Deterministic symmetric approximation of Sidon window–table witnesses

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Abstract

A finite joint window–table law in the Sidon-set method of Carter, O'Bryant and Tao can be approximated by a single signed symmetric pair on a finite rational grid. The window reproduces the mean autocorrelation in the integral and uniform norms. Its convolution with the table reproduces the symmetrized covering profile in both norms after extension of the plateau. Both original pointwise constraints hold exactly at every stage, and both actual energy costs converge along the same sequence. Smooth compactly supported pairs have the same property. A two-cell input forces the sufficient cardinality threshold to grow at least quadratically in the logarithm of the inverse profile error when the product budget is retained. Together with the sharp bound of Akwei and Williams, this identifies the infimum over the deterministic symmetric intersection as $4\sqrt{2}/3$. An explicit single rational pair gives $2\sqrt{\alpha\beta} = 1.88562173355792\ldots$ with sufficient integer threshold 12595. Its two exact costs improve those of the finite two-window example of Akwei and Williams, whose threshold is 473.

1 Which features of a random witness can one pair retain?

Carter, O'Bryant and Tao associate a coefficient $b = 2\sqrt{\alpha\beta}$ to a finite joint law of windows and tables. It governs the $k^{3/2}$ correction to a diameter lower bound with leading term k^2 for a Sidon set of size k. The finite threshold also depends on window width [2, Theorem 2.3]. Their Conjecture 7.1 asks whether the unrestricted infimum is strictly smaller than each of the deterministic and the symmetric infima. Akwei and Williams have already disposed of both strict inequalities and identified the common value as $4\sqrt{2}/3$. Their constructions use a deterministic asymmetric ramp for one restricted class and symmetric mixtures for the other. They also prove a positive gap for a single fixed nonnegative symmetric window, allowing signed random tables [1].

We ask for a single pair which is deterministic and symmetric and which retains the data of *each given input*. This is a simultaneous approximation problem for functions and their costs. A spectral factorization of the window alone leaves the table and its pointwise lower bound undetermined.

The input. Take $v, \alpha, \beta > 0$, $\sigma \geq 2v$, and probabilities p_i on finitely many pairs of bounded integrable real functions (w_i, t_i). Each window has integral one and is supported in $[0, v]$. Each table is supported in $[-v, \sigma]$ and equals one on $(0, \sigma - v)$. Put $\widetilde{w}(x) = w(-x)$ and set

$$A = \textstyle\sum_i p_i w_i * \widetilde{w}_i, \qquad B = \sum_i p_i \|w_i\|_2^2,$$
$$C = \textstyle\sum_i p_i w_i * t_i, \qquad H = \sum_i p_i \|t_i\|_2^2 - \sigma.$$

The remaining conditions are $A \geq 0$ on $\mathbb{R}$, $C \geq 1$ on $(0, \sigma)$, $B \leq \beta$ and $H \leq \alpha$ [2, Definition 2.1]. The convolutions are continuous by continuity of translation in L^2, so the cover extends to the endpoints. The same index labels both functions in the mixed convolution. The additional requirement $w_i, t_i \geq 0$ defines the strong class.

The target profile. Reflect both functions of each pair simultaneously about their prescribed centres $v/2$ and $(\sigma - v)/2$. Giving the pair and its reflection weight $p_i/2$ each preserves A, B, H, the geometry and feasibility. Its cover is $(C(x) + C(\sigma - x))/2$. Thus symmetry of the two *expectations*, one of the readings in [2, §7.1], already costs nothing for each individual feasible triple.

Use separate translations to centre both functions at zero. The covered interval is now $[-L, L]$, initially $L = \sigma/2$. For larger L, move each table shoulder with its endpoint and fill the middle with ones. Its support is contained in $[-L - v/2, L + v/2]$, and it equals one for $|x| < L - v/2$. This adds the same length to its squared norm and the covered interval. Write C_L^+ for the cover of this extended reflected law. Equivalently it is the even part of the extended input cover. Two even factors have an even convolution, which explains the choice of target.

Theorem 1. *For each input law there are $D_j \to \infty$ and deterministic even WT pairs (V_j, T_j) of covering length $2D_j$, with rational breakpoints and heights on finite uniform rational grids (constant on open cells, rational at the boundaries), such that*

$$\|V_j * V_j - A\|_1 + \|V_j * V_j - A\|_\infty \longrightarrow 0, \qquad \|V_j * T_j - C_{D_j}^+\|_1 + \|V_j * T_j - C_{D_j}^+\|_\infty \longrightarrow 0,$$
$$\|V_j\|_2^2 \longrightarrow B, \qquad \|T_j\|_2^2 - 2D_j \longrightarrow H.$$

At every stage $V_j * V_j \geq 0$ on $\mathbb{R}$ and $V_j * T_j \geq 1$ on $[-D_j, D_j]$.

2 Proof of Theorem 1

In this section w_i denotes a centred input window in the reflected law, and $t_{i,L}$ its extended table. An offset used in sampling may have a continuous distribution. The resulting WT witness consists of one pair.

Proof. Sampling with a common offset. Choose a rational $h > 0$. Put $b_h = h^{-1}\mathbf{1}_{[-h/2,h/2]}$. Use the joint probability space of the input atom i and an independent uniform $u \in [-h/2, h/2]$. For $k \in \mathbb{Z}$ define

$$x(k) = h(b_h * w_i)(kh + u), \qquad y(k) = t_{i,L}(kh - u).$$

Here $L = Nh$ is eventually taken arbitrarily large. Translates of the averaging interval tile the line up to null sets, so $\sum_k x(k) = 1$ for every i, u. The sequences x have common finite support and $|x(k)| \leq \sqrt{h}\,\|w_i\|_2$. For sequences, write $*$ for discrete convolution and $\widetilde{x}(k) = x(-k)$. Integrating over the offset and summing over its translated cells gives the identities

$$\mathbb{E}(x * \widetilde{x})(k) = h(b_h * b_h * A)(kh) \geq 0, \qquad \mathbb{E}(x * y)(k) = (b_h * C_L^+)(kh), \tag{1}$$
$$h^{-1}\mathbb{E}\sum_k x(k)^2 = \sum_i p_i \|b_h * w_i\|_2^2, \qquad h\,\mathbb{E}\sum_k y(k)^2 = 2L + H.$$

Opposite offsets in the two sequences are essential for the second convolution identity. The auxiliary offset is integrated out. The output law has one atom.

Give weight $\theta \in (0, 1)$ to the extra channel $x = \delta_0,\ y = \mathbf{1}_{\{|k| \leq N\}}$, retaining the old channels with total weight $1 - \theta$. Use $\mathbb{E}_\theta$ for this joint law. View $\widehat{x}(\xi) = \sum_k x(k)e^{-ik\xi}$ as a trigonometric polynomial. On the unit circle put

$$a = \mathbb{E}_\theta|\widehat{x}|^2, \qquad \widehat{s} = \sqrt{a}, \qquad \widehat{K} = \widehat{x}/\sqrt{a}, \qquad U = \mathbb{E}_\theta(K * y).$$

The real even trigonometric polynomial a is at least θ. It extends to a strip in which its real part exceeds $\theta/2$, so its positive square root and reciprocal are bounded and analytic on a smaller strip. Contour shift in the Fourier coefficient formula gives exponential decay of s and of each K, uniform in the channel because the support and bound of x are uniform. These bounds depend on h, θ and the input, never on N. The sequence s is real even. Every K is real, and the joint

pairing $(i,u)\mapsto(i^{\mathrm{refl}},-u)$ makes U even. Since $\widehat{x}(0)=1$ for every channel, $\sum s=\sum K=1$. Fourier multiplication and Cauchy–Schwarz on the probability space now give

$$\begin{aligned} s*s=\mathbb{E}_\theta(x*\widetilde{x})\ge 0, \qquad s*U=\mathbb{E}_\theta(x*y), \\ h^{-1}\sum s^2=(1-\theta)\sum_i p_i\|b_h*w_i\|_2^2+\theta/h, \qquad h\sum U^2\le 2L+(1-\theta)H+\theta h. \end{aligned} \tag{2}$$

Indeed $|\mathbb{E}_\theta\widehat{x}\,\widehat{y}|^2/a\le\mathbb{E}_\theta|\widehat{y}|^2$. Parseval proves the last bound. For every j, averaging $|y(j)|$ over the offset gives at most one plus $h^{-1/2}$ times the sum of the two input shoulder L^2 norms. Thus $\sup_{j,N}\mathbb{E}_\theta|y(j)|<\infty$. Summing against the uniform exponential bound on $|K(k)|$ gives $\|U\|_\infty=O_{h,\theta}(1)$. By (1), $(s*U)(k)\ge 1$ for $|k|\le N-1$.

A rational finite window. Fix h,θ throughout this step. Take M through the squares of even positive integers, $R=M/4$, and $N=50M$. Normalize the restriction of s to $[-R,R]$ to have sum one (its sum tends to one), then approximate by a rational even sequence s_R of sum one on that same interval: round its positive-index coefficients first and use the central coefficient for the sum. Choose the total ℓ^1 rounding error at most M^{-8}. Exponential decay gives

$$\|s_R-s\|_1=O(M^{-8}), \quad \sum k^2|s_R(k)|=O(1), \qquad s_R*s_R\ge -O(M^{-8}).$$

Let $q_M(k)$ be proportional to $M^{-1}(1-(k/M)^2)_+^3$, normalized to sum one, and set $\eta=M^{-3/2}$, $v_M=(1-\eta)s_R+\eta q_M$. These sequences and weights are rational. The twice continuously differentiable function of the lattice coordinate whose samples define q_M has second derivative $O(M^{-3})$, since its normalization tends to a positive constant. Evenness cancels the first moment of s_R; Taylor's inequality therefore gives $s_R*q_M\ge q_M-O(M^{-3})$ at every index. For $|k|\le 5M/4$, $(q_M*q_M)(k)\ge c/M$ with an absolute $c>0$ for large M: at least $M/4-O(1)$ summands have both indices of modulus $\le 3M/4$. Consequently

$$(v_M*v_M)(k)\ge c\eta^2/M-O(\eta M^{-3})-O(M^{-8})>0 \qquad (|k|\le 5M/4).$$

Farther out, only the nonnegative q_M*q_M term can survive. Thus $v_M*v_M\ge 0$ globally, $\sum v_M=1$, $\operatorname{supp} v_M\subset[-M,M]$ and $\|v_M-s\|_1=O(M^{-3/2})$.

A rational finite table. Since $\sum K=1$, exponential tails imply uniformly in $N=50M$

$$U(k)=1+O(e^{-cM}) \quad (|k|\le N-M), \qquad U(k)=O(e^{-cM}) \quad (|k|\ge N+M).$$

Indeed all shifts of fewer than $M-O(v/h+1)$ cells sample the plateau, respectively the exterior, of the input tables. The remaining shifts are bounded in expectation by the uniform first-moment estimate above. Set T_0 equal to one and zero on these two regions. Round its other values from U, in reflected pairs, to rationals with error $\le M^{-8}$. Then $\|T_0-U\|_\infty=O(M^{-8})$ and $h\sum T_0^2\le h\sum U^2+O_{h,\theta}(M^{-7})$; setting the exterior to zero only decreases energy. Put

$$d=\max_{|k|\le N-1}(1-(v_M*T_0)(k))_+, \quad J=\{k:\,||k|-N|\le 4M\}, \qquad Z=T_0+d\,v_M*\mathbf{1}_J.$$

The finite rational maximum satisfies $d=O(M^{-3/2})$ by (2) and the norm bounds. The covering correction is $d(v_M*v_M)*\mathbf{1}_J\ge 0$ everywhere. A deficient index must lie within $2M$ of N or $-N$, since $v_M*T_0=1$ for $|k|\le N-2M$. At such an index the entire support of v_M*v_M, translated to that index, belongs to J. Its sum is one, so the correction there is exactly d. We have obtained $v_M*Z\ge 1$ on $\{|k|\le N-1\}$, and

$$h\sum Z^2\le 2L+(1-\theta)H+\theta h+O_{h,\theta}(M^{-1/2}).$$

Indeed the table alteration is $O(M^{-3/2})$ in sup norm on $O(M)$ cells, including in the energy cross term.

Returning to actual functions. On the cell centered at kh take heights $v_M(k)/h$ and $Z(k)$ for V,T, assigning the average of the adjacent values at each boundary. These pointwise representatives are even and rational. Convolution of centered cell indicators is a triangle, so

$V * V$ is the linear interpolant of $(v_M * v_M)(k)/h$ at kh; $V * T$ is the interpolant of $(v_M * Z)(k)$. Thus the inequalities hold at every real shift in the claimed domains. Take $D = (N-1)h$ and declared width $v' = 12Mh$. The window support has radius at most $(M + 1/2)h$; the table support has radius at most $(N + 5M + 1/2)h \le D + v'/2$, and the table is exactly one for $|x| < D - v'/2$. Also $v' \le D$. Translating the two functions to centers $v'/2$ and $D - v'/2$ gives the original WT coordinates. The actual table surplus is therefore bounded above by $(1-\theta)H + \theta h + 2h + O_{h,\theta}(M^{-1/2})$. The term $2h$ is the loss in covering length.

To see convergence of the profiles, first let M grow with h, θ fixed. The error from replacing $s * s$ by $v_M * v_M$ is $O_{h,\theta}(M^{-3/2})$ in ℓ^1 and ℓ^∞. The affine interpolation operator takes these errors to $o(1)$ in both asserted norms. The perturbed delta term in $V * V$ has mass θ and height θ/h. The functions $b_h * b_h * A$ and their interpolants converge to A uniformly and in L^1, by uniform continuity and a common compact support for $h \le 1$. Thus $\|V\|_2^2 = (V * V)(0) \to A(0) = B$ on choosing $\theta = o(h)$.

Similarly the error in $v_M * Z - s * U$ is $O_{h,\theta}(M^{-3/2})$ uniformly at all knots. Both knot sequences agree with the unit plateau or with zero outside the two bands $||k| - N| \le 6M + O_h(1)$. For the first sequence this follows from its support and plateau, and for the second from the exact mixed convolution in (2). After interpolation their L^1 difference is $O_{h,\theta}(M^{-1/2})$. The extended, reflected input covering has the form $C_L^+(x) = F(x + L) + F(L - x) - 1$ for all sufficiently large L, with a common continuous function F equal to zero on $(-\infty, -v]$ and one on $[v, \infty)$. Indeed, near each endpoint the input window sees only that shoulder and the adjacent plateau. Reflection identifies the two shoulder profiles. The middle has cover exactly one by the per-atom mass. This representation proves a uniform modulus of continuity in x and in L. Convolution by b_h, linear interpolation on mesh h, and replacement of L by $D = L - h$ therefore change the reference profile by $O(\omega_F(2h))$; the extra channel changes it by $O(\theta)$. These changes occur only within $O(v + h)$ of the two ends, so they vanish in L^1 as well.

Finally, if the current surplus $H_0 = \|T\|_2^2 - 2D$ is below H, choose two exterior cells centered at $\pm(N + 11M/2)h$ on an odd refinement of the grid. For large M these cells lie outside $\operatorname{supp} T$, within the permitted table support, and more than the window radius from $[-D, D]$. On the two cells of length r add the same rational height approximating $\sqrt{(H - H_0)/(2r)}$ from above closely enough that the new surplus lies in $[H, H + 1/M]$. At cell boundaries take the common even representative as before. This does not alter the cover on $[-D, D]$. Its effect in either the uniform or the integral norm is at most $\|V\|_p \sqrt{2r(H - H_0 + 1/M)}$ for $p = \infty, 1$, made $< 1/M$ in both norms by taking the odd refinement sufficiently fine. The old cells are unions of cells of that refinement. If $H_0 \ge H$, keep the old table. In both cases $H \le \|T\|_2^2 - 2D \le \max(H, H_0) + 1/M$. Taking rational $h \downarrow 0$, $\theta = o(h)$, and for each pair a sufficiently large M also satisfying $Mh \to \infty$, proves all four convergences for the same deterministic pair. □

3 Smooth witnesses, fixed geometry and compact roots

Corollary 2. *In Theorem 1 one may replace the rational-grid condition by $V_j, T_j \in C_c^\infty(\mathbb{R})$. The same four limits and pointwise feasibility conditions hold for these even functions.*

Proof. For an output with half-length D and width v', convolve both factors with a nonnegative even smooth probability density supported on $[-\delta, \delta]$. Their two convolutions are the old profiles averaged with the self-convolution of that density. Positivity persists everywhere and the cover is at least one through $|x| \le D - 2\delta$. Use half-length $D' = D - 2\delta$ and width $v'' = v' + 6\delta$. The table support radius increases by at most δ, exactly the increase in $D' + v''/2$. Its mandatory plateau shrinks within the region still identically one. The construction has $v' < D$, so $v'' \le D'$ for small δ. Both squared norms approach their previous values by L^2 continuity, and the reduction in covering length adds 4δ to the surplus before accounting for the change in table energy. Each compact continuous profile converges uniformly and in the integral norm to its old value as $\delta \downarrow 0$. The common shoulder modulus controls $C_{D'}^+ - C_D^+$ in both norms, since their difference vanishes

away from the two shoulders. For each fixed pair choose δ so all these errors are at most $1/j$. □

Corollary 3. *Let $\mathcal{E}$ be the set of actual cost pairs (H, B) of all feasible laws, allowing all admissible geometries. Let $\mathcal{E}_g$ and $\mathcal{E}_s$ be the corresponding sets for single even rational-grid and single even smooth compactly supported pairs. Then in $\mathbb{R}^2$,*

$$\overline{\mathcal{E}_g} = \overline{\mathcal{E}_s} = \overline{\mathcal{E}}.$$

The unrestricted, deterministic, symmetric and deterministic symmetric infima of $2\sqrt{\alpha\beta}$ all equal $4\sqrt{2}/3$. The single-pair rational-grid and smooth subclasses each have the same infimum.

Proof. Each restricted set lies in $\mathcal{E}$. Theorem 1 and Corollary 2 put each point of $\mathcal{E}$ in both closures. Given one feasible input and $\varepsilon > 0$, a single output meets both budgets $\beta + \varepsilon, \alpha + \varepsilon$, by the two actual limits. Class inclusion and continuity of the objective give equality of the infima. The numerical value follows from [1]. □

For the product objective alone one may even fix *any* admissible geometry (v_0, σ_0). Dilation $w(x) \mapsto \lambda w(\lambda x),\ t(x) \mapsto t(\lambda x)$, with $\lambda = v/v_0$, sends the budgets to $(\alpha/\lambda, \lambda\beta)$. At width v_0, Lemma 2.7 of [2] permits any $\sigma_0 \geq 2v_0$ by rejoining the two shoulders with a constant plateau. Reflection and the rescaled costs are preserved. For rational geometry, rational grids are preserved after common refinement. These operations preserve each individual threshold $4(1 + v\sqrt{\beta/\alpha})^2$, as well as its product. They change the separately prescribed profile and costs of a given input. Theorem 1 gives no uniform bound on the finite-size thresholds along its sequence.

Proposition 4. *Take the window with heights $2/3, 1/3$ on the two successive unit cells of $[0, 2]$, and the table $\mathbf{1}_{[-2,4]}$. For a single even WT window of declared width v', let $\varepsilon = \|V * V - A\|_\infty$ in centred coordinates and $S = \max(2, v')$. For $S \geq 4$,*

$$\varepsilon \geq \frac{5}{648eS}\, 2^{-S}.$$

For any sequence with $\varepsilon_j \to 0$ and $\limsup_j \alpha_j\beta_j \leq 10/9$, its sufficient cardinality thresholds in [2, Theorem 2.3, (1)] satisfy

$$\liminf_{j\to\infty} \frac{4(1 + v_j'\sqrt{\beta_j/\alpha_j})^2}{\log^2(1/\varepsilon_j)} \;\geq\; \frac{10}{9(\log 2)^2}.$$

Proof. The input is strong. Its cover equals one on $[0, 4]$, its table surplus is 2 and its window energy is $5/9$. Write $a = \log 2$. With Fourier kernel e^{-izx},

$$\widehat{A}(z) = \left(\frac{\sin(z/2)}{z/2}\right)^2 \frac{(2 + e^{iz})(2 + e^{-iz})}{9}.$$

There is exactly one zero, simple, inside each circle $|z - (\pi + ia)| = r$, $0 < r \leq 1/4$. On it write $z = \pi + ia + h$. We have $|z| < 7/2$ and $|\sin(z/2)| \geq \cos(1/8) > 7/8$. The remaining factors have moduli at least $2 - e^r/2 \geq 4/3$ and $2|1 - e^{-ih}| \geq 2r(1 - re^r/2) \geq 5r/3$. Thus $|\widehat{A}| \geq 5r/81$ on the circle.

The transform of $V * V$ is an entire square. By Rouché's theorem its difference from $\widehat{A}$ has modulus at least $5r/81$ somewhere on that circle. The difference in physical space is supported in $[-S, S]$. Both autocorrelations are nonnegative of mass one; hence

$$\|A - V * V\|_1 = 2\int_{-2}^{2} (A - V * V)_+ \leq 8\varepsilon, \qquad 5r/81 \leq 8\varepsilon e^{(a+r)S}.$$

Fixed r forces $S_j \to \infty$. Taking $r = 1/S$ gives the explicit bound and $\liminf v_j'/\log(1/\varepsilon_j) \geq 1/a$. Evaluation at zero gives $\beta_j \geq 5/9 - \varepsilon_j$, so $\liminf \beta_j/\alpha_j = \liminf \beta_j^2/(\alpha_j\beta_j) \geq 5/18$. Substitution proves the threshold assertion. Without control of the product, increasing the table budget to $\max(\alpha_j, (v_j')^2\beta_j)$ makes the same functions admissible with sufficient threshold at most 16. □

The zero-parity criterion for exact compact even roots and examples are due to Ehm, Gneiting and Richards [4]. Proposition 4 quantifies its consequence for one prescribed profile with product 10/9. The width theorem of Akwei and Williams concerns proximity of the product to 8/9 [1]. Approximation by general reflection-convolution factors [6] and finite spectral factorization [8] do not impose the WT table constraints. Multiple-kernel boundary inequalities appear in [5].

4 Finite rational witnesses

We compare the exact rational data in [2, 3, Appendix A.3, Claim 6]. Its six windows have 200 cells each. Direct summation of the reversed table rows verifies every one of the 401 covering knots at spacing $h = 1/20$, with minimum exactly one. Interpolation between consecutive knots is affine. The recovered actual budgets are $\alpha = 0.94481304378915084\ldots$ and $\beta = 0.94125802624178783\ldots$. All comparisons below use integer cross multiplication before rounding.

Witness	Width	Length	$2\sqrt{\alpha\beta}$	Least integer $\geq k_0$
Carter–O'Bryant–Tao, six atoms	10	20	1.88606771963731	483
Single signed pair	313.45	1056	1.88606745373460	394032
Regenerated single pair	313.45	680	1.88606745374163	394032
Single pair with shorter shoulders	64.95	129.9	1.88606745217749	17334
Akwei–Williams, two atoms	7	14	1.88562178309372	473
Single pair from these two atoms	2503/64	2503/32	1.88562173355792	12595
Williams, 2048 centred boxes	59	118	1.88561813935992	28520

Here $k_0 = 4(1 + v\sqrt{\beta/\alpha})^2$ is sufficient in [2, Theorem 2.3]. The width-seven pair of nonnegative symmetric atoms is due to [1]. The three single pairs derived from A.3 improve both of its rational budgets and have larger sufficient thresholds. Strong-witness applications in [2, Theorems 2.4–2.5] require additional nonnegativity.

For the shorter pair, let W_i be the rational cell masses of the first single window. Their outermost nonzero indices are ± 649; thus $1299h$ is the minimum admissible width of this particular window. The supplied preliminary right shoulder $t^0(d)$ is one for $d < -410$ and zero for $d > 410$. With interval indicators on $\mathbb{Z}$, set

$$Q_j = \binom{720}{360 + j} \mathbf{1}_{\{|j| \leq 60\}} \Big/ \sum_{a=-60}^{60} \binom{720}{360 + a},$$

$$T_k = t^0(|k| - 1299) + (Q * \mathbf{1}_{[-500,40]})(|k| - 1299)/250000000000.$$

On the cells centred at ih use heights W_i/h and T_i, with the mean of the traces at each boundary. Integer dot products verify $W * W \geq 0$, $W * Q \geq 0$, and $\min_{|k| \leq 1299}(W * T)_k = 1$. The last inequality is also checked by direct convolution of the completed table. The costs follow from $h^{-1} \sum W_i^2$ and $h \sum T_i^2 - 2598h$. The smaller table reduces the two A.3 costs by $2.6795019934\ldots \times 10^{-7}$ and $1.4152520608\ldots \times 10^{-11}$, respectively. A separate integer fixed-point generator starts directly from the six A.3 atoms and reproduces the third row byte for byte. Feasibility is verified using exact rational arithmetic independently of the discovery transform.

One window from the two-window input. On 32768 half-offset frequencies, take a signed even square root of the two windows' mixed power in centred half-cell coordinates [1]. Divide the mixed edge-difference transform by it, invert, truncate and mix a sampled seventh-power bump into the window. Interval arithmetic certifies rounding; the integer arrays alone suffice for the feasibility check.

Write $m = 512,\ r = 10012,\ s = 3072$. The grid cells are $[i, i + 1]/m$ up to their endpoints. Trim only the trailing zeros of the supplied window half-array, mirror it and divide by its total mass to obtain W_i for $-r \leq i < r$. For right-end offsets j, the preliminary table $t^0(j)$ equals

one for $j < -5120$, uses the supplied 10240 integers divided by 10^{13} for $-5120 \le j < 5120$, and vanishes thereafter. Set

$$Q_j = \frac{(s^2 - j^2)_+^{12}}{\sum_\ell (s^2 - \ell^2)_+^{12}}, \quad J = [s - 1 - r, s] \cap \mathbb{Z}, \quad T(j) = t^0(j) + \frac{(Q * \mathbf{1}_J)(j)}{2 \cdot 10^9}.$$

Take $v = 2r/m,\ \sigma = 2v$. Around the right cover endpoint put $T(j)$ on the cells with offsets $-r \le j < r$. Reflect this shoulder to the left and fill the $2r$ middle cells with ones. Translate to the original coordinates and average the two traces at every cell boundary. The resulting table consists of $6r$ cells. The window uses heights mW_i.

Integer convolution gives $W * W, W * Q \ge 0$. In original coordinates the cover at k/m is coefficient $2r+k-1$ of the full window–table product. Its minimum over $0 \le k \le 4r$ is exactly one, confirmed by a second convolution order. The squared norms give $0 < \alpha < \alpha_{\mathrm{AW}}$ and $0 < \beta < \beta_{\mathrm{AW}}$ for the tabulated two-window energies. The objective improves by $4.9535801563\ldots \times 10^{-8}$. The strict bounds

$$1.88562173355 < 2\sqrt{\alpha\beta} < 1.88562173356, \qquad 12594 < 4(1 + v\sqrt{\beta/\alpha})^2 < 12595$$

follow by squaring positive quantities. With $q = \alpha\beta \in (8/9, 1)$, the remainder beyond $k^2 - 2\sqrt{q}\,k^{3/2}$ in [2, Theorem 2.3, (2)] is $qk + \sqrt{q}(1-q)\sqrt{k} + q(q-2) - 1 \ge (8/9)k - 2 > 0$ at this threshold. Thus every Sidon set S of cardinality $k \ge 12595$ satisfies

$$\operatorname{diam}(S) \ \ge \ k^2 - 1.885621734\, k^{3/2}.$$

At $k = 117^2$, rounding up the full bound [2, Theorem 2.3, (1)] gives 184380155 for A.3, 184380863 for the two windows and 184380864 for our single pair. Its right side decreases with $u = b/234$, so rational coefficient enclosures suffice. Williams' 2048 boxes give $b < 1.885618140$ [7]. Their boundary depth 30 embeds at width 59, giving $k \ge 28520$ [1]. The WT infimum is $1.88561808316412\ldots$.

A A compact independent lower-bound test

Carter, O'Bryant and Tao test the covering against an interval and two endpoint atoms to prove $\alpha\beta > 1/2$ [2, Theorem 5.4]. Averaging the interval over a random inward trim gives the following rational certificate. The sharp bound of [1] for the same product is $8/9$, larger by $4291/562500$.

Lemma 5. *For every signed WT witness, $\alpha\beta \ge 495709/562500$.*

Proof. Dilate to $\beta = 1$, so the new table budget is $\alpha\beta$. Denote this budget temporarily by a. In the original left-endpoint coordinates A is even, integrates to one and satisfies $A(0) \le 1$. Put $l = 14/15,\ b = 2/3$. Let X have mass $11/20$ at zero and density $3(7 + 16x)/100$ on $(0, 1)$. Lengthen the table so that $\sigma > 2l + 2v$, and put

$$\mu = \mathbb{E}_X \mathbf{1}_{[lX,\sigma - lX]}\, dx + b(\delta_0 + \delta_\sigma).$$

Pairing the cover with μ gives $\mathbb{E}\langle t, \mu * \widetilde{w}\rangle \ge \mu(\mathbb{R})$. Fubini gives $\mathbb{E}\|\mu * \widetilde{w}\|_2^2 = \iint A(x-y)\, d\mu(x) d\mu(y)$. For $|s| \le v$, the length of overlap of the two interval terms, shifted by s, is $\sigma - l(X + X') - \max(|s|, l|X - X'|)$, where X' is an independent copy. There are four equal interval–endpoint terms, each contributing $b\,\mathbb{E}\int_{lX}^{\infty} A(s)\, ds$. The two unequal endpoints have zero interaction. Expanding $\mathbb{E}\|t - \mu * \widetilde{w}\|_2^2 \ge 0$ and using $A(0) \le 1$ therefore gives

$$a \ge \int A(s) P(|s|/l)\, ds, \quad P(r) = l(\mathbb{E}\max(r, |X - X'|) - 2\mathbb{E}X) + 2b(1 + \Pr(X > r)) - 2b^2.$$

On $0 \le r \le 1$, writing $F(r) = \Pr(|X - X'| \le r)$, direct integration gives

$$\mathbb{E}X = \tfrac{53}{200}, \quad \Pr(X > r) = \tfrac{3}{100}(1 - r)(8r + 15), \quad F(r) = \tfrac{192r^4 + 39r^2 + 6744r + 3025}{10000}.$$

Indeed if $h(x) = 3(7+16x)/100$, then $F(0) = (11/20)^2$ and $F'(r) = 2(11/20)h(r) + 2\int_0^{1-r} h(y)h(y+r)\,dy$. Since $\mathbb{E}\max(r, |X - X'|) = 1 - \int_r^1 F(y)\,dy$, one obtains

$$P(r) = \frac{991418 + r(4032r^4 + 1365r^2 - 5940r + 2625)}{1125000}.$$

Using the tangent line to r^4 at $5/8$, the quartic in parentheses is at least $15(5824r^2 - 8544r + 3325)/64 > 0$. For $r \geq 1$, P continues as an affine function of positive slope l. Thus $P \geq P(0) = 495709/562500$. Integrating against $A \geq 0$ of mass one proves the claim. □

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