity. All the other coordinates of T thus use D as well, and activity writes U as the graph of $M : D \to D$, in a suitable basis e_0, e_1 of E. Every eigenspace has dimension at most one by the fixed-line statement.

- In a Jordan chain $Mw_j = \tau w_j + w_{j-1}$, write $w(t) = \sum w_j t^j$ and $\xi(t) = e_0 + \tau e_1 + te_1$. The coefficient at every chain position of $w(t) \otimes \xi(t)$ belongs to U. The grid-syzygy reconstruction over the truncated ring gives $\xi(t) = \lambda(t) e_{w(t)}$, with $\lambda(0) \neq 0$. Three different eigenvectors, a chain of length two plus another eigenvector, or a chain of length three would respectively put the images under R of

$$x^2, y^2, z^2; \qquad x^2, xy, z^2; \qquad x^2, xy, y^2 + 2xz$$

into the same two-dimensional E, with x, y, z independent. Each triple is independent on three variables, contradicting the injectivity of R there. Thus $\dim D \leq 2$. If $\dim E = 2$ and $\dim D = 1$, every component of T would be a multiple of one square; hence $\dim D = 2$.

For two distinct eigenvalues the entire space of tensors with derivatives in U is spanned by $x^2 \otimes e_x, y^2 \otimes e_y$, by comparing coefficients of binary quadrics. For a single length-two block, write

$$U = \langle x \otimes \xi_0,\ y \otimes \xi_0 + x \otimes \xi_1 \rangle, \quad \xi_0 = \lambda_0 e_x, \quad \xi_1 = \lambda_0 R(xy) + \lambda_1 e_x.$$

Coefficient comparison gives the tensor span of $x^2 \otimes \xi_0$ and $xy \otimes \xi_0 + \frac{1}{2} x^2 \otimes \xi_1$. Up to nonzero scalars and the first tensor, these are the responses of x^4 and $x^3 y$:

$$A(x^4) = 12 e_x(A) x^2, \qquad A(x^3 y) = 6 e_x(A) xy + 3R(xy)(A) x^2.$$

The one-coefficient case gives simply the response of a fourth power: every derivative of T is a simple tensor with the same coefficient line and the first bullet makes the derivative directions proportional. At coefficient dimension zero put $B = 0$. Subtracting the appropriate B leaves a quartic annihilated by I, hence a member of P_4, by the quadratic presentation in Proposition 1.1. Any variable needed by B occurs in the joint support of its response: in the semisimple case the two nonzero square responses select both eigenvectors; in the nontrivial Jordan case the xy response selects the second one. This proves the proposition.

The positional consequence

Let $L_1, \ldots, L_{10}$ span a nine-dimensional U, and

$$V_4 = \langle L_{i_1} L_{i_2} L_{i_3} L_{i_4} : i_1 < i_2 < i_3 < i_4 \rangle.$$

Lemma 1.3, applied to the full space of four-position products, gives at least $9 - (10 - 4) = 3$ essential variables in each nonzero member. The same assertion holds after base change and in factor rank ten.

Suppose that $0 \neq F \in V_4$ and $\dim IF \leq 2$. Apply the proposition after base change. The variables in B belong to U because the quadrics IF live there, so $F_0 = F - B$ is supported on U. Every nonzero quartic target has at least sixteen essential variables, by projection to a four-by-four board supporting a nonzero coefficient. Hence $F_0 = 0$, and the positional observation gives $F = 0$, a contradiction. Thus every nonzero $F \in V_4$ has $\dim IF \geq 3$ over every characteristic-zero field. No condition on the unique circuit length was used.

9.1 The symmetric-matrix estimate

Assume the factor rank is at least eight. Give the positions additive integer weights separating all pair weights and all four-set weights (distinct powers of two suffice to break ties). The leading edges of H form a graph G on the ten positions, and the leading edges of W form a family D. Its cardinality is $s = \dim W$. The forbidden graph $E = \binom{[10]}{2} \setminus G$ consists of the pivots of $Z = H^\perp$ in the opposite order. For $A \in G$, $C \in D$ with $A \cap C = \varnothing$, the leading terms of the two corresponding forms multiply to a nonzero leading term indexed by $A \cup C$. Thus

$$\dim(HW) \geq |N(G, D)|, \qquad N(G, D) = \{A \cup C : A \in G,\ C \in D,\ A \cap C = \varnothing\}.$$

Here $I = \operatorname{Ann}(P)_2$. Write $V_j = \operatorname{im}\sigma_j$, $f_j = \dim V_j$, and assume $W \subset \psi(S_2)$. The space $W^\perp$ contains $\ker\sigma_2$, so $K' = \sigma_2(W^\perp)$ has dimension $k = f_2 - s$. If $F_0 \in (HW)^\perp$, Leibniz's rule gives $I\sigma_4(F_0) \subset K'$. On putting $\mathcal{L} = \{F \in V_4 : IF \subset K'\}$, we obtain

$$\dim(HW) \geq f_4 - \dim\mathcal{L}, \qquad \dim\mathcal{L} \leq \binom{k}{2}. \tag{11}$$

We give the physical ingredients of the second inequality explicitly. For $t = \dim(K' \cap P_2)$, the symmetric bilinear form $B_F(A, A') = AA'F$ on I factors through $(K'/(K' \cap P_2))^*$, of dimension $k - t$. The required small-support facts are as follows. On a support of dimension at most ten, the quartics annihilated by I^2 form a space of dimension at most ten, and every nonzero such quartic has $\dim IF \geq 6$. Also, for a quartic with $\dim IF = 1$ there are $F_0 \in P_4$, $\ell \neq 0$ and $c \neq 0$ with $F = F_0 + c\ell^4$, $IF = K\ell^2$. These follow from Sections 6 and 9. In the first input the physical weight calculation is especially small: on a coordinate board with $m \leq 10$ cells the dimension is m times the number of 3×3 rectangles plus twice the number of 2×4 and 4×2 rectangles. In a 3×3 block the relevant forms are multiples of its cubic permanent by linear forms, giving nine independent derivatives under I. In the 2×4 case, differentiating in two of the four majority-row columns and restricting to the four columns isolates the six complementary two-column subpermanents. Row and column weight degeneration bounds an arbitrary support by a coordinate board.

If $t \leq 5$, $F \mapsto B_F$ is injective by the six-dimensional lower bound. If $t \geq 6$, the kernel has dimension at most ten, giving at most $\binom{k-5}{2} + 10 \leq \binom{k}{2}$ dimensions. For $1 \leq t \leq 5$, the target symmetric matrices alone give the desired bound. For $t = 0$ their rank equals $\dim IF$, and no nonzero $F \in V_4$ can have rank one: $\ell^2 \in \operatorname{Sym}^2 U$ implies $\ell \in U$, while a nonzero $F_0 \in P_4$ has at least sixteen essential variables by restriction to a nonzero 4×4 subpermanent. Finally, in factor rank $\rho \geq 8$ each nonzero element of V_4 has at least $\rho - 6 \geq 2$ essential variables by Lemma 1.3. The projective variety of rank-one symmetric matrices in k variables has dimension $k - 1$; an injective linear space avoiding it has vector dimension at most $\binom{k}{2}$ (the cases $k \leq 1$ follow directly). This proves (11).

We also use the following improvement. If every nonzero $F \in V_4$ has $\dim IF \geq 3$, then $\dim\mathcal{L} \leq \binom{k-1}{2}$ for $k \geq 3$. For $t = 0, 1, 2$, the nonzero symmetric matrices have rank at least $3 - t$. The locus of symmetric matrices of rank at most $2 - t$ in $k - t$ variables has codimension $\binom{k-1}{2}$ (for $t = 2$ use the dimension of the whole target). For $3 \leq t \leq 5$ the target has dimension at most $\binom{k-2}{2}$. For $t \geq 6$, the preceding kernel bound gives $\binom{k-5}{2} + 10 \leq \binom{k-1}{2}$. For $k < 3$ the space is zero. All statements used for the rank-locus arguments hold after extending scalars to an algebraic closure. Kernel and image dimensions of the original linear maps descend to K.

9.2 Simultaneous limits of factors and allowed quadrics

Choose a basis of original factors x_i and order it with weights 3^i. The initial product is a nonzero scalar multiple of x^α with $\alpha_i \geq 1$. Transport the restriction $J \subseteq \operatorname{Sym}^2 U^*$ of the physical annihilator by the contragredient weights. Its limit J_0 is a monomial subspace of codimension at most $Z(\dim U)$. Echelon representatives lift every linear combination of its pivots to a member of the transported J with that limit. The matrix $(DEQT)_{D,E}$ on the fixed degree-four operator space has rank equal to the central apolar rank even when the apolar Hilbert function jumps. A nonzero minor at (x^α, Q_0) therefore lifts to a nonzero minor polynomial on the original J and hence on the original physical annihilator. Include these polynomials in the simultaneous choice of Q.

Lemma 5 (Two repeated variables in a nonuniform core)**.** *Let a finite list of nonzero linear factors have a nonuniform matroid after deletion of all coloops. A basis of original factors and an ordering of its coordinate weights can be chosen so that the initial product has at least two distinct variables of exponent at least two.*

Proof. Let r be the core rank. A circuit of size at most r exists, and a basis of its span extends by original factors to a hyperplane of rank $r-1$. At least r positions lie inside, counting the extra circuit element. At least two lie outside, since otherwise the unique exterior element would be a coloop. Complete an original basis of the hyperplane by one exterior factor. Give this exterior coordinate the lowest weight, so it is initial in every exterior factor. At least r interior factors use only $r-1$ independent coordinates, and hence another initial variable repeats. Adjoin the deleted coloops to the basis. Each is a separate factor of exponent one, and the core is supported on the first r coordinates. These choices are compatible with the simultaneous operator limit above. □

10 Central ranks on six, seven and eight factor directions

The local statement

Let K be any field of characteristic zero, and give the variables degree one. Put

$$A = K[x, y, z_1, \ldots, z_6]/(x^3, y^3, z_1^2, \ldots, z_6^2).$$

There are 30 nonzero quadratic monomials, comprising the three monomials in x, y, twelve mixed monomials with one z_i, and fifteen $z_i z_j$ with $i < j$. Let $J_0 \subseteq A_2$ contain every quadratic monomial except for a specified set of at most six.

Proposition. Some $Q \in J_0$ satisfies

$$\operatorname{rank}(Q : A_4 \longrightarrow A_6) \geq 104.$$

Both source and target have dimension 113, by the Hilbert series $(1+t+t^2)^2(1+t)^6$.

Call a quadratic monomial *available* when it is outside the specified missing set. The proof only uses linear combinations of available monomials.

Three constructions cover all missing sets

In the graph on the six variables z_i, join p, q when pq is available. Renaming the six variables, and exchanging x and y, preserves all hypotheses.

1. If the graph has a perfect matching M, and either xy or both x^2, y^2 are available, take

$$Q = xy + \sum_{pq \in M} pq \quad \text{or} \quad Q = x^2 + y^2 + \sum_{pq \in M} pq,$$

   using whichever option is available.

2. It is enough that y^2, xp, xq, rs, tu are available, for six distinct single variables p, q, r, s, t, u. Take

$$Q = x(p+q) + y^2 + rs + tu.$$

3. If x^2, xy, y^2 are all missing, at most three further monomials are missing. Select an available edge $z_5 z_6$ such that on the other four vertices at most one mixed monomial xz_i, yz_i is missing. To see this, if three mixed monomials are missing, all pure-single edges are available, and two appropriately chosen endpoints cover at least two missing mixed monomials. If two are missing, at most one pure-single edge is missing, so select an available edge incident to one affected vertex. If at most one mixed monomial is missing, any available edge works. After labeling the remaining four vertices, and exchanging x, y if necessary, the available mixed monomials support

$$Q = q + z_5 z_6, \qquad q = \sum_{i=1}^{4} \ell_i z_i, \qquad (\ell_1, \ell_2, \ell_3, \ell_4) = (x, y, x+y, x-y).$$

   Indeed, the two permitted zero coefficients of this choice are yz_1 and xz_2, and either can contain the single missing mixed monomial after labeling.

To see that these constructions exhaust the possibilities, first suppose the graph on the six single variables has no perfect matching. The complete graph on six vertices decomposes into five perfect matchings, so at least five of the missing monomials are pure-single edges. A direct one-factorization labels the vertices by ∞ and $\mathbb{Z}/5$: in matching a, use ∞a and $(a-1)(a+1)$, $(a-2)(a+2)$. At most one missing monomial is elsewhere. Consequently, after exchanging the double variables if needed, x is joined to all singles and y^2 is available. At least nine single edges remain, hence at least two are disjoint: a pairwise intersecting family in a simple graph is a star or is contained in a triangle. Use the two disjoint edges for rs, tu and the leftover vertices for p, q in construction 2.

Now suppose the single graph has a perfect matching but construction 1 is unavailable. Then xy and at least one of x^2, y^2 are missing. If both squares are missing, construction 3 applies. Otherwise label the available square y^2. At most four other monomials are missing. If x is joined to all but at most one single vertex, some matching edge has both ends joined to x; use its ends as p, q in construction 2. If x misses at least two singles, then at most two single edges are missing, and x still has at least two neighbors. Pick any two as p, q. The remaining four vertices have an available perfect matching because K_4 decomposes into three pairwise edge-disjoint perfect matchings. This again gives construction 2.

Ranks for the first two constructions

A homogeneous string for a degree-two operator consists of one basis vector in each of grades $d, d+2, \dots, d+2a$, with each mapped to the next, and the last mapped to zero. Its center is $d+a$. Two strings with centers c, c' tensor under the sum of raising operators into strings all centered at $c+c'$. Indeed, over a characteristic-zero field use the usual divided string basis for $\mathfrak{sl}_2$. The Clebsch–Gordan summands start in degrees $d+d'+2j$, have $a+b-2j+1$ positions, and so have the asserted center, for $0 \le j \le \min(a,b)$. In particular, for strings centered in degree c, the map from grade $c-1$ to grade $c+1$ is an isomorphism. In a given parity class, maps toward the center are injective and maps away from it are surjective.

On two square-zero single variables p, q, the operator pq has the string $1, pq$ and two singletons p, q, all centered in grade 1. On $K[x,y]/(x^3, y^3)$ multiplication by xy decomposes into the diagonals of the 3×3 exponent grid, each centered in grade 2. For x^2+y^2 the same center follows by using in each single double-variable algebra the string $1, x^2$ and singleton x. Therefore construction 1 has strings centered in grade 5 and rank 113.

For construction 2, the factor $K[p,q]/(p^2, q^2)$ decomposes under $p+q$ as $\langle 1, p+q, pq\rangle$ and $K(p-q)$. On the first part tensored with $K[x]/(x^3)$, multiplication by $x(p+q)$ has the same degree-two diagonal strings on a 3×3 grid, centered in degree 2. On $(p-q)K[x]/(x^3)$ it is zero, leaving one singleton in each of grades 1, 2, 3. The complementary algebra

$$C = K[y, r, s, t, u]/(y^3, r^2, s^2, t^2, u^2)$$

has raising operator $y^2+rs+tu$, centered strings of grade 3, and Hilbert function $(1, 5, 11, 14, 11, 5, 1)$. All products of properly centered strings are centered at 5. Of the three singletons, the grade-2 singleton also centers at 5, and the grade-3 singleton yields $C_1 \to C_3$, which is injective. The grade-1 singleton yields the surjection $C_3 \to C_5$, with kernel of dimension $14 - 5 = 9$. Thus construction 2 has rank $113 - 9 = 104$.

Rank for the third construction

Set $M = K[x, y, z_1, \dots, z_4]/(x^3, y^3, z_1^2, \dots, z_4^2)$. Its socle bidegree is $(4,4)$ for the degrees in x, y and the singles; $\dim M_4 = 36$. In the last two variables the two degree-1 singletons contribute twice the rank of $q : M_3 \to M_5$. On the string $1, z_5 z_6$ the matrix is

$$M_4 \oplus M_2 \longrightarrow M_6 \oplus M_4, \qquad (u, v) \longmapsto (qu, u + qv).$$

Eliminating u yields the rank formula

$$\operatorname{rank}(Q : A_4 \to A_6) = 36 + \operatorname{rank}(q^2 : M_2 \to M_6) + 2\operatorname{rank}(q : M_3 \to M_5).$$

For $q : M_3 \to M_5$, the four bidegree blocks in increasing x, y-degree have ranks $2, 12, 12, 2$. The first follows since the four ℓ_i span two dimensions, the last by perfect-pairing duality. For the $(1, 2)$ block, write an element as $\sum_{i<j} u_{ij}(x, y)z_iz_j$ with u_{ij} linear. In x, y-degree two no truncation occurs. Its product by q vanishes exactly when

$$\ell_i u_{jk} + \ell_j u_{ik} + \ell_k u_{ij} = 0 \qquad (\{i, j, k\} \subset \{1, 2, 3, 4\}).$$

Dividing through by $\ell_i\ell_j\ell_k$, each of the four triangle sums of $h_{ij} = u_{ij}/(\ell_i\ell_j)$ is zero. Adding two triangle sums and subtracting the other two shows $2(h_{ij} - h_{kl}) = 0$ for each pair of opposite edges. Because the four ℓ_i are pairwise coprime, $\ell_i\ell_j \mid u_{ij}$, forcing $u_{ij} = 0$. The block therefore has rank 12. Its Gorenstein dual is the $(2, 1)$ block, which also has rank 12. This proves $\operatorname{rank} q = 28$ in the required degrees.

For $q^2 : M_2 \to M_6$, the extreme bidegree blocks have rank 3: the products of two distinct ℓ_i span the quadrics in x, y, and the other block is dual. For the middle block, write its input basis as (xz_i, yz_i) and output basis as $(x^2yz_{[4]\setminus j}, xy^2z_{[4]\setminus j})$, ordered by i, j. After removing the scalar 2, the diagonal blocks vanish, and block (j, i) is multiplication by $\prod_{k\notin\{i,j\}} \ell_k$ from degree 1 to degree 3 in $K[x, y]/(x^3, y^3)$. Its first six rows and columns are

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

The leading 4×4 block has determinant 1 and squares to minus the identity. Its Schur complement is $\left(\begin{smallmatrix} 0 & -2 \\ 2 & 0 \end{smallmatrix}\right)$, with determinant 4. Consequently the middle block has rank at least 6, and the rank in construction 3 is at least $36+(3+6+3)+2(28) = 104$. Keeping the entire eight-dimensional middle matrix gives determinant 9 after the scalar 2 is removed, and rank 106, though the six-dimensional minor already proves the proposition.

The other rank-eight exponent type

The same construction treats the second rank-eight monomial limit. Put

$$A' = K[x, z_1, \ldots, z_7]/(x^4, z_1^2, \ldots, z_7^2).$$

Its degree-four dimension is 98. Among its 29 effective quadratic monomials at most six may be missing. Then some allowed Q has rank at least 74 from degree four to degree six.

If x^2 is available, one of the seven edge-disjoint near-perfect matchings of K_7 survives the six missing positions. For odd v, label K_v by $\mathbb{Z}/v$; the matching leaving a isolated uses the pairs $(a - j, a + j)$, $1 \le j \le (v - 1)/2$. Take $Q = x^2$ plus the three matching-edge monomials. On the three pairs all strings are centered in grade 3, with Hilbert function $H_i = \binom{6}{i}$. The two strings for x^2 lie in grades $(0, 2)$ and $(1, 3)$, and the unmatched variable supplies singletons of grades 0 and 1. The only kernel from degree 4 to degree 6 occurs for the first string and the grade-0 singleton, of dimension $(H_4 + H_2) - (H_6 + H_4) = 14$. Thus the rank is 84.

If x^2 is missing, there are at most five other missing positions. Choose two singles p, q joined to x and two disjoint available edges rs, tu on the other five vertices. This selection always exists. If at least one mixed position is missing, at most four pure-single edges are missing. After choosing any two of the at least two neighbors of x, one of five edge-disjoint near-perfect matchings of the

remaining K_5 survives. If no mixed position is missing, pick any two disjoint available edges first (indeed one near-perfect matching of K_7 survives), then choose two of the other three as p, q. Set $Q = x(p+q) + rs + tu$, leaving the seventh single variable unused.

On $K[x,p,q]/(x^4,p^2,q^2)$, the strings of $x(p+q)$ have the following grades. The first six are diagonals of the 4×3 exponent grid for x and $p+q$; the last four arise from $(p-q)K[x]/(x^4)$. For a grade list S, the other two pairs have centered strings and Hilbert function $H_i = \binom{4}{i}$, taken zero outside $0 \le i \le 4$. With the unmatched variable in grade $e = 0, 1$, the kernel contribution is $\max(0, \sum_{d \in S}(H_{4-d-e} - H_{6-d-e}))$. Indeed the entire tensor summand is centered, so the indicated map has maximal rank.

Grades in the string	Kernel when $e=0$	Kernel when $e=1$
$(0,2,4)$	1	0
$(1,3)$	4	0
(2)	5	0
$(1,3,5)$	0	0
$(2,4)$	0	0
(3)	0	0
(1)	4	5
(2)	5	0
(3)	0	0
(4)	0	0

The kernel has dimension 24, yielding rank $98 - 24 = 74$.

All rank-six and rank-seven monomial profiles

For the other eight exponent partitions, three effective quadratic coordinates may be missing. For a positive exponent vector α of total ten, write $A_\alpha = K[a,b,c,\ldots]/(a^{\alpha_1+1}, b^{\alpha_2+1}, \ldots)$. Variables are labeled by the letters in the displayed exponent order. The rank in this section always means the rank from degree four to degree six in this algebra.

Four disjoint supports

In each row the four displayed quadrics have pairwise disjoint supports in the effective quadratic monomial basis. Thus at least one survives any three missing coordinates. Their ranks appear in the same order.

Exponents	Four quadrics	Ranks
$(3,3,1^4)$	ab, $(a+c)^2+b^2$, $af+be+cd$, $ad+bf+ce$	$36,36,33,33$
$(3,2,2,1^3)$	$ac+b(d+e)$, $ab+c(d+e)$, $a^2+bc+de$, $a(d+e+f)+b^2+c^2$	$44,44,46,45$
$(2,2,2,2,1,1)$	$ad+bc+ef$, $a(c+e+f)+bd$, $a^2+b(e+f)+cd$, $ab+c(e+f)+d^2$	$61,54,57,57$
$(2,2,2,1^4)$	$a(f+g)+bc+de$, $ac+b(e+g)+df$, $ab+c(e+f)+dg$, $a^2+b^2+c(d+g)+ef$	$77,77,77,77$

Four conditional constructions

The single variables (exponent one) may be relabeled for each displayed choice. Each list is in the order used in its coverage proof.

Exponents	Quadrics	Ranks
$(5,1^5)$	a^2, $a(b+c+d)$	$30,20$
$(4,2,1^4)$	a^2, ab, $a(c+d+e)+b^2$, $a(c+d+e)+bf$	$36,30,30,30$
$(4,1^6)$	a^2, $a(b+c+d+e+f)$, $a(b+c+d+e)+fg$	$50,42,42$
$(3,2,1^5)$	$ab+cd+ef$, $a^2+bc+de+fg$, $b^2+a(c+d+e)+fg$, layered operator below	$62,62,60,64$

- In type $(5,1^5)$, when a^2 is missing at least three of the five single vertices remain joined to a.
- In type $(4,2,1^4)$, take a^2 or ab whenever available. If both are missing, at most one other position is missing. When b^2 survives, at least three singles remain joined to a. Otherwise all four singles are joined to both double variables, which gives the last construction.
- In type $(4,1^6)$, when a^2 is missing at most two of the six mixed positions are missing. If at least five survive use five of them. When exactly four survive the only other missing coordinates are the two mixed positions, so the edge between the two remaining singles survives.
- In type $(3,2,1^5)$, when ab is available one of the five edge-disjoint near-perfect matchings of the five singles survives the three missing coordinates. Use its two edges for the first choice. If ab is missing but a^2 survives, supplement each of the same five near-perfect matchings by the edge from b to its unmatched vertex. These five triples have disjoint supports and at most two other holes remain. This supplies $a^2+bc+de+fg$ after relabeling. If both ab, a^2 are missing while b^2 is available, only one further coordinate can be missing. Use $b^2+a(c+d+e)+fg$, choosing a triple that avoids a missing mixed edge, or choosing its complementary pair to avoid a missing single edge.

If a^2, ab, b^2 are all missing, every mixed edge is available. Write $B_5 = K[s_1,\dots,s_5]/(s_i^2)$, take $L=\sum s_i$, $M=\sum t_i s_i$ for five distinct scalars t_i, and use $Q=aL+bM$. On the five blocks indexed by the input degree $0,1,2,3,4$ in a,b, the ranks are $2,15,30,15,2$. The outside pair and next pair are dual under the socle pairing. In input core degree zero, the two functionals are the independent L, M on $(B_5)_4$, hence give rank two. In core degree two use the bases (a^2, ab, b^2) and (a^3, a^2b, ab^2). The three-by-three block matrix is triangular with diagonal $L:(B_5)_2\to(B_5)_3$, an isomorphism by the degree-one string rule. This gives rank thirty.

In core degree one the transposed kernel consists of triples $x,y,z\in(B_5)_1$ satisfying $Lx+My = Ly+Mz=0$. A solution of the first equation obeys $x_i+x_j+t_iy_j+t_jy_i=0$ for all distinct indices. Subtracting the equations for ik, il, jk, jl with alternating signs gives

$$\frac{y_i-y_j}{t_i-t_j} = -\frac{y_k-y_l}{t_k-t_l} \qquad (\{i,j\}\cap\{k,l\}=\varnothing).$$

The graph on the pairs of a five-element set, with adjacency given by disjointness, is connected, since two meeting pairs have a common disjoint pair. It has the odd cycle $12, 34, 15, 23, 45, 12$. Thus all the displayed slopes vanish. All y_i equal some d, and the original equations force $x_i=-dt_i$. The solution pairs are the multiples of $(M,-L)$. The second equation consequently forces y to be a multiple of both independent forms L, M, so $x=y=z=0$. This proves rank fifteen and finishes the rank-sixty bound for every hole pattern.

Evaluating the displayed ranks

The following finite binomial formula computes every entry of the last column in both tables. It also gives a convenient paper check where some of the strings have different centers.

For a group of variables of exponents β, put

$$h_i = [t^i]\prod_{b\in\beta}(1+t+\cdots+t^b), \qquad D=|\beta|_1.$$

The sum of the variables acts with h_i-h_{i-1} strings of degrees $i, i+1,\dots,D-i$ for $0\le i\le\lfloor D/2\rfloor$ and $h_{-1}=0$. This is the degree-one Clebsch–Gordan rule on each truncated variable. A squared sum splits each such string into the even and odd subsequences. For the product of the sums on two disjoint groups, take one degree-one string from each group. Their rectangular exponent grid splits into diagonals with step $(1,1)$; these are the degree-two strings of the product. For an unused variable of cap b, use $b+1$ one-position strings in degrees $0,1,\dots,b$. All remaining sums in the two tables involve disjoint groups of variables.

For explicit arithmetic record a degree-two string as (d, m) if it starts in degree d and has m nonzero steps. When adding the operators on two independent groups replace the pair of strings $(d, m), (e, n)$ by

$$(d + e + 2j, m + n - 2j) \qquad 0 \le j \le \min(m, n).$$

A resulting string contributes one to the indicated rank precisely when $d \le 4$, $d + 2m \ge 6$, and d is even. All other strings contribute zero. These formulas only involve strings on groups of at most five single variables.

For example, the second witness $a(b + c + d)$ in type $(5, 1^5)$ has a separate degree in a and the three chosen singles. In their square-zero algebra multiplication by $b + c + d$ on successive degrees has ranks $1, 3, 1$. Fix the unused two singles with total degree $j = 0, 1, 2$, of multiplicity $\binom{2}{j}$. For input degree four, all three single-degree ranks contribute when $j = 0, 1, 2$ (the needed power of a lies between 0 and 4). Hence the rank is $(1 + 2 + 1)(1 + 3 + 1) = 20$. For an example with centers, the witness $a(f + g) + bc + de$ for $(2, 2, 2, 1^4)$ has on a, f, g the five diagonals of the 3×3 grid, all centered at 2, and singletons in grades 1, 2 and 3 coming from $(f - g)K[a]/(a^3)$. On the remaining b, c, d, e all strings have center 3, with Hilbert coefficients $(1, 4, 8, 10, 8, 4, 1)$. The only kernel has dimension $10 - 4 = 6$; the input dimension $[t^4](1 + t + t^2)^3(1 + t)^4 = 83$ gives rank 77.

The resulting universal lower bounds in exponent order are

Factor rank	Exponent types	Lower ranks given here
6	$(5, 1^5), (4, 2, 1^4), (3, 3, 1^4), (3, 2, 2, 1^3), (2^4, 1^2)$	$20, 30, 33, 44, 54$
7	$(4, 1^6), (3, 2, 1^5), (2^3, 1^4)$	$42, 60, 77$
8	$(3, 1^7), (2^2, 1^6)$	$74, 104$

10.1 The weight of a short circuit

Lemma 2.7, applied to this same Q with ten factors, gives $w \ge 28, 21, 6$ for circuits of sizes $2, 3, 4$, respectively. These inequalities allow further dependencies among all outside positions. In particular $w < 5$ implies girth at least five.

11 At most six factor directions

In the notation of the decic filtration let k be the dimension of the old quadratic intersection, u the width, L the old quartic intersection, and R the rank of the common physical quadratic from degree four to six. The gross increment is $G = 105 + R/2 - \dim L$, and the net increment satisfies $g \ge G - \delta$, $0 \le \delta \le u$. If the ten factors span at most six directions, then

$$G \ge 14u/3.$$

This also gives absolute estimates at small widths.

Restrict to the polynomial operator ring on the factor span. Let H be the restricted physical quadratic annihilator. The eight-cell threshold in Section 6 gives $H^2 = \mathrm{Sym}^4$. Every element of L maps under H into the k-dimensional old quadratic space. Its symmetric form on H therefore factors through the restriction of H to that old space, of rank at most k. The symmetric form determines the quartic by $H^2 = \mathrm{Sym}^4$. In particular $\dim L \le \binom{k+1}{2}$. The potential old and new quadratic directions satisfy $k + u \le \binom{\rho+1}{2}$, where ρ denotes factor-span dimension.

For $\rho \le 5$, every quartic lies in a space of dimension at most 70. Thus $G \ge 35$, enough for $u \le 7$. At $8 \le u \le 15$ we have $k \le 7$, hence $G \ge 105 - 28 = 77 \ge 14u/3$. At widths one through five, $g \ge 35 - u \ge 30$.

For $\rho = 6$ we have $R \ge 20$ by the rank-six monomial constructions in the central-rank section together with the simultaneous factor and operator degeneration. Also H has codimension at most three by the six-cell rectangle bound: two distinct rectangles on six cells must share two

adjacent cells, giving a two-by-three board with at most three rectangles. Let W be the quadratic subspace annihilating the old intersection in the polynomial ring. It has dimension $21 - k \geq u$. The products HW annihilate L. As the quartic polynomial layer has dimension 126,

$$G \geq \dim HW - 11.$$

For $1 \leq u \leq 9$, use a subspace of W of dimension u. Macaulay's lex-segment theorem [9] bounds its full two-step polynomial shadow below by that of the lexicographically first u quadrics in six variables. Restricting the multiplier space from all quadrics to H costs at most $3u$ dimensions. The elementary counts are

u	1	2	3	4	5	6	7	8	9
Full polynomial shadow	21	36	46	52	55	56	71	81	87
Lower bound for $\dim HW$	18	30	37	40	40	40	50	57	60

For the first six columns the lex family is $x_1\langle x_1, \ldots, x_u\rangle$; the full count is $56 - \binom{8-u}{3}$. The next three input quadrics are x_2^2, x_2x_3, x_2x_4. After the first six, they add respectively 15, 10, 6 new quartics avoiding x_1, by counting the complementary two factors in nondecreasing order. In column six use the dimension-four subfamily to retain 40. Subtracting 11 from the last row gives $G \geq 14u/3$. At widths one through five, subtracting a further u gives $g \geq 6, 17, 23, 25, 24$, respectively.

For $10 \leq u \leq 21$, we have $G \geq 115 - \binom{22-u}{2}$. The convex function $\binom{22-u}{2} + 14u/3$ has values $338/3$ and 98 at the two endpoints, both less than 115. Thus $g \geq 11u/3$. At widths two through five the preceding estimates also imply the respective absolute bounds $11, 13, 15, 17$.

The absolute bound at unit width

For factor rank six and $u = 1$ we prove $g \geq 9$. If there is a coloop, choose it as one basis variable and all other basis variables from the remaining factors. Every four-position product has degree at most one in the coloop variable. It follows that $h_4 \leq \binom{9}{4} - \binom{7}{2} = 105$. The central estimate $R \geq 20$ yields $G = 105 + R/2 - \dim L \geq 10$.

Suppose there is no coloop. If the list is uniform, Lemma 3.3 gives $f_4 = \binom{9}{4} = 126$. The nondegenerate positional form gives $R \geq 2f_4 - 210 = 42$. Otherwise Lemma 5 yields a monomial limit with at least two exponents above one. Among the rank-six profiles in Section 10, the only one with a central bound below 30 is $(5, 1^5)$, which has just one such exponent. Therefore $R \geq 30$ in every coloop-free case. The rank condition is imposed on the same physical Q as the other central conditions. At $u = 1$ choose any nonzero quadric in the polynomial annihilator of K_0. Its products with the restricted physical quadratic space have dimension at least $21 - 3 = 18$, by injectivity of multiplication in the polynomial ring. Thus $\dim L \leq 126 - 18 = 108$, $G \geq 105 + 15 - 108 = 12$, and $g \geq 11$. In particular the common bound $g \geq 9$ holds at unit width in rank six.

12 Seven factor directions

Write c for the number of coloops, $a = 28 - c$, $k = \dim K_0 \leq a - u$, and $A = 105 + R/2$. The two common estimates are $G = A - \dim L$ and $G \geq \dim(HW) + \max(0, A - f_4)$. The two seven-dimensional diagrams in Appendix A give $P_7(c + u)$ and $B_c(u)$, respectively. There are 210 ordinary quartics in seven variables. Thus the corresponding bounds for G are $A - 210 + P_7(c + u)$ and $B_c(u) + \max(0, A - f_4)$. The additional bounds there are $A - 114$ at $(c, u) = (2, 5)$ and $A - 129 + v(u)$ for $c = 3$, where $v(1) = 11, v(2) = 18, v(u) = 25$ for $u \geq 3$. The square of the physical quadratic space spans all quartics on seven variables by Section 6, giving also $\dim L \leq \binom{k+1}{2}$.

The only remaining physical inputs here are central ranks, the one-position template, square saturation, and the light-girth exclusions. The upper bounds for f_4, listed in increasing coloop

count, are

$$(210, 182, 155, 129, 104, 80, 57).$$

They follow by splitting the coloops and choosing the smaller of the subproduct count and the full symmetric-power dimension in the core. For arbitrary weights use central ranks $(60, 60, 60, 42, 42, 42, 42)$. To see the first three, remove the coloops. If the core is nonuniform, Lemma 5 gives exponents $(3, 2, 1^5)$ or $(2^3, 1^4)$. Their common physical rank is at least 60 by the symbolic witnesses in the central-rank constructions. For a uniform core of rank $7 - c$, Lemma 3.3 gives all its position dimensions. Tensoring the coloops gives $f_4 = 210, 182, 155$ for $c = 0, 1, 2$, respectively, and $R \geq 2f_4 - 210 \geq 100$. For every rank-seven list the three possible monomial exponent profiles give at least 42, including $(4, 1^6)$. At weight less than five the short-circuit weight bounds force primal girth at least five, and $c \leq 2$. Indeed $c = 3$ leaves a rank-four uniform seven-point core. The 35 independent indexed four-products on that core have actual fourth dimension at most the third symmetric-power dimension 20, giving a radical of dimension at least 15, which survives after tensoring the coloops and is paid by the weight. Larger c would leave a core of rank at most three and a circuit of size at most four.

Here is a direct light-rank count with no list of Gale multiplicity profiles. In the rank-three dual, discard the c zero columns; the $n = 10 - c$ remaining columns are projective point classes with weights m_i. Each projective line contains total weight at most $q = 5 - c$, since its complement contains a primal circuit. Lemma 3.2 gives

$$210 - f_4 = \sum_{|S|=3} (3 - r(S)) - \sum_{|S|=2} \binom{3 - r(S)}{2} + \sum_{|S|=1} \binom{3 - r(S)}{3}$$

for dual ranks, including its zero columns. Put $P = \sum_i \binom{m_i}{2}$. Among subsets with no zero columns, the loss equals

$$D = \sum_i \left[\binom{m_i}{2} (n - m_i - 1) + 2\binom{m_i}{3} \right] + T,$$

where T counts collinear triples from three distinct point classes, with positional multiplicity. A pair of classes i, j has at most $q - m_i - m_j$ weighted extensions, so $3T \leq \sum_{i<j} m_i m_j (q - m_i - m_j)$. Substitution and $n - q = 5$ give

$$6D \leq (q - 3)n^2 + 7n + \sum_i m_i^2 (m_i - 1).$$

For $c = 0$, if every weight is at most two, the final sum is at most 20; if the maximum is three, at most one triple class and three double classes give at most 30. Thus $D \leq 50$. A weight four leaves six singletons. No triple of distinct classes through that heavy point is collinear. At any singleton, its five other singleton neighbors contribute at most $\binom{4}{2} + \binom{1}{2} = 6$ pairs on pencil lines. Hence $T \leq 6 \cdot 6/3 = 12$, and the repeated-class terms total 38; again $D \leq 50$.

For $c = 1$, subsets meeting the zero column add $28 + P$. A weight three forces the others to be singletons, giving $P = 3$ and $\sum m_i^2 (m_i - 1) = 18$. If all weights are at most two there are at most four double classes, giving $P \leq 4$ and moment at most 16. Thus the integral loss is at most 58. For $c = 2$, at most one weight is two and all others are one. The displayed moment bound gives $D \leq 10$. The contributions meeting zero columns are $2n + 2(\binom{n}{2} + P) - 3 - 2n + 2 = 55 + 2P \leq 57$. The total loss is at most 67. We have proved $f_4 \geq 160, 152, 143$, whence the same common quadratic gives $R \geq 2f_4 - 210 \geq 110, 94, 76$, respectively.

For clarity, the following table supplies intervals and endpoint margins for the two rates. Put $b = 13$ without a weight restriction and $b = 14$ when $w < 5$. For the small interval $1 \leq u \leq l$ take the positional bound $B_c(u) + \max(0, A - f_4^+)$, where f_4^+ is the displayed upper bound. Use $A - 129 + v(u)$ for $c = 3$, and $A - 114$ at $(c, b, u) = (2, 14, 5)$. Use the polynomial bound $A - 210 + P_7(u + c)$ on $l < u < t$, and the symmetric bound $A - \binom{a-u+1}{2}$ on $t \leq u \leq a$. The

columns are lower bounds for $3G - bu$.

c	b	l	t	1 to l	$l+1$ to $t-1$	t	a
0	13	8	18	1	0	6	41
0	14	6	16	10	4	22	88
1	13	7	17	5	13	19	54
1	14	6	15	3	34	12	78
2	13	5	15	1	3	12	67
2	14	5	15	6	21	21	65
3	13	5	15	1	12	18	53
4	13	5	13	1	33	11	66
5	13	10	11	8	−−	1	79
6	13	9	10	90	−−	14	92

The first margin follows by substitution in the short list B_c of the two seven-dimensional diagrams in Appendix A (zero outside its stated range). For the middle margin put $s = u + c$. Each used interval starts at $s = 7, 8, 9$ or 10, and ends by 17. The values of $3P_7(s) - 13s$ at those four starts are $161, 202, 225, 233$ and are at least 244 for $11 \le s \le 17$. For $b = 14$ the values are $154, 194, 216, 223$ and the subsequent values are at least 233. Its minimum on each interval is therefore its left value. On the last interval the bound is concave in u, so the two displayed endpoint margins prove the claim throughout.

For the five absolute widths, the positional bound at $u = 1$ may also count the good edges disjoint from the sole source edge. There are at least $\binom{6}{2} - c - 3 = 12 - c$: the complete eight-vertex template has fifteen disjoint edges and at most $c + 3$ of these are missing. Taking the largest available gross estimate above and subtracting u gives the following net bounds.

$c\backslash u$	1	2	3	4	5
0	11	17	19	18	24
1	10	15	16	18	22
2	9	13	13	17	17
3	8	13	19	18	17
4	29	20	19	18	25
5	52	44	43	42	41
6	74	67	66	65	64

The column minima are $8, 13, 13, 17, 17$. In particular $g \ge 8$ at unit width.

13 Eight factor directions and at most four coloops

Let G be the gross gain at an actual prefix of new quadratic width u for the representation's fixed common quadratic over a characteristic-zero field. For $0 \le c \le 4$ coloops one has $3G \ge 13u$. At widths one through five the net gains are at least $10, 14, 13, 17, 17$, respectively. If the weight is below five then $c \le 3$ and $3G \ge 14u$. Section 14 treats five, six and seven coloops.

We use the fixed physical/positional pairing, the same-prefix response identity, the quadratic contact bounds $Z(4) = Z(5) = 1$, $Z(6) = Z(7) = 3$, $Z(8) = 6$, and the small-support quartic response inequality $\dim L \le \binom{k}{2}$ for the actual old space. The required exclusion of pure fourth powers follows from Lemma 1.3. The ideal-square estimate and rank-one response theorem supply the symmetric-form bound. The following bounds on the old quartics also hold on the ambient physical support.

Cell boards and old quartics

Let the base field have characteristic zero, let U be an eight-dimensional physical support, and write $I = \operatorname{Ann}(\operatorname{perm}_{10})_2$. For a subspace $K \subseteq \operatorname{Sym}^2 U$ of dimension at most k, define

$$L(U, K) = \{F \in \operatorname{Sym}^4 U : IF \subseteq K\}.$$

Here perm_{10} is the ten-by-ten permanent.

Use a row-column torus with distinct cell weights and degenerate U to an eight-cell coordinate space. For example take row and column weights from disjoint powers of two. The subspace L, its support U, and K have simultaneous Grassmannian limits. The incidence $IL \subseteq K$ is closed and the torus preserves I. Thus it suffices to bound the dimension on every eight-cell board D. One may first extend scalars to the algebraic closure.

Let $H \subseteq \operatorname{Sym}^2 D^*$ be the restriction of I and let $W = K^\perp$. The products HW annihilate L. The quadratic generators consist of same-row and same-column products, including squares, and differences of the diagonals of rectangles. A pair crossing an incomplete rectangle is consequently in H. Choose the monomial order making the increasing diagonal of each rectangle its leading term. Weights ij on cell (i, j), with ties settled by any monomial order, suffice. The possible missing quadratic monomials are decreasing diagonals of complete rectangles. Leading products bound the original product dimension from below.

Every board on eight cells permits row and column orders and a partition into two sets X, Y of three coordinates and two remaining coordinates, such that every missing quadratic is xy for $x \in X, y \in Y$.

- With at most three rectangles the missing diagonals give a triangle-free graph. A triangle requires three distinct rows and columns and all six off-diagonal cells completing its three rectangles, hence nine cells. A triangle-free graph with at most three edges is, after removing isolated vertices, a subgraph of a three-edge matching, a three-leaf star, a four-vertex path, or a two-edge path and a disjoint edge. Each has a bipartition with both sides of size at most three.
- If no two rows or columns have three common neighbors, distinct rectangles share at most one edge in the board's bipartite incidence graph. Already three rectangles use at least $12 - 3 = 9$ cells by inclusion-exclusion. Any board with at least four rectangles therefore contains 2×3, up to transposition. There are just two more cells. Another rectangle needs two edges incident to a new row or column. The two remaining edges either add a fourth complete column or occupy two columns of a new row. This gives 2×4 or 3×3 with one cell removed.
- On 2×4 all decreasing diagonals run between the last three upper entries and the first three lower entries. On 3×3 minus one cell, put the missing cell in the center. The five decreasing diagonals form the edge between the northeast and southwest corners, two leaves at the northeast corner, and two leaves at the southwest corner. Both bipartition sizes are three.

Use the polynomial ring on

$$X = \{x_1, x_2, x_3\}, \qquad Y = \{y_1, y_2, y_3\}, \qquad Z = \{s, t\}.$$

Replace the available quadratic monomials by the smaller space H_0 of all quadrics except the nine products $x_i y_j$. Let O consist of the k old quadratic monomials missing from the leading monomials of W, enlarging it if necessary. A quartic monomial survives if each quadratic divisor whose complement is in H_0 belongs to O. Their number bounds $\dim L$.

We may make O stable under replacing a neutral letter s, t by any letter in $X \cup Y$, under replacing t by s, and under left shifts inside either three-set. To see this, compress the complementary family of quadratic monomials and H_0 in the reverse direction along the chosen pair of variables. On each fiber of fixed other exponents and fixed total degree in these two variables this replaces exponent sets by initial intervals. A nonempty sumset of two fibers has at least the sum

of their sizes minus one elements. After compression all the intervals giving the same output fiber have the same endpoint and are nested. Their product size cannot grow. These directions leave H_0 fixed, and the complement O moves in the indicated direction. Repeating terminates by a weight sum on the finite family.

In the three-plus-three core just obtained one may keep three additional multipliers. Number both sets from one to three. The missing edges lie in the half graph $i + j \le 4$. For at most three rectangles the triangle-free three-edge graph embeds in this half graph. A three-edge matching uses the anti-diagonal, the degree-three star is present, and the four-vertex path and a two-edge path disjoint from an edge are also subgraphs. For 2×4 the three degrees of the earlier omitted diagonals are $3, 2, 1$, exactly the half graph. The center-missing 3×3 board gives a double star, supported on the first row and first column of the half graph. Thus the three good mixed monomials

$$M = \{x_2y_3,\ x_3y_2,\ x_3y_3\}$$

are available in every case. The same compressions preserve this smaller forbidden diagram as well: move missing old edges toward the lower indices and from neutral into active positions, and shift the good edges in the opposite direction.

Write a, b, d for the numbers of old XX, XY, YY quadrics. Pure quartics on one three-set require all their quadratic divisors to be old. For $a = 0, \dots, 6$ the largest possible count is

$$q(a) = 0, 1, 2, 5, 6, 9, 15.$$

Indeed the Macaulay representation $a = \binom{h}{2} + r$, $0 \le r < h$, gives the two-step upper bound $\binom{h+2}{4} + \binom{r+2}{3}$. The same list follows by listing the shifted diagrams on the six available quadratic monomials.

Bidegree $(2, 2)$ contributes at most ad since both pure pairs must be old. For $(3, 1), (1, 3)$ the required pairs are the mixed ones. The mixed Ferrers diagram λ inside 3×3, of area b, therefore contributes

$$\sum_i \binom{\lambda_i + 2}{3} + \sum_i \binom{\lambda'_i + 2}{3}.$$

The core contributes at most $P(a + d) + J(b)$, where

$$P(v) = \max_{\substack{a+d=v \\ 0 \le a,d \le 6}} \{q(a) + q(d) + ad\}, \qquad J(b) = \max_{|\lambda|=b} \sum_i \left[\binom{\lambda_i + 2}{3} + \binom{\lambda'_i + 2}{3} \right].$$

argument	0	1	2	3	4	5	6	7	8	9	10	11	12
P	0	1	3	5	9	13	19	23	29	38	45	54	66
J	0	2	6	13	17	24	32	39	48	60			

The longer line takes at most four comparisons per entry, by $a \leftrightarrow d$. For the second line the diagrams, up to transposition, have row lengths $\varnothing, 1, 2, 3, 21, 31, 22, 32, 311, 33, 321, 331, 332, 333$; trailing zeros are omitted.

No old ss, st, tt is possible at $k \le 21$ because stability would force all 21 core quadrics in addition to this pair. Let N_s, N_t be the core neighbors of s, t in O, with $N_t \subseteq N_s$, and write $m = |N_s| + |N_t|$. For each $v \in N_s$ all six v-to-core quadrics are old. If $h = |N_s|$, at least $6h - \binom{h}{2}$ core edges are old. Thus $h \le 3$ through $k = 21$, and $m \le 6$. For $k \le 19$ one has $m \le 4$ because $m \ge 5$ requires $h = 3$ and forces at least fifteen old core edges. Three neutral factors cannot occur in a surviving quartic since a neutral pair would then have a complement in H_0.

For $z = s, t$ write $x_z = |N_z \cap X|$, $y_z = |N_z \cap Y|$. With precisely one neutral z, triples of pure core type XXX, YYY contribute at most $\binom{x_z+2}{3} + \binom{y_z+2}{3}$. For XXY the Y letter must neighbor z and the XX pair must be old, giving $y_z a$. The XYY type similarly gives $x_z d$. With two neutrals

the core must be XY, since the neutral pair is absent from O. Both core letters must neighbor both neutral occurrences, giving at most $x_s y_s + 2x_t y_t$ in total. A bound for the extra contribution is therefore

$$E = \sum_{z=s,t} \left[\binom{x_z+2}{3} + \binom{y_z+2}{3} + y_z a + x_z d \right] + x_s y_s + 2x_t y_t. \tag{E2}$$

For $k = 13, \dots, 19$ the old quartic space now satisfies

$$\dim L \;\le\; (69, 73, 79, 86, 92, 96, 100)_k. \tag{E3}$$

The subscript here means consecutive entries starting at $k = 13$.

Here is a scalar count for (E3). Use the quantities a, b, d, x, y, X, Y where x, y are the two neighbor counts of s and X, Y those of t. Define $m = x + y + X + Y$ and $g(v) = \binom{v+2}{3}$. In the relaxed core with all nine mixed multipliers missing an upper bound is

$$T = q(a) + q(d) + ad + J(b) + g(x) + g(y) + g(X) + g(Y) + (y+Y)a + (x+X)d + xy + 2XY. \tag{E4}$$

Use the two lists q, J above. For checking the small table below take $x \ge y,\ X \le x,\ Y \le y$; if $x = y$ take $X \ge Y$. The conditions on the old core stars are

$$a \ge 3x - \binom{x}{2}, \qquad d \ge 3y - \binom{y}{2}, \qquad b = k - m - a - d \ \in [3(x+y) - xy, 9], \qquad 0 \le a, d \le 6.$$

For a fixed star type and b, put $v = k - m - b$ and substitute $d = v - a$ in (E4). One takes at most seven conditional integer values, with a running from $\max(3x - \binom{x}{2}, v - 6)$ to $\min(6, v - 3y + \binom{y}{2})$. In the unmarked $m = 0$ column the previous $P(v)$ list gives the answer directly.

k	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$
13	69	66	55	—	—
14	73	71	67	56	—
15	79	76	73	64	62
16	83	86	81	71	70
17	90	90	93	81	77
18	98	97	97	92	87
19	105	105	105	99	98

Only the final three rows require the additional multipliers. In these rows any excess above the asserted respective targets $92, 96, 100$ occurs at the following core counts. Interchange X, Y as names for the two core groups to put $a \ge d$ in this table. Each entry is the largest excess over the indicated row target at those core counts across all three budgets and neutral counts, again from (E4).

(a,b,d)	5,9,3	5,9,4	5,9,5	6,3,6	6,4,6	6,5,6	6,6,5	6,6,6	6,7,6	6,8,5	6,9,3	6,9,4
required deduction	1	1	3	1	1	5	2	5	5	2	5	5

For example at $k = 17$ the only excess has $(x, y, X, Y) = (1, 0, 1, 0)$, $(a, b, d) = (6, 3, 6)$. At $k = 18$ the core possibilities up to transposition are $(6, 9, 3), (6, 6, 6), (6, 5, 6), (6, 4, 6)$. At $k = 19$ the only excess at $m = 2$ comes from $(a, b, d) = (5, 9, 3), (6, 5, 6), (6, 6, 5)$ up to transposition, and the two final m-columns already pass.

All twelve deductions follow without counting the individual source diagrams.

- If $a = d = 6$, all 36 monomials of bidegree $(2, 2)$ were counted. The products of M with the $9 - b$ old-missing mixed pairs now disappear. A product of two finite nonempty sets of ordinary monomials has at least the sum of their sizes less one distinct members, by ordering

the monomials and following one increasing row and one increasing column. This removes at least $11 - b$. For $b = 7$ the two old-missing mixed pairs are the last cell and its predecessor in the last row or column. Their products with M have five distinct members: the two sets of three products share only the mixed product of these two missing pairs. This gives five at that last endpoint as well.

- If $b = 9$, every $(3, 1)$ and $(1, 3)$ monomial was counted. With $e > 0$ pure pairs missing in one group, their products with M remove at least $2e + 1$. Indeed sorting the three multipliers by their letter in the other group gives one singleton and one two-element set. Their products with the e missing pure pairs have sizes at least e and $e + 1$ on disjoint fibers. Add the contributions of the two groups when both have missing pure pairs.
- For $(a, d) = (6, 5)$, the only missing pure pair is y_3^2. If $b = 8$, the only missing mixed pair is x_3y_3. The two previously allowed monomials $x_3^2y_2y_3$ and $x_2y_3^3$ each have an old-missing pair complementary to a member of M. If $b = 6$, $x_3^2y_2y_3$ is still lost from $(2, 2)$. If any missing mixed pair uses y_j, $j < 3$, multiplying it by x_2y_3 removes a second, distinct $(2, 2)$ monomial, since its Y pair is old and all X pairs are old. If all three missing pairs use y_3, their three products with x_3y_2 are distinct previously counted monomials. This supplies the required deduction in the final type.

These are deductions from the actual count bound (E4), since in each case the monomials removed came from bidegrees entirely counted in the corresponding upper bound. At $b = 8, a = 6, d = 5$ all Ferrers diagrams of area eight have the single stated missing corner, so the monomial in $(1, 3)$ was counted in $J(8)$.

For $k = 20, 21$ the identical scalar expression (E4), before deducting any of M, gives the following rows in increasing order of m:

$$(114, 112, 114, 112, 107, 98), \qquad (126, 121, 121, 121, 121, 109, 106).$$

Thus at $k = 20$ the old space has dimension at most 114. At $k = 21$ the maximum 126 requires $m = 0$, hence every one of the twenty-one core pairs is old and the 126 quartics are exactly all quartics on the six core letters. If the original term has three coloops this last maximum improves to 125.

For three coloops the three independent differential directions dual to the coloop factors span a three-space A. Every product of four members of A kills every quartic of the term's physical derivative space: in coloop factor coordinates each such quartic is at most linear in each of those three variables. The incidence $\mathrm{Sym}^4(A)L = 0$ is closed with the three-space A in its Grassmannian. This incidence may be carried through every torus and monomial limit above. The fiber compressions admit linear realizations when this extra incidence is required. On a two-variable fiber of fixed total degree, apply the shear $x \mapsto x + z$ and take leading terms favouring z. For distinct x-exponents $i_1, \ldots, i_p$, the coefficients at x-powers $0, \ldots, p - 1$ form the matrix $\binom{i_\alpha}{j}$, $0 \le j < p$, with nonzero Vandermonde determinant divided by $\prod j!$. Thus this linear degeneration realizes precisely the interval compression of the multiplier-complement family. The shears used after passage to H_0 preserve H_0. For instance moving a core differential toward a neutral differential adds only allowed monomials to allowed quadrics. Carry L contragrediently and A covariantly through the same changes and limits. If equality 126 held, the limit L would contain all quartics on the six core variables. Each $D \in A_{\lim}$ has $D^4L = 0$, hence is zero on all six core linear variables by testing fourth powers. Their annihilator has dimension two, contradicting $\dim A_{\lim} = 3$. This proves the improvement to 125 on the actual old space.

The common position graph

Choose an eight-factor basis by deleting two nonparallel Gale columns. Write $d_i = y_i + g_i$ in the squarefree position algebra, where g_i belongs to the two-space on the deleted positions. It vanishes at each coloop, and in live positions its line gives the original nonzero Gale direction up to one invertible change of basis. The quadratic filtration has scalar leading levels ij and colored

star levels $i*$ with coefficient line Kg_i. Write D, E for the good and source graphs on the eight basis vertices together with $*$, and $F = \binom{[9]}{2} \setminus D$. There are at least $36 - c - Z(8)$ good pivots and at least u source pivots, allowing deletion to $|E| = u$. The c edges from $*$ to coloops occur in neither graph. When the direct-summand core is placed first, the live vertices and $*$ have at most $Z(8-c)$ missing edges. This follows by projection onto that core, with quadratic codimension at most its physical contact. For genuine R, the central rank, put $A = 105 + R/2$. If f is the fourth position dimension, the nondegenerate 210-dimensional central form gives $A \geq f$. The same-prefix inequalities are

$$G \geq A - \dim L, \qquad G \geq N(D,E), \qquad N(D,E) = |\{d \cup e : d \in D,\ e \in E,\ d \cap e = \varnothing\}|\,.$$

At a three-vertex level each endpoint i supplying a star witness gives the line of g_i. Two nonparallel witnesses increase its dimension from one to two. The positional radical is paid once through $A \geq f$.

Shifting the two graphs simultaneously toward earlier vertices cannot increase their four-set shadow, by the two-vertex switch for disjoint-edge products. A shifted graph is a strict partition into row lengths $h_i \leq 9 - i$; row i ends at vertex $i + h_i$. A nine-vertex complete graph has 36 edges. For a fixed number of holes the good partitions are the complements of strict partitions of that size in the reversed order.

Lemma 6 (Direct count for shifted pairs)**.** *For shifted graphs H, S on any ordered set $[n]$, let $h(i)$ be the endpoint of row i of H, write $h(i) = 0$ for a missing row, and put $H_{ij} = \mathbf{1}_{ij \in H}$. For source S put $s(i)$ for its endpoint. At three smallest vertices $a < b < c$ the allowed last vertices form an interval whose upper endpoint is*

$$\max\{h(a)S_{bc}, h(b)S_{ac}, h(c)S_{ab}, s(a)H_{bc}, s(b)H_{ac}, s(c)H_{ab}\}.$$

Subtract c, take the positive part and sum over $a < b < c \leq n - 1$. This counts every shadow element once, by its first three indices.

For a set of test diagrams one obtains the maximal diagrams avoiding them by starting with 87654321. To impose avoidance of a test h, retain each current bound v which already avoids it. Replace a bound containing h by the alternatives indexed by the positive rows i of h. In alternative i, keep rows preceding i, and bound row $j \geq i$ by $\min(v_j, \max(0, h_i - 1 - j + i))$. Delete alternatives contained in another. This is precisely the assertion that a strictly decreasing diagram avoiding h must first fall short in some row.

For any simple graph M one can shift toward a specified vertex v to obtain a star there of degree at least its graphic matroid rank, with v chosen in its nonisolated support. In the component of v, shift neighbors towards v. The edge to a neighbor survives such a shift, and all of that neighbor's other neighbors become neighbors of v unless already adjacent. Following a spanning tree makes v adjacent to every other vertex of the component. For a different component, first shift one root toward v and then repeat on the new neighbors. At most one vertex from each component is lost, giving degree at least the size of a spanning forest. Further shifts towards the same extreme in the total vertex order never decrease its degree. Shifts away from the extreme but between other vertices preserve its degree. This reasoning also applies to shifting the missing complement, which is the opposite shift of the good graph.

In particular a six-edge graph of graphic rank at most three must be K_4, and a ten-edge graph of rank at most four must be K_5. Components of ranks r_i have at most $\sum_i \binom{r_i+1}{2}$ edges, with equality at the extremal total only on a single complete component.

All girth-five terms with at most three coloops

Write d_i for the nonzero Gale class sizes. Their sum is $10 - c$ and each is at most $5 - c$. The rank-two formula is

$$f = 210 - 2\binom{c}{3} + \binom{c}{2} - \sum_i \left(\binom{c + d_i}{3} - \binom{c}{3} \right).$$

Filling classes to capacity by convexity gives $f \geq 190, 190, 187, 181$ for $c = 0, 1, 2, 3$. The old width is $k \leq 36 - c - u$. The physical bounds just proved, combined with $\binom{k}{2}$, give $3G \geq 14u$ from $u \geq 16, 15, 13, 12$, respectively. For verification, $\min[3(f - \dim L) - 14(36 - c - k)]$ using these upper bounds over the indicated k-ranges is $4, 18, 1, 0$, attained at $k = 20, 20, 21, 21$. For $k \leq 12$ only the two endpoints of the concave quadratic require substitution. The graph proof is therefore needed only up to $u = 15, 14, 12, 11$.

Delete Gale columns from two largest distinct classes. The caps on multiplicity among the live basis columns are $4, 3, 2, 2$. They follow from the original cap and the fact that every original class that attains size five, four, or three, at $c = 0, 1, 2$ respectively, can be met by one of the two deleted columns.

For $u \leq 11$ relax to nine missing edges. The seven possible shifted good diagrams, ordered for reference below, are

$$87651, \quad 87642, \quad 87543, \quad 876321, \quad 875421, \quad 865431, \quad 765432.$$

Their sizes are all 27. The following table gives, for each test, its least disjoint shadow among these seven diagrams. A test covers an input containing it up to area $\lfloor 3N/14 \rfloor$.

test	1	21	3	5	42	51	6	52	431
least shadow	12	24	19	30	35	39	34	43	40
test	62	8	432	531	621	81	5321	542	65
least shadow	47	49	42	49	52	55	53	53	54

Use the avoiding-corner rule and direct shadow formula above. The maximal diagrams avoiding all tests sufficient at the indicated width are as follows.

width	maximal avoiders
1–2	empty
3–4	2
5	4
6	41, 321
7	5, 41, 321
8	7, 43, 421
9	71, 431
10	54, 71, 521, 4321
11	8, 74, 532, 541, 4321

Only $321 = K_4$ of area six, $4321 = K_5$ of area ten, and 74 of area eleven can fail. Their respective shadows with the seven fixed multipliers are

$$(27, 33, 39, 39, 39, 44, 47), \qquad (45, 45, 44, 55, 54, 51, 55), \qquad (67, 63, 56, 62, 56, 55, 51).$$

The last diagram fails only when the missing graph is an eight-leaf star and one more edge. For the first two widths, if the original input is not the displayed clique then the graphic-rank shift lemma above makes its first degree exceed three or four, respectively. Thus the clique repairs below apply on the original labeled graph.

The true cliques at six and ten

Suppose E is K_a on A, with $a = 4, 5$, and B its complement. Write $e = |F \cap \binom{B}{2}|$ and $d_A(v) = |N_F(v) \cap A|$. Four-sets with exactly two members of A contribute $\binom{a}{2}(\binom{|B|}{2} - e)$. Those with three members of A contribute $|B|\binom{a}{3} - \sum_{v \in B} \binom{d_A(v)}{3}$. Those entirely in A contribute $\binom{a}{4}$ minus the number of four-cliques in $F[A]$.

At $a = 4$, this count is at least 28 except when $e = 9$. For $e = 8, 7, 6$ there are at most $1, 2, 3$ remaining missing edges, respectively, giving lower bounds $33, 39, 44$. For $e \leq 5$ there are at most nine edges to distribute, with $d_A(v) \leq 4$; the cubic loss is at most $4 + 4 = 8$ and at most one complete four-set is lost. If $e = 9$, the count is 27. Only $c = 3$ permits nine missing edges. All three structural edges must lie in B, so $*$ and the three coloops lie in B. All four vertices of A are live, all their good folded star edges occur, and one triple has witnesses on distinct Gale lines by the cap of two. This gives the missing dimension.

At $a = 5$, the count is at least 47 for $e \leq 5$. At $e = 5, 4$ the remaining four or five missing edges give cubic loss at most $4, 10$ and no lost four-clique. For $e \leq 3$, nine edges with $d_A(v) \leq 5$ give cubic loss at most $10 + 4 = 14$ and at most five four-cliques. Only $e = 6$ needs help. The old count is at least 44 since at most three missing edges are outside B and at most one triple is lost. If $* \in B$, there are at least $5 - c$ good star endpoints in A. At $c = 0, 1, 2$ this exceeds the respective class cap. At $c = 3$, the actual flag gives $\deg_F(*) \leq 3 + Z(5) = 4$; three bad star edges are already within B, so there are at least four good endpoints in A. Thus two good endpoints on distinct lines have each of the three other vertices of A as a third member of a witnessing triple, giving three extra dimensions. If $* \in A$, all four other vertices of A are live because their star edges are in the source. Every coloop is in B and all at most c missing edges outside B are precisely its structural star edges. If the four live vertices supply a nonparallel pair, then with any of the four B-vertices the two complementary ordinary edges are good. This gives four extra dimensions. At $c = 0$ alone, the four live vertices could all lie on one line. The cap four then puts every vertex of B on another line, and each triple with two live vertices from A and a vertex of B has all three as witnesses. This again gives at least three extra dimensions. Both cases reach 47.

Preventing an eight-leaf star in the terminal diagram

The original missing graph has maximum degree at most seven. This follows by its size at $c = 0, 1$. At $c = 2$ a degree eight would give a star centered at $*$ by the two structural edges, contradicting the first flag. At $c = 3$ all but at most one missing edge would meet the vertex of degree eight. The three structural edges force that vertex to be $*$, again contradicting the first flag. One can fill F to nine missing edges while preserving maximum degree at most seven. With fewer than nine edges at most two vertices have degree seven, and there are many absent edges avoiding them.

For any nine-edge simple graph on nine vertices with maximum degree at most seven, there is an order of shifts toward later vertices after which the last four vertices contain at least five edges. If there is a vertex v of degree at least three, choose three of its neighbors and two distinct edges off v. Color the three neighbors in three distinct colors. The two off-vertex edges can be made proper with different unordered color pairs. If they meet, color their three vertices injectively, extending the compatible assignments on the overlap with the three chosen neighbors. If they are disjoint, at most one has both ends already fixed. For an edge with at most one fixed end at least two unordered color pairs are available. Give v a fourth color and complete the coloring arbitrarily. At least three different color pairs incident to the fourth color and two other color pairs are therefore represented. Choose one vertex of each color as a representative in the last four positions and shift all remaining vertices to the representative of their color. A represented pair of distinct colors becomes an edge between the representatives: at each endpoint shift either the new edge is created or it already exists. Further right shifts cannot reduce the number of edges in the last four. If all degrees are at most two initially, the graph is a union of cycles covering all nine vertices. An adjacent-vertex shift in a cycle of length at least four produces a degree-three vertex. If all cycles are triangles, shifting a vertex between two triangles gives degree four. The shifted pair has degree at most four, so two edges off the selected vertex are still available. In this preliminary shift choose its target as v, and place its donor below that target when the four representatives are chosen. In particular the preliminary shift and the quotient shifts may also be used as a finite sequence of arbitrary two-vertex switches before choosing a final stabilizing order, since each switch separately cannot increase the good/source shadow.

An eight-leaf star with one extra edge has at most four edges in any four vertices. It cannot be the resulting shifted graph. This settles the area-eleven exception.

At width 12 for $c = 0, 1, 2$, and width 13 when needed for $c = 0, 1$, relax instead to eight missing edges with the missing graph not a star. This is possible from the preceding degree and flag bounds. A nonstar graph with at least three edges contains a triple of edges having no common vertex. These three form a triangle, a three-edge path, three independent edges, or a two-edge path plus a disjoint edge. In every case one can assign their vertices three colors so that all three unordered color pairs appear. Shift to three last representatives as before. The shifted graph therefore contains a triangle in its last three vertices. Its good diagram is one of

$$87652, \quad 87643, \quad 876421, \quad 875431, \quad 865432.$$

The omitted sixth diagram has the full missing star. At width 14, needed for zero or one coloop, seven missing edges suffice. The five possible good diagrams are $87653, 876521, 876431, 875432, 8654321$.

test	631	541	65	84	831	651	642	6321	76	5431
least shadow in the eight-missing list	58	56	58	70	66	64	64	61	61	63
least shadow in the seven-missing list	62	57	60	71	68	67	67	63	61	67

The maximal avoiding diagrams at width 12 are $83, 74, 821, 5321$. At width 13 they are $83, 821, 75, 741, 732, 543, 5421$. At width 14 they are $83, 821, 741, 543, 5421, 76, 7321$. All have area strictly smaller than the relevant width. For the last width $u = 15, c = 0$, relax to seven missing edges: the five minima in the unrestricted width-fifteen row below are $75, 73, 74, 71, 72$, all at least 70. This closes every light width.

The intermediate weight threshold

In fact all the preceding light arguments require only girth at least five and $c \leq 3$. They do not use a further numerical bound on the weight. The common-quadratic short-circuit lemma gives weights at least $28, 21, 6$ for circuits of sizes two, three and four, respectively. A term of weight $w < 5$ therefore has girth at least five. Its nonzero Gale multiplicities satisfy $d_i \leq 5 - c$, ruling out $c \geq 5$. If $c = 4$, the remaining six positions form the uniform rank-four core. Its fourth position dimension is 15 by Lemma 3.3. The actual core fourth dimension equals its actual second dimension and is at most 10. Thus already the pure core part of the full degree-four position space has untwisted radical of dimension at least five. Generally, writing f and h for ordinary and actual fourth dimensions, restriction of the common ambient nondegenerate pairing gives $R \geq 2f - 210$, and the weight is $105 - h + R/2$. Hence $w \geq f - h \geq 5$, excluding $c = 4$. It follows that every rank-eight term with $w < 5$ has $G \geq 14u/3$, supplying also the intermediate target $G \geq 9u/2$.

Central constants and the quadratic radical

Write $A = 105 + R/2$ for the common central contribution, $f = f_4$ for the degree-four position dimension and $\rho_2 = f_2 - h_2$ for the degree-two position radical. The old width satisfies

$$f_2 = 36 - c, \qquad k \leq 36 - c - \rho_2 - u.$$

The source graph has at least u pivots whether or not the positional pairing has a radical. The common nondegenerate form on 210 positions gives $A \geq f$.

The constants in the following table cover all Gale multiplicities; a stated lower bound on ρ_2 may be used even if the actual radical is larger.

c	case	A at least	ρ_2 at least
0	except $7,3$	175	0
0	$7,3$	174	1
1	all	171	0
2	except $5,3$	171	0
2	$5,3$	166	1
3	except $4,3$	165	0
3	$4,3$	158	1
4	all	157	0

To check this, let the nonzero Gale multiplicities be d_i, with sum $10-c$. The rank-two formula is

$$f = 210 - 2\binom{c}{3} + \binom{c}{2} - \sum_i \left(\binom{c+d_i}{3} - \binom{c}{3} \right).$$

In the absence of a proportional factor pair each $d_i \le 7-c$. Discrete convexity makes $7-c,3$ the worst distribution for $c \le 3$. Excluding it, the worst distribution with a part of size $7-c$ is $7-c,2,1$. Alternatively fill all parts to the smaller cap $6-c$ before starting another part. These endpoint substitutions give the first seven rows. Two projective Gale classes represent a direct sum of two primal circuits, because the dependence space has disjoint generators on the two classes. At the three distinguished distributions there is a separate three-factor circuit. Its binary cubic has apolar dimensions $1,2,2,1$, while its quadratic position dimension is three. Tensoring with the other factors retains its degree-two radical, yielding $\rho_2 \ge 1$.

With a proportional pair, $d_i \le 8-c$, and the same convex calculation gives $f \ge 154,153,151,147,140$ for $c = 0,1,2,3,4$. The positional space is in the symmetric subspace of the pair transposition after equal rescaling. This 154-dimensional ambient central form is nondegenerate: the original positional quadratic is invariant and its inverse commutes with the involution. Hence $R \ge 2f - 154$ and $A \ge f + 28$, sufficient for each listed row. Finally for $c = 4$ Lemma 5 gives the joint degeneration of the nonuniform six-factor core to exponent type $(2,2,1^6)$, giving $R \ge 104$ by the three explicit quadratic constructions in Section 10. The uniform core has six distinct Gale classes and $f = 172$.

For $u \ge 17-c$, put $k = 36-c-\rho-u$ for the stated lower bound ρ on ρ_2. Use the smaller of $\binom{k}{2}$ and (E3) wherever applicable. For $k \le 12$ the difference $3(A - \binom{k}{2}) - 13(36-c-\rho-k)$ is concave and needs only its two endpoints. For $k = 13,\dots,19-\rho$ use (E3) directly. The minima of $3G-13u$ over these upper-width branches, in table order, are $2,12,3,16,14,11,3,0$. The minimum in every row occurs at $k = 17$.

Small widths and the reversed coloop projection

It remains to prove $3G \ge 13u$ for $u \le 16-c$. For $u \le 12$ relax to ten missing edges on the folded nine vertices. With the graph notation and disjoint-shadow formula above, the eight shifted multipliers are

$$8765,\ 87641,\ 87632,\ 87542,\ 86543,\ 875321,\ 865421,\ 765431.$$

The minimum shadow in each column is taken over strict partitions of u with parts at most eight. The numbers of choices for $u = 1,\dots,12$ are $1,1,2,2,3,4,5,6,7,8,9,10$, the coefficients of $\prod_{j=1}^{8}(1+t^j)$. At $u = 13,14,15,16$ below they are $11,12,13,13$. An entry can be checked by the three-smaller-vertices endpoint formula, or by listing the four-sets whose complementary pairs meet the two small diagrams.

u	8765	87641	87632	87542	86543	875321	865421	765431
1	11	11	11	11	12	11	12	13

u	8765	87641	87632	87542	86543	875321	865421	765431
2	16	16	16	18	19	18	19	21
3	16	19	21	21	22	24	22	25
4	21	24	26	25	22	28	26	26
5	21	26	31	29	30	31	29	30
6	21	27	33	33	34	34	32	34
7	33	35	37	35	34	40	35	35
8	41	40	39	39	39	43	43	43
9	45	42	39	42	40	44	47	44
10	45	45	45	44	41	47	51	45
11	51	53	51	52	49	48	54	49
12	59	57	54	56	52	53	57	52

The only violating shifted pairs are 8765 with 32 or 321 at widths five and six, and 86543 with 4321 at width ten. In the former two rows the shifted missing graph is K_5. At $c \leq 3$ the actual graph has at most nine missing edges and can be extended to ten without making a clique. At $c = 4$ choose a legitimate order in the folded position filtration that gives first priority to two factors lying among the four coloop positions. Projection onto the pure coloop list has degree-two image of dimension six. Its physical quadratic defect is at most $Z(4) = 1$, because those four factors span only four physical variables. At least five of the six coloop pairs are therefore good pivots. This priority can be obtained by counting coloop positions first, then ordinary basis positions, then a generic additive separating weight. Each individual positional basis element retains its usual leading pair or its single colored star, since coloop factors have no outside component. Trim to ten missing edges preserving a good ordinary coloop pair, which is possible since the full folded graph has 36 edges. The four structural missing star edges would force any missing K_5 to have vertices $*$ and the four coloops, a contradiction. In every case the actual padded missing ten-edge graph is not K_5, so it has graphic rank at least five. The spanning-forest shift produces a last degree at least five and excludes the offending shifted clique.

At width ten, the graphic-rank shift on the source forces $E = K_5$ on a labeled set A if failure is possible. Write $B = A^{\mathrm{c}}$, $e = |F \cap \binom{B}{2}|$ and use the true clique count from the light proof above. This time use the original core-first flag. At $e \leq 5$ the result is at least 44. Indeed at $e = 5$ there are at most five missing edges outside B, which cost at most $\binom{5}{3} = 10$ triples and no four-clique. At $e = 4$ they cost at most $10 + 1$. At $e \leq 3$, at most ten missing edges in total cost at most $20 + 5$, against the baseline of at least 75. The case $e = 6$ starts from 45. To fall below 44 there must be four missing cross edges all from a single $b \in B$ to four vertices of A; with three or fewer extra holes at most one triple can disappear. Thus $c = 4$, the actual missing graph is the K_4 on B together with this four-edge star, and its unique vertex of degree at least four is b, of degree seven. The four structural holes force $b = *$. But the core-first flag on the four live vertices and the folded star has at most $Z(4) = 1$ internal hole. Hence $\deg_F(*) \leq 4 + 1$, a contradiction. This establishes the width-ten bound without any Gale coloring.

Girth is not needed for the first physical core flag: deleting the actual coloops gives the rank-$(8 - c)$ support, and the quadratic position projection has relative defect at most its physical contact. If the original position source has more than u pivots, retain any u of them for the graph argument.

At the remaining four small widths the following one-row minima suffice, in the shifted multiplier orders used above.

u	number of missing edges	minima over shifted sources by permitted multiplier
13	9, exclude the full star plus one edge	63, 63, 59, 58, 61, 60
14	8, exclude the full star	67, 69, 67, 67, 64
15	7	75, 73, 74, 71, 72
16	6	83, 82, 78, 77

The first row is needed only at $c \le 3$ and the second at $c \le 2$. The three- and four-color shift arguments of the light proof above apply: the necessary stars are excluded on the original graph by the first core flag and the structural holes. They rely on the count of live positions and physical contact but not on their projective multiplicities. At $c = 0, 1$ a nine-edge extension preserving maximum degree at most seven and an eight-edge extension that is not a star are always possible within the indicated actual hole budgets. The lower bounds in all four rows exceed $13u/3$.

The first table together with the width-five repair also gives the gross lower bounds $11, 16, 16, 21, 22$ at absolute widths one through five. Subtracting the true target intersection increment, at most u, gives $10, 14, 13, 17, 17$.

14 Eight factor directions and at least five coloops

Let $c \in \{5, 6, 7\}$ be the number of coloops of a ten-factor product of rank eight. At every prefix the same physical quadratic of the decic argument gives $G \ge 13u/3$. At $u = 1, \dots, 5$ the net gain also satisfies the five absolute bounds, including $g \ge 8$ at unit width. The argument uses two small multiplication diagrams and a single 20-row certificate.

Write D for the actual apolar algebra of the $10 - c$ remaining factors on $8 - c$ variables. The product algebra is $D \otimes B_c$, where B_c is squarefree. Put $A_* = 105 + R/2$, where R is the central rank of the common quadratic. Here h_4 is the actual degree-four dimension. For the original input $X \subseteq (D \otimes B_c)_2$ of width u, the same-prefix estimates give

$$G = A_* - \dim L, \quad G \ge A_* - h_4 + \dim UX, \quad \dim L \le \binom{k}{2}, \quad g \ge G - u.$$

Here U denotes the image of the physical quadratic annihilator and $k = h_2 - u$. The six-hole support bound gives $\operatorname{codim} U \le 6$ on these eight directions. The projection of U onto the full core layer D_2 is surjective. Its failure would give a nonzero target quadric supported on at most three variables. Restricting to any rectangle with a nonzero coefficient of a target quadric gives a nonsingular four-variable quadratic, ruling this out.

The truncated polynomial calculation

Suppose b of eight coordinates have squares annihilating the old spaces after an initial passage and let all other variables be free. In the two cases below $b = 6, 7$ and $q = 36 - b$ is the effective dimension in degree two. Upper-bound the old quartics by taking the annihilator of the old quadrics of dimension at least u in this polynomial quotient and multiplying it by the physical space, of codimension at most six. It suffices to use any u leading input monomials. When an extra vanishing square arises as the initial term of a core relation, take the same initial flag for the physical space and the old spaces before this calculation. Equivalently use the quotient first, then take its initial quadratic and quartic layers. Discarding other annihilating relations in higher degree can only enlarge the old quartic space. Thus if P quartics are covered and the truncated polynomial quartic layer has size d, then $G \ge A_* - d + P$.

List square-zero coordinates first and simultaneously shift both quadratic families left. For two free variables, compressed exponent intervals have minimum sumset. For two square-zero variables use disjoint-union compression. If the earlier variable has square zero and the later one is free, a positive-degree fiber has possible earlier exponents 0 and 1. Shift toward 1. Two singleton fibers both moved to 1 give zero. A two-element fiber times a nonempty fiber has a terminal surviving interval no larger than its original product. Taking unions uses nested intervals. The two families can therefore be taken to be shifted. Row i, starting from zero, begins at $j = i + 1$ for $i < b$ and at $j = i$ otherwise. A word gives the initial number of occupied entries in each successive row. Missing trailing rows are zero.

With exactly six holes, the multiplier words at $b = 6$ are 7643211, 765321, 76542, with common subdiagram 76432. At $b = 7$ they are 764321, 76532, 76541, with common subdiagram 76431. In

the following table every input of width at least the left endpoint contains one listed subdiagram. The last column counts its distinct products with the common multiplier subdiagram, in the same order.

b	Widths	Input subdiagrams	Products
6	1–7	1	9
6	8–10	41	24
6	11–13	63, 532	36, 37
6	14–16	741, 652, 6431, 5432	52, 48, 51, 49
7	1–10	1	8
7	11–16	53	30

To check inclusion, trim to the left endpoint. The size-eight inputs are 431, 521, 53, 62, 71. The size-eleven inputs are 5321, 542, 632, 641, 65, 731, 74. The size-fourteen inputs are 5432, 6431, 6521, 653, 7421, 743, 752, 761. These lists follow by nonincreasing row endpoints. An entry of the last column can be read by taking every sum of two indicated pairs and counting the different sorted four-tuples, omitting repeated indices smaller than b.

For $b = 6$, $d = 129$ and $A_* \geq 152$ suffice: the four row minima $9, 24, 36, 48$ plus 23 dominate $\lceil 13u/3 \rceil$ at the four right endpoints. For $b = 7$, $d = 99$ and $A_* \geq 142$ suffice: the two minima plus 43 dominate the respective endpoints. At $u \geq 17$ instead maximize the convex expression $13u/3 + \binom{q-u}{2}$ on $[17, q]$. Both endpoints are at most the respective A_*.

Six or seven actual coloops

At seven coloops the core is a cube in one variable. The established monomial central rank $R \geq 74$ gives $A_* \geq 142$, so apply the $b = 7$ diagram.

At six coloops the core is a concise binary quartic with second dimension $m \leq 3$. For $m \leq 2$ its quadratic annihilator contains a nonzero quadric. After a core change of basis its leading term is a square, giving the $b = 7$ diagram. The universal rank-eight monomial limit gives $R \geq 74$ throughout.

For $m = 3$, filter the physical multipliers first by highest core degree and then by generic weights on the individual coloop positions. This preserves the tensor algebra. All core quadrics are present in its leading space and at least nine of the fifteen pure coloop edges are present. The latter contain two disjoint edges, as an intersecting edge family on six vertices is a star or a triangle. Choose a strong binary core linear form ℓ. Its existence follows directly from the nonzero Hessian of a concise binary quartic and the nonzero fourth evaluation. In the limit use $Q = \ell^2 + t_1t_2 + t_3t_4$. The active algebra on the core and those four slots has socle degree eight; all strings for Q have center four. Indeed squaring the strong linear raising operator on a socle-four binary algebra splits its odd-length strings into step-two strings of the same center. Each two-slot edge has a two-term string and two stationary strings centered at one. Tensoring under the sum preserves the summed center by the characteristic-zero two-string rule. The two unused coloop variables have dimensions $1, 2, 1$ by degree. Thus the full rank from degree four to six is at least $2h_2^{\mathrm{act}} + 2h_3^{\mathrm{act}}$. These active dimensions are $14 + m$ and $18 + 4m$, by multiplying $(1 + 2t + mt^2 + 2t^3 + t^4)(1 + t)^4$. At $m = 3$ the rank is 94, giving $A_* \geq 152$. The nonzero minor lifts from the leading multiplier space to the actual one; its open condition can be imposed in the choice of the common quadratic. The $b = 6$ diagram now applies.

Five actual coloops

The nonzero Gale labels of the five-factor rank-three core form a rank-two list. If their projective classes are all different, any original three factors are independent. Lemma 3.3 gives the full ten-dimensional Boolean third layer. The second layer has dimension six and no trace radical, since no core factor is a coloop. The ordinary degree-four dimension of the full product is 150

(the core ordinary layers are $1, 3, 6, 10, 5$ through degree four). Hence $A_* \geq f_4 = 150$ by restriction of the common position pairing. If a nonzero Gale direction repeats, the matroid of the core is nonuniform, since the complement of the parallel pair is a dependent original triple. Lemma 5 gives the monomial limit $(2, 2, 1^6)$, furnishing $R \geq 104$ by the three explicit quadrics of the central-rank constructions. Thus in this alternative $A_* \geq 157$.

If $\dim D_2 < 6$, the core contributes an additional initial vanishing square. This is the second alternative and the $b = 6$ diagram applies. Assume from now on $\dim D_2 = 6$. Then the actual Hilbert vector of the core is $(1, 3, 6, 6, 3, 1)$ and of the whole product the fourth dimension is $h_4 = 128$. We use the characteristic-zero strong Lefschetz theorem in codimension three and socle degree five [2]. With a strong final variable f in a reverse lexicographic core basis d, e, f, all six cubics fD_2 and all three quartics f^3D_1 are independent and are the first standard monomials in their degrees. Thus a core degree-three monomial is standard precisely if it contains f, and a core degree-four monomial precisely if it contains f^3. The two lower degrees are unrestricted.

Filter by core degree and then by coloop weights, retaining the actual core slices, as above. Write v_i for the dimension of the retained mixed coefficients at coloop i. Missing pure edges plus $\sum(3 - v_i)$ is at most six. There is a center C with two available pure edges to a pair of vertices A, such that the other pair B both have $v_i > 0$. Two zero mixed slices consume six deficiencies and all pure edges are available. With one zero, at most three pure edges are missing; use it as center if it has two neighbors, otherwise use its unique neighbor as center. With no zero, at least four pure edges remain and a graph on five vertices with four edges has a degree-two vertex or higher. This proves the assertion. Use generic remaining basis choices in the core so the two nonzero mixed slices both have leading letter d, while f stays strong. In fact they can be made nonzero on the same first coefficient by taking a plane spanning e, f containing neither entire slice; f can be chosen generally first. The leading multipliers consequently include all six core quadrics, the two edges AC, and the two mixed edges Bd.

The single certificate for the five-coloop case

A type is the tuple of exponents of d, e, f followed by the counts in classes A, B, C of sizes $2, 2, 1$. Its multiplicity is $\binom{2}{A}\binom{2}{B}$ where the subscripts here mean the class counts. List all standard degree-four types lexicographically from zero; the labels use two digits in base 62, with alphabet 0–9, A–Z, a–z. Add a link from a degree-two type to a degree-four type if their difference is one of the allowed multipliers just listed (in class counts). Along a type link one can sample nested coloop subsets uniformly, so any indicated integral charge lifts to a fractional charge uniform on individual monomials. Each quadratic is allowed to send 13 and each quartic to receive 3. The following allocations total 338 out of the available $31 \cdot 13 = 403$. Each entry gives its receiving label and charge.

Source type	Multiplicity	Quartic allocations
000011	2	08:6, 0M:6, 0V:6, 0d:3, 0i:5
000020	1	00:6, 09:3, 0N:3, 0W:1
000101	2	0A:6, 0O:6, 0X:6, 0e:8
000110	4	01:6, 0B:12, 0P:12, 0Y:12, 0f:6, 0l:4
000200	1	0C:3, 0Q:3, 0Z:3, 0h:4
001001	1	0D:3, 0R:3, 0a:3, 0i:1, 0n:3
001010	2	04:12, 0E:6, 0S:3, 0b:2, 0j:3
001100	2	06:3, 0F:6, 0T:2, 0c:6, 0l:8, 16:1
002000	1	0G:3, 0U:3, 0o:6, 0q:1
010001	1	0r:6, 0w:3
010010	2	0I:12, 0S:3, 0b:4, 0s:3, 0x:4
010100	2	0K:3, 0T:4, 0u:12, 0y:6
011000	1	0x:2
020000	1	

Source type	Multiplicity	Quartic allocations
100001	1	0z:6, 14:3
100010	2	0e:4, 10:3, 15:6
100100	2	0g:3, 0p:6, 12:12, 16:5
101000	1	0k:6, 0q:2
110000	1	0t:6
200000	1	11:6

The table only uses class multiplicities, legal differences and addition. Every input monomial occupies one source type. Distinct leading products give independent outputs. For any input of dimension u, at most $403 - 338 = 65$ units can fail to leave its leading family; its product neighborhood therefore has dimension at least $(13u - 65)/3$. With $A_* - h_4 \geq 22$ this proves $3G \geq 13u + 1$.

For the absolute bounds, the gain satisfies $g \geq A_* - h_4 - u$. With five coloops $h_4 = 38 + 15 \dim D_2$, hence $A_* - h_4 \geq 22$. With six coloops $h_4 = 68 + 15m$, so the difference is at least 39 for $m = 3$ and at least 44 for $m \leq 2$. With seven coloops $h_4 = 98$, giving at least 44. Even the smallest difference 22 gives $g \geq 21, 20, 19, 18, 17$ at the first five widths. These estimates use the full original physical space on every characteristic-zero field, allowing the temporary scalar extension for a nonzero-minor argument.

15 The independent-factor estimate

Let T be a product of ten independent linear forms in the physical variables of the generic 10×10 permanent. Use those factors as coordinates $y_1, \ldots, y_{10}$ on their span U. Let $I = \mathrm{Ann}(P)_2$, let B be the squarefree algebra on the ten positions, and let $H \subset B_2$ be the position image of I. For every $W \subset B_2$, writing $s = \dim W$, we prove

$$\dim HW \;\geq\; \frac{149}{32}s \quad (0 \leq s \leq 32), \qquad \dim HW \;\geq\; \frac{14}{3}s \quad (33 \leq s \leq 45).$$

The argument uses one deletion and three diagrams.

The response bound of Section 9, applied over the algebraic closure and descended by rank, gives $\dim IF \geq 3$ for every nonzero positional quartic. For $k = 45 - s$ the symmetric-matrix estimate then gives $\dim HW \geq 210 - \binom{k-1}{2}$ (with zero subtracted for $k < 3$). For $k \leq 12$, $\binom{k-1}{2} \leq 14k/3$, proving the asserted bound for $s \geq 33$.

Three multiplier diagrams

Deleting any chosen factor coordinate gives an injection on $H^\perp$. Indeed a squarefree quadratic vanishing after that deletion would be divisible by the deleted coordinate, hence of rank at most two. Every nonzero quadratic in $H^\perp \subseteq P_2$ restricts on some physical 2×2 square to a nonzero multiple of the rank-four subpermanent. Giving the edges avoiding the chosen coordinate first priority in the dual order consequently gives H a leading graph G containing the entire star at that coordinate. Let D be the leading graph of W in the same primal order. Distinct union monomials give

$$\dim HW \geq |N(G, D)|, \qquad N(G, D) = \{e \cup f : e \in G,\ f \in D,\ e \cap f = \varnothing\}.$$

There are at most ten missing edges, by the physical contact bound. We may delete additional good edges away from the chosen star until exactly 35 good edges remain. Put the center of the star first and simultaneously shift both graphs towards earlier vertices. Such shifts cannot enlarge the disjoint-union shadow. This follows on each two-vertex fiber by pairing a newly shifted early occurrence with the original alternative when the late occurrence remains in the other factor. The good star stays complete.

A shifted graph on ten vertices has strictly decreasing positive row lengths $b_1, \dots, b_r$. Its edges in row i have other endpoint $i+1, \dots, i+b_i$. A full first star forces $b_1 = 9$; the remaining distinct positive row lengths sum to 26 and lie between 1 and 8. Listing the partitions gives the following three graphs sufficient for a lower bound:

$$G_1 = (9,8,7,6,5), \qquad G_2 = (9,8,6,5,3,1), \qquad G_3 = (9,7,6,5,4,3,1).$$

Indeed the two end possibilities are G_1, G_3 themselves. The six remaining tails after the 9 are $(8,7,6,4,1), (8,7,6,3,2), (8,7,5,4,2), (8,6,5,4,3), (8,7,5,3,2,1), (8,6,5,4,2,1)$, each containing the tail of G_2.

An elementary bound from strict row lengths

Suppose D has area $s \le 32$, exactly r positive rows, and largest possible row length M. For $1 \le i \le r$ put $t = r - i + 1$. Its i-th row is at least

$$b_i = \max\left\{t, \left\lceil \frac{s - (i-1)M + \binom{i-1}{2} + \binom{t}{2}}{t} \right\rceil\right\}.$$

The first $i-1$ rows contribute at most $(i-1)M - \binom{i-1}{2}$; if row i has length b, the last t rows contribute at most $tb - \binom{t}{2}$. If the first row is specified as 9, set $b_1 = 9$. Write $L = (b_1, \dots, b_r)$ for this lower diagram. The possible r are 1 through 7.

For each lower diagram L, evaluate $\mathcal{N}(G, L) = |N(G, L)|$ by Lemma 6 on ten vertices.

The next table gives, separately for each of the three fixed diagrams, the minimum of $32\mathcal{N}(G, L) - 149s$ on each indicated interval. The two omitted widths use the sharper elementary row bounds below. Thus each entry is a minimum of at most 21 explicitly specified single-variable integer values.

Positive rows r	First row	Widths s	G_1	G_2	G_3
1	at most 9	1–9	427	331	459
2	at most 9	3–17	397	323	547
3	at most 9	6–24	419	134	486
4	at most 9	10–30	186	18	254
5	at most 8	15–30	490	168	234
5	equal to 9	19–32, omit 31	192	32	234
6	at most 8	21–32	821	213	0
6	equal to 9	24–32, omit 32	725	53	85
7	at most 8	28–32	1312	448	32
7	equal to 9	30–32	1216	384	96

In the two remaining cases, split by whether row two is at most seven or equal to eight. Apply the same row-length formula to the tail after the prescribed first row, setting row two to eight in the latter case.

(r, s)	Second row	Lower diagram L	$\mathcal{N}(G_1, L)$	$\mathcal{N}(G_2, L)$	$\mathcal{N}(G_3, L)$
$(5, 31)$	at most 7	$(9,7,6,5,4)$	155	153	151
$(5, 31)$	equal to 8	$(9,8,6,4,1)$	154	147	166
$(6, 32)$	at most 7	$(9,7,6,5,3,1)$	170	151	150
$(6, 32)$	equal to 8	$(9,8,6,4,2,1)$	169	150	167

For example the zero in the preceding table occurs when $s = 32$ and $L = (8,7,6,5,4,2)$. Every covered four-set there uses one of the admissible complementary edges, and the count for G_3 is 149. The two tables prove $32 \dim(HW) \ge 149s$ throughout the required interval.

16 Rounding at independent width 32

Suppose ten factors in a summand are independent. For the same common physical quadratic the position space is the actual algebra. The quartic gross bound $G = 210 - \dim L$ is an integer, because the independent summand has weight zero. The true increment satisfies $g + \delta \geq G$ on this same prefix. Let u be the new physical quadratic width and δ the increment in its intersection with the target, so $0 \leq \delta \leq u$.

The three fixed diagrams above give $32G \geq 149u$ below width 33 and $3G \geq 14u$ at widths 33 through 45. For $u \leq 31$ the distance from $149u/32$ to $14u/3$ is $u/96 < 1/3$. An integer strictly larger than $14u/3 - 1/3$ is at least $14u/3$. At $u = 32$ the guaranteed integer is 149, and the desired gross value is $149 + 1/3$. In the last ordering the actual target is $g \geq 11\delta/3$. Even at this width it holds whenever $\delta \leq 31$, since $149 \geq 14 \cdot 31/3$. Therefore a loss is possible only at

$$u = \delta = 32, \qquad g \geq 117.$$

Its size is at most $1/3$. The sum of all target increments in any one ordering is 2025. There can be at most $\lfloor 2025/32 \rfloor = 63$ such steps; their entire cost is at most 21. This does not require a bound for the sum of the unrelated widths u. Earlier orderings use only $g \geq 10u/3,\ 7u/2$, both immediate from $149/32 - 1 > 7/2$.

For the absolute estimates at $u = 1, \ldots, 5$, the leading multiplier graph misses at most ten of the 45 edges. A single input edge therefore has at least $\binom{8}{2} - 10 = 18$ distinct disjoint multiplier outputs. Two different input edges have at most seven common four-set outputs if they meet, and at most one otherwise. Consequently the gross bound is 18 at $u = 1$ and at least 29 at every $2 \leq u \leq 5$. Subtracting $\delta \leq u$ supplies each of the five absolute estimates, including $g \geq 8$ at unit width.

17 Nine factor directions and a single circuit

There are ten factors spanning a space of dimension nine. Write g for the unique circuit length and $c = 10 - g$ for the number of coloops. Thus $0 \leq c \leq 8$. In the squarefree ten-position coordinate algebra let σ_j send a squarefree position monomial of degree j to the corresponding product of physical factors. The kernel is $\ell\,\mathrm{SF}_{j-1}(C^c)$, with ℓ the unique relation, supported on the circuit C. Indeed a squarefree polynomial divisible by ℓ has quotient independent of every circuit variable and squarefree in the others. Consequently

$$f_2 = 45 - c, \qquad f_4 = 210 - \binom{c}{3}.$$

The radical of the degree-two position pairing has dimension $r_2 = \mathbf{1}_{c=7}$. For a circuit of length at least three, the algebra of its relation hyperplane has its only power-span deficit in degree one: in every higher degree the squarefree kernel of physicalization would have to be a multiple of ℓ with quotient independent of all circuit positions. Its only pairing radical is accordingly one-dimensional in degree $g - 1$. Tensoring with the squarefree algebra of the c coloops gives radical dimension $\binom{c}{2-g+1}$ in degree two. For a proportional pair ($g = 2$) the invariant subalgebra of the transposition has a perfect pairing and no degree-two radical. The potential source width u runs from 1 through $f_2 - r_2$.

Fix the common physical quadratic and an actual prefix. Write $H = \psi(I_2)$ for the position image of the permanent's quadratic annihilator and W for that of the degree-two operators annihilating the old quadratic intersection. Write $w = \dim W = u + r_2$; here w is a **dimension**, and w_i denotes the summand weight of this proof. The old-intersection quotient bound gives the following two estimates on the same gross gain G:

$$G \geq \dim(HW) + \epsilon_c, \qquad G \geq f_4 - \binom{f_2 - w - 1}{2} + \epsilon_c, qquad \epsilon_c = 28\mathbf{1}_{c=8}.$$

Binomial coefficients with top less than the bottom are zero. The first estimate is the compensation for the degree-four position radical, proved above: $w_i - r_4 = 105 - f_4 + R/2 \geq 0$. In the proportional-pair case $R = f_4 = 154$: rescale the two positions equally, use symmetry of the position image under their interchange, and restrict the central isomorphism furnished by the common quadratic to the invariant subalgebra. Thus $w_i - r_4 = 28$ there. For the second estimate, duality on position quartics gives $\dim(HW) \geq f_4 - \dim \mathcal{L}$ with $\mathcal{L} = \{F \in V_4 : I_2 F \subseteq K'\}$, $V_4 = \operatorname{im} \sigma_4$, and $\dim K' = f_2 - w$. The rank-two quartic response theorem in the quartic response classification gives $\dim(I_2 F) \geq 3$ for every nonzero $F \in V_4$. The genuine symmetric-matrix estimate then yields the displayed $\binom{f_2 - w - 1}{2}$ bound. Its independent inputs are the dimension and minimum derivative rank of the low-support quartics annihilated by the square of I_2, proved in the ideal-square classification. The proof of the rank-two response theorem uses the universal linear syzygies of the permanent quadrics and works over the algebraic closure; the resulting dimension inequalities descend to each characteristic-zero base field.

For $c \leq 5$, the objective is $G \geq 14u/3$. For $c = 6, 7, 8$, circuits of lengths $4, 3, 2$ force $w_i \geq 6, 21, 28$, respectively. They cannot enter either low-weight branch, and only $G \geq 13u/3$ is needed. In both statements g, as this proof denotes the net increment, is at least $G - \delta$ with $0 \leq \delta \leq u$.

The one protected vertex and the structural edges

Write $Z = H^{\perp}$. It has dimension $c + z'$ where $z' = \dim(\operatorname{im} \sigma_2 \cap P_2) \leq 9$. No nonzero formal member of Z is supported just on the star of a circuit position i: it would physicalize to $L_i v$, of rank at most two, whereas every nonzero target quadric has rank at least four. A member of $\ker \sigma_2$ cannot be supported on this star either, by its formula above and $g \geq 2$. Hence the projection of Z away from any such position is injective. Prioritize the complementary positions in the dual flag. Its complementary good pivots for H contain all nine edges at vertex 0, taken to be the protected circuit position. Since $z' \leq 9$, keep $h = 36 - c$ good edges including this star. This deletion only weakens the sought inequality. Use a common additive weight for the good and source leading monomials. A disjoint pair of them yields its four-set as a nonzero leading product. Thus $\dim HW$ is at least the number of distinct good/input disjoint-edge unions.

There is room to impose one more priority when choosing the dual flag. Put a *different* circuit position last, with label 9, and label the coloops by $9 - c, \ldots, 8$. Give priority first to avoiding the protected vertex, then to incidence with vertex 9. For each coloop p, the relation ℓs_p has leading edge $p9$ in this dual flag. These distinct pivots force all c specified edges to be absent from both the good and source leading sets: both original spaces lie in $(\ker \sigma_2)^{\perp}$. The excluded edges form an upper set in the componentwise order on ordered vertex pairs. Standard simultaneous shifts of both families toward earlier vertices thus preserve their absence, as well as preserving the good star at 0 and not increasing the disjoint-union shadow. The proof of this shifting fact is the two-vertex argument above. For $c \geq 6$ we impose the resulting cap $9 - c$ on the last-vertex degrees of both shifted graphs. The other cases only need the single protected vertex. No second protected circuit position is used.

18 Small widths and exceptional circuit counts

Write c for the number of coloops, $z \leq 9$ for the actual quadratic contact, u for the prefix width, $r = \mathbf{1}_{c=7}$, and $w = u + r$. The shifted good and source graphs have at least $45 - c - z$ and exactly w edges. The good graph has a complete star at vertex zero. The last vertex has degree at most $9 - c$ in both graphs. Write N for the number of disjoint good/source edge unions, and G for the gross physical gain. Then $G \geq N + 28\mathbf{1}_{c=8}$. The physical quadratic bound proves the target directly above the respective cutoffs $32, 31, 29, 27, 26, 25, 24, 23, 22$. The rates required below are $3G \geq 14u$ for $c \leq 5$ and $3G \geq 13u$ otherwise.

Here is the upper-width calculation. Let $r = \mathbf{1}_{c=7}$, $F = 210 - \binom{c}{3}$, $f = 45 - c$, $b = 14$ for $c \le 5$ and $b = 13$ otherwise. With the convention that $\binom{j}{2} = 0$ when $j < 2$, the bound just proved has margin

$$J_c(u) = 3\left(F + 28\mathbf{1}_{c=8} - \binom{f-u-r-1}{2}\right) - bu.$$

This is discretely concave, including its linear tail. The endpoint values at $t_c + 1$ and $f - r$ suffice throughout the indicated upper interval:

c	0	1	2	3	4	5	6	7	8
t_c	32	31	29	27	26	25	24	23	22
$J_c(t_c+1)$	3	17	12	1	6	2	11	15	13
$J_c(f-r)$	0	14	28	39	44	40	63	44	65

Thus the graph argument need only cover $u \le t_c$.

Thirty-five good edges and integer rounding

The three-diagram estimate of Section 15 applies to any shifted good graph on ten positions having a complete star and at least 35 edges. At $w \le 32$ its graphical conclusion is $32N \ge 149w$. For $w \le 31$,

$$14w/3 - 149w/32 = w/96 < 1/3.$$

An integer strictly greater than $14w/3 - 1/3$ is at least $14w/3$. This covers every $c + z \le 10$, including all graph widths with $c = 1$. For $c = 7$ the graph widths have $w \le 24$ and the required rate is weaker.

For $c = 0$, the only extra case is $w = 32$. Every strictly positive margin in the three-diagram table gives $N \ge 150$, since $32N - 149 \cdot 32$ is a multiple of 32. The only zero is for the lower graph with strict row lengths 9765431 and the input with lengths 876542, using one-based rows and vertices. Take exactly 36 good edges preserving the first star. If row two has length seven, the remaining rows after the first sum to 27 in distinct parts at most seven, hence consist of $7, 6, 5, 4, 3, 2$. If row two has length eight, the tails after $9, 8$ have sum 19 in distinct parts at most seven and are $65431, 75421, 7543, 76321, 7642, 7651$. Each contains 6531 componentwise or starts 765. Hence one of the two positive-margin diagrams $986531, 98765$ applies. Row two cannot be shorter than seven, since the largest possible remaining sum would be 21. In the second-row-seven case the old lower graph gains the edge 79. Together with input edge 68 this produces the previously absent four-set 6789, raising 149 to 150. On this quartet the only input edges are $67, 68$, whose respective good complements $89, 79$ were both absent. This proves the exact local rate at every $c = 0$ width.

The proportional-pair case

Set $c = 8$. Delete good edges preserving the first star until 28 remain. Vertex 9 has only the edge 09 in the good graph. Its source star is the same edge or empty. The other eight vertices, excluding 0 and 9, form the core. There are 19 core good edges. Let v be the number of core input edges, d the number of input edges joining 0 to the core, and $p \in \{0, 1\}$ the input indicator of 09. Then $u = v + d + p$, $v \le \binom{d}{2}$, and $p = 1$ forces $d = 8$. We need $u \le 22$.

Quartets avoiding 0 and 9 contribute at least v. Each core input edge has at least $15-(28-19) = 6$ disjoint good edges, and each four-set has at most six input-edge witnesses. Quartets containing both 0 and 9 contribute at least $\max(v, 19p)$. Quartets containing 0 alone correspond to core triples. Write T for their number. Pair each of the 19 good edges with the d possible input neighbors outside its endpoints. Each triple has at most three witnesses, so $3T \ge 19(d-2)$. Every triple containing an internal input edge qualifies as well, hence

$$T \ge 56 - K_3(28 - v), \qquad K_j\left(\binom{k}{2} + t\right) = \binom{k}{j} + \binom{t}{j-1}, \quad 0 \le t < k.$$

The clique bound for a prescribed number of edges is the degree-two Kruskal–Katona theorem.

When $p = 1$, the bounds $d = 8,\ v \le 13,\ T \ge 38$ give margin $3G - 13u \ge 138 - 10v \ge 8$. When $p = 0$, it suffices that $3T \ge 13d - 84 + 7v$. The first bound gives this for $7v \le 6d + 46$. Otherwise $v = 12, 13, 14, 15$; the largest possible d is respectively $6, 7, 8, 7$, using $v + d \le 22$ and $v \le \binom{d}{2}$. The clique formula gives $T \ge 36, 36, 40, 43$. Thus $3T$ is at least $108, 108, 120, 129$ against requirements at most $78, 98, 118, 112$. This proves the full $c = 8$ rate.

The graph giving equality at $c = 6$

Take K_8, distinguish its vertex 0, and add two exterior vertices each joined only to 0. Let D be any shifted source graph of width $w \le 24$. An edge between the exterior vertices would force every edge of K_{10}, so it is absent. Let a be the number of internal inputs, $p \le 7$ the internal degree of 0, and $b \ge d$ the exterior input degrees. Then $w = a + b + d$. If $b > 0$ we have $p = 7,\ a \ge \binom{8}{2} - \binom{8-b}{2}$, and consequently $b \le 3$. The shadow is at least

$$70 - K_4(28 - a) + H(b) + H(d) + (b - 1)_+, \quad H(0) = a - p, \qquad H(j) = 56 - \binom{8 - j}{3} \quad (j > 0).$$

Inside the eight vertices count all quartets containing any source edge. With a single exterior vertex of degree $j > 0$, every internal triple meeting its source prefix qualifies. Triples using its good edge already meet that prefix through 0. At $j = 0$ only the good edge is available. With both exterior vertices the other core vertex can be any input neighbor except 0.

For $4 \le m \le 21$, use $3K_4(m) \le 7m - 28$. The difference is concave between successive triangular numbers. At interval endpoints $4, 5, 6, 9, 10, 14, 15, 20, 21$ it equals $0, 7, 11, 29, 27, 43, 32, 37, 14$. If $b = d = 0$ and $a \ge 7$, this gives $3N \ge 13a$. For $a \le 7$, the minimum upper shadow on eight vertices occurs at a edges of a star and equals $\binom{7}{3} - \binom{7-a}{3} \ge 5a$, by concavity. If $b > 0,\ d = 0$, substitution followed by $a \le 24 - b$ gives

$$3N - 13w \ge 114 - 7b - 3\binom{8 - b}{3} \ge 2.$$

If $b \ge d > 0$, the same calculation gives

$$3N - 13w \ge 231 - 4b - 7d - 3\left[\binom{8 - b}{3} + \binom{8 - d}{3}\right] \ge 10.$$

The right sides are minimized at $b = 1$, respectively $b = d = 1$, because the positive exterior degrees are at most three. Equality occurs when $a = 24,\ b = d = 0,\ p = 7$. Four missing core edges cannot contain a full quadruple, so the 70 internal quartets and the two exterior sets of 17 quartets give exactly 104. This proves the full interval for this extremal good graph and its supergraphs.

Seven small widths simultaneously

For $c \le 5$ there are at least $e = 22$ good edges on the nine vertices left after removing the first star. For $c = 6, 7$ the corresponding values are 21 and 20. Write d for the length of the first source row and $v = w - d$ for the number of other inputs, all on those d vertices. Split the e remaining good edges into counts A, B, C inside the d neighbors, across the cut, and outside. The triples obtained from a first-star input have count at least

$$(d - 2)A/3 + (d - 1)B/2 + dC.$$

Count witnesses on triples with three, two, or one of the neighbors. Each internal edge gives $d - 2$ witnesses, each across-edge $d - 1$, and each outside edge d. Their maximum multiplicities are

three, two, and one. To minimize this linear expression fill A up to $\binom{d}{2}$, then B to $d(9-d)$, then C. Round up to the next integer. If $v > 0$, one input outside the first star supplies at least $e - 15$ additional quartets avoiding the distinguished vertex.

For $e = 22$, this gives gross bounds $14, 21, 22, 28, 29, 29, 35$ at $w = 1, \ldots, 7$. For $e = 21$ the bounds are $13, 19, 19, 25, 25, 25, 33$. For $e = 20$, $w = 2, \ldots, 8$ they are $17, 18, 23, 23, 23, 30, 30$. In each entry the possible d satisfy $d \le w \le d + \binom{d}{2}$. In the $e = 21, w = 6, d = 3$ case the three inputs off the first star form a triangle; in the $e = 20, w = 8, d = 4$ case four shifted edges on four vertices likewise contain a triangle. If x is the number of good edges on the six vertices outside the triangle, $x = 0$ is impossible. The triangle would support all good edges, yet the terminal vertex has residual degree at most two or one respectively. Their capacity is only 20 or 19. Thus $x \ge 1$. The number of quartets outside the initial star obtained from this triangle is at least

$$3x + \lceil (e - 3 - x)/3 \rceil \ge 9.$$

Each disjoint good edge gives three. Each other external vertex with a good edge to the triangle yields its union with the triangle. This improves the two borderline entries. All counts therefore attain the required rate through width seven. At widths one to five they also meet the absolute net gains after deducting at most u, including $g \ge 8$ at unit width. The already proved $c = 8$ rate and gross credit cover that count.

19 Disjoint edge unions in shifted graphs

Let G, D be two shifted graphs. The number $N(G, D)$ counts four-element sets that are the union of disjoint edges from the two graphs. If the strict row diagram of G has $m - 1$ positive rows then its first m vertices form a clique. All its other vertices are mutually independent. The outside neighborhoods are initial segments of the clique, with respective sizes $h_1 \ge \cdots \ge h_s$. This follows by moving each endpoint of an edge left as long as it stays distinct from the other endpoint.

Let a be the number of input edges within the clique and b_i the size of the input neighborhood of exterior vertex i within the clique. These neighborhoods are initial segments too. Write J for the graph of input edges between two exterior vertices, and $F(m, b) = \binom{m}{2} - \binom{m-b}{2}$. Since the input is shifted, every core edge meeting $[b_i]$ is present. In particular $a \ge F(m, b_i)$.

For n vertices, a graph of e edges and the first p vertices, define

$$A = \min(e, \tbinom{p}{2}), \qquad B = \min(e - A, p(n - p)), \qquad C = e - A - B,$$
$$T(n, e, p) = \lceil (p - 2)A/3 + (p - 1)B/2 + pC \rceil .$$

At $p = 0$ this gives zero. Among triples containing a distinguished vertex of $[p]$ whose complementary pair is an edge, the number is at least T. Indeed, for the actual numbers of edges inside, across and outside $[p]$ the linear expression follows by counting witnesses, whose respective multiplicities are at most three, two and one. Its coefficients increase, so filling the first two bins in order gives the displayed lower bound.

For $i < j$ define P_{ij} to be the number of ordinary unordered pairs $\{x, y\}$ in the clique with a matching from exterior vertices i, j to distinct x, y, one good edge and one input edge. In zero-based indices $x < y$, the explicit count is the sum over $x < y < m$ of the indicator of

$$(x < h_i,\ y < b_j) \ \vee\ (x < b_j,\ y < h_i) \ \vee\ (x < h_j,\ y < b_i) \ \vee\ (x < b_i,\ y < h_j).$$

Four disjoint layers give the following purely combinatorial inequality. Put $K_4(\binom{e}{2}+t) = \binom{e}{4}+\binom{t}{3}$

with $0 \le t < e$.

$$\begin{aligned} N(G,D) \;\ge\; & \binom{m}{4} - K_4\left(\binom{m}{2} - a\right) \\ & + \sum_i \left[\binom{m}{3} - \binom{m-b_i}{3} + T\big(m-b_i,\ a - F(m,b_i),\ (h_i-b_i)_+\big)\right] \\ & + \sum_{i<j} \begin{cases} \binom{m}{2} & ij \in J, \\ P_{ij} & ij \notin J, \end{cases} \\ & + \sum_{i<j<k} \max\big(\{0\} \cup \{h_\ell : \ell \in \{i,j,k\},\ \{i,j,k\} \setminus \{\ell\} \in J\}\big). \end{aligned}$$

In the first layer every clique quartet containing an input edge qualifies, so the missing ones are cliques of the complementary core input graph. In the second layer, with exterior vertex i, every triple meeting its input prefix qualifies because the complementary edge lies in the complete good clique. The other triples are disjoint from this prefix. Choose their good neighbor in the remaining good prefix and take a remaining core input edge. The sizes are exactly those used in T. With two exterior vertices their input edge, if present, accepts every core pair. Otherwise the two cross-edges are necessary, giving P_{ij}. For three exterior vertices the exterior input must be a pair and the remaining exterior vertex needs a core good neighbor. The union of the possible prefixes has cardinality equal to the stated maximum. Four exterior vertices cannot contribute.

20 Endpoint inequalities for an eight-vertex clique

Use the one-clique notation. For $0 \le a \le \binom{m}{2}$, put $f_k = \binom{m}{2} - \binom{m-k}{2}$ and $Y_k = \binom{m}{4} - \binom{m-k}{4}$. Let U_m be the polygonal function interpolating (f_k, Y_k). Its slope on the k-th interval is $\binom{m-k-1}{3}/(m-k-1)$, which decreases. The function is concave. The number of core quartets containing an input edge is at least $U_m(a)$. By Kruskal–Katona its minimum for a given $a = f_k + j$ is

$$Y_k + \binom{m-k-1}{3} - \binom{m-k-1-j}{3}.$$

The last two terms form a concave function of j, hence lie above the chord. This proves the claim.

Write $\tau(n,e,p) = (p-2)A/3 + (p-1)B/2 + pC$ with the bins A, B, C filled in that order as in the split-clique inequality; no rounding is taken. For a fixed outside profile (h_i, b_i), replace the first layer of the split-clique formula by $U_m(a)$ and replace T by τ. The resulting margin is concave on each interval of a between the following explicit breakpoints:

$$f_{b_i} + \binom{(h_i-b_i)_+}{2}, \qquad f_{b_i} + F\big(m-b_i, (h_i-b_i)_+\big).$$

Indeed, the τ-terms are affine between consecutive breakpoints and the other nonconstant term is concave. Only the two interval endpoints and these breakpoints need be compared.

Suppose the multiplier has maximum clique K_8. It has two exterior vertices. The $c = 6$ minimal multiplier has exterior degrees $(1,1)$, already handled for every width in Section 18. A $c = 7$ minimal multiplier has only 29 edges, and every K_8 with a full good star would have at least 30. It remains to prove the long-circuit rate with exterior degrees $(2,1)$ at widths up to 25 and with $(3,1)$ or $(2,2)$ at widths through 29. Every larger good graph in the relevant window contains one of these, since it has 31 or at least 32 edges. Widths at most seven were proved separately. For $(3,1), (2,2)$ only widths 26–29 remain after $(2,1)$.

There is no exterior input edge because it would force all ten positions to be a clique. For exterior input degrees $b \ge d \ge 0$ and a width range $[L, M]$, the interval of core edge counts is

$$a_{\min} = \max(f_b, L - b - d), \qquad a_{\max} = \min(28, M - b - d).$$

When $f_b + b > M$ the profile is empty. The following triangular arrays give lower bounds on $3N - 14(a + b + d)$. The b-th row is ordered by $d = 0, \dots, b$.

b	$(2,1)$, widths 8–25	$(3,1)$, widths 26–29	$(2,2)$, widths 26–29
0	4	19	34
1	6,17	5,5	23,17
2	17,31,73	2,5,50	17,17,23
3	46,61,103,133	5,11,58,87	47,50,58,87
4	empty	23,31,78,107, empty	65,70,81,110, empty

For direct pencil verification, every printed margin is the value at $a_{\max}$. In the two right columns the only other required endpoint is $a_{\min}$. In the left column include $a = 13$ in addition to the endpoints if $b = 0, 1$ and 13 belongs to the interval. All other τ-breakpoints lie at or outside the two endpoints. The polygon values at $a = 0, 7, 13, 18, 22, 25, 27, 28$ are $0, 35, 55, 65, 69, 70, 70, 70$; linear interpolation supplies the rest.

For example at good degrees $(3, 1)$, input degrees $(2, 0)$, the endpoints are $a = 24, 27$. At 27 the core value is 70, the two one-exterior values are $56 - \binom{6}{3} + (27 - 18) = 45$ and $27 - 7 = 20$, and the two-exterior value is 1. Their sum 136 gives margin $408 - 14 \cdot 29 = 2$. At 24 the corresponding values are $209/3, 42, 17, 1$, giving margin 25. Thus even the smallest entry can be checked with two ordinary substitutions. This certifies all the eight-clique rows analytically.

21 Endpoint inequalities for cliques of size six and seven

Windows

Write H for the trimmed number of good edges and m for the good clique size. At rate $3N \geq 14u$, prove $H = 31$ on $8 \leq w \leq 25$, $H = 32$ at $w = 26$, $H = 33$ at $w = 27$, and $H = 34$ at $w = 28, 29$. Each later case inherits the earlier widths by deleting a maximal non-star edge in a shifted diagram. At rate $3N \geq 13u$, prove $H = 30$ on $8 \leq w \leq 24$ with $w = u$ and terminal degree at most 3, and $H = 29$ on $9 \leq w \leq 24$ with $w = u + 1$ and terminal degree at most 2. Together with the three-coloop-count, small-width and eight-clique proofs, these cases prove all stated local rates.

At $m \leq 5$ there are at most four positive rows, of lengths bounded by $9, 8, 7, 6$. They attain 30 only by taking all four in full, violating terminal cap 3. The only four-row diagram of size 29 is 9875, whose terminal degree three violates cap 2. For $H \geq 31$, four rows cannot suffice. At $m \geq 9$ the full first star requires at least $\binom{9}{2} + 1 = 37 > 34$ edges. Thus $m = 6, 7, 8$. For $H = 29$ the eight-clique case is impossible since $28 + 2 > 29$.

If there is an exterior input edge, shiftedness gives K_{m+2} in the first $m + 2$ positions. This can happen here only at $H = 34, m = 6, w = 28, 29$. The good exterior degrees are $5, 5, 5, 4$, of sum 19. Within the first eight vertices the missing good edges are a triangle, so all 70 quartets admit a good edge and a complementary input. The next two vertices have five and four good neighbors in the first eight. Each accepts all triples meeting its good neighborhood, giving $56 - \binom{3}{3} = 55$ and $56 - \binom{4}{3} = 52$. The sum 177 exceeds $14 \cdot 29/3$. Thus the trees may assume no exterior input edge.

The leaf calculation

Let $[L, M]$ be the graph-width window. There are $s = 10 - m$ exterior vertices. Their good degrees form a nonincreasing vector h and their input degrees a nonincreasing vector b. Write $S = \sum b_i$; a counts internal input edges. Then

$$1 \leq h_i \leq m - 1, \quad \sum h_i = H - \binom{m}{2}, \quad 0 \leq b_i \leq m, \quad F(m, b_1) \leq a \leq \binom{m}{2}, \quad L \leq a + S \leq M.$$

The terminal cap bounds h_s, b_s where specified. Write k for 14 or 13 and $r = w - u$.

Each leaf supplies coordinatewise lower bounds $g \le h, d \le b$ and sum bounds $l \le S \le v$. Restrict both exterior neighborhoods to the printed prefixes, retaining every internal input edge. Use the split-clique formula at good and source prefixes g, d, with empty exterior input graph. In this formula use the concave function $U_m(a)$ interpolating $(F(m,t), \binom{m}{4} - \binom{m-t}{4})$ instead of the first line, and the same real linear expression τ before its ceiling instead of T. Call the resulting lower bound $B_m(g, d; a)$. Containment of the restricted graphs makes it valid for the original pair. The leaf suffices once

$$\Delta(a) = 6B_m(g, d; a) - 2k\big(\min(M, a + v) - r\big) \ge 0$$

on

$$A_- = \max(F(m, d_1), L - v) \le a \le A_+ = \min(\binom{m}{2}, M - l).$$

The endpoints to insert are $A_-, A_+, M - v$ and for each i the two values

$$F(m, d_i) + \binom{(g_i - d_i)_+}{2}, \qquad F(m, d_i) + F\big(m - d_i, (g_i - d_i)_+\big).$$

Discard values outside the interval and repeated endpoints. Between consecutive listed numbers Δ is concave, since U_m is concave and the other terms are affine there. Nonnegativity at the displayed endpoints proves it for the full interval. The factor 6 clears all denominators.

Reading the trees

At each split, a zero bit means the inequality holds and a one bit means it fails. An asterisk denotes the root. The symbols h_i, b_i, S are the actual parameters, indexed from one. Each branch partitions the integers into two exhaustive alternatives. At the leaves the first two numeric columns are g, d, the third is $[l, v]$, and the final two columns give every required a and the corresponding $\Delta(a)$, in order. An empty interval marks an impossible branch.

The following interval rule verifies the prefix bounds. For a nonincreasing vector with coordinate intervals $[l_i, u_i]$, first replace l_i by $\max_{j \ge i} l_j$ and u_i by $\min_{j \le i} u_j$. If the total is in $[T_-, T_+]$, entry i is at least the smallest z in its interval satisfying

$$\sum_{j<i} u_j + \sum_{j \ge i} \min(u_j, z) \ge T_-.$$

It is at most the largest z satisfying

$$\sum_{j \le i} \max(l_j, z) + \sum_{j>i} l_j \le T_+.$$

These necessary conditions follow by filling the other coordinates as large or as small as the proposed z allows. Apply the rule to h with its fixed sum and to b with the current bounds on S. For b, also impose

$$\max\left(\sum l_i, L - \binom{m}{2}\right) \le S \le \min\Big(\sum u_i, M - F(m, l_1)\Big).$$

Repeat whenever the total bounds tighten. Every step is necessary for an actual vector. Here each list has at most four entries. The recorded endpoints and margins allow the finite cases to be checked without searching for a minimizer.

31 good edges, clique size 6

Split level 0. `*=b1≤2`.
Split level 1. `0=S≤4`, `1=S≤11`.
Split level 2. `00=b1≤0`, `01=S≤5`, `10=S≤4`, `11=b1≤3`.
Split level 3. `000=h2≤4`, `001=b2≤1`, `011=h2≤4`, `101=b2≤0`.
Split level 4. `0010=b4≤0`, `0111=S≤6`, `1011=b1≤5`.
Split level 5. `00100=S≤3`, `10110=b2≤4`.
Split level 6. `001000=b1≤1`, `101100=S≤5`.
Split level 7. `0010000=b2≤0`, `1011000=h2≤4`, `1011001=b4≤1`.
Split level 8. `00100001=b3≤0`, `10110010=S≤10`, `10110011=h2≤4`.
Split level 9. `101100100=h3≤4`, `101100101=h1≤4`.
Split level 10. `1011001000=b1≤4`, `1011001001=S≤8`, `1011001011=h4≤2`.
Split level 11. `10110010000=b2≤1`, `10110010011=S≤9`, `10110010110=h3≤4`.
Split level 12. `101100100001=h1≤4`, `101100100111=b4≤0`, `101100101101=b2≤3`.
Split level 13. `1011001000011=S≤7`.
Split level 14. `10110010000111=b2≤3`.
Split level 15. `101100100001110=S≤9`.
Split level 16. `1011001000011100=h2≤4`.

Path	Good	Input	Sum range	Endpoints	Doubled margins
0000	4443	0000	0–0	8,12,14,15	16,48,82,144
0001	5531	0000	0–0	8,10,12,15	4,4,24,84
00100000	4431	1000	1–1	7,8,12,14,15	17,18,26,48,86
001000010	4431	1100	2–2	6,8,12,14,15	38,26,22,38,70
001000011	4431	1110	3–3	5,6,8,12,14,15	70,58,40,24,28,54
0010001	4431	2000	2–3	9,10,12,14,15	15,10,6,22,54
001001	4431	2110	4–4	9,10,12,14,15	59,48,32,30,50
00101	4431	1111	4–4	5,6,8,12,14,15	150,132,102,62,54,74
0011	4431	2200	4–4	9,10,12,14,15	56,42,26,30,50
010	4431	2110	5–5	9,10,12,14,15	31,20,4,2,22
0110	4443	2210	6–8	9,10,12,14,15	52,38,22,26,58
01110	5531	2210	6–6	9,12,15	63,24,30
01111	5531	2221	7–8	9,12,15	142,76,46
100	4431	3000	3–4	12,14,15	2,12,38
1010	4431	5000	5–6	15	54
10110000	4443	3100	5–5	12,14,15	76,86,124
10110001	5531	3100	5–5	12,13,15	19,24,40
101100100000	4432	3110	6–7	12,14,15	8,12,32
1011001000010	4444	3200	6–10	12,14,15	14,0,38
10110010000110	5432	3200	6–7	12,13,14,15	2,7,15,32
101100100000111000	5443	3210	8–9	12,13,14,15	36,35,37,60
101100100000111001	5532	3210	8–9	12,13,15	0,2,12
1011001000011101	5432	3320	10–10	12,13,14,15	26,16,9,8
101100100001111	5432	4400	8–10	14,15	30,32
10110010001	4432	5100	6–10	15	14
10110010010	5551	3100	6–8	12,13,15	7,6,10
101100100110	5551	3200	9–9	12,13,15	12,8,6
1011001001110	5551	4310	10–10	14,15	24,14
1011001001111	5551	3211	10–10	12,13,15	65,52,32
1011001010	4444	4310	11–11	14	20
101100101100	5542	4310	11–11	14	11
1011001011010	5551	4320	11–11	14	5
1011001011011	5551	4410	11–11	14	11
10110010111	5433	4310	11–11	14	8
101100110	4443	3222	9–11	12,14,15	52,20,56

Path	Good	Input	Sum range	Endpoints	Doubled margins
101100111	5531	3222	9–11	12,13,14,15	94,78,65,80
101101	4431	5500	10–10	15	20
10111	4431	6100	7–10	15	2
110	4431	3333	12–12	12,13	72,44
111	empty				

31 good edges, clique size 7

Split level 0. `*=S`$\leq$`7`.
- Split level 1. `0=b1`$\leq$`0`, `1=b1`$\leq$`3`.
- Split level 2. `00=h1`$\leq$`5`, `01=b2`$\leq$`2`.
- Split level 3. `010=b3`$\leq$`1`.
- Split level 4. `0100=b1`$\leq$`4`.
- Split level 5. `01000=S`$\leq$`6`.
- Split level 6. `010000=S`$\leq$`1`.
- Split level 7. `0100001=b1`$\leq$`1`.
- Split level 8. `01000010=b3`$\leq$`0`, `01000011=S`$\leq$`2`.
- Split level 9. `010000111=b2`$\leq$`0`.
- Split level 10. `0100001110=b1`$\leq$`3`, `0100001111=h1`$\leq$`4`.
- Split level 11. `01000011111=h2`$\leq$`2`.
- Split level 12. `010000111111=b2`$\leq$`1`.
- Split level 13. `0100001111110=b3`$\leq$`0`.
- Split level 14. `01000011111100=b1`$\leq$`2`.

Path	Good	Input	Sum range	Endpoints	Doubled margins
000	431	000	0–0	8,15,18,21	10,21,42,102
001	621	000	0–0	8,11,15,21	17,20,36,72
0100000	421	100	1–1	7,9,11,18,21	30,20,18,20,44
010000100	421	110	2–2	6,9,11,18,21	94,70,62,22,28
010000101	421	111	3–3	6,9,11,18,21	180,138,118,36,24
010000110	421	200	2–2	11,12,18,21	38,34,16,22
01000011100	421	300	3–3	15,18,21	48,24,12
01000011101	421	400	4–4	18,21	32,2
01000011110	432	210	3–6	11,12,15,18,19,21	22,15,3,6,14,86
010000111110	622	210	3–6	11,17,19,21	16,8,8,62
010000111111000	531	210	3–3	11,14,15,20,21	106,73,66,52,60
010000111111001	531	310	4–5	15,16,20,21	40,32,8,38
01000011111101	531	211	4–6	11,14,15,19,20,21	112,61,48,14,32,62
0100001111111	531	220	4–6	11,14,15,19,20,21	88,46,36,2,20,50
010001	421	421	7–7	18	8
01001	421	500	5–5	20	8
0101	421	222	6–7	11,12,18,19	192,170,44,56
011	421	330	6–7	15,18,19	74,14,26
10	421	332	8–9	15,16,17	132,106,108
11	empty				

32 good edges, clique size 6

Split level 0. `*=b1`$\leq$`4`.
- Split level 1. `0=b2`$\leq$`3`.
- Split level 2. `00=b1`$\leq$`3`.

Path	Good	Input	Sum range	Endpoints	Doubled margins
000	5442	3332	11–12	14,15	49,64
001	5442	4321	11–12	14,15	10,34
01	5442	4420	11–12	14,15	4,28
1	5442	5200	11–11	15	16

32 good edges, clique size 7

Split level 0. `*=b1`$\leq$`3`.
  Split level 1. `0=b3`$\leq$`1`, `1=b1`$\leq$`4`.
  Split level 2. `00=h3`$\leq$`2`, `10=h3`$\leq$`1`.
  Split level 3. `000=S`$\leq$`5`, `100=S`$\leq$`6`.
  Split level 4. `0001=h1`$\leq$`5`, `1001=h1`$\leq$`5`.

Path	Good	Input	Sum range	Endpoints	Doubled margins
0000	531	210	5–5	21	4
00010	541	320	6–7	19,20	1,22
00011	631	320	6–7	19,20	4,22
001	433	210	5–7	19,21	28,112
01	431	222	6–9	17,18,20	35,46,82
1000	541	410	5–6	20,21	16,46
10010	551	420	7–8	18,19	22,34
10011	641	420	7–8	18,19	10,28
101	432	410	5–8	18,21	22,94
11	431	500	5–6	20,21	22,46

33 good edges, clique size 6

Split level 0. `*=b1`$\leq$`4`.
  Split level 1. `0=b2`$\leq$`3`.

Path	Good	Input	Sum range	Endpoints	Doubled margins
00	5543	3332	12–13	14,15	66,84
01	5543	4420	12–13	14,15	24,54
1	5543	5200	12–12	15	30

33 good edges, clique size 7

Split level 0. `*=b1`$\leq$`3`.
  Split level 1. `0=b3`$\leq$`1`, `1=h3`$\leq$`1`.
  Split level 2. `00=h2`$\leq$`4`, `10=S`$\leq$`7`, `11=S`$\leq$`8`.
  Split level 3. `101=b1`$\leq$`4`, `111=h3`$\leq$`2`.

Path	Good	Input	Sum range	Endpoints	Doubled margins
000	432	320	6–7	20,21	24,48
001	551	320	6–7	20,21	0,36
01	431	222	6–9	18,21	18,72
100	651	400	6–7	20,21	3,42
1010	651	420	8–9	18,19	6,18
1011	empty				
110	432	400	6–8	19,21	0,60
1110	542	410	9–9	18	0
1111	433	410	9–9	18	24

34 good edges, clique size 6

Split level 0. `*=b1≤5`.

Path	Good	Input	Sum range	Endpoints	Doubled margins
0	5554	4320	13–15	14,15	1,34
1	5554	6310	13–14	15	130

34 good edges, clique size 7

Split level 0. `*=b3≤1`.
- Split level 1. `0=b1≤5`.
- Split level 2. `00=S≤9`.
- Split level 3. `000=h3≤2`.
- Split level 4. `0000=h2≤5`.
- Split level 5. `00001=b3≤0`.
- Split level 6. `000010=b1≤4`.

Path	Good	Input	Sum range	Endpoints	Doubled margins
00000	652	310	7–9	19,20,21	11,10,52
0000100	661	430	7–8	20,21	35,22
0000101	661	520	7–9	20,21	13,28
000011	661	311	7–9	19,20,21	32,22,40
0001	543	310	7–9	19,20,21	11,22,70
001	empty				
01	541	600	7–8	21	16
1	541	322	7–14	15,16,18,20,21	7,12,28,58,82

30 good edges, clique size 6

Split level 0. `*=b1≤2`.
- Split level 1. `0=b3≤1`, `1=b1≤4`.
- Split level 2. `00=S≤4`, `10=S≤5`.
- Split level 3. `000=b1≤0`, `101=b2≤1`.
- Split level 4. `0001=S≤2`, `1011=S≤7`.
- Split level 5. `00011=b1≤1`, `10111=b2≤3`.
- Split level 6. `101110=b4≤1`.
- Split level 7. `1011100=b3≤2`.
- Split level 8. `10111000=h4≤2`.
- Split level 9. `101110000=h4≤1`.

Path	Good	Input	Sum range	Endpoints	Doubled margins
0000	4431	0000	0–0	8,12,14,15	8,36,68,114
00010	4431	1000	1–2	6,8,12,14,15	4,10,26,52,92
000110	4431	1110	3–4	5,6,8,12,14,15	60,50,36,28,36,64
000111	4431	2100	3–4	9,10,12,14,15	46,40,34,48,76
001	4431	2110	5–6	9,10,12,14,15	33,24,12,14,36
01	4431	2220	6–8	9,10,12,14,15	62,44,20,4,14
100	4431	3000	3–5	12,14,15	8,22,50
1010	4431	3110	6–7	12,14,15	10,6,22
10110	4431	3200	6–7	12,14,15	4,6,22
1011100000	5541	3210	8–10	12,13,14,15	16,8,3,36
1011100001	5432	3210	8–10	12,13,14,15	4,5,9,48

Path	Good	Input	Sum range	Endpoints	Doubled margins
101110001	4433	3210	8–10	12,14,15	4,18,66
10111001	4431	3330	9–11	12,13,14,15	20,0,6,24
1011101	4431	3222	9–12	12,14,15	48,66,90
101111	4431	4400	8–10	14,15	0,24
11	4431	5000	5–9	15	18

30 good edges, clique size 7

Split level 0. `*=b1`$\leq$`2`.
- Split level 1. `0=b2`$\leq$`1`, `1=b2`$\leq$`1`.
- Split level 2. `00=b2`$\leq$`0`, `10=b1`$\leq$`3`.
- Split level 3. `000=b1`$\leq$`0`.

Path	Good	Input	Sum range	Endpoints	Doubled margins
0000	321	000	0–0	8,11,15,21	5,8,24,90
0001	321	100	1–2	6,7,11,15,21	23,18,10,14,44
001	321	110	2–4	6,7,11,15,20,21	52,44,24,4,0,24
01	321	220	4–6	11,15,18,20	98,42,6,30
100	321	300	3–5	15,19,21	32,6,42
101	321	400	4–6	18,20	24,60
11	321	320	5–9	15,19	6,36

29 good edges, clique size 6

Split level 0. `*=b1`$\leq$`2`.
- Split level 1. `0=b3`$\leq$`1`, `1=b1`$\leq$`4`.
- Split level 2. `00=S`$\leq$`4`, `10=S`$\leq$`6`.
- Split level 3. `000=b1`$\leq$`0`, `101=b3`$\leq$`0`.
- Split level 4. `0001=S`$\leq$`2`, `1011=S`$\leq$`7`.
- Split level 5. `00011=b1`$\leq$`1`, `10111=S`$\leq$`8`.
- Split level 6. `101111=b2`$\leq$`2`.
- Split level 7. `1011111=h1`$\leq$`4`.
- Split level 8. `10111111=b1`$\leq$`3`.

Path	Good	Input	Sum range	Endpoints	Doubled margins
0000	4421	0000	0–0	9,14,15	26,82,122
00010	4421	1000	1–2	7,8,9,14,15	15,15,19,60,94
000110	4421	1110	3–4	5,8,9,14,15	86,56,54,50,72
000111	4421	2100	3–4	9,10,14,15	36,36,44,66
001	4421	2110	5–6	9,10,14,15	44,38,22,38
01	4421	2220	6–8	9,10,14,15	88,70,18,22
100	4421	3000	3–6	12,14,15	8,10,32
1010	4421	4300	7–8	14,15	30,28
10110	4421	3110	7–7	12,14,15	36,20,30
101110	4421	3210	8–8	12,14,15	28,6,10
1011110	4421	3221	9–10	12,14,15	54,8,26
10111110	4442	3310	9–12	12,14,15	20,56,98
101111110	5421	3320	9–11	12,13,14,15	40,20,29,44
101111111	5421	4310	9–10	14,15	14,32
11	4421	5000	5–9	15	14

29 good edges, clique size 7

Split level 0. `*=b1`$\leq$`2`.
Split level 1. `0=b2`$\leq$`1`, `1=b2`$\leq$`1`.
Split level 2. `00=b1`$\leq$`0`, `10=b1`$\leq$`4`.
Split level 3. `001=S`$\leq$`2`, `100=h1`$\leq$`5`.
Split level 4. `0011=h1`$\leq$`5`, `1001=S`$\leq$`4`.

Path	Good	Input	Sum range	Endpoints	Doubled margins
000	311	000	0–0	9,15,21	26,26,56
0010	311	100	1–2	7,15,21	26,10,4
00110	321	110	3–4	6,7,11,15,20,21	78,70,50,30,26,50
00111	611	110	3–4	6,16,20,21	96,42,14,32
01	311	220	4–6	11,15,18,20	124,68,32,56
1000	321	300	3–6	15,18,21	32,14,68
10010	611	300	3–4	15,18,20,21	54,24,8,26
10011	611	310	5–6	15,18,19	56,8,20
101	empty				
11	311	320	5–9	15,19	32,62

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