The cost of a final edge exchange after Pósa rotations in cubic Cayley graphs

Peiwei Wu Qian Peng*
The University of Melbourne

29 November 2026

hexagon:2611.01334v1

Abstract

For every integer $t \geq 0$, we construct a Hamiltonian bipartite cubic Cayley graph on $72t + 156$ vertices and a path missing one vertex. If a single exchange must make the path immediately extendible to the missing vertex, its minimum size lies between $3t + 2$ and $6t + 13$, even when arbitrary Pósa rotations are allowed beforehand. If rotations are also allowed after the exchange, the minimum exchange size is exactly two. Any repair with an exchange of size k and s subsequent rotations satisfies $k + s \geq 3t + 2$. An explicit repair uses seven initial rotations, a two-edge exchange, and $6t + 13$ subsequent rotations. The lower bound combines confinement of both endpoints with a signed height function. The main results are formalized in Lean 4.

1 Two ways to extend a prescribed path

If a path misses just one vertex v, its endpoints determine whether it can immediately be extended. Pósa's rotation [3] moves an endpoint while preserving all vertices already visited. Even in a Hamiltonian graph, all paths reachable by rotations from a given path may have both endpoints outside the neighbourhood of v. We ask how many edges of such a path must be replaced in one exchange to expose v, and how the answer changes when rotations may follow the exchange.

All graphs below are simple and undirected. In an ordering $(x_0, \ldots, x_\ell)$ a right rotation at x_i, where $i < \ell - 1$ and $x_\ell x_i$ is an edge, reverses the suffix strictly after x_i. A left rotation is defined by reversal. Write $\mathcal{R}(P)$ for the paths reachable from P by rotations at either end, including P itself. We regard a path and its reversal as the same undirected path and choose an orientation when specifying a move.

An ordering of $V(P)$ *exposes* v if it is a path with an endpoint adjacent to v. For simple paths R, Q on the same vertices, set

$$d_E(R, Q) = |E(R) \setminus E(Q)| = \tfrac{1}{2}|E(R) \triangle E(Q)|.$$

An exchange from R to Q simultaneously replaces $d_E(R, Q)$ path edges. Extension by the edge to v is a separate final step. Define

$$\rho(P, v) = \min\{d_E(R, Q) : R \in \mathcal{R}(P),\ V(Q) = V(P),\ Q \text{ exposes } v\},$$

with value $+\infty$ if there is no such pair. This is the cost when the exchange must expose v immediately.

*Corresponding author.

Theorem 1. *For each integer $t \geq 0$ there are a Hamiltonian bipartite cubic Cayley graph G on $n = 72t + 156$ vertices, a vertex v, and a path P spanning $G - v$ with the following properties.*

1. *The cost of a final exchange satisfies*

$$3t + 2 \leq \rho(P, v) \leq 6t + 13.$$

2. *With arbitrary rotations before and after one exchange, the minimum exchange size for completing P is exactly two.*

3. *Any repair with an exchange of size at most k and s subsequent rotations satisfies $k + s \geq 3t + 2$. Seven initial rotations, a two-edge exchange and $6t + 13$ subsequent rotations suffice.*

Thus a final exchange has linear cost in the order of the graph. Moving the exchange earlier reduces its size to two, at the price of a linear number of subsequent rotations. Both statements concern the same missing vertex of the same prescribed path.

Shields [4] studied endpoint rotations in Hamilton cycle searches on cubic Cayley graphs. In the language of path reconfiguration, a rotation is also called a backbite. Kazazi [1] studied when these moves connect the Hamiltonian paths of a graph. Here the relevant paths span the punctured graph $G - v$, and the target consists of orders that expose v. Our earlier work [5] supplied a sixty-vertex rooted path whose free end stays confined under rotations. We use two copies of that path and vary the distance to the missing vertex. Our technical preprint [6] proved the lower bound $3t + 1$ and gave the two-edge repair with $6t + 13$ subsequent rotations, implying a final-exchange upper bound of $6t + 15$. Here we sharpen the final-exchange bounds and prove that two is the least possible exchange size when subsequent rotations are allowed.

2 The Cayley graph and the prescribed path

Fix an integer $t \geq 0$ and put $m = 12t + 26$. Let $a = (0\,1)$ and $b = (0\,1\,2)$ in S_3. Write the vertices of G_m as (h, g) in $(\mathbb{Z}/m\mathbb{Z}) \times S_3$. The three neighbours of (h, g) are

$$(h, ga), \qquad (h + 1, gb), \qquad (h - 1, gb^{-1}). \tag{1}$$

We use the same rule on $\mathbb{Z} \times S_3$ before reducing the height modulo m. For numerical path descriptions we use the following permutations in one-line notation, composing products from right to left.

$$(g_0, g_1, g_2, g_3, g_4, g_5) = (012, 102, 120, 021, 210, 201).$$

In particular $g_0 = e$, $g_1 = a$ and $g_5 = b^{-1}$.

Lemma 2. *G_m is a connected, bipartite, cubic Cayley graph on $6m$ vertices and has a Hamilton cycle.*

Proof. The elements $A = (0, a)$, $B = (1, b)$ and B^{-1} are distinct and nonidentity. Their Cayley graph is therefore simple and cubic. The words B^3 and $(AB)^2$ give $(3, e)$ and $(2, e)$, hence the subgroup they generate contains $(1, e)$. Together with A and B this generates the whole direct product. Since m is even, height parity plus permutation parity is a well-defined bipartition.

Retain all edges in the B-direction except those between heights 0 and 1. The result consists of six paths, one from $(1, g)$ to $(0, gb^{m-1})$ for each g. Add the A-matching on each of those two levels. Going up one path, across the matching at height 0, back down a second path and across the matching at height 1 sends g to $gb^{m-1}ab^{-(m-1)}a = gb^{2(m-1)}$. As $m \equiv 2 \pmod 3$, this permutation cycles through the three labels of one parity. The intervening paths have the three opposite-parity labels at height 1. All six paths thus form one cycle. □

These hosts also appear as generalized Petersen graphs in Kuzman [2, Theorem 4.7(ii)], with parameter $p = m/2 \equiv 1 \pmod 3$. Indeed, with i taken modulo $3m$, put $u_i = (i, b^i)$ and $w_i = (i, b^i a)$, reducing the height modulo m. Since 3 and m are coprime, these are the two permutation-parity classes. Equation (1) gives edges $u_i u_{i+1}$, $u_i w_i$, and $w_i w_{i+m-1}$. This identifies G_m with $GP(3m, m-1)$.

2.1 Two rooted paths

Take the rooted path F of Wu and Peng [5, Section 2] on heights $0, \ldots, 9$. Write the code of (h, g_i) as $6h + i$. In order from its root $(0, e)$ to its free end, F is

$$0, 1, 9, 16, 14, 23, 24, 25, 33, 40, 43, 51, 58, 56, 48,$$
$$47, 38, 30, 31, 39, 46, 49, 57, 59, 50, 42, 41, 32, 34, 37,$$
$$45, 52, 55, 54, 53, 44, 36, 35, 26, 28, 21, 13, 12, 11, 2,$$
$$4, 7, 15, 22, 20, 29, 27, 19, 18, 17, 8, 10, 3, 5, 6.$$

Lemma 3. *When only the free end of F rotates, every reachable free end has all three neighbours in F, even in the infinite strip.*

Proof. Here is the finite calculation for the displayed word. Start with the set containing F. For each unprocessed word $(x_0, \ldots, x_{59})$ in the set, insert

$$(x_0, \ldots, x_i, x_{59}, x_{58}, \ldots, x_{i+1}) \qquad (0 \le i \le 57, \;\; x_i \in N(x_{59})),$$

where N is given by (1), and mark the word as processed. The calculation terminates after 30 words and 60 directed rotations. The last entries of these words form the set

$$\{6, 11, 15, 16, 18, 20, 23, 28, 30, 39, 42, 44, 51\}.$$

Their heights lie between 1 and 8. Every neighbour therefore has height between 0 and 9, all of whose vertices occur in F. □

We call this copy of F the first room. The second room is $T(F)$, where

$$T(h, g) = (-12t - 7 - h, ga).$$

The relation $aba = b^{-1}$ shows that T is an involutory graph automorphism interchanging upward and downward steps.

2.2 Joining the roots

Starting at $(0, e)$, step down and traverse $3t$ pairs of consecutive levels, in decreasing height order. Within each pair the induced graph is a 12-cycle. In fact the steps horizontal, down, horizontal, up return to the upper level with the label multiplied by $ab^{-1}ab = b^2$. Three repetitions traverse all six labels on both levels. If the entry into the upper level is (h, g), remove the cycle edge from this entry to $(h - 1, gb^{-1})$ and traverse the remaining eleven edges. A downward step from its last vertex reaches the next upper level with label gb^{-2}. Let S be the resulting word on the $6t$ levels -1 through $-6t$, exclusive of the root. Its length is $36t$. The initial downward step has label b^{-1} and the $3t$ pairs multiply it by $b^{-6t} = e$. Thus entry into the next block is at $(-6t - 1, b^{-1})$. At $t = 0$ this is the neighbour directly below the root, and S is the empty word.

The central block has six levels. For this block only, put $j = -6t - 1 - h$ and encode (h, g_i) by $6j + i$. Take the word C

$$5, 3, 7, 6, 2, 4, 9, 11, 0, 1, 10, 8, 12, 13, 22, 20, 17, 15,$$
$$19, 18, 14, 16, 21, 23, 26, 30, 31, 27, 29, 32, 34, 25, 24, 35, 33.$$

Its first vertex is the specified entry and its last is the image of the first under T. Checking consecutive entries in (1) gives a path. Its only missing vertex is code 28, namely $v = (-6t-5, g_4)$.

For a word X let X^{rev} denote its reversal. The desired path is the reduction modulo m of the concatenation

$$P = F^{\mathrm{rev}}\, M\, T(F), \qquad M = S\, C\, T(S^{\mathrm{rev}}). \tag{2}$$

The first half of the construction supplies the joins up to C. Reflection supplies the remaining joins since the last vertex of C is the reflected first vertex. The five successive pieces occupy disjoint sets of levels and contain respectively $60, 36t, 35, 36t, 60$ vertices. Together their levels lie in the m consecutive integers from $-12t-16$ to 9. Reduction identifies none of these levels. Thus P is simple and spans $G_m - v$.

Figure 1 shows how the length of the connecting path varies with t.

F^{rev}	S	C	$T(S^{\mathrm{rev}})$	$T(F)$
60	$36t$	35 of 36	$36t$	60

Figure 1: The five blocks in path order, with their vertex counts. The missing vertex lies on one of the six central levels. Reduction modulo m places the two outer blocks on consecutive intervals of heights.

Lemma 4. *For every $R \in \mathcal{R}(P)$, both endpoints have least nonnegative height in $\{0, \dots, 19\}$. The omitted vertex v has height $6t+21$ in these coordinates.*

Proof. Orient P as in (2). After any number of rotations the ordering still has the form $F_1^{\mathrm{rev}}\, M\, T(F_2)$, where the F_i are reachable from F by free-end rotations. Indeed, by Lemma 3, every possible pivot for the moving end belongs to its own copy of F. Reduction modulo m preserves this property because the three strip neighbours project onto the three quotient neighbours. The rotation acts on that copy alone, even when its pivot is the root. This proves the description by induction. The levels of F and $T(F)$ reduce to $0, \dots, 9$ and $10, \dots, 19$, respectively. The representative $-6t-5$ of the missing height reduces to $6t+21$. □

3 The cost of a final exchange

For a nonempty path R, denote by $s_w(R)$ the sum of the values of a weight w at its endpoints, counting the endpoint of a singleton twice.

Lemma 5. *Let R, Q be simple paths with the same nonempty vertex set. If w is integer-valued and $|w(x) + w(y)| \le 1$ on every graph edge xy, then*

$$|s_w(R) - s_w(Q)| \le 2 d_E(R, Q).$$

Proof. The sum of $w(x)+w(y)$ over edges xy of R equals $2\sum_{x \in V(R)} w(x) - s_w(R)$. Subtract the corresponding identity for Q. Common edges cancel. Each of the $2d_E(R,Q)$ remaining edges contributes at most one in absolute value. □

Proposition 6. *If $R \in \mathcal{R}(P)$ and Q is a path on $V(P)$ with an endpoint adjacent to v, then $d_E(R, Q) \ge 3t + 2$.*

Proof. Along an upward edge $(h, g) \mapsto (h+1, gb)$ the product gb^{-h} is constant until the height wraps around. This lets us distinguish the reachable endpoints from other labels at the same height. Set $D(q) = 2$ if $q = g_4$, $D(q) = 1$ if q is even, and $D(q) = 0$ for the other odd permutations. For the least nonnegative representative h of each height, define

$$f(h, g) = \max\{0, \min\{h - 18 + D(gb^{-h}),\, m - h\}\}.$$

On a horizontal edge the two arguments of D have opposite parity, so their values differ by one. On an upward step from $h < m-1$, the argument gb^{-h} is unchanged. The two terms in the minimum therefore change by at most one on either type of edge. On the step across the height modulus the value at $m-1$ is at most one and the value at 0 is zero. Thus f is 1-Lipschitz on G_m.

At heights at most 16 the function vanishes identically. By the endpoint list in Lemma 3, the only possible endpoints above height 16 are in the reflected room. Those at height 17 have even labels, so $D = 1$ there. Those at height 18 have labels g_1 and g_3. Since $b^{-18} = e$, both labels have $D = 0$. It follows from Lemma 4 and its proof that f vanishes at both ends of every $R \in \mathcal{R}(P)$.

At $v = (6t+21, g_4)$ the correction is 2 since $3 \mid (6t+21)$. Both terms in the minimum equal $6t+5$, so $f \geq 6t+4$ on every neighbour of v. Give f opposite signs on the two colour classes to obtain w. For each edge xy the absolute value of $w(x) + w(y)$ is the absolute difference of the f-values. We have $s_w(R) = 0$, and the $6m-1$ vertices of Q give its endpoints the same colour. Their contributions add in absolute value, one having magnitude at least $6t+4$. Lemma 5 yields $2d_E(R, Q) \geq 6t+4$. □

4 Rotations after an exchange

Let P and v be as in Section 2. Write $s_2(t)$ for the least number of rotations following an exchange of at most two edges that suffice to expose v, with unrestricted rotations before the exchange.

Theorem 7. *For every $t \geq 0$, seven rotations of P, one exchange removing two edges and inserting two edges, and $13 + 6t$ further rotations expose v. For any sequence consisting of arbitrary initial rotations, one exchange of at most k edges, and s final rotations, exposure implies*

$$k + s \geq 3t + 2.$$

Consequently $3t \leq s_2(t) \leq 13 + 6t$.

Proof. Encode a vertex of height h and label i by $6h + i$, taking $0 \leq h < m$. All endpoint and pivot codes below refer to the current ordering, read from left to right. Rotate the left endpoint at the seven successive pivots

$$14,\ 21,\ 31,\ 41,\ 43,\ 53,\ 46.$$

Delete the edges $\{48, 56\}, \{46, 49\}$ and insert $\{48, 49\}, \{39, 46\}$. The reordered path has left endpoint 56 and the same junction with the middle word M. Starting from this ordering, perform the following rotations, one column at a time:

end	L	L	L	L	R	L	L	L	L	L	L	R	R
old endpoint	56	66	75	78	109	76	83	88	99	102	112	104	109
pivot	65	67	77	79	106	74	81	91	101	110	115	106	108

The resulting word is $M^{\mathrm{rev}}W$, with W ending in 116. To verify the table, keep M as a single signed letter. Every pivot lies in one of the two rooted paths, so the successive reversals require only their two orderings and the orientation of M.

We now move the t six-level blocks at the start of M^{rev} past the left endpoint. Each block consists of three twelve-vertex slices. If the current ordering has the form

$$a\,U\,x\,y\,V\,b\,J\,z$$

with first slice $a\,U\,x\,y\,V\,b$, a right rotation at x using zx, followed by a left rotation at b using ab, gives

$$J\,z\,x\,U^{\mathrm{rev}}\,a\,b\,V^{\mathrm{rev}}\,y.$$

The next unprocessed slice, when present, now begins the word J. The new right endpoint is y. For block j, add $36j$ to every code in the following array:

slice	z	a	x	y	b
1	116	123	125	126	130
2	126	133	134	143	141
3	143	148	144	152	151

The neighbour rule verifies zx and ab in each row. The first and last entries of the slice are a, b, and y immediately follows x. Translation by six levels preserves these relations. Since the last new right endpoint $152 + 36j$ equals $116 + 36(j + 1)$, induction crosses all t blocks in $6t$ rotations. What remains at the left end is the central vertex of code $123 + 36t$, adjacent to v of code $130 + 36t$. This also covers $t = 0$, when the thirteen room rotations already expose v. Prepending v gives a Hamilton path.

Finally let R be the path just before the exchange and Q the exposed path after the last rotation. Because d_E is half a symmetric-difference distance, it satisfies the triangle inequality. A rotation changes at most one undirected edge, so the triangle inequality gives $d_E(R, Q) \le k + s$. Proposition 6 applies to this pair and yields $3t + 2 \le k + s$. □

Corollary 8. *For every $t \ge 0$, the minimum single-exchange size allowing completion of P with arbitrarily many rotations on either side is two.*

Proof. We use the odd-order bipartite case of Kazazi's observation that a one-edge flip preserves the rotation component [1, Section 1]. In a bipartite graph the endpoints u, v' of a path of odd order lie in the same colour class. Consider such a path with one edge xy removed, oriented so that its two components run from u to x and from y to v'. A replacement edge producing a path on the same vertices must join an endpoint of one component to an endpoint of the other. Reinserting xy restores the original path. Joining u to y or x to v' gives a Pósa rotation up to reversal. The final possibility uv' is excluded by bipartiteness. This argument also covers a singleton component, when two of the listed choices coincide.

Every rotation preserves the odd order $72t + 155$ of P. Thus an exchange of at most one edge, even between two sequences of rotations, stays in its undirected rotation component. An exposed path in this component would give a final exchange of cost zero, contradicting Proposition 6. Theorem 7 attains two. □

Completion of the proof of Theorem 1. It remains to sharpen the upper bound on ρ. Adding the exchange size to the subsequent rotation count gives $6t + 15$ before accounting for cancellations in the initial part of the schedule. Let R be the ordering after seven rotations. Let $H = M^{\mathrm{rev}}W$ be the ordering after the exchange and the thirteen room rotations. The two rooted paths in R contain 118 internal edges. Of the 119 edges in W, exactly 106 belong to those paths. The room edges of R missing from W are precisely

$$(46, 49),\ (48, 56),\ (65, 66),\ (67, 75),\ (74, 83),\ (76, 79),$$
$$(77, 78),\ (81, 88),\ (91, 99),\ (101, 102),\ (108, 116),\ (110, 112).$$

Here each pair denotes an undirected edge in the code $6h + i$. Reversing M preserves all its internal edges. Its junction with the vertex of code 0 is present in both paths, so at most one other junction of R is lost. Hence $d_E(R, H) \le 12 + 1$. The remaining $6t$ rotations change at most $6t$ more edges. Their exposed output therefore lies at distance at most $6t + 13$ from R. □

The exact value of $\rho(P, v)$ for this family remains open, as does the least number of post-exchange rotations at budget two. Does the ratio $\rho(P, v)/t$ converge as $t \to \infty$? Theorem 1 places any limit between three and six.

Formal verification. The Lean 4 development for this manuscript identifies the Cayley generators, constructs the path for every t, and checks the rooted rotations and the repair schedule. Its file `PaperVerification.lean` states the bounds and the optimal exchange size. The cited archives contain the earlier formal development and the rooted-path certificate [6, 5].

Reproducibility. The accompanying script `anc/verify.py` uses only the Python standard library. It reproduces the rooted-path enumeration and replays the exchange and rotations for the parameters specified by `-max-t`. The default is $0 \le t \le 50$.

Author contributions and use of AI. Peiwei Wu and Qian Peng jointly formulated the research question, guided the comparison of the two repair protocols, and contributed to writing the manuscript. They used OpenAI Codex to help search the literature, develop and check the constructions and proofs, formalize the results in Lean 4, and draft and edit the text. The human review covered the problem formulation and the statements of the principal results. The detailed proofs and Lean code have not been checked in full by a human.

References

[1] A. Kazazi, Connectivity of the reconfiguration graph of Hamiltonian paths, 2026. arXiv:2610.05068.

[2] B. Kuzman, On cubic rainbow domination regular graphs, *Discrete Appl. Math.* **373** (2025), 26–38. doi:10.1016/j.dam.2025.04.046. arXiv:2410.10662.

[3] L. Pósa, Hamiltonian circuits in random graphs, *Discrete Math.* **14** (1976), 359–364. doi:10.1016/0012-365X(76)90068-6.

[4] I. B. Shields, *Hamilton Cycle Heuristics in Hard Graphs*, Ph.D. thesis, North Carolina State University, 2004. NC State University Libraries.

[5] P. Wu and Q. Peng, Two-ended Pósa rotations can get stuck in cubic Cayley graphs, v1.0.0, 2026. doi:10.5281/zenodo.24621624.

[6] P. Wu and Q. Peng, Linear edge-exchange cost after Pósa rotations in cubic Cayley graphs, v1.19.0, 2026. doi:10.5281/zenodo.24639140.