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A COUNTEREXAMPLE TO THE HIRSCH CONJECTURE IN DIMENSION 14

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Abstract. We give a bounded counterexample to the Hirsch conjecture in dimension 14: a simple polytope with 28 facets whose graph has diameter 15. It is defined by 28 explicit inequalities with integer coefficients of absolute value at most 4,626, and it has 27,968 vertices. Santos disproved the bounded conjecture in 2010 in dimension 43, and the smallest dimension we found in work by other authors is 19. Our polytope comes from a 5-dimensional prismatoid of width 6 with 19 vertices. A second width-6 prismatoid has 20 vertices, answers Santos's 2010 question and yields a counterexample in dimension 15. From the printed coordinates, Lean 4 proves both prismatoid widths and the exact graph diameters of the two polytopes using only its three standard axioms (Section 7).

Statement of provenance. Valen Venkatesh directed this work, provided the compute, and chose to publish this paper and its verification files. The proofs and the Lean code came out of an AI-assisted workflow that he designed, in which Claude Fable 5.2 (Anthropic, in Claude Code) and GPT-6.2 Astra (OpenAI, in Codex) worked together. The two models also wrote the search programs and the exact checkers and did the literature searches, and each checked what the other had produced. The first draft of this paper was written by an LLM.

Appendix B gives commands to reproduce the computations used to verify the stated results. Section 7 says what Lean 4 (version 4.36.0) proves, what it trusts and what lies outside it.

The dimension-15 result was first published on GitHub, at `https://github.com/valenvenk/solved-problems`, in commit `6ec8e14` on 27 November 2026, and the geometric Lean proof of its pole distance in commit `fd043e3` on the same day. The dimension-14 result was first published there in commit `0d3216d` on 28 November 2026. This is an unrefereed proof.

1. Introduction

In 1957 Warren M. Hirsch proposed a bound on the number of edge steps between vertices of a convex polyhedron. George B. Dantzig published the question in 1963 [5, pp. 160, 168], as quoted by Klee and Walkup [10, p. 54]. The bound is $n-d$ for a d-dimensional polyhedron with n facets. Klee and Walkup gave an unbounded 4-dimensional polyhedron with eight facets and diameter five in 1967 [10]. We study the bounded case. Santos's 2010 counterexample, published in 2012, has dimension 43, 86 facets and diameter at least 44 [15]. His construction uses a 5-dimensional prismatoid with 48 vertices and width six. The table lists starting prismatoids and the dimensions obtained from them by the strong d-step theorem.

vertices v	dimension $v-5$	source
48	43	Santos [15, Theorem 3.1]
28	23	Matschke, Santos and Weibel [11, Corollary 2.9]
25	20	Matschke, Santos and Weibel [11, Theorem 2.14]
24	19	Świrszcz et al. [17, Section 4.1 and Appendix A.13]
20	**15**	this paper, Section 5
19	**14**	this paper, Sections 3 and 4

The 20-dimensional counterexample of Matschke, Santos and Weibel has 40 facets and 36,425 vertices. They give vertex coordinates for its polar [11, Theorem 1.3 and Table 1]. Świrszcz et al. give coordinates for their 24-vertex prismatoid, which yields a counterexample in dimension 19 [17]. In work by other authors, our literature searches of 26 and 28 November 2026 found no bounded counterexample below dimension 19 and no explicit inequalities or vertex coordinates for one below dimension 20. Davies et al. give 5-prismatoids with $16k+24$ vertices and width at least $5+k$, for every $k \geq 1$ [6, Theorem 1.1 and Lemma 3.6].

When he announced his counterexample, Santos asked whether there are 5-prismatoids with, say, 20 vertices that need more than five steps from one base to the other. He observed that a positive answer would give counterexamples in dimension 15 instead of 43 [14, Section 3, item 2, p. 535]. The prismatoid in Table 3 answers this question with 20 vertices and width six (Theorem 5). The prismatoid in Table 1 has 19 vertices and width six (Theorem 1).

The Hirsch bound holds for bounded polytopes in dimensions $d \leq 3$ and whenever $n-d \leq 6$ [9, 10, 3]. It also holds for $n \leq 12$ [3, 2]. In particular, $(d,n) = (4,12)$ and $(5,12)$ both have maximum diameter seven [2]. So a bounded counterexample has $d \geq 4$ and $n-d \geq 7$, and the smallest dimension of one is at most 14 by Corollary 2. A convex 5-prismatoid of width six with v vertices gives, by the strong d-step theorem, a counterexample of dimension $v-5$ with $2(v-5)$ facets. Hence $v \geq 12$.

Criado and Santos found four topological 4-prismatoids with 14 vertices and width larger than five [4, Section 5, Tables 2–5]. Here the dimension is that of the abstract complex modelling the lateral boundary of a geometric 5-prismatoid. Pfeifle gave nonrealizability certificates for these four combinatorial types [13]. Gouveia, Macchia and Wiebe ruled out convex realizations of all four and of the 36 types with 15 vertices in the same collection [8, Proposition 19].

Theorem 1. *The convex hull Q of the 19 points of Table 1 is a 5-dimensional prismatoid with the bases $Q \cap \{x_5 = 1\}$ and $Q \cap \{x_5 = -1\}$. It has 19 vertices and 175 facets, and its width is 6.*

Corollary 2. *There is a polytope of dimension 14 with 28 facets whose graph has diameter at least 15.*

Applying the strong d-step theorem (Theorem 9) with $(d,n,l) = (5,19,6)$ gives dimension $19-5 = 14$, $2(19-5) = 28$ facets and diameter at least $6+19-2\cdot 5 = 15$, as in Corollary 2. The following theorem gives an explicit example defined by inequalities.

Theorem 3. *Let $p_1, \ldots, p_{28} \in \mathbb{Z}^{14}$ be the rows of Table 5, and let*

$$P = \{\, y \in \mathbb{R}^{14} : p_i \cdot y \leq 1 \text{ for } i = 1, \ldots, 28 \,\}.$$

Then P is a simple polytope of dimension 14 with exactly 28 facets, 27,968 vertices and 195,776 edges. The points $e_{14} = (0, \ldots, 0, 1)$ and $-e_{14}$, its poles*, are vertices of P, and their distance in the graph of P is exactly 15. Thus the diameter of P is at least $15 > 28 - 14$.*

Proposition 4. *The diameter of P is exactly* 15*. Among the* 391,090,528 *unordered pairs of distinct vertices of P, exactly two are at distance* 15*. One is $\{e_{14}, -e_{14}\}$. In the other, one vertex is tight on rows 1, 2, 3, 4, 5, 6, 7, 8, 12, 16, 21, 23, 25, 27 of Table 5, and the other is tight on the complementary* 14 *rows.*

Theorem 5. *The convex hull Q_{20} of the 20 points of Table 3 is a* 5*-dimensional prismatoid with the bases $Q_{20} \cap \{x_5 = 1\}$ and $Q_{20} \cap \{x_5 = -1\}$. It has* 20 *vertices and 190 facets, and its width is* 6*.*

Corollary 6. *There is a polytope of dimension* 15 *with* 30 *facets whose graph has diameter at least* 16*.*

Theorem 7. *Let $q_1, \ldots, q_{30} \in \mathbb{Z}^{15}$ be the rows of Table 6, and let*

$$P_{15} = \{\, y \in \mathbb{R}^{15} : q_i \cdot y \leq 1 \text{ for } i = 1, \ldots, 30 \,\}.$$

Then P_{15} is a simple polytope of dimension 15 *with exactly* 30 *facets, 49,056 vertices and 367,920 edges. Its diameter is $16 > 30 - 15$, and its poles $e_{15} = (0, \ldots, 0, 1)$ and $-e_{15}$ form the only unordered pair of vertices at this distance.*

Corollary 6 follows from Theorem 5 by Theorem 9 with $(d, n, l) = (5, 20, 6)$. Theorem 7 gives an explicit example. The product $P \times [0, 1]$ gives another example for Corollary 6. By Theorem 3, it is simple, has dimension 15 and has 30 facets. Graph distances in the factors add, so $(e_{14}, 0)$ and $(-e_{14}, 1)$ are $15 + 1 = 16$ edges apart. Unlike Q, the prismatoid Q_{20} has a directed two-cycle in its incidence pattern (Section 6).

The proofs use exact arithmetic on the listed points and inequalities (Sections 3 to 7). Section 8 describes the searches.

On the separate polynomial Hirsch question, a November 2026 preprint by OpenAI reports a family of bounded simple rational polytopes, one for every sufficiently large facet count n, whose graph diameters grow faster than every fixed power of n [12].

2. Prismatoids and the strong d-step theorem

Definition 8 (Santos [15, Definitions 2.1 and 2.5])**.** A *prismatoid* is a polytope Q with two designated parallel facets Q^+ and Q^- whose union contains every vertex. These facets are its *bases*. Its *width* is the minimum number of ridge crossings in a path of facets from Q^+ to Q^-. A ridge is a face of codimension two, and two distinct facets are adjacent in the dual graph when they share a ridge.

Theorem 9 (Santos [15, Theorem 2.6], strong d-step theorem for prismatoids)**.** *Let Q be a prismatoid of dimension d with n vertices and width l. Then there is a prismatoid Q' of dimension $n - d$ with $2n - 2d$ vertices and width at least $l + n - 2d$. In particular, if $l > d$ then the polar of Q' (taken about an interior point) has dimension $n - d$, has $2(n - d)$ facets and has diameter larger than $n - d$.*

Polarity exchanges facets with vertices and ridges with edges. The width of Q' is therefore the distance between the two vertices dual to its bases.

When $n > 2d$, at least one base is not a simplex. Santos suspends a vertex of the opposite base, replacing it by the endpoints of a segment in a new coordinate direction. A sufficiently small, suitably chosen perturbation of the nonsimplex base makes it a facet again. The proof allows a preliminary generic perturbation within this base before one of its vertices moves in the new direction. The resulting prismatoid has dimension $d + 1$, $n + 1$ vertices and width at least $l + 1$. After $n - 2d$ steps each base has exactly as many vertices as the final dimension, so both bases are simplices.

3. The prismatoid with 19 vertices

	x_1	x_2	x_3	x_4	x_5		x_1	x_2	x_3	x_4	x_5
1	−156	1288	−605	−748	1	11	1002	1372	2057	−1475	−1
2	407	1452	−108	−1278	1	12	2443	−1097	1923	−1732	−1
3	74	798	658	−163	1	13	−1859	−313	−735	1526	−1
4	1898	1293	−3680	−579	1	14	2792	−674	2264	−2166	−1
5	−1423	1937	4626	−542	1	15	−1485	−2135	−1436	1934	−1
6	−1059	522	4423	−320	1	16	−902	−2261	−1591	1698	−1
7	−1415	−2203	846	1113	1	17	−2103	2499	450	690	−1
8	671	1074	−1346	−142	1	18	−1301	−2765	−1440	2000	−1
9	611	994	−2039	−539	1	19	−139	−2650	−424	993	−1
10	2112	1757	−3953	−653	1						

Table 1. The 19 vertices of the prismatoid Q. The base Q^+ is the convex hull of the points 1 to 10, the base Q^- that of the points 11 to 19.

Proof of Theorem 1. Each height class has affine dimension four, so its supporting hyperplane cuts out a facet and the two classes together span dimension five. Every facet contains five affinely independent input points. We enumerate all such frames, keep the hyperplanes with all points on one side, and identify each facet by the full set of input points on it. Intersections of facets give all proper nonempty faces. The singleton faces account for every input point, and counting faces by dimension gives $f(Q) = (19, 147, 410, 455, 175)$. We join two facets when their intersection has dimension three. This gives 455 edges in the dual graph. Breadth-first search from the positive base finds

$$1,\ 23,\ 40,\ 52,\ 36,\ 17,\ 6$$

facets at the distances $0, 1, \ldots, 6$, and it puts the negative base at distance six (Section 7). □

All 173 *lateral* facets of Q, those other than the bases, are simplices. Classifying them by their vertex counts in Q^+ and Q^- gives 23 of type $(4,1)$, 65 of type $(3,2)$, 63 of type $(2,3)$ and 22 of type $(1,4)$. So the bases are simplicial 4-polytopes with 10 and 9 vertices and with 23 and 22 facets.

Remark 10. (a) Table 1 is the search output with each of the 76 horizontal coordinates rounded to the nearest multiple of 1/3,293 and multiplied by 3,293, and with one integer vector subtracted in each base. The two translations interpolate affinely in the height x_5, so this is a rounding followed by an invertible affine map. The search output and the table have the same facets, as sets of vertex labels. Rounding the search output to five decimal places also gives width 6; four decimal places give width 5. (b) Deleting any one vertex from Table 1 gives an 18-vertex prismatoid of width 5.

4. The polytope of dimension 14

Starting with Q in the coordinates of Table 1, we use nine one-point suspensions followed, at each step, by a single-vertex perturbation. We check the width of each resulting hull directly. A step (s, m, K) inserts a coordinate z immediately before the height. Row s receives $z = -K$, and a copy of that row with $z = K$ is appended. Row m on the opposite base receives $z = 1$. All other rows receive $z = 0$. Up to the scale of z, this is a suspension to $z = \pm 1$ with a

perturbation of size $1/K$. Row numbers start at one. Table 2 lists each step's parameters and the resulting hull's computed facet count and width.

dimension	vertices	facets	width	$(s,\ m,\ K)$
5	19	175	6	start
6	20	320	7	(18, 1, 64)
7	21	548	8	(10, 15, 4096)
8	22	951	9	(12, 6, 512)
9	23	1,654	10	(9, 13, 4096)
10	24	2,852	11	(17, 3, 512)
11	25	5,310	12	(4, 19, 512)
12	26	9,332	13	(14, 2, 64)
13	27	16,088	14	(8, 11, 4096)
14	28	27,968	15	(16, 7, 4096)

TABLE 2. The nine lifting steps. The vertices are numbered as in Table 1; vertices that a step adds get the next number.

The final rows $p_1, \ldots, p_{28}$ in Table 5 are the vertices of a simplicial 14-prismatoid Q' with 27,968 facets and width 15. Its two bases have last coordinates 1 and -1. For $Q' = \operatorname{conv}\{p_i\}$ the polar inequalities are exactly $p_i \cdot y \le 1$, since a linear functional on a convex hull takes its maximum at an input point. The following proof also checks that the origin is interior to Q'.

Proof of Theorem 3. Dimension and boundedness. The strict inequalities at the origin give P nonempty interior in $\mathbb{R}^{14}$. Exact elimination finds that the 14 rows with last coordinate 1 are linearly independent, and so are the 14 rows with last coordinate -1. The file of the inequalities lists positive integers w_i with $\sum_i w_i p_i = 0$ (Section 7). Normalizing these weights puts 0 in Q'. As $0 \in Q'$, the row rank makes Q' full-dimensional. If 0 lay on its boundary, a nonzero supporting normal u there would satisfy $p_i \cdot u \le 0$ for all i. The positive dependence forces every $p_i \cdot u = 0$, contradicting the rank. Hence Q' contains a ball about 0, and its polar P is bounded.

Facets. At e_{14} precisely the 14 positive-height rows are tight, and at $-e_{14}$ precisely the 14 negative-height rows are tight. Each set of tight normals is independent. At either point the slacks $1 - p_i \cdot y$ of its tight rows form local affine coordinates, in which the feasible region near the point is a nonnegative orthant. Thus the point is a vertex and each coordinate hyperplane gives a distinct incident facet. The two sets of rows are strict at opposite vertices. Together they give 28 different facets, and the defining system has only 28 rows.

Graph and distance. The same slack coordinates give all the edges at a vertex with 14 independent tight rows. Each edge increases just one of the 14 slacks from zero. Along that ray the edge ends at the first inactive inequality to become tight. The exact ratio test checks that an endpoint exists and that its entering row is unique. The entering normal has nonzero component along the ray, so replacing the released row preserves independence. The other inactive rows stay strict. Starting at e_{14}, this argument applies inductively to every vertex reached.

Breadth-first search visits each set of tight rows once. Such a set determines a unique point. Since every edge at each reached vertex is explored and the graph of a polytope is connected, the 27,968 vertices reached are all the vertices of P. Each is incident with exactly 14 facets, so every vertex is simple and P is simple. Each has degree 14, giving $14\,|V(P)|/2 = 195{,}776$ edges. The successive layer sizes from e_{14} are

1, 14, 92, 406, 1307, 2836, 2912, 6112, 6336, 3520, 1824, 1629, 798, 164, 16, 1.

The last layer consists of $-e_{14}$ alone, at distance 15. □

Proof of Proposition 4. By the proof of Theorem 3, the vertices of P correspond to the 27,968 sets of 14 tight rows found there, and two vertices are adjacent exactly when they have 13 tight rows in common. Breadth-first search from every vertex gives the distance between each of the 391,090,528 pairs. No distance is larger than 15. Exactly two pairs are at distance 15, namely those of the statement, and 146 pairs are at distance 14 (Section 7). □

Remark 11. The tight rows at $-e_{14}$ and at e_{14} are complementary, and an edge exchanges one tight row. So a walk of 14 edges from $-e_{14}$ to e_{14} would increase the number of tight rows shared with e_{14} by one at every edge. Starting at $-e_{14}$ and allowing only steps that increase this number, we reach

1, 14, 104, 446, 1267, 2448, 3520, 3808

vertices after $0, 1, \ldots, 7$ edges and no vertex after 8 edges (Section 7). So the poles are more than 14 edges apart. As P is simple, this lower bound needs the edges at these 11,608 vertices only. A walk of 15 edges from $-e_{14}$ to e_{14} exchanges the rows (leave, enter)

(26,21), (12,27), (20,25), (15,23), (19,4), (11,10), (16,9), (4,7), (13,8), (24,6), (22,5), (28,4), (17,3), (14,2), (18,1).

Its edge 8 exchanges row 4 for row 7, two rows tight at e_{14}. Every other edge increases the number shared with e_{14} by one.

A second sequence of nine lifts starts from the search output rounded to five decimal places. After an affine change of coordinates, the polar of the resulting prismatoid is another simple 14-polytope with 28 facets (`certs/polytope14_second_facets.txt`). It has 27,175 vertices, its integer coefficients have absolute value at most 2,246,778, and its vertices e_{14} and $-e_{14}$ are at distance 15. They are the unique pair attaining its diameter of 15.

5. The prismatoid with 20 vertices

	x_1	x_2	x_3	x_4	x_5		x_1	x_2	x_3	x_4	x_5
1	0.0979	−0.1440	−1.4891	−0.0300	1	11	−0.1550	0.8880	0.3191	0.1859	−1
2	0.1408	0.0273	0.2086	0.3752	1	12	0.5152	−0.1393	−0.2658	−0.1260	−1
3	0.1135	−0.0868	−0.1412	−0.9556	1	13	0.3457	−1.8411	0.3472	−0.3512	−1
4	0.1295	0.0946	0.1188	0.3908	1	14	0.4275	0.5609	−0.3511	−0.0169	−1
5	−0.2031	0.2182	0.4820	−0.4924	1	15	−0.4434	0.5371	0.2437	0.1982	−1
6	−0.0042	0.0994	0.9333	0.3815	1	16	−0.0040	−1.1009	0.0398	−0.1532	−1
7	0.0622	−0.1858	−1.4509	0.3417	1	17	−0.3807	0.9310	0.5584	0.2358	−1
8	0.1520	−0.2906	−0.0398	−0.8737	1	18	−0.3561	−0.2062	−0.2527	0.0523	−1
9	0.1300	−0.0188	−0.5112	0.3770	1	19	−0.3556	−0.1855	−0.2623	0.0351	−1
10	−0.3244	0.1726	1.1275	0.1722	1	20	0.4161	0.5445	−0.3645	−0.0634	−1

Table 3. The 20 vertices of the prismatoid Q_{20}. The base Q_{20}^+ is the convex hull of the points 1 to 10, the base Q_{20}^- that of the points 11 to 20.

Proof of Theorem 5. The computations in the proof of Theorem 1, applied to the points of Table 3, give $f(Q_{20}) = (20, 159, 446, 495, 190)$, 495 edges in the dual graph and distance six between the bases. □

The 188 lateral facets of Q_{20} are simplices, with 26 of type $(4,1)$, 70 of type $(3,2)$, 68 of type $(2,3)$ and 24 of type $(1,4)$. So the bases are simplicial 4-polytopes with 10 vertices and with 26 and 24 facets.

Remark 12. (a) Rounding the four horizontal coordinates of the original search output to four decimal places gives Table 3. Rounding that same output to three decimal places gives width 5. (b) Deleting any one vertex from Table 3 gives a 19-vertex prismatoid of width 5. In particular, Q does not arise from Q_{20} by the deletion of a vertex. (c) A second 20-vertex prismatoid of width six has f-vector $(20, 161, 452, 501, 192)$. Its facet count differs from that of Q_{20}, so the two face lattices are not isomorphic.

The polytope of dimension 15. We lift Q_{20} ten times by steps of the form of Section 4, after the following normalization of Table 3. For each horizontal coordinate, multiply by the least common multiple of its denominators (10,000 for each). In each base subtract $\lfloor(\min+\max)/2\rfloor$, with the extrema taken over that scaled coordinate in the base. The two offsets interpolate affinely in the height x_5, so together with the positive scalings these changes give an invertible affine map. Table 4 lists each step's parameters and the resulting hull's computed facet count and width.

dimension	vertices	facets	width	$(s,\ m,\ K)$
5	20	190	6	start
6	21	349	7	(16, 9, 4096)
7	22	640	8	(4, 18, 4096)
8	23	1,187	9	(20, 6, 256)
9	24	2,204	10	(2, 19, 4096)
10	25	3,837	11	(13, 8, 256)
11	26	6,338	12	(7, 17, 256)
12	27	10,968	13	(11, 10, 256)
13	28	18,208	14	(3, 12, 4096)
14	29	30,208	15	(15, 1, 256)
15	30	49,056	16	(5, 14, 4096)

TABLE 4. The ten lifting steps for Q_{20}. The vertices are numbered as in Table 3; vertices that a step adds get the next number.

The final rows $q_1, \ldots, q_{30}$ in Table 6 are the vertices of a simplicial 15-prismatoid with 49,056 facets and width 16, and P_{15} is its polar.

Proof of Theorem 7. The computations in the proofs of Theorem 3 and Proposition 4, applied to the rows of Table 6, give 49,056 vertices, each incident with exactly 15 facets, and $15 \cdot 49{,}056/2 = 367{,}920$ edges. The successive layer sizes from e_{15} are

1, 15, 102, 474, 1593, 3831, 5504, 10144, 9856, 6208, 3072, 3837, 3203, 1016, 184, 15, 1.

The last layer consists of $-e_{15}$ alone, at distance 16. Among the 1,203,221,040 pairs of distinct vertices, this is the only pair at distance 16, and no distance is larger. As in Remark 11, walks from $-e_{15}$ that increase the number of tight rows shared with e_{15} at every edge have at most 8 edges. □

The coordinates of Q_{20} and the inequalities for P_{15} appeared in the first edition of this paper on 27 November 2026. A Lean proof of the pole distance using a compiler-assisted table check followed the same day.

6. The incidence patterns of the two prismatoids

A sufficient condition of Matschke, Santos and Weibel concerns transversal pairs of geodesic normal maps on S^3. A directed arrow goes from a facet of one map to a facet of the other if a vertex of the first lies in the interior of the second. This is the *incidence pattern*. Absence of directed two-cycles in the subgraph induced by facets containing vertices of the other map, the *reduced* incidence pattern, implies width greater than three for the pair of maps and hence greater than five for the prismatoid [11, Lemma 1.2, Definition 2.2 and Proposition 2.3].

Simplicial lateral facets imply the required transversality. Let U, V be faces of the two bases, of dimensions r, s, whose closed spherical normal cells meet. A common horizontal normal, with its height coefficient chosen to tie the two supporting levels, exposes a proper face containing U and V. That face lies in a lateral simplex facet. Hence $E = \mathrm{conv}(U \cup V)$ is a simplex face of dimension $r + s + 1$. Projection onto the horizontal coordinates identifies the normal cone of E with the intersection of the two base normal cones: for each horizontal normal in the intersection, tying one point of U with one of V uniquely determines the height coefficient. The intersection of the spherical cells therefore has dimension $5 - (r + s + 1) - 1 = 3 - r - s$. The cells themselves have dimensions $3 - r$ and $3 - s$, as required by [11, Section 2.1].

Facets of a base's normal map correspond to base vertices, and vertices of the map to base facets. Each facet F of one base lies in exactly one lateral facet, which has one more vertex $\varphi(F)$ in the other base [6, Proposition 2.4]. The vertex of the map given by F lies in the interior of the cell of $\varphi(F)$. So there is an arrow from the cell of a vertex p of Q^+ to the cell of a vertex q of Q^- exactly when a lateral facet of type $(4, 1)$ contains p and q, and an arrow from the cell of q to the cell of p exactly when a lateral facet of type $(1, 4)$ contains them. This is also the row-indexed description of Criado and Santos [4, Definition 3.9].

The prismatoid Q. No two cells have arrows in both directions. The reduced incidence pattern has 4 upper and 4 lower cells, those of the rows 1, 4, 6, 8 and of the rows 14, 15, 17, 19 of Table 1, and 16 arrows. It is the pattern that Matschke, Santos and Weibel describe for Santos's 48-vertex prismatoid [11, Section 2.1]: the eight cells form four pairs A, B, C, D, and the arrows are all those from A to B, from B to C, from C to D and from D to A. A reduced incidence pattern without directed two-cycles has at least eight cells: its minimum out-degree is at least two, so Proposition 2.4 of [11] applies.

The prismatoid Q_{20}. The reduced incidence pattern has 6 cells of the map of Q_{20}^+ and 4 cells of the map of Q_{20}^-, and 25 arrows. The cells of the rows 2 and 13 of Table 3 have arrows in both directions: for each of the 3 facets $\{2, 6, 7, 8\}$, $\{2, 6, 7, 9\}$ and $\{2, 7, 8, 9\}$ of Q_{20}^+, the lateral facet through it has row 13 as its fifth vertex, and the lateral facet through the facet $\{11, 13, 15, 17\}$ of Q_{20}^- has row 2 as its fifth vertex. So the sufficient condition does not apply to Q_{20}, and Theorem 5 needs the width calculation.

Matschke, Santos and Weibel reported that the condition held for every large-width example then known to them in the 3-sphere [11, Section 2.1]. Criado and Santos later gave topological examples with two-cycles [4, Remark 3.11]. For convex examples, Świrszcz et al. report that none of their sample of 100 width-six constructions satisfied the no-two-cycle criterion [17, Section 4.1.1].

Remark 2.6 of the arXiv version of Davies et al. [6] states an equivalence for a graph on *base facets*. An upper facet F and a lower facet G form a two-cycle there exactly when $q = \varphi(F) \in G$ and $p = \varphi(G) \in F$. These are precisely the witnesses for a two-cycle on the cells of p, q in the normal maps. In Q_{20} the three displayed facet pairs witness the same cell two-cycle. Its width is six, so the implication from width at least $d + 1$ to absence of two-cycles fails already for a convex 5-prismatoid with simplicial lateral facets. The sufficient direction of Proposition 2.3

of [11] applies as stated. Proposition 13 below says what a path of length five needs beyond a two-cycle.

7. Verification

Where the files are. The first dimension-14 verification files appeared in the result folder at commit `0d3216d` (28 November 2026). In the result folder, `certs/` holds the points and inequalities as plain text, `verify/` the checkers, `lean/` the Lean project, `search/` the lifting program and the two programs for Proposition 13, and `sources/` the published examples with 24, 25, 28 and 48 vertices.

The certificates, checkers and Lean proofs that this paper describes are in the result folder at commit `3f08bf6` (28 November 2026). The current edition has the PDF of this version and its sources. The script `build/make_tables.py` reruns the checkers and regenerates Tables 1–6 and the computed numbers of Sections 3 to 7, including those of Remarks 10 and 12.

How to repeat the checks. The checkers need Python 3.10 or newer, use only its standard library, and compute with exact integers and rationals. For example, in the result folder:

```
$ python3 verify/check_diameter.py certs/polytope14_facets.txt
   vertices 27968, each on exactly 14 of the hyperplanes; edges 195776
   pairs of different vertices: 391090528
   diameter 15, attained by 2 pairs
   walks from -e_d increasing tight-row overlap with e_d at each edge reach 1 14 104 446 1267
   2448 3520 3808 0 vertices after 0, 1, 2, ... edges
   no walk of 14 edges joins e_d and -e_d
   PASS
```

Appendix B lists the Python checks and the command to build, audit and replay the Lean proofs.

The prismatoids. One program (`verify_prismatoid.py`) tests the supporting hyperplanes spanned by five affinely independent input points. A second (`verify_prismatoid_b.py`) constructs the hull by incremental insertion. A third (`verify_prismatoid_c.py`) constructs the common refinement of the normal fans of the bases. Its rays correspond to the lateral facets. The distance between the two sets of base-facet normals in the spherical refinement, plus the two steps incident to the bases, gives the width [11, Lemma 1.2]. For Q all three computations give 175 facets and width 6, and for Q_{20} they give 190 facets and width 6. A fourth checker (`spark_check_width.py`) intersects the facets to enumerate all nonempty proper faces and obtains the displayed f-vectors. The independent script `check_patterns.py` checks Section 6 by enumerating the tetrahedral facets of each base and maximizing their outward normals on the opposite base.

The polytope. Three checkers examine the explicit polytope from its two descriptions.

(1) The inequality checker (`check_polar15.py`) uses the positive weights for boundedness and enumerates the vertices and edges directly from the inequalities by the pivots in the proof of Theorem 3.

(2) An incremental hull algorithm (`verify_polytope15.py`) enumerates the facets of the polar prismatoid Q' from its vertices, identifies ridges and measures the distance between its bases.

(3) A third program (`verify_simplicial_width.py`) starts at a base and proposes adjacent simplex facets. For every candidate F it checks affine independence of F together with a reference point r outside it, solves for affine coordinates in this frame and checks the solution by multiplication. The coordinate of r is strictly positive at every input point outside F, so F is a supporting simplex facet. Each of its ridges lies in two listed

facets. Since a ridge has just two incident facets and the dual graph is connected, the nonempty list is complete. The program also checks the two parallel bases and their distance in this graph.

A fourth program, `check_diameter.py`, computes the distance between every pair of vertices for Proposition 4 and Theorem 7. It starts from the inequalities and shares no code with the other checkers. A search proposes a nonempty list of tight-row sets. In exact integer arithmetic the program checks that each set consists of d independent rows whose common point satisfies all other inequalities strictly. Each subset of $d-1$ tight rows belongs to exactly two listed sets. At each listed vertex, the segments to these neighbours therefore follow all d edge directions, and each ends at a listed vertex. For any linear functional, choose a listed vertex maximizing it over the list. The functional is nonincreasing on all d directions there, which generate the feasible cone in slack coordinates. Thus it is maximized over the feasible region at that vertex. By separation, the region equals the convex hull of the list, so it is bounded and all its vertices have been found. A breadth-first search tracks every source separately with bit sets and counts each unordered pair at its first distance. The numbers of vertices and edges and the layer sizes from e_d agree with those of the inequality checker of item (1). The program also prints the counts of Remark 11. For its lower bound alone, `check_pole_walks.py` follows only edges that increase the number of tight rows shared with the far pole.

Formal verification. Allamigeon, Canu and Strub previously verified bounded counterexamples of dimensions 20 and 23 in Coq [1]. For the two polytopes in Appendix A, Lean proves that the printed inequalities are irredundant and define bounded simple polytopes. In dimension 14 it counts 28 facets, 27,968 vertices and 195,776 edges. The poles are 15 edges apart, the diameter is 15, and exactly two unordered pairs attain it, the poles and the pair specified in Proposition 4. In dimension 15 it counts 30 facets, 49,056 vertices and 367,920 edges. The poles are the unique pair at the diameter of 16.

For the two starting hulls, Lean counts 19 vertices and 175 facets, and 20 vertices and 190 facets, respectively. It proves that the width of each is six by exhibiting a walk of six ridge crossings between the bases and ruling out every shorter walk. These are component results for Theorems 1 and 5. Neither theorem is stated in full as a single Lean declaration. The kernel replays the proofs. Their only permitted axioms are propositional extensionality, classical choice and quotient soundness. The program `verify/check_lean_data.py` checks that the matrices, points and rows in the Lean sources are those of the certificates and of Proposition 4. The Lean proofs omit, in particular, the lifts connecting the prismatoids to the polytopes, Santos's strong d-step theorem, Proposition 13, the assertion that every lateral facet of Q_{20} is a simplex, and the rounding and deletion experiments.

8. How the examples were found, and what did not work

Section 7 checks the examples from their coordinates. Appendix B also gives commands to test Proposition 13 on examples and to check the six-dimensional prismatoid described below.

The search. In a 5-prismatoid with simplicial lateral facets, a path of length five between the bases passes through lateral facets of types $(4,1),(3,2),(2,3),(1,4)$ in that order: the first and last types are forced, and adjacent lateral facets exchange one vertex. So no path is shorter than five, and the width is at least six exactly when no path has length five. The number of five-step paths between the bases is the *energy*. The search estimates it in floating point from the two base normal fans and keeps configurations of estimated energy zero. It tries inserting one or two vertices, moving vertices at random and deleting vertices while the estimated energy stays zero.

Twenty vertices. Starting from the 28-vertex coordinates of [11, Corollary 2.9], the search reached 20 vertices. Runs from the 24-vertex example of [17] found no smaller width-six prismatoid. For each of our two 20-vertex inputs the minimum energy after a single deletion is 4. Starting with row 8 removed from Table 3, four walks at 19 vertices tested about $1.25 \cdot 10^8$ proposals in total, and none reported an energy below 4.

Nineteen vertices. The prismatoid Q came from a new descent with the same program, started from Santos's 48-vertex prismatoid [15] after each horizontal coordinate was moved by Gaussian noise of standard deviation 0.18. For Q the minimum energy after a single deletion is again 4, for the rows 2 and 5 of Table 1.

For a d-prismatoid with simplicial lateral facets, Criado and Santos observed that a path of length d between the bases requires an edge whose star contains facets incident to both bases [4, Proposition 3.8]. In dimension five we count these paths for each such edge. Orient paths from the positive base to the negative base. A vertex in the positive base is *positive* and a vertex in the negative base is *negative*.

Proposition 13. *Let R be a 5-prismatoid whose lateral facets are simplices.*

1. *For every path of length five between the bases in the dual graph, there is exactly one edge pq of R contained in all four intermediate facets, with p positive and q negative.*
2. *For any edge pq joining the two bases, with p positive, its link is a triangulated 2-sphere. Call a triangle of it* pure *if its vertices all have the same sign and* mixed *otherwise. The paths of length five whose intermediate facets contain pq are in bijection with the pairs of adjacent mixed triangles A, B of the link such that A has two positive vertices and shares an edge with a pure positive triangle, and B has two negative vertices and shares an edge with a pure negative triangle.*
3. *Call an edge of the link a* type change *if it lies in a mixed triangle with two positive vertices and in a mixed triangle with two negative vertices. The number of type changes of a link is even, and it is not two.*

Proof. (1) Let $R^+, X_1, X_2, X_3, X_4, R^-$ be such a path. As in the paragraph on the search above, X_i has $5 - i$ positive vertices. From X_i to X_{i+1} the facet loses one positive vertex and gains one negative vertex, so it keeps all its negative vertices. Hence the negative vertex q of X_1 lies in X_2, X_3 and X_4. In the other direction the facet keeps all its positive vertices, so the positive vertex p of X_4 lies in X_3, X_2 and X_1. These are the only vertices common to X_1 and X_4. They span an edge of the simplex X_1, hence an edge of R.

(2) The link is the boundary of the three-dimensional face figure at pq. Every facet containing pq is lateral and hence a simplex. Removing p, q from its vertex set gives a link triangle, with one fewer positive vertex. Two such facets share a ridge exactly when their triangles share an edge. Thus a path gives successive triangles of types $+++$, $++-$, $+--$, $---$. Conversely, given the middle pair in (2), the only edge the $++-$ triangle can share with a pure positive triangle is its positive edge. This edge has exactly one other incident triangle. The negative edge of the $+--$ triangle likewise determines its pure neighbour. Adjoining p, q recovers the four facets. The first meets R^+ and the last meets R^- in a tetrahedral ridge, so the pair gives exactly one path.

(3) Join mixed triangles across edges with opposite-coloured endpoints. Each mixed triangle has two such edges, each with one other incident triangle. Its two neighbours are distinct since two different triangles cannot share two edges. This graph is a disjoint union of cycles. Each change switches the triangle type, so each cycle has an even number of changes. If a cycle had exactly two, it would have one run of each type. The $++-$ triangles in their run retain the same negative vertex c at every step: two consecutive ones share an edge with a negative

endpoint. Likewise the $+--$ triangles retain the same positive vertex d. Both transition edges would then have endpoints c, d. There is only one such edge in a simplicial complex, while the two transitions cross distinct edges of the cycle. This excludes two on each cycle and therefore on the whole link. □

Thus a link has zero type changes or an even number at least four. Only changes whose two mixed triangles have the pure neighbours in part (2) contribute paths, so a link can contribute fewer paths than it has type changes: with row 1 removed from Table 3, the link of the edge between rows 7 and 11 has four type changes and one path. So the proposition does not by itself explain the minimum of four. For each deletion that leaves energy four (rows 8, 19 and 20 of Table 3, rows 8 and 20 of the second 20-vertex example, and the rows 2 and 5 of Table 1), the four paths come from all four type changes of a single link. For Q, both 20-vertex inputs and their 59 single deletions, `search/links_exact.py` compares the facet-graph path count with the sum over links in exact arithmetic. On 411 configurations `search/check_links.py` compares both exact counts with a third, the number of five-step walks from the positive base to the negative base in the facet graph. It tests the two 20-vertex inputs with their single deletions, 240 random deletions of two to five rows, and the examples with 24, 25, 28 and 48 vertices [17, 11, 15] with their single deletions. Where a lateral facet is not a simplex, it first moves each horizontal coordinate by at most 10^{-9}. The counts agree in every case, and every link tested satisfies part (3).

A directed two-cycle in the incidence pattern of Section 6 gives a pure positive and a pure negative triangle in the link of one edge pq. By part (2) a path of length five through that edge needs adjacent mixed triangles of opposite types, each adjacent to a pure triangle of its majority sign. In the link of the edge between rows 2 and 13 of Q_{20}, every adjacent pair of opposite mixed types lacks at least one required pure neighbour.

Dimension 6. A 6-prismatoid of width seven with 19 vertices would yield a counterexample in dimension 13. A search from a perturbed lift of Q_{20} gave a 21-vertex example of width seven (`certs/prismatoid21_dim6.txt`, coordinates with six decimals). A descent from a perturbed, lifted and repaired copy of the 28-vertex example of [11] also reached 21 vertices. Those searches found no smaller width-seven example and give no lower bound for the method. The first lift of Q in Table 2 has 20 vertices and width seven.

9. Questions

(1) What is the smallest number of vertices of a 5-prismatoid of width 6? It is between 12 and 19 inclusive.

(2) What is the smallest dimension of a bounded counterexample to the Hirsch bound? It lies between 4 and 14 inclusive. In dimension four the question for unrestricted facet count remains open [7]. A counterexample yields one in every higher dimension by taking products with a segment. Each product adds one to the dimension, two to the number of facets, and one to the diameter.

(3) Is there a d-prismatoid with n vertices and width larger than d with $n - d < 14$? By Theorem 9 it would give a counterexample of dimension $n - d$. It needs $d \geq 5$: prismatoids of dimension at most four have width at most their dimension [16], [11, Section 1].

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Appendix A. The inequalities of the two polytopes

i	$p_{i,1}$	$p_{i,2}$	$p_{i,3}$	$p_{i,4}$	$p_{i,5}$	$p_{i,6}$	$p_{i,7}$	$p_{i,8}$	$p_{i,9}$	$p_{i,10}$	$p_{i,11}$	$p_{i,12}$	$p_{i,13}$	$p_{i,14}$
1	−156	1288	−605	−748	1	0	0	0	0	0	0	0	0	1
2	407	1452	−108	−1278	0	0	0	0	0	0	1	0	0	1
3	74	798	658	−163	0	0	0	0	1	0	0	0	0	1
4	1898	1293	−3680	−579	0	0	0	0	0	−512	0	0	0	1
5	−1423	1937	4626	−542	0	0	0	0	0	0	0	0	0	1
6	−1059	522	4423	−320	0	0	1	0	0	0	0	0	0	1
7	−1415	−2203	846	1113	0	0	0	0	0	0	0	0	1	1
8	671	1074	−1346	−142	0	0	0	0	0	0	0	−4096	0	1
9	611	994	−2039	−539	0	0	0	−4096	0	0	0	0	0	1
10	2112	1757	−3953	−653	0	−4096	0	0	0	0	0	0	0	1
11	1002	1372	2057	−1475	0	0	0	0	0	0	0	1	0	−1
12	2443	−1097	1923	−1732	0	0	−512	0	0	0	0	0	0	−1
13	−1859	−313	−735	1526	0	0	0	1	0	0	0	0	0	−1
14	2792	−674	2264	−2166	0	0	0	0	0	0	−64	0	0	−1
15	−1485	−2135	−1436	1934	0	1	0	0	0	0	0	0	0	−1
16	−902	−2261	−1591	1698	0	0	0	0	0	0	0	0	−4096	−1
17	−2103	2499	450	690	0	0	0	0	−512	0	0	0	0	−1
18	−1301	−2765	−1440	2000	−64	0	0	0	0	0	0	0	0	−1
19	−139	−2650	−424	993	0	0	0	0	0	1	0	0	0	−1
20	−1301	−2765	−1440	2000	64	0	0	0	0	0	0	0	0	−1
21	2112	1757	−3953	−653	0	4096	0	0	0	0	0	0	0	1
22	2443	−1097	1923	−1732	0	0	512	0	0	0	0	0	0	−1
23	611	994	−2039	−539	0	0	0	4096	0	0	0	0	0	1
24	−2103	2499	450	690	0	0	0	0	512	0	0	0	0	−1
25	1898	1293	−3680	−579	0	0	0	0	0	512	0	0	0	1
26	2792	−674	2264	−2166	0	0	0	0	0	0	64	0	0	−1
27	671	1074	−1346	−142	0	0	0	0	0	0	0	4096	0	1
28	−902	−2261	−1591	1698	0	0	0	0	0	0	0	0	4096	−1

Table 5. The vectors $p_1, \ldots, p_{28}$ of Theorem 3. They are also the vertices of the prismatoid Q' of dimension 14.

i	$q_{i,1}$	$q_{i,2}$	$q_{i,3}$	$q_{i,4}$	$q_{i,5}$	$q_{i,6}$	$q_{i,7}$	$q_{i,8}$	$q_{i,9}$	$q_{i,10}$	$q_{i,11}$	$q_{i,12}$	$q_{i,13}$	$q_{i,14}$	$q_{i,15}$
1	1841	−1078	−13083	2524	0	0	0	0	0	0	0	0	1	0	1
2	2270	635	3894	6576	0	0	0	−4096	0	0	0	0	0	0	1
3	1997	−506	396	−6732	0	0	0	0	0	0	0	−4096	0	0	1
4	2157	1308	2996	6732	0	−4096	0	0	0	0	0	0	0	0	1
5	−1169	2544	6628	−2100	0	0	0	0	0	0	0	0	0	−4096	1
6	820	1356	11141	6639	0	0	1	0	0	0	0	0	0	0	1
7	1484	−1496	−12701	6241	0	0	0	0	0	−256	0	0	0	0	1
8	2382	−2544	1410	−5913	0	0	0	0	1	0	0	0	0	0	1
9	2162	174	−3304	6594	1	0	0	0	0	0	0	0	0	0	1
10	−2382	2088	13083	4546	0	0	0	0	0	0	1	0	0	0	1
11	−1909	13431	2222	2436	0	0	0	0	0	0	−256	0	0	0	−1
12	4793	3158	−3627	−683	0	0	0	0	0	0	0	1	0	0	−1
13	3098	−13860	2503	−2935	0	0	0	0	−256	0	0	0	0	0	−1
14	3916	10160	−4480	408	0	0	0	0	0	0	0	0	0	1	−1
15	−4793	9922	1468	2559	0	0	0	0	0	0	0	0	−256	0	−1
16	−399	−6458	−571	−955	−4096	0	0	0	0	0	0	0	0	0	−1
17	−4166	13861	4615	2935	0	0	0	0	0	1	0	0	0	0	−1
18	−3920	2489	−3496	1100	0	1	0	0	0	0	0	0	0	0	−1
19	−3915	2696	−3592	928	0	0	0	1	0	0	0	0	0	0	−1
20	3802	9996	−4614	−57	0	0	−256	0	0	0	0	0	0	0	−1
21	−399	−6458	−571	−955	4096	0	0	0	0	0	0	0	0	0	−1
22	2157	1308	2996	6732	0	4096	0	0	0	0	0	0	0	0	1
23	3802	9996	−4614	−57	0	0	256	0	0	0	0	0	0	0	−1
24	2270	635	3894	6576	0	0	0	4096	0	0	0	0	0	0	1
25	3098	−13860	2503	−2935	0	0	0	0	256	0	0	0	0	0	−1
26	1484	−1496	−12701	6241	0	0	0	0	0	256	0	0	0	0	1
27	−1909	13431	2222	2436	0	0	0	0	0	0	256	0	0	0	−1
28	1997	−506	396	−6732	0	0	0	0	0	0	0	4096	0	0	1
29	−4793	9922	1468	2559	0	0	0	0	0	0	0	0	256	0	−1
30	−1169	2544	6628	−2100	0	0	0	0	0	0	0	0	0	4096	1

TABLE 6. The vectors $q_1, \ldots, q_{30}$ of Theorem 7. They are also the vertices of a prismatoid of dimension 15.

APPENDIX B. COMMANDS AND EXPECTED OUTPUT

Run these Python commands from the result folder (Section 7). Each exits with code 0. Selected output follows each command. Together they take a few minutes.

The prismatoid Q (Table 1)

```
$ python3 verify/verify_prismatoid.py certs/prismatoid19.txt 19
  ACCEPT 19 vertices, 175 facets, width 6
$ python3 verify/verify_prismatoid_b.py --summary certs/prismatoid19.txt 19
  ACCEPT 19 vertices, 175 facets, 455 ridges, width 6; facets at distance 0, 1, 2, ... from
  the upper base: 1 23 40 52 36 17 6
$ python3 verify/verify_prismatoid_c.py certs/prismatoid19.txt 19
  ACCEPT 19 vertices (10+9), base facets 23+22, overlay 173 rays and 410 edges, fan distance
  4, width 6
$ python3 verify/spark_check_width.py certs/prismatoid19.txt 19
  PASS 19 vertices (10+9), f-vector [19, 147, 410, 455, 175], width 6
```

The search output behind Table 1, and the same output with five decimals

```
$ python3 verify/verify_prismatoid.py certs/spark_prismatoid19.txt 19
  ACCEPT 19 vertices, 175 facets, width 6
$ python3 verify/verify_prismatoid.py certs/spark_prismatoid19_rounded.txt 19
  ACCEPT 19 vertices, 175 facets, width 6
```

The prismatoid Q_{20} (Table 3), its search output, and the second prismatoid with 20 vertices

```
$ python3 verify/verify_prismatoid.py certs/prismatoid20.txt 20
  ACCEPT 20 vertices, 190 facets, width 6
$ python3 verify/verify_prismatoid_b.py --summary certs/prismatoid20.txt 20
```

```
   ACCEPT 20 vertices, 190 facets, 495 ridges, width 6; facets at distance 0, 1, 2, ... from
   the upper base: 1 26 42 55 39 23 4
$ python3 verify/verify_prismatoid_c.py certs/prismatoid20.txt 20
   ACCEPT 20 vertices (10+10), base facets 26+24, overlay 188 rays and 445 edges, fan distance
   4, width 6
$ python3 verify/spark_check_width.py certs/prismatoid20.txt 20
   PASS 20 vertices (10+10), f-vector [20, 159, 446, 495, 190], width 6
$ python3 verify/verify_prismatoid.py certs/prismatoid20_raw.txt 20
   ACCEPT 20 vertices, 190 facets, width 6
$ python3 verify/spark_check_width.py certs/spark_prismatoid20_flex28-a.txt 20
   PASS 20 vertices (10+10), f-vector [20, 161, 452, 501, 192], width 6
```

The incidence patterns (Section 6)

```
$ python3 verify/check_patterns.py
   19 rows: 4 + 4 cells, 16 arrows, Santos pattern, no two-cycle
   20 rows: 6 + 4 cells, 25 arrows, sole two-cycle (2, 13), three facet-pair witnesses
```

The polytope P (Table 5): its inequalities, then the prismatoid Q'

```
$ python3 verify/check_polar15.py --dimension 14 certs/polytope14_facets.txt
   POLAR 27968 vertices, distance 15
   Active inequality exchanges (leave, enter), one-based: (1, 18) (2, 14) (3, 17) (4, 28) (5,
   22) (6, 24) (8, 13) (7, 4) (9, 16) (10, 11) (4, 19) (23, 15) (25, 20) (27, 12) (21, 26)
$ python3 verify/check_diameter.py certs/polytope14_facets.txt
   vertices 27968, each on exactly 14 of the hyperplanes; edges 195776
   pairs of different vertices: 391090528
   diameter 15, attained by 2 pairs
   walks from -e_d increasing tight-row overlap with e_d at each edge reach 1 14 104 446 1267
   2448 3520 3808 0 vertices after 0, 1, 2, ... edges
   no walk of 14 edges joins e_d and -e_d
   PASS
$ python3 verify/check_pole_walks.py --start minus certs/polytope14_facets.txt
   from -e_d: 1 14 104 446 1267 2448 3520 3808 0
$ python3 verify/verify_polytope15.py --dimension 14 certs/polytope14.txt
   ACCEPT dimension 14, 28 vertices, 27968 facets, width 15
$ python3 verify/verify_simplicial_width.py certs/polytope14.txt 15
   ACCEPT dimension 14, 28 vertices, 27968 facets (all simplices), 195776 ridges, width 15
```

The second polytope of dimension 14 (end of Section 4)

```
$ python3 verify/check_polar15.py --dimension 14 certs/polytope14_second_facets.txt
   POLAR 27175 vertices, distance 15
$ python3 verify/check_diameter.py certs/polytope14_second_facets.txt
   diameter 15, attained by 1 pair
   no walk of 14 edges joins e_d and -e_d
   PASS
$ python3 verify/verify_polytope15.py --dimension 14 certs/polytope14_second.txt
   ACCEPT dimension 14, 28 vertices, 27175 facets, width 15
```

The polytope P_{15} (Table 6)

```
$ python3 verify/check_polar15.py certs/polytope15_facets.txt
   POLAR 49056 vertices, distance 16
$ python3 verify/check_diameter.py certs/polytope15_facets.txt
   diameter 16, attained by 1 pair
   no walk of 15 edges joins e_d and -e_d
   PASS
$ python3 verify/verify_polytope15.py certs/polytope15.txt
   ACCEPT dimension 15, 30 vertices, 49056 facets, width 16
```

```
$ python3 verify/verify_simplicial_width.py certs/polytope15.txt 16
   ACCEPT dimension 15, 30 vertices, 49056 facets (all simplices), 367920 ridges, width 16
```

Proposition 13, and the example of dimension 6 (Section 8)

```
$ python3 search/links_exact.py certs/prismatoid19.txt certs/prismatoid20.txt \
    certs/spark_prismatoid20_flex28-a.txt
    both counts agree in all 20 configurations; no link has exactly two type changes
    both counts agree in all 21 configurations; no link has exactly two type changes
    both counts agree in all 21 configurations; no link has exactly two type changes
$ python3 search/check_links.py
   the three counts agree in all 411 configurations; every link has zero or an even number of
   at least four type changes
$ python3 verify/verify_polytope15.py certs/prismatoid21_dim6.txt --legacy --control-width 7
   CONTROL dimension 6, 21 vertices, 382 facets, width 7
```

The data of the Lean proofs are the data of the certificates

```
$ python3 verify/check_lean_data.py
   PASS: the Lean matrices, points and second pair are those of the certificates and of
   Proposition 4
```

With the paper sources in the folder, `python3 build/make_tables.py` runs these programs again, also on every single deletion and rounding of Remarks 10 and 12 and on every lifting step of Tables 2 and 4. It takes about ten minutes and must leave `build/tables/` unchanged.

From the `lean/` directory run the following command with Python, Elan and Lake installed. It uses Lean 4.36.0 and fetches the pinned Mathlib cache if needed. The compiler reports two style warnings, an unused pattern variable in `CoverCheck.lean` (line 43) and a deprecated lemma name in `Ridge.lean` (line 17). With the dependencies already cached, the run took 10 minutes 29 seconds on a 20-core NVIDIA GB10, with up to 16 workers running concurrently.

```
$ python3 verify-public.py --workers 18
   PASS: 16 required results; 13787 local theorems, three allowed axioms
   PASS: 164 proof modules compiled and replayed
```

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