Exact covering deficits and exclusive residue classes for divisor families

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Abstract

Choose one congruence class for each nonunit divisor of $3^a qr$, where $q, r \geq 5$ are distinct primes. We determine the minimum number of missed residues for every $a \geq 1$. The excess above the signed independence-polynomial count is zero at height one, three at height two and $4 \cdot 3^{a-2}$ thereafter. The first three ternary digits force the entire excess, and an explicit assignment attains it at every height. More generally, for a family closed under nonunit divisors, we characterize when all pairs with noncoprime moduli can be separated simultaneously. A cycle on each prime support gives an optimizer whenever the criterion holds.

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1 The extremal function

For $N > 1$, let $u(N)$ be the minimum size of the complement in $\mathbb{Z}/N\mathbb{Z}$ of one residue class for every divisor $d > 1$ of N. Maximizing the number covered over subfamilies of these divisors gives the same quantity after taking the complement, since additional classes can only increase coverage. The Erdős–Selfridge question asks whether $u(N) = 0$ for some odd $N > 1$. Here the period has three specified prime coordinates. Digit boxes in this setting go back to Berger, Felzenbaum and Fraenkel [5]. McNew and Setty study the corresponding maximal covered proportion and its relation to covering numbers [1]. The values $u(315) = 74$ and $u(945) = 191$ already appear in the literature and computational archives [2, 3].

Theorem 1. *Let $a \geq 1$, let $q, r \geq 5$ be distinct primes, and put $n = 3^a$, $M = (n-1)/2$, $L = (n+1)/2$. Then*

$$u(nqr) = Lqr - n(q+r) + M + D_a, \qquad D_a = \begin{cases} 0 & a = 1, \\ 3 & a = 2, \\ 4 \cdot 3^{a-2} & a \geq 3. \end{cases}$$

In particular $u(9qr) = 5qr - 9q - 9r + 7$. The term before D_a is the signed independence-polynomial count of the gcd graph on nonunit divisors, scaled by nqr. Indeed an independent set has pairwise disjoint prime supports. Summing over the partitions of each used support gives normalized coefficients $1, -1, 0, 1$ in support sizes zero through three. Substitution of the prime weights $M/n, 1/q, 1/r$ gives $1 - M/n - 1/q - 1/r + M/(nqr)$. At height one this recovers the three-prime squarefree formula considered by Öztürk [4]. At greater heights a repeated ternary coordinate forces intersections. We count them directly. In Section 4 we characterize the divisor families in which every such intersection can be avoided. The criterion also explains the transition between heights one and two in Theorem 1.

2 A three-level lower bound

By the Chinese remainder theorem, each ternary residue is the base of a q-by-r grid. Write digits in order from least significant to most significant. At depth $i = 1, \dots, a$ denote by C_i, X_i, Y_i, Z_i the chosen prefixes for $3^i, 3^i q, 3^i r, 3^i qr$. Over a leaf on which a prefix is active, it removes respectively a whole slice, one row, one column or one point. First remove the row and column of the moduli q, r. The residual grid has $A = q - 1$ rows and $B = r - 1$ columns. The modulus qr can remove one further point.

At a leaf, let p, u, v, s count active prefixes of types C, X, Y, Z. Define the slack δ by expressing its uncovered count as

$$(1 - p)(AB - 1) - Bu - Av - s + \delta.$$

Each prefix family has $M = \sum_{i=1}^{a} 3^{a-i}$ incidences over the leaves with multiplicity. Summing all terms other than δ gives $L(AB - 1) - M(A + B + 1) = Lqr - n(q + r) + M$.

Lemma. *If $p \geq 1$, then $\delta = (p - 1)(AB - 1) + Bu + Av + s$. If $p = 0$, then $\delta \geq mk$ for every $0 \leq m \leq \min(u, A)$, $0 \leq k \leq \min(v, B)$.*

Proof. A pure prefix covers the whole slice. Otherwise, any m active rows and k active columns together cover at most $Bm + Ak - mk$ cells. Coincidences or lines outside the residual grid can only reduce the union. Other rows and columns cover at most B and A additional cells each. The root point and s other points cover at most $1 + s$ more. □

Write $h_i = 3^{a-i}$ for the number of leaves below a depth-i prefix. Every slack is nonnegative, proving the required lower bound at $a = 1$. For $a \geq 2$ consider the three first digits of C_1, X_1, Y_1.

If C_1 equals either of the other two, the slack is at least four on each of h_1 leaves. If all three differ, C_2 must meet an earlier pure, row or column prefix, contributing at least $4h_2 = h_1 + h_2$. This uses $A, B \geq 4$.

The remaining possibility is $X_1 = Y_1 = H \neq C_1$. The shared first digit contributes at least h_1, which suffices for $a = 2$. For $a \geq 3$, consider C_2, X_2, Y_2. A prefix under C_1 adds at least four per leaf there. One under H increases the local minimum from one to at least two. Either gains the required h_2. If all three are under the third first digit and two coincide, the gain there is again at least h_2. If all three differ, every leaf now belongs to some earlier pure, row or column prefix. The class C_3 then forces slack at least four on its cylinder, where at most one per leaf was previously charged. This gains at least $3h_3 = h_2$, proving the lower bound.

3 Attaining all heights

At $a = 1$ take $(C_1, X_1, Y_1, Z_1) = (0, 1, 2, 1)$. At $a = 2$ take $(0, 1, 1, 1)$ at the first level and $(20, 21, 22, 21)$ at the second. For $a \geq 3$ let b_i be the word consisting of i twos, including the empty word b_0. Concatenation in the following table denotes a ternary word.

Depth i	C_i	X_i	Y_i	Z_i
1	0	1	1	1
2	20	10	21	21
$i \geq 3$	$b_{i-1}0$	$b_{i-1}1$	$21b_{i-3}0$	$b_{i-1}1$

Give both root lines digit 0, the root point digits $(1, 1)$, and every nonroot point digits $(2, 2)$. Give each nonroot row digit 3 except X_2, which takes 4 when $a \geq 3$. Give columns digit 3 at depths one and two and digit 4 thereafter.

The pure prefixes are disjoint and avoid every nonpure positive-depth prefix. The point prefixes are mutually disjoint and their grid coordinates avoid the lines. Nested row or column prefixes have different grid digits. Thus the slack is zero over pure prefixes and exactly uv on each remaining leaf. For $a = 1$ the row and column prefixes are disjoint. For $a = 2$ the only

intersection is $X_1 \cap Y_1$. For $a \geq 3$ these are exactly $X_1 \cap Y_1$ and $X_2 \cap Y_1$: later columns lie under 21, away from the later rows under 22. Their total sizes are $0, 3, h_1 + h_2$ in the respective regimes. This attains the lower bound and proves Theorem 1.

4 When noncoprime classes can be separated

Let $\mathcal{D}$ be a finite collection of integers greater than one, closed under taking divisors greater than one, and put $N = \operatorname{lcm} \mathcal{D}$. On the primes dividing N form the graph in which p and q are adjacent if $pq \in \mathcal{D}$. Let $b_p = 1 + \deg(p)$ and let h_p be the largest exponent of p in a modulus. Call an assignment *exclusive* if classes for distinct moduli with a common prime factor are disjoint.

Theorem 2. *An exclusive assignment exists exactly when $p \geq b_p$ at every vertex and $h_p = 1$ wherever $p = b_p$. For the full divisor family of $N = \prod_{i=1}^{k} p_i^{a_i}$, this says $p_i \geq k$ for every i, with $a_i = 1$ at equality.*

Proof. The moduli p and pq for all its neighbors q have pairwise gcd p. Their classes must use b_p different first digits at p. If these exhaust the alphabet, every class modulo p^2 meets one of them. Divisor closure makes p^2 compulsory whenever $h_p \geq 2$.

Conversely, label distinct digits at p by p and each neighbor. When $p > b_p$, reserve a further digit e_p. For a nonempty prime support S occurring in $\mathcal{D}$, fix one cycle σ_S on S, allowing a one-cycle. For a modulus $\prod_{p \in S} p^{a_p}$ prescribe at p the word consisting of $a_p - 1$ copies of e_p followed by the digit for $\sigma_S(p)$. The successor belongs to the local alphabet by divisor closure. Where no escape digit is available the word has length one. The Chinese remainder theorem gives a class with these prescriptions.

Two compatible words at a common prime have the same length and terminal symbol. If two classes with intersecting supports meet, the two successors agree on the intersection. Starting at any point of the intersection and following either cycle then visits its entire support inside the intersection. Hence the supports agree, and the lengths at all their primes agree as well. These are the same modulus. □

For completeness we give the density achieved by this construction. Let G be the graph on $\mathcal{D}$ joining noncoprime pairs, and put

$$Z(T) = \sum_{J \subseteq T \text{ independent in } G} (-1)^{|J|} \prod_{d \in J} d^{-1}.$$

The independence-polynomial optimum for an exclusive assignment is prior work [6, 7]. Theorem 2 determines exactly when the hypothesis holds for divisor-closed families. Schroeder also gives a sufficient condition for arbitrary moduli based on the number of moduli through each prime. In a squarefree full family that count is 2^{k-1}; the local alphabet in Theorem 2 has size k.

Proof of the density formula. In the constructed assignment, an independent intersection has density $\prod_{d \in J} 1/d$ by CRT and all other intersections are empty. Inclusion–exclusion gives uncovered density $Z(T)$. It is positive for every T. Indeed in each connected component on m primes, the largest prime is greater than m, so it has an escape digit. Root a spanning tree there. Use only escape digits at the root and at each other prime choose the digit of its tree parent as first digit. A matched support would have to follow parent edges up to the root, where no word matches.

For any assignment let $H(T)$ be its uncovered density. Given $d \in T$, write $T' = T \setminus \{d\}$ and $U = T \setminus N_G[d]$. The class for d is independent of the whole family on U, whose period is coprime to d. Thus $H(T) \geq H(T') - H(U)/d$, while $Z(T) = Z(T') - Z(U)/d$. Induct on $|T|$ to prove $H(T)/H(T') \geq Z(T)/Z(T') > 0$. Multiplying earlier ratios along a chain from U to T' gives $H(U)/H(T') \leq Z(U)/Z(T')$. Substitution proves the induction step, starting with empty-set density one. This is Shearer's ratio argument. Multiplication along a full chain proves optimality. □

For the full divisor family set $X_i = \sum_{e=1}^{a_i} p_i^{-e}$. An independent collection partitions its union of prime supports into blocks and chooses one exponent at each prime. Thus in the exclusive range the complementary Bell expression of McNew and Setty [1] gives

$$u(N)/N = \sum_{I \subseteq [k]} b_{|I|} \prod_{i \in I} X_i, \qquad b_h = \sum_{\pi \text{ a partition of } [h]} (-1)^{|\pi|}.$$

Here $b_0 = 1$. In particular $u(pqr) = pqr - pq - pr - qr + 1$ for three distinct odd primes. Theorem 1 computes the precise excess when this family loses exclusivity at the repeated prime 3. The qualifications on the use of the Bell expression in general covering problems are discussed by Gokalp [8].

Verification and provenance

A second lower-bound check enumerates only prefixes at depths $1, \ldots, b$ with $b = \min(a, 3)$. If the truncated counts are p_0, u_0, v_0, every descendant has slack at least $15(p_0 - 1) + 4(u_0 + v_0)$ for $p_0 > 0$, and at least $u_0 v_0$ for $p_0 = 0$. In the latter case $u_0, v_0 \leq 3$ and a later pure prefix incurs at least $4(u_0 + v_0)$. Enumerating the 729 possible single-family profiles at depth three and the five pure-union shapes up to tree symmetry gives a minimum of 12. At depths one and two the minima are 0 and 3. The verification program also reconstructs the CRT classes and counts their union directly. A second implementation obtains the same three minima with integer dynamic programming. At a node it assigns each remaining depth/type label to one of the three children, carrying the counts already active. At a leaf it returns the lower bound above. Additivity across the children and induction on the height give the exact minimum over all prefix prescriptions. This computation uses no tree-isomorphism reduction. The written case split proves the lower bound independently of either enumeration. Lean checks the prefix separation and the cycle-support implication in Theorem 2. The arithmetic reduction and the extremal counts have written proofs. Chengyi Fang initiated, funded and directed the work and is the contact for corrections. AI systems derived and independently rechecked the arguments, wrote the programs and prepared the manuscript. The sponsor has not independently checked the proofs.

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