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Finitary collapse of Bernoulli shifts from an algebraic splitting

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Abstract

We construct a countable group for which all nontrivial Bernoulli shifts with countable discrete bases are finitarily conjugate, regardless of their base entropies. For Polish bases the conjugacies are homeomorphisms between invariant sets of full measure. The construction turns an algebraic existence result due to OpenAI into a reversible extraction of independent coordinates, and the group can be taken to be finitely presented. Accompanying Lean certificates, registered by Palomar, establish the measurable isomorphism of all nontrivial Bernoulli shifts of such a group; the finitary statements are not formalized.

1 Statement and proof plan

Attribution and scope. This note develops a dynamical construction from the algebraic data of Hypothesis 1.1. The existence of those data is credited to OpenAI's characteristic-two construction [1]; the pinned source and the exact dependency are recorded in Appendix A. The purpose here is to explain how extraction, almost-sure recovery, and absorption turn that input into Bernoulli conjugacies, and then to give the stronger all-base finitary extension using the stated coding results of Meyerovitch and Spinka [2].

Context as of 25 October 2026. For every countably infinite group, Bernoulli shifts with equal base Shannon entropy are measurably isomorphic by Seward's theorem [8]. The unequal-entropy question is different: Bowen's sofic entropy invariant rules out a measurable conjugacy between finite-uniform Bernoulli shifts of unequal base entropy over a sofic group [9]. Neither equal values of Rokhlin entropy nor factor maps in both directions supply the inverse required for a conjugacy.

Related public approaches should also be acknowledged. The Astra research archive develops finite defect extraction with a residual factor and a conditional random equivariant basis criterion [10]. The SauersML group-approximation project gives a conditional route through approximate right inverses [11]. In the versions consulted, these approaches leave a residual factor, an unconstructed basis, or an additional exactification hypothesis. They contain some antecedents to the splitting and extraction strategy.

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[2] Human-facing version to come soon; this manuscript has not been fully human-checked. Please contact author for more information.

Formal verification and availability. The Lean build is available in the same GitHub repository as the compiled PDF. Its closed finite-uniform certificate constructs a concrete countable group and a finite alphabet with more than one symbol, and gives a measure-preserving equivariant conjugacy on invariant conull sets between the uniform Bernoulli shift on that alphabet and the shift on its square. The algebraic witness is imported from the pinned OpenAI proofs. This certifies the finite-uniform measurable existence statement; it does not certify this note's all-base extension, the integer coding inputs, finitary continuity, or the later corollaries. The stronger statements are presented here with their written proofs and explicit external dependencies.

Note added on 26 November 2026. Since this note was written, two Lean developments have been registered in the Palomar registry. The finite-uniform certificate described above is entry PALOMAR-2026-10-27-000030 [12]. A second development, registered on 12 November 2026 as entry PALOMAR-2026-11-12-000001 [13], certifies that all nontrivial Bernoulli shifts over standard probability spaces of the direct product of the algebraic source group with the lamplighter group are measurably isomorphic. It proves this by a different argument from the one in this note. The finitary and topological statements, the finitely presented and graph formulations, and the integer coding inputs remain uncertified. Registration means that Palomar's mechanical verification passed and that its automated review identified no blocking problem; it is not peer review. The abstract was shortened on the same date. This paragraph and the shortened abstract were written by Claude Fable 5.2; the rest of the text is that of 25 October 2026.

All groups are countable and discrete. All products carry their product Borel structures and the probability measures specified below. An *isomorphism of probability-preserving actions*, or *conjugacy*, means an equivariant measure-preserving bijection between invariant conull Borel sets, with measurable inverse.

If (L, λ) is a probability space, its Bernoulli G-shift is (L^G, λ^G) with

$$(a \cdot x)(g) = x(a^{-1}g) \qquad (a, g \in G).$$

The base Shannon entropy of the uniform measure on a finite set of size r is $\log r$. All logarithms are natural.

> **Hypothesis 1.1: The algebraic input**
>
> There are a finite field K, a countable group N, positive integers d, m, κ, and subgroups $J_1, \dots, J_d \le N$ of order κ, where $\kappa \cdot 1_K \neq 0$, with the following properties. Put
>
> $$p_v = \kappa^{-1} \sum_{u \in J_v} u \in K[N], \qquad P = \operatorname{diag}(p_1, \dots, p_d).$$
>
> There are matrices
>
> $$U \in \operatorname{Mat}_{m \times d}(K[N]), \qquad T \in \operatorname{Mat}_{d \times m}(K[N])$$
>
> such that
>
> $$UP = U, \qquad PT = T, \qquad UT = I_m, \qquad d < \kappa m. \tag{1.1}$$
>
> Here $K[N]$ is the algebra of finite formal K-linear combinations of elements of N, with multiplication induced by the group law.

Write $q = |K|$ and set

$$\begin{gathered} M = \kappa m, \qquad \Delta = M - d > 0, \qquad A = \bigoplus_{j \in \mathbb{Z}} C_2, \\ \Lambda = A \rtimes \langle s \rangle = C_2 \wr \mathbb{Z}, \qquad D = N \times A, \qquad G = N \times \Lambda. \end{gathered} \tag{1.2}$$

where s shifts the lamp indices by one. The height homomorphism $G \to \mathbb{Z}$ kills D and sends s to 1. In particular, $D \cap \langle s \rangle = \{e\}$. We regard N and A as the corresponding subgroups of D and then of G, and fix an enumeration of the lamp indices $\mathbb{Z}$.

Definition 1.2: Finitary regularity

For Polish probability bases, a conjugacy is *topo-finitary* if it is a homeomorphism between invariant conull Borel subsets of the product spaces. For countable discrete bases this means that, at almost every input, each output label is determined by finitely many input labels, in both directions. We call this *finitary*. The finite inspection set can depend on the input; no uniform bound or moment bound is required. For a nondiscrete output alphabet, continuity does not require an entire output value to be determined by finitely many input labels.

Theorem 1.3: All-base finitary collapse

Under Hypothesis 1.1, any two nontrivial Bernoulli G-shifts with Polish probability bases are topo-finitarily conjugate. In particular, any two nontrivial countable-discrete Bernoulli bases give finitarily conjugate G-shifts, including bases of unequal entropy.

Here nontrivial means that the probability measure is not concentrated at one point. All nontrivial standard probability bases are covered measurably, since their standard Borel spaces admit Polish realizations. The first step is the following explicit finite-uniform construction.

Theorem 1.4:

Under Hypothesis 1.1, the following G-actions, with uniform product measures, are measurably conjugate:

$$(K^{d+1})^G \;\cong\; (K^{\mathbb{N}_{>0}})^G. \tag{1.3}$$

Consequently

$$(K^{d+1})^G \;\cong\; (K^{2(d+1)})^G.$$

The latter two bases are finite and have distinct entropies $(d+1)\log q$ and $2(d+1)\log q$.

The external construction in Appendix A supplies Hypothesis 1.1. The group in this theorem is an enlargement of its group N. We make no assertion here that the original group N has the same conjugacy.

What must be proved. An algebraic splitting will let us repeatedly extract independent coordinates. That alone does not ensure invertibility: information might remain forever in the unextracted input. We therefore arrange that each original coordinate stops depending on that input after finitely many stages, almost surely. This is the main argument, in Section 5.

Here is the structure of the proof. A single independent marker $\Omega = K^G$ supplies fresh random orderings inside each left D-coset. For every finite $H \le A$, the algebra gives an exact splitting of d coordinates per H-coset into $M > d$ such coordinates and an auxiliary factor. At each stage we enlarge H, retain only one packet per larger coset, and extract all other packets. We then apply enough algebraic splittings to obtain Δ ordinary Bernoulli coordinates per group element. The chance that any prescribed original coordinate still depends on the retained input falls by a factor of at least two. After this exhaustion, elementary countable-product isomorphisms absorb the auxiliary factors.

2 Coordinates, retained markers, and finite rows

For a finite subgroup $H \le A$ and an integer $r \ge 1$, define

$$X_H(r) = (K^r)^{G/H}, \qquad X_H(\infty) = (K^{\mathbb{N}_{>0}})^{G/H}, \qquad \mathsf{B}_\infty = X_{\{e\}}(\infty).$$

The set G/H consists of the left cosets gH. A scalar coordinate of $X_H(r)$ is indexed by (gH, i), $1 \le i \le r$. These are compact metrizable abelian groups under addition, with their Haar probability measures, namely independent uniform K-labels. The left G-action permutes the cosets.

We use the word *packet* for the r scalar labels at one coset. Every coset gH is contained in a unique left D-coset. A map *preserves D-cosets* if its output on one such coset uses only data in that coset. Marker digits themselves may lie elsewhere; the random variables indexed by this D-coset will be specified explicitly.

Definition 2.1: Isomorphisms over a marker

Let $\Omega = K^G$, with uniform product measure $\mathbb{P}$, and let X, Y be probability-preserving G-spaces independent of Ω. The notation $X \cong_\Omega Y$ means that there is a conjugacy

$$(\omega, x) \longmapsto (\omega, \phi_\omega(x)) \quad \text{from } \Omega \times X \text{ to } \Omega \times Y,$$

where the map and inverse are jointly measurable, and, for almost every ω, ϕ_ω carries the probability measure on X to that on Y. Equivariance means

$$\phi_{a\omega}(ax) = a\phi_\omega(x) \qquad (a \in G). \tag{2.1}$$

The same single marker occurs on both sides.

Lemma 2.2: Products over the same marker

Suppose $X_j \cong_\Omega Y_j$ for $j \geq 1$. Applying all the fiber maps coordinatewise gives $\prod_j X_j \cong_\Omega \prod_j Y_j$. These relative isomorphisms can also be composed and inverted.

Proof. Intersect the countably many invariant conull sets of allowed markers. For each marker in this intersection, the product map is a bijection with its coordinatewise inverse. Both are measurable because all coordinate maps are measurable. On cylinder sets the product measure is preserved, hence it is preserved on the product Borel sigma-algebra. Equation (2.1) holds coordinatewise. Integrating in ω proves preservation of the full product measure. Composition and inversion have the same properties. □

Definition 2.3: Rows and inverse sections

A *row* on $X_H(r)$ is a finitely supported linear functional

$$\ell(x) = \sum_{(b,i)\in F} c_{b,i} x_i(b), \qquad F \subseteq (G/H) \times \{1, \ldots, r\} \text{ finite.}$$

Its support consists of the nonzero coefficients, after collecting terms with the same index. A linear map has row bound C if every output scalar coordinate is a row with at most C terms. Composing maps of row bounds C_1, C_2 gives row bound at most $C_1 C_2$.

If $\phi : X \to X' \times Y$ is a linear isomorphism, its *inverse section for X'* is $x' \mapsto \phi^{-1}(x', 0)$. It describes the part of the old input depending on the new input when the extracted factor Y is set to zero. It is not asserted to preserve probability measure.

3 Fresh randomness from one finite-base marker

Fix a bijection $\beta : K \to \{0, \ldots, q-1\}$ and a bijection

$$\rho : \mathbb{N}_{\geq 0} \times \mathbb{N}_{>0} \longrightarrow \mathbb{N}_{>0}.$$

For $g \in G$ and a stream index $r \geq 0$, put

$$\xi_g^{(r)}(\omega) = \sum_{j=1}^{\infty} q^{-j} \beta\big(\omega(gs^{\rho(r,j)})\big). \tag{3.1}$$

These are measurable $[0,1]$-valued functions of the marker.

Lemma 3.1:
For each fixed left D-coset C, the family $\{\xi_g^{(r)} : g \in C,\ r \geq 0\}$ consists of independent uniform random variables on $[0,1]$. Moreover

$$\xi_{ag}^{(r)}(a\omega) = \xi_g^{(r)}(\omega).$$

Proof. Write $C = g_0 D$. The elements $g_0 u s^j$, for $u \in D$ and $j \geq 1$, are pairwise distinct. Indeed, equality for (u, j) and (v, k) would give $v^{-1}u = s^{k-j}$. Applying the height homomorphism gives $j = k$, and then $u = v$. The digit sequences used in (3.1), for all the indicated (g, r), are disjoint sequences of independent uniform digits. A sequence of such base-q digits has the uniform distribution on $[0,1]$: each interval specified by its first j digits has probability q^{-j}, and these intervals determine the distribution. This also proves independence of the family. The covariance identity follows directly from $(a\omega)(ags^j) = \omega(gs^j)$. □

For a finite $H \leq A$, give an H-coset b the priority

$$\pi_{H,r}(b) = \min_{g \in b} \xi_g^{(r)}. \tag{3.2}$$

Distinct H-cosets inside a fixed D-coset have independent priorities with the same continuous distribution. A finite collection of these cosets can therefore be ordered by increasing priority, and every member is equally likely to come first. Ordering the individual elements of a finite coset uses $H = \{e\}$.

There are only countably many finite subgroups of A, cosets, stream indices, and pairs of priorities being compared. The event on which all these comparisons are tie-free has probability one and is G-invariant. Call it Ω_{ord}. For measurability statements on the whole probability space, break any ties using a fixed enumeration of the countable index sets. This convention has no effect on Ω_{ord}, where every ordering is covariant.

Remark 3.2:
The random fields belonging to different left D-cosets need not be independent. We will never require this. The recovery estimate for a coordinate conditions only on earlier streams indexed by its own D-coset. We also do not condition on the entire marker in that estimate: once the marker is fixed, all the selections are fixed.

4 The algebraic splitting on each finite coset space

4.1 The action of the group algebra

For $a = \sum_{u \in N} a_u u \in K[N]$, define an operator on $K^{G/H}$ by

$$(\mathcal{R}_a f)(gH) = \sum_{u \in N} a_u f(guH). \tag{4.1}$$

This is well-defined: replacing g by gh, $h \in H$, does not change guH, since $H \leq A$ commutes with N. Each sum is finite. Direct substitution gives

$$\mathcal{R}_a \mathcal{R}_b = \mathcal{R}_{ab}.$$

For a matrix B over $K[N]$, use $(\mathcal{R}_B f)_i = \sum_j \mathcal{R}_{B_{ij}} f_j$. These continuous linear operators commute with the left G-action, preserve D-cosets, and respect matrix multiplication. We henceforth write the matrices themselves for their operators on G/H.

For every v, $p_v^2 = p_v$. Its image consists exactly of the functions constant on each coset of HJ_v. The inclusion in this space follows by reindexing the subgroup average, and a function already constant there is fixed by that average. Because $H \cap J_v = \{e\}$ and the two subgroups commute, each HJ_v-coset contains exactly κ H-cosets. It follows that

$$V_H := \mathrm{im}(P : X_H(d) \to X_H(d)) = \prod_{v=1}^{d} K^{G/(HJ_v)}. \tag{4.2}$$

This identification preserves Haar probability measure.

Lemma 4.1: A split product

Put $Q = P - TU$ and let $Y_H = \mathrm{im}(Q : X_H(d) \to X_H(d))$. Then

$$V_H \longrightarrow X_H(m) \times Y_H, \qquad v \longmapsto (Uv, Qv)$$

is a continuous linear G-equivariant isomorphism, with inverse $(y, z) \mapsto Ty + z$. It sends Haar measure to product Haar measure.

Proof. The identities in (1.1) give

$$Q^2 = Q, \quad PQ = QP = Q, \quad UQ = 0, \quad QT = 0.$$

For example, $(TU)^2 = TU$, while $PTU = TUP = TU$, so $(P - TU)^2 = P - TU$ in every characteristic. If $v \in V_H$, then $TUv + Qv = Pv = v$. Conversely, for $y \in X_H(m)$ and $z \in Y_H$, $Ty + z \in V_H$ and

$$U(Ty + z) = y, \qquad Q(Ty + z) = z.$$

All maps are continuous because their coordinate formulas are finite. The image Y_H is compact, hence closed. A continuous group isomorphism between compact groups carries normalized Haar measure to normalized Haar measure; Haar measure on a product is the product of the Haar measures. This proves the measure assertion. □

4.2 Turning subgroup-constant labels into ordinary coset labels

Take κ independent copies of the splitting in Lemma 4.1. Define the fixed compact G-space

$$Z_H = Y_H^{\kappa},$$

with product Haar measure. Fix a stream index r. In each HJ_v-coset b, order its κ constituent H-cosets as $b_1, \dots, b_\kappa$ by (3.2). Define

$$F_{H,r,\omega} : V_H^{\kappa} \longrightarrow X_H(d), \qquad [F_{H,r,\omega}(v)]_v(b_j) = v_v^{(j)}(b). \tag{4.3}$$

Thus the label in copy j is placed at the jth ordered coset. For a fixed marker this is a bijection of scalar coordinate slots, hence a continuous linear Haar-preserving isomorphism. Its map and inverse are jointly measurable in the marker, and they satisfy (2.1) on Ω_{ord}.

Combining $F_{H,r,\omega}^{-1}$ with the κ copies of the algebraic split, and identifying $(X_H(m))^{\kappa}$ with $X_H(M)$, gives

$$X_H(d) \cong_{\Omega} X_H(M) \times Z_H. \tag{4.4}$$

Fix the track order in this identification once and for all. Separating the first d tracks gives

$$X_H(d) \cong_{\Omega} X_H(d) \times X_H(\Delta) \times Z_H. \tag{4.5}$$

Conditional on the marker, all factors on the right have their product Haar law. In particular, the new $X_H(d)$ input is independent of the extracted factors. This statement concerns the full split map; its inverse section alone does not have this measure property.

4.3 A uniform finite bound on inverse rows

Define the finite integer

$$C_0 = \max\left(1, \max_{1\le v\le d} \sum_{i=1}^{m} |\operatorname{supp}(T_{vi})|\right). \tag{4.6}$$

Let $\widehat{T}$ apply T separately in the κ copies. The inverse section for the $X_H(M)$ factor in (4.4) is

$$B_{H,r,\omega} = F_{H,r,\omega}\widehat{T}.$$

Each row of this map uses at most C_0 scalar coordinates: F chooses one coordinate in one copy, and that row of T has at most C_0 terms. Collisions of cosets or cancellation can only reduce this bound. It is uniform in H, r, ω.

Let $\iota_d : X_H(d) \to X_H(M)$ insert the input into the first d tracks, with zero in the other tracks. Then

$$S_{H,r,\omega} = B_{H,r,\omega}\iota_d \tag{4.7}$$

is the inverse section for the first factor of (4.5). Iterate that same split k times, each time on its first factor. We obtain

$$X_H(d) \cong_\Omega X_H(d) \times X_H(k\Delta) \times Z_H^k. \tag{4.8}$$

For a fixed marker its inverse section is $S^k_{H,r,\omega}$, with row bound C_0^k. All these maps preserve D-cosets, and their coefficients on a given D-coset depend only on stream r indexed by that coset. The same stream may be reused in all k iterations: conditional product Haar measure is preserved at each iteration.

5 Growing finite blocks and recovery of every coordinate

5.1 Choose all sizes deterministically

Let A_a denote the subgroup of A generated by the first a lamps in the fixed enumeration, so $|A_a| = 2^a$ and $A_0 = \{e\}$. Define increasing integers a_n, finite subgroups $H_n = A_{a_n}$, and row bounds R_n recursively. Start with $a_0 = 0$ and $R_0 = 1$. Having chosen a_n and R_n, choose $a_{n+1} > a_n$ so that

$$L_n := 2^{a_{n+1}-a_n} = [H_{n+1} : H_n] \ge 2R_n. \tag{5.1}$$

Then set

$$h_{n+1} = |H_{n+1}|, \qquad k_n = h_{n+1}, \qquad R_{n+1} = R_n C_0^{k_n}. \tag{5.2}$$

Every choice is finite. The order is important: R_n determines a sufficient block size, that size determines k_n, and only then do we define R_{n+1}. There is no requirement to bound R_{n+1} before choosing the current block.

5.2 One stage of the construction

At stage n, the remaining input is $x_n \in X_{H_n}(d)$. Each H_{n+1}-coset B is a union of L_n cosets of H_n. Order them using stream $2n$:

$$b_1(B), \dots, b_{L_n}(B).$$

Retain the d-packet at $b_1(B)$ and extract the other packets in their listed order. This is an isomorphism over Ω

$$X_{H_n}(d) \cong_\Omega X_{H_{n+1}}(d) \times W_n, \qquad W_n = X_{H_{n+1}}(d(L_n - 1)). \tag{5.3}$$

The inverse section for its first factor is the insertion map

$$(I_{n,\omega}v)_i(b) = \begin{cases} v_i(B), & b = b_1(B), \quad b \subset B, \\ 0, & b \neq b_1(B), \quad b \subset B. \end{cases} \tag{5.4}$$

Each row of $I_{n,\omega}$ has at most one nonzero coefficient.

Next apply (4.8) to the retained factor, with $H = H_{n+1}$, stream $2n+1$, and $k = k_n$. The complete stage is

$$X_{H_n}(d) \cong_\Omega X_{H_{n+1}}(d) \times \mathcal{Y}_n, \qquad \mathcal{Y}_n = W_n \times E_n \times Z_{H_{n+1}}^{k_n}, \tag{5.5}$$

where

$$E_n = X_{H_{n+1}}(k_n\Delta) = X_{H_{n+1}}(h_{n+1}\Delta). \tag{5.6}$$

Write its fiber map as $\phi_{n,\omega}(x_n) = (x_{n+1}, y_n)$, with $y_n \in \mathcal{Y}_n$. Its inverse section for the new input is

$$J_{n,\omega} = I_{n,\omega} S_{H_{n+1},2n+1,\omega}^{k_n}. \tag{5.7}$$

Composition is read from right to left: the algebraic section acts first; insertion into the one selected old packet acts afterwards. This map has row bound $C_0^{k_n}$.

Lemma 5.1: A fixed Bernoulli output at every stage

For every n, $E_n \cong_\Omega X_{\{e\}}(\Delta)$.

Proof. In each H_{n+1}-coset B, order its h_{n+1} individual group elements by their stream-$2n$ values. Regard the $h_{n+1}\Delta$ labels at B as h_{n+1} ordered packets of Δ labels, and place packet j at the jth element. This is a measurable covariant bijection of scalar slots. For every tie-free marker it is a linear homeomorphism and preserves uniform product measure. □

The reuse of stream $2n$ here causes no difficulty: for a fixed marker this operation is just a coordinate permutation. Since every full stage preserves conditional product Haar measure, the extracted ordinary Bernoulli factors are independent of one another, the other extracted factors, and the retained marker.

5.3 The dependence of an original coordinate

Let $\Pi_{0,\omega}$ be the identity on $X_{H_0}(d)$, and recursively put

$$\Pi_{n+1,\omega} = \Pi_{n,\omega} J_{n,\omega}.$$

Thus $\Pi_{n,\omega} : X_{H_n}(d) \to X_{H_0}(d)$ is the inverse section for the input left after the first n stages, with all extracted factors set to zero. For an original scalar coordinate $\alpha = (g, i) \in G \times \{1, \dots, d\}$, define the row

$$\ell_{n,\alpha,\omega} = \mathrm{ev}_\alpha \, \Pi_{n,\omega}. \tag{5.8}$$

The row is zero exactly when that original coordinate no longer depends on the remaining input.

Fix this original coordinate and let $C = gD$. Define

$$\mathcal{F}_n^C = \sigma(\xi_u^{(r)} : u \in C,\ 0 \le r < 2n). \tag{5.9}$$

All probability statements in the next lemma are on the original marker space $(\Omega, \mathbb{P})$, before restricting to a conull set.

Lemma 5.2: Recovery estimate

The row $\ell_{n,\alpha,\omega}$ has at most R_n nonzero scalar coefficients. It is supported on H_n-cosets inside C, and its coefficients are $\mathcal{F}_n^C$-measurable. Moreover

$$\mathbb{P}(\ell_{n+1,\alpha,\omega} \neq 0 \mid \mathcal{F}_n^C) \le \frac{R_n}{L_n} \mathbf{1}_{\{\ell_{n,\alpha,\omega} \neq 0\}} \le \frac{1}{2} \mathbf{1}_{\{\ell_{n,\alpha,\omega} \neq 0\}} \quad \text{almost surely.} \tag{5.10}$$

Proof. The assertions on support and measurability follow by induction. At $n = 0$ we have the fixed coordinate evaluation, of support size one. The section $J_{n,\omega}$ has row bound $C_0^{k_n}$, preserves D-cosets, and uses only streams $2n$ and $2n+1$ within the relevant coset. The recursion (5.2) therefore gives all three assertions at $n+1$. Measurability of a row means measurability of its coefficient at each of the countably many possible coordinate indices.

Condition now on $\mathcal{F}_n^C$. The row $\ell_n = \ell_{n,\alpha,\omega}$ is fixed. Let $\mathcal{S}_n$ be the set of H_n-cosets on which at least one of its coefficients is nonzero. Then $|\mathcal{S}_n| \le R_n$. By Lemma 3.1, the stream-$2n$ priorities within C are independent of $\mathcal{F}_n^C$. In every H_{n+1}-coset, the L_n constituent H_n-cosets have independent, identically distributed continuous priorities. Thus each specified H_n-coset is selected with conditional probability $1/L_n$.

If no member of $\mathcal{S}_n$ is selected, then $\ell_n I_{n,\omega} = 0$ by (5.4). In that event

$$\ell_{n+1,\alpha,\omega} = \ell_n I_{n,\omega} S^{k_n}_{H_{n+1},2n+1,\omega} = 0,$$

regardless of the second stream or the algebraic spreading. A conditional union bound gives $|\mathcal{S}_n|/L_n$ as an upper bound on survival. If $\ell_n = 0$, its successor is also zero. Combining these facts proves (5.10). No independence between the selections in different blocks is needed. □

Taking expectations and iterating gives

$$\mathbb{P}(\ell_{n,\alpha,\omega} \neq 0) \le 2^{-n}. \tag{5.11}$$

The events on the left are decreasing in n, because a zero row stays zero. Continuity of probability for decreasing events therefore yields

$$\mathbb{P}(\ell_{n,\alpha,\omega} \neq 0 \text{ for every } n) = 0.$$

There are countably many original scalar coordinates. Consequently there is a Borel conull set $\Omega_* \subseteq \Omega_{\mathrm{ord}}$ on which *every* original coordinate row becomes zero at a finite stage. All sections are covariant, so the property quantified over all coordinates is invariant under G. Thus Ω_* is G-invariant.

5.4 The infinite extraction has a measurable inverse

Proposition 5.3:

The successive stage maps give an isomorphism over the marker

$$X_{H_0}(d) \cong_\Omega \prod_{n \ge 0} \mathcal{Y}_n. \tag{5.12}$$

For every $\omega \in \Omega_*$, the fiber map is a continuous linear isomorphism of compact groups.

Proof. Fix $\omega \in \Omega_*$. Starting from x_0, recursively apply $\phi_{n,\omega}$ to obtain x_{n+1} and y_n. Define

$$\mathcal{E}_\omega(x_0) = (y_0, y_1, \ldots).$$

This map is linear and continuous: each output coordinate is obtained by finitely many continuous stage maps. For every n, composing the first n stages gives a continuous linear isomorphism

$$X_{H_0}(d) \longrightarrow X_{H_n}(d) \times \prod_{j<n} \mathcal{Y}_j. \tag{5.13}$$

Injectivity. The inverse of (5.13) expresses each original coordinate as

$$x_0(\alpha) = f_{n,\alpha,\omega}(y_0, \ldots, y_{n-1}) + \ell_{n,\alpha,\omega}(x_n), \tag{5.14}$$

where $f_{n,\alpha,\omega}$ is a continuous linear function of the extracted data. For each α choose a stage with $\ell_{n,\alpha,\omega} = 0$. That coordinate is then determined by the extracted data alone. Hence two inputs with the same infinite extraction agree at every coordinate.

Surjectivity. The image of $\mathcal{E}_\omega$ is compact and therefore closed in $\prod_n \mathcal{Y}_n$. Its projection onto every finite initial product $\prod_{j<n} \mathcal{Y}_j$ is onto, by (5.13). Every basic open set in the infinite product restricts only finitely many factors and is therefore met by the image. The image is dense as well as closed, so it is the whole product. This proves that $\mathcal{E}_\omega$ is a compact-group isomorphism, and it consequently preserves Haar probability measure.

Joint measurability of the inverse. Each coefficient of $\ell_{n,\alpha,\omega}$ is measurable in ω. Equality of this row to zero is a countable intersection of coefficient equalities, and is therefore a Borel event. On Ω_* define

$$\tau_\alpha(\omega) = \min\{n \geq 0 : \ell_{n,\alpha,\omega} = 0\}.$$

This is a finite Borel integer-valued function. The inverse coordinate is given by the first term of (5.14), with $n = \tau_\alpha(\omega)$. Each fixed-n expression is jointly measurable, being part of a finite composition of the stage inverses. Partitioning according to the countably many values of τ_α proves joint measurability of this coordinate, and hence of the full inverse. The forward map is jointly measurable for the same finite-stage reason.

Finally all stage maps are covariant, so their infinite extraction and its inverse are covariant on Ω_*. Fiberwise preservation of Haar measure, integrated against $\mathbb{P}$, proves the required measure-preserving conjugacy. □

6 Absorbing the remaining factors

By Lemma 5.1, the product of the factors E_n in (5.12) is, over the same marker, the ordinary Bernoulli shift B_∞: there are countably many sets of $\Delta > 0$ tracks, and a fixed bijection of their indices with $\mathbb{N}_{>0}$ interleaves them. Thus

$$X_{H_0}(d) \cong_\Omega \mathsf{B}_\infty \times \prod_{n\geq 0} (W_n \times Z_{H_{n+1}}^{k_n}). \tag{6.1}$$

We now remove the factors on the right by explicit isomorphisms.

Lemma 6.1: Finite-coset coordinate interleaving

For every finite $H \leq A$,

$$X_H(\infty) \cong_\Omega \mathsf{B}_\infty.$$

Consequently $\mathsf{B}_\infty \times X_H(r) \cong_\Omega \mathsf{B}_\infty$ for every finite r.

Proof. Write $h = |H|$. In each H-coset b, order its elements as $g_1, \ldots, g_h$ using stream 0. For $x \in X_H(\infty)$, define

$$y_k(g_i) = x_{h(k-1)+i}(b) \qquad (k \geq 1,\ 1 \leq i \leq h).$$

This is a covariant permutation of scalar coordinate slots, with its explicit inverse given by the same formula read backwards. It is a linear Haar-preserving homeomorphism for each allowed marker and is jointly measurable in both directions. On $X_H(\infty)$, one can interleave any additional r tracks with the countably many existing ones using a fixed bijection $\mathbb{N}_{>0} \sqcup \{1, \ldots, r\} \to \mathbb{N}_{>0}$. Conjugating that operation by the first isomorphism proves the second. □

Lemma 6.2: Absorption of an algebraic remainder

For every finite $H \leq A$ and every finite integer $k \geq 1$,

$$\mathsf{B}_\infty \times Z_H^k \cong_\Omega \mathsf{B}_\infty.$$

Proof. Take countably many copies of (4.4), using the same marker and, for example, stream 0 for every copy. By Lemma 2.2 this gives

$$(X_H(d))^{\mathbb{N}_{>0}} \cong_\Omega (X_H(M))^{\mathbb{N}_{>0}} \times Z_H^{\mathbb{N}_{>0}}.$$

Since $d, M \geq 1$, fixed track interleavings identify each of the first two countable products with $X_H(\infty)$. Lemma 6.1 therefore supplies an isomorphism

$$\theta_H : \mathsf{B}_\infty \cong_\Omega \mathsf{B}_\infty \times Z_H^{\mathbb{N}_{>0}}. \tag{6.2}$$

To absorb k more copies, use the following composition, throughout over the same marker:

$$\mathsf{B}_\infty \times Z_H^k \xrightarrow{\theta_H \times \mathrm{id}} \mathsf{B}_\infty \times Z_H^{\mathbb{N}_{>0}} \times Z_H^k \xrightarrow{\text{interleave}} \mathsf{B}_\infty \times Z_H^{\mathbb{N}_{>0}} \xrightarrow{\theta_H^{-1}} \mathsf{B}_\infty.$$

The middle arrow is a permutation of the copies of the *same* probability-preserving action Z_H. This is an actual isomorphism; no cancellation of factors is being inferred. □

Completion of Theorem 1.4. First split the countably many scalar tracks of B_∞ into countably many infinite sets. This identifies B_∞ with a countable product of independent copies of itself. Allocate one copy to each W_n and one to each $Z_{H_{n+1}}^{k_n}$. Lemma 6.1 absorbs $W_n = X_{H_{n+1}}(d(L_n - 1))$ into its allocated copy, and Lemma 6.2 absorbs $Z_{H_{n+1}}^{k_n}$ into the other allocated copy. Taking the product of these isomorphisms over the single marker, then interleaving the resulting countable family of B_∞ factors, yields

$$\mathsf{B}_\infty \times \prod_{n \geq 0} (W_n \times Z_{H_{n+1}}^{k_n}) \cong_\Omega \mathsf{B}_\infty. \tag{6.3}$$

Combining (6.1) and (6.3) proves $X_{H_0}(d) \cong_\Omega \mathsf{B}_\infty$. Restoring the retained marker gives

$$(K^{d+1})^G = \Omega \times X_{H_0}(d) \cong \Omega \times \mathsf{B}_\infty \cong \mathsf{B}_\infty.$$

The final arrow interleaves the single marker coordinate with the countably many data coordinates at each group element. This proves (1.3).

Let $\Phi : (K^{d+1})^G \to \mathsf{B}_\infty$ denote this conjugacy. Then

$$(K^{2(d+1)})^G \cong (K^{d+1})^G \times (K^{d+1})^G \xrightarrow{\Phi \times \Phi} \mathsf{B}_\infty \times \mathsf{B}_\infty \xrightarrow{\text{interleave}} \mathsf{B}_\infty \xrightarrow{\Phi^{-1}} (K^{d+1})^G$$

is the required conjugacy of two finite-base shifts. Since $q \geq 2$ and $d \geq 1$, their stated Shannon entropies are positive and unequal. □

7 Sparse initial data and finitary regularity

The construction above has two useful freedoms: its initial finite subgroup need not be trivial, and its marker need not be uniform. We record both, including the joint continuity needed below.

Fix $0 < p < 1$, let ν_p assign probabilities $p, 1-p$ to $0, 1$, and write $\Omega_p = (\{0,1\}^G, \nu_p^G)$. In this section relative isomorphisms retain Ω_p instead of the uniform marker. Set

$$\epsilon_p = -p \log p - (1-p) \log(1-p).$$

Lemma 7.1: Sparse-start relative exhaustion

For every $a \geq 0$ and $H = A_a$, there is a topo-finitary conjugacy

$$\Omega_p \times X_H(d) \;\cong\; \Omega_p \times \mathsf{B}_\infty$$

which fixes the marker. For each allowed marker its data map is a linear homeomorphism of compact groups preserving Haar measure.

Proof. First supply the random streams. In each disjoint digit column used in Section 3, group successive biased bits into pairs. Discard 00 and 11, and read 01 and 10 as fair bits 0 and 1. An accepted pair occurs with probability $2p(1-p) > 0$, independently between pairs. Almost surely infinitely many are accepted. Conditional on their positions, the accepted bits are independent and fair, so they remain so after forgetting the positions. Their binary expansion is uniform on $[0,1]$. Disjoint input columns give independent streams inside each fixed D-coset. Covariance is unchanged.

Every prescribed finite prefix of accepted bits requires finitely many original bits. The binary-expansion map is continuous. Thus a finite collection of strict priority comparisons is certified by finitely many marker coordinates. Intersect the countably many events of infinite acceptance and absence of ties to obtain an invariant conull marker set.

Start the recursion of Section 5 at $a_0 = a$, $H_0 = H$, $R_0 = 1$. An original coordinate is now (gH, i). Its initial row still has one coefficient, supported in gD. Choose $L_n \geq 2R_n$ exactly as before and keep $k_n = |H_{n+1}|$. This is the absolute subgroup order, not the relative index $[H_{n+1} : H_n]$. Therefore Lemma 5.1 still extracts Δ ordinary G-tracks at each stage. All row estimates, compactness, measurability, and absorption arguments apply without another change. They give the asserted relative measurable compact-group isomorphism.

It remains to check joint continuity, including in the marker. For a fixed finite stage and output scalar, expand its formula into its full finite computation tree. Each algebraic operator has finite support; each ordering concerns a finite coset. At the given marker, finitely many strict comparisons determine every coordinate index in this tree. Their finite certificates freeze all its branches. The entire linear row, before collecting terms or cancellations, is then fixed. This proves continuity of every finite-stage coordinate map, both forward and backward, on the tie-free domain.

For an inverse coordinate α, choose any stage n at which $\ell_{n,\alpha,\omega} = 0$. Freeze the finite tree of this whole row, not just the terms which survive cancellation. The same row is zero for every allowed marker in that neighborhood. Freeze also the finite tree of the recovery function $f_{n,\alpha,\omega}$ in (5.14). On this neighborhood the inverse coordinate is one fixed finite linear function of extracted data. It is therefore continuous. The first vanishing stage need not be locally constant: the single chosen witness stage suffices.

Finally, the absorption maps of Section 6 are coordinate permutations and countable products of finite compositions of elementary splits and their inverses. Each requested coordinate uses only one such finite composition. There is no infinite recursive application along one coordinate. Consequently these maps and their inverses are jointly continuous on the common invariant conull domain. Their composition with extraction proves the lemma. □

8 From sparse data to every Bernoulli base

We now use two precise inputs from integer finitary coding. For a probability base λ, write $H(\lambda)$ for its Shannon entropy, equivalently the supremum of the entropies of its finite measurable partitions. This is the sum of $-t \log t$ over its atoms when it is purely atomic, and is infinite if it has a nonatomic part.

Lemma 8.1: Integer coding inputs

The following consequences of Meyerovitch–Spinka [2] will be used.

(i) Equal-entropy iid integer processes with Polish bases admit a topo-finitary conjugacy, also when their entropy is infinite.

(ii) Let W be iid Bernoulli-θ on $\mathbb{Z}$, $0 < \theta < 1$. Given W, let X_n independently be uniform on K^d when $W_n = 1$ and the fixed symbol $*$ otherwise. If a finite probability base λ has

entropy $\theta d \log q$, then (W, X) is finitarily isomorphic over W to W times an independent iid λ process.

Identification of the cited inputs. Part (i) is [2, Theorem 3.15]. For (ii), apply [2, Theorem 3.2]. Both compared processes are conditionally independent over the aperiodic ergodic process W, with relative entropy $\theta d \log q$. Their conditional laws on any finite set are determined by the corresponding finite W-block. Thus the theorem's relative dependence bound is zero and its stop-finitary conditional-law hypothesis holds. Its relative isomorphism fixes W. We use the resulting bijection, not only a factor map. □

Remark 8.2: Keeping the topology of a Polish base

The topological reduction underlying part (i) can be made explicit. For a Polish probability space (L, λ), choose a countable metric-ball basis (U_j) with $\lambda(\partial U_j) = 0$. This is possible because, at each center in a countable dense set, only countably many spheres have positive measure. The set $L_0 = L \setminus \bigcup_j \partial U_j$ is conull and G_δ. Each $U_j \cap L_0$ is clopen in L_0. Consequently $x \mapsto (\mathbf{1}_{U_j}(x))_j$ is a homeomorphism of L_0 onto its image in $\{0,1\}^{\mathbb{N}}$: the basis separates points and also makes the inverse continuous. This realizes a conull part of the original topology without replacing it by a discrete topology. The integer coding assertion itself remains the cited theorem.

The next elementary observation lets us apply integer coding along the left cosets $g\langle s \rangle$, parametrized by right multiplication by powers of s.

Lemma 8.3: Extension from a subgroup

If two Bernoulli shifts over a countable subgroup $B \leq \Gamma$ are topo-finitarily conjugate, their Bernoulli shifts over Γ are topo-finitarily conjugate. Finitary conjugacy of discrete bases is preserved as well.

Proof. Apply the same B-equivariant code on every left coset gB, identified with B by $b \mapsto gb$. Changing this identification by a left translation in B does not change the output, by B-equivariance. Left translation by Γ carries these cosetwise rules to one another, so the result is Γ-equivariant. Different cosets have independent product laws. Thus the product of the codes preserves probability and has the product of their inverses. Intersect the countably many good coset domains and their translates. The resulting domain is invariant and conull. Product continuity gives topo-finitary regularity; a discrete output reads a finite set in its one coset. □

Lemma 8.4: Finitary sparse-label homogenization

Let $H = A_a$ have order $h \geq 2$. There is a finite probability base (L_H, λ_H) with $H(\lambda_H) = d \log q / h$ and a finitary conjugacy over Ω_p

$$\Omega_p \times X_H(d) \;\cong\; \Omega_p \times (L_H, \lambda_H)^G.$$

Proof. Write $A = H \oplus A'$ and choose distinct $a_j \in A'$, $j \geq 1$. Obtain a uniform priority η_g from the biased marker sequence $(\omega(ga_j))_{j \geq 1}$ using the pair extractor from Lemma 7.1. For a fixed coset gH, the coordinates gh_0a_j, $h_0 \in H$, $j \geq 1$, are distinct. Hence its h priorities are independent uniforms. Select their unique minimum. Call the selected set $S_H(\omega)$. It is covariant, has exactly one element in every H-coset, and selects each fixed element with probability $1/h$. Each selection has a finite marker certificate because the finite collection of priorities has a strict gap.

Store a coset label only at its selected representative:

$$w(g) = \begin{cases} x(gH), & g \in S_H(\omega), \\ *, & g \notin S_H(\omega). \end{cases}$$

This is a relative finitary bijection to its constrained image: recover $x(gH)$ by finding the selected element in the finite coset. Given the whole marker, the labels at selected elements are independent uniform K^d-labels.

On each line $g\langle s\rangle$ define

$$W_n^g = \mathbf{1}_{\{gs^n \in S_H(\omega)\}}, \qquad X_n^g = w(gs^n) \quad (n \in \mathbb{Z}).$$

The coordinates determining W_n^g belong to gs^nD. These D-cosets have different integer heights for different n. Therefore W^g is iid Bernoulli-$1/h$. Given W^g, the variables X_n^g have exactly the sparse product law in Lemma 8.1(ii).

Choose a finite base of entropy $c_H = d\log q/h$. Such a base exists for every finite c_H: on a sufficiently large finite alphabet, entropy varies continuously from zero to the logarithm of its size. Let $(W, X) \mapsto (W, \varphi_W(X))$ be the relative integer code from Lemma 8.1(ii), and set

$$z(g) = \varphi_{W^g}(X^g)_0.$$

Integer equivariance makes this a consistent rule on each line, independent of its chosen origin. Left G-translation carries the inputs and lines covariantly to one another, so the rule is G-equivariant.

Here the full conditional law deserves attention. Fix the entire marker. Every selected position uses a distinct original H-coset label. Consequently the sparse label processes on different lines are independent under this conditional law, although the lines' marker processes can be correlated. By disintegration of the integer isomorphism, for almost every W the fiber code and its inverse preserve the specified conditional probability laws. Almost every marker makes every line good, since there are countably many lines and each W^g has the required iid marginal. The product of these fiber codes has conditional output law λ_H^G, independent of the full marker.

Invert separately on each line, then recover the coset labels. The joint law and inverse just verified prove a conjugacy over the entire marker. Each integer output reads finitely many sparse labels and selection indicators. Each indicator has a finite marker certificate. A finite union of these inspection sets is finite. The inverse satisfies the same argument. Countable intersection of the good domains makes them invariant and conull. No independence from the exhaustion streams is needed: both conjugacies fix the marker and preserve each conditional data law. □

Proof of Theorem 1.3. Combine Lemmas 7.1 and 8.4 to obtain

$$V_{p,H}^G \cong_{\mathrm{tf}} \Omega_p \times \mathsf{B}_\infty, \qquad V_{p,H} = (\{0,1\} \times L_H, \nu_p \times \lambda_H),$$

where

$$H(V_{p,H}) = \epsilon_p + \frac{d\log q}{|H|}.$$

This positive finite entropy can be made arbitrarily small by taking p small and H large.

The Polish bases of $\Omega_p \times \mathsf{B}_\infty$ and B_∞ both have infinite entropy. Apply Lemma 8.1(i) and then Lemma 8.3 with $B = \langle s\rangle$ to obtain

$$V_{p,H}^G \cong_{\mathrm{tf}} \mathsf{B}_\infty. \tag{*}$$

This use of integer coding retains topology; an arbitrary nonatomic measure-space isomorphism would not suffice.

Now let (L, λ) be a nontrivial Polish probability base and write $b = H(\lambda) \in (0, \infty]$. If $b = \infty$, the same equal-entropy integer coding and extension immediately give $L^G \cong_{\mathrm{tf}} \mathsf{B}_\infty$. If $b < \infty$, choose $V = V_{p,H}$ with entropy $a < b$, and a finite base R of entropy $b - a$. Equal-entropy integer coding, its extension to G, the conjugacy $(*)$, and one more infinite-entropy integer coding give

$$L^G \cong_{\mathrm{tf}} (V \times R)^G \cong_{\mathrm{tf}} \mathsf{B}_\infty \times R^G \cong_{\mathrm{tf}} \mathsf{B}_\infty.$$

All good domains can be restricted to invariant conull Borel sets, and compositions are taken on the preimages of these sets. Compose the code for one base with the inverse code for the other to prove the theorem. When both bases are countable and discrete, continuity of each output label means constancy on a finite cylinder neighborhood in the conull domain. Thus the final conjugacy is finitary, even if intermediate alphabets were nondiscrete. □

9 Finitely presented groups and a graph formulation

Lemma 8.3 transfers the all-base conclusion to every countable overgroup of G. For the algebraic source in Appendix A, N is finitely presented. We can preserve a direct-product construction while making the auxiliary factor finitely presented. Consider the fixed group

$$B = \langle a, u, t \mid a^2 = 1,\ [a, a^t] = 1,\ [u, t] = 1,\ a^u = aa^t \rangle,$$

where $x^y = y^{-1}xy$ and $[x, y] = x^{-1}y^{-1}xy$. This familiar metabelian overgroup of the lamplighter group appears in Cleary–Riley [3]. We give the required facts explicitly.

Lemma 9.1:
The finitely presented group B is metabelian and contains $C_2 \wr \mathbb{Z}$ as the subgroup generated by a, t.

Proof. Write $a_j = a^{t^j}$. We first prove $[a_i, a_j] = 1$ for all integers i, j. Conjugating by powers of t reduces this to nonnegative distances $j - i$. Distances zero and one follow from the presentation. Suppose the claim holds through distance $n \geq 1$. Conjugating $[a_0, a_n] = 1$ by u gives $[a_0a_1, a_na_{n+1}] = 1$, since u commutes with t. Every commutator between these four elements except possibly $[a_0, a_{n+1}]$ is already trivial. Cancelling the other factors therefore gives $[a_0, a_{n+1}] = 1$. Induction proves the claim. Thus $A_0 = \langle a_j : j \in \mathbb{Z} \rangle$ is abelian of exponent two.

The relation $a_j^u = a_ja_{j+1}$ shows that conjugation by every nonnegative power of u carries each a_j into A_0. For any two elements $a^{u^it^j}$, simultaneously conjugate by a sufficiently large power of u to make both u-exponents nonnegative. Their images lie in A_0 and commute, so the original pair commutes. The subgroup generated by all these conjugates is therefore abelian. It is normal under u, t and contains a, so is the normal closure of a. The quotient is $\mathbb{Z}^2$. This proves that B is metabelian, and hence amenable.

It remains to exclude extra relations among a, t. Over the rational function field $\mathbb{F}_2(z)$, the matrices

$$a \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad t \longmapsto \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \qquad u \longmapsto \begin{pmatrix} (1+z)^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

satisfy all four defining relations. They send a_j to $\left(\begin{smallmatrix} 1 & z^j \\ 0 & 1 \end{smallmatrix}\right)$. Distinct Laurent monomials are linearly independent over $\mathbb{F}_2$, and the diagonal entry detects each power of t. Thus the homomorphism from the lamplighter presentation into $\langle a, t \rangle$ is injective as well as onto. □

Corollary 9.2:
For the finitely presented algebraic source N, the explicit direct product $N \times B$ has the all-base conclusions of Theorem 1.3. Here B is a fixed finitely presented amenable group, independent of N. There is also a two-generator finitely presented group Γ with these conclusions.

Proof. The direct product $N \times B$ is finitely presented and contains $N \times \Lambda$. Apply Lemma 8.3. Theorem IV of Higman–Neumann–Neumann [4], including its finite-relator conclusion, embeds $N \times B$ in a two-generator finitely presented group Γ. Apply the same lemma again. □

The group Γ is not virtually abelian: it contains the finite lamp groups C_2^n for all n, so its finite subgroups have unbounded orders. A finitely generated virtually abelian group has a torsion-free abelian subgroup of finite index, and each finite subgroup injects into the set of its cosets. Its finite subgroups therefore have uniformly bounded orders, a contradiction. Leemann–de la Salle [5, Theorem 1.1] therefore supplies a finite-degree Cayley graph $\mathcal{G}$ of Γ whose full automorphism group is exactly its left translation group.

Corollary 9.3:

There is a connected locally finite vertex-transitive graph $\mathcal{G}$ on which all nontrivial iid vertex-label processes with countable discrete bases are finitarily conjugate under $\mathrm{Aut}(\mathcal{G})$. In particular this holds for every two independent site-percolation processes with parameters strictly between zero and one.

Proof. Apply Corollary 9.2 on the graphical regular representation just described. Every finite inspection set lies in a finite graph ball, so the codes have almost surely finite coding radii. □

The coding can change the geometry of occupied clusters. No monotonicity, deterministic radius bound, or integrable radius bound is asserted. For a discrete-valued finitary factor of an iid process with a countable discrete base, composition with the source conjugacy gives a finitary realization from Bernoulli-p bits for every $0 < p < 1$. The same holds for a topo-finitary discrete-valued factor of a Polish-base iid process, by composition of continuous maps. This converts the entire source reversibly before taking the given factor; it does not supply an inverse to an arbitrary factor.

The distinction is relevant to the factor-of-iid viewpoint in Lyons [6]. Mutual Bernoulli factor maps are already known for every countable nonamenable group by Bowen [7]. Here recovery of every input coordinate gives the additional two-sided conclusion.

10 What the construction uses

The auxiliary group $\Lambda = C_2 \wr \mathbb{Z}$ is fixed independently of the algebraic input. It is metabelian, hence amenable. The proof uses its finite lamp subgroups and the height homomorphism; it does not require a free-product factor. More generally, the same formulas work in any countable ambient group containing $D = N \times A$ and an infinite-order element s with $D \cap \langle s \rangle = \{e\}$. This intersection condition gives both marker-address injectivity and disjointness of the D-cosets along each s-line. No step requires the H_n to be normal in the ambient group.

In the finite-uniform engine, the retained marker contributes exactly one uniform K-coordinate per element of G. Its apparent abundance of randomness inside a D-coset comes from the disjoint marker coordinates indexed by us^j. No independent infinite-base G-Bernoulli system is included in the input.

The remaining input has d coordinates per H_n-coset. Its coordinate density per group element is $d/|H_n|$, which tends to zero. This decreasing density makes recovery of every coordinate possible even when the inverse rows have very large finite supports. The choice $k_n = |H_{n+1}|$ offsets the decreasing density in the extracted Bernoulli factor: each stage still contributes Δ ordinary tracks.

The finite groups H_n have orders that may vanish in K. We never average over them. They are used only for finite partitions, orderings, and coordinate permutations. The only subgroup averages are over the J_v, whose order is invertible by hypothesis.

The core finite-uniform construction uses no entropy classification. The all-base extension uses precisely the two integer finitary-coding inputs in Lemma 8.1. No assertion that zero Rokhlin entropy alone implies Bernoulli isomorphism is needed. Continuity is asserted on invariant conull sets. A

continuous conjugacy between the entire two-symbol and four-symbol full shifts is impossible: it would biject their group-fixed configurations, of which there are respectively two and four.

A The precise external algebraic dependency

The algebraic existence input is due to OpenAI. We use [1], *A Counterexample to Kaplansky's Direct-Finiteness Conjecture in Characteristic Two*, manuscript dated September 23, 2026. We use its rectangular construction before the subsequent packing into a scalar one-sided inverse. The implication proved above requires the actual data of Hypothesis 1.1, not merely an abstract failure of direct finiteness.

The source's subsection "Subgroup averages and a rectangular identity" constructs a countable group N with embedded subgroups, denoted there by K_v, each of order $\kappa = \ell^2$ for an odd prime ℓ. Rename them J_v here. Its field K is finite of characteristic two, so κ is invertible. With source vectors $x_v \in K^m$, its matrices are

$$U_{i,v} = -\kappa(x_v)_i p_v, \qquad T_{v,i} = p_v(x_v)_i.$$

The displayed rectangular identities in that subsection are $UP = U$, $PT = T$, and $UT = I_m$. The source's number t of local subgroups is our d. These formulas also give $C_0 \le \kappa m$ in (4.6), although our proof needs only finiteness of C_0.

The strict inequality needed here is slightly stronger than the weak inequality displayed in the source's incidence criterion. It follows from its terminating prescription. Write Q_0 for the number of affine points in that prescription. The selected parameters satisfy

$$m \ge \frac{2Q_0}{(\ell+1)^2}, \qquad d = Q_0 - m.$$

Consequently

$$m(\ell^2+1) \ge \frac{2(\ell^2+1)}{(\ell+1)^2} Q_0 > Q_0,$$

because $2(\ell^2+1) - (\ell+1)^2 = (\ell-1)^2 > 0$. Rearranging gives $d < \ell^2 m = \kappa m$, as required.

Thus the cited existence construction supplies all of Hypothesis 1.1, and Theorem 1.4 applies to its data. The existence of that incidence construction and the embedding of its local groups are the external results invoked here. This note does not reproduce their proofs or independently certify a formal verification of them. Every subsequent operator, probability, exhaustion, and absorption argument is given in the body of this note.

The same source's local-group embedding construction gives a finite presentation of this N: it has finitely many generators and the finitely many local multiplication-table relations of length at most three. Corollary 9.2 uses this finite presentation, together with the classical embedding theorems cited there.

> **Remark A.1: The algebraic input can be binary**
>
> For a characteristic-two instance of Hypothesis 1.1, let $f = [K : \mathbb{F}_2]$. Choose an $\mathbb{F}_2$-basis of K and represent multiplication by each field element as an f-by-f binary matrix. Applying this representation to every coefficient gives a unital algebra homomorphism $K[N] \to \mathrm{Mat}_f(\mathbb{F}_2[N])$. Expanding U, T, P into blocks therefore preserves their three identities. Each average p_v has coefficients in $\mathbb{F}_2$ because κ is odd, so its block is $p_v I_f$. The expanded diagonal averages correspond to repeating each J_v exactly f times. The new dimensions are $d' = fd$, $m' = fm$, and $d' < \kappa m'$ still holds. Thus the construction may be carried out with binary data.

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