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THE PERMANENT OF THE COMPLEMENT OF THE PROJECTIVE PLANE OF ORDER THREE IS MINIMAL

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ABSTRACT. Every 13×13 zero–one matrix with all row and column sums equal to nine has permanent at least 64,803,969. Equality characterizes the complement of the incidence matrix of the projective plane of order three, up to independent row and column permutations. This establishes the $(13, 9, 6)$ case of Ryser's permanent conjecture for symmetric designs. For complements with at most seven four-cycles we give a complete finite enumeration. For the other 110 possible four-cycle counts, we bound the moments of the positive roots of the rook polynomial and certify a lower bound using exact Laguerre quadrature and interval-verified proof trees.

1. INTRODUCTION

Let Λ_v^k be the set of $v \times v$ zero–one matrices with every line sum equal to k. If Λ_v^k contains an incidence matrix of a symmetric (v, k, λ) design, Ryser's conjecture states that some such incidence matrix minimizes the permanent over Λ_v^k [1, 2]. The survey of Cheon and Wanless records the result for $v < 13$. For $v = 13$, Lavi gives the design values and reports strict bounds for non-designs within two switches of either design [3]. A switch interchanges the two 2×2 permutation matrices in a submatrix. Lavi also proves the lower bound 64,839,709 for the permanent of the complement of a non-design with defect statistic at most five. That statistic is the four-cycle count C below, by (1). Our low-cycle census reproduces this overlap and extends to $C = 6, 7$. The other possible values of C are handled by the proof trees, giving a bound on all of Λ_{13}^9.

An independent technical note [4] gives short-cycle deviation identities for general symmetric design parameters and studies a signed loop expansion. The four-cycle defect identity used here is an instance of that general identity. The global comparison below uses sixth rook-root moment bounds and exact quadrature on the resulting feasible sets.

Theorem 1. *For every $B \in \Lambda_{13}^9$, $\operatorname{per} B \geq 64{,}803{,}969$. Equality holds precisely when independent permutations of the rows and columns of B give the complement of an incidence matrix of* $\mathrm{PG}(2, 3)$.

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Write $A = J - B$, so $A \in \Lambda_{13}^4$. Let c_{ij} be the number of common ones of rows i and j and set $C = \sum_{i<j} \binom{c_{ij}}{2}$, the number of four-cycles of the bipartite incidence graph. Each row satisfies $\sum_{j \neq i} c_{ij} = 12$. Since $0 \leq c_{ij} \leq 4$ and $\binom{c}{2} \leq 3c/2$ in this range, $0 \leq C \leq 117$. We prove strictness for the nonzero values in two parts, with a complete search for $C \leq 7$ and continuous certificates for $8 \leq C \leq 117$.

2. The design case and the low four-cycle census

If $C = 0$, all twelve overlaps in each row are one. Thus $AA^{\mathsf{T}} = 3I + J$. The same argument on columns holds because counting four-cycles by columns gives the same C. The incidence structure is a projective plane of order three. To see uniqueness, delete a line and its points. The resulting affine plane has four parallel classes. Regard the first two as the coordinate lines on $\mathbb{F}_3^2$. The third class gives a Latin square of order three, whose reduced form is unique, so its lines can be written $x + y = b$. A line of the fourth class is the graph of a permutation of $\mathbb{F}_3$. Each such permutation is affine, $y = ax + b$ with $a \in \{1, 2\}$. Meeting every line of the third class forces $a = 1$. This determines the affine structure and hence its completion by one point for each direction. In particular, the uniqueness is up to independent row and column permutations. Using the thirteen nonzero triples over $\mathbb{F}_3$ whose first nonzero coordinate is one, the standard incidence rule is a zero dot product. The subset recurrence below gives permanent 64,803,969 for its complement.

Proposition 2. *For $C = 0, 4, 6, 7$ the numbers of classes, up to row and column permutations and transposition, are $1, 2, 11, 41$ respectively. There are no other classes with $C \leq 7$. All non-designs in this range have complementary permanent at least* 64,839,709*.*

Proof. Let M be the maximum row overlap. It lies in $\{1, 2, 3, 4\}$. Anchor a pair with overlap M in rows 0 and 1. Relabel the columns so the first support is $\{0, 1, 2, 3\}$ and the second consists of the first M of those columns together with $4 - M$ new ones. Column 0 has two further incident rows, which we place next. The permutations preserving these two anchor rows and column 0 act independently in the four blocks determined by membership in the anchor rows, with column 0 removed. The third row may therefore be taken to use an initial segment in each block. The fourth row is unrestricted except for containing column 0 and meeting the necessary overlap and capacity conditions.

After the first four rows, select a column with smallest positive remaining demand, breaking ties by its label. Place all as-yet unplaced rows incident to that column next, in nondecreasing order of their bit masks. The next target column is selected after the current demand has been filled. Any completion may be ordered this way, including a completion with repeated rows. The enumeration visits every weight-four candidate mask consistent

with these rules. It rejects a partial matrix if a column capacity is exceeded, if a column cannot reach degree four with the remaining rows, if a pair exceeds the anchored M, or if accumulated four-cycles exceed seven.

For the remaining pruning, put $d_{ij} = c_{ij} - 1$ off the diagonal. The zero row sums of the defect matrix and the identity

$$\sum_{i<j} d_{ij}^2 = 2C \tag{1}$$

follow from the line sums. If r rows have been placed, the missing defects for a placed row must sum to the negative of its current defect sum. Among $m = 13 - r$ integers of that given sum, the sum of squares is minimized by entries differing by at most one. Terms smaller than -1 cannot occur, so a requested sum below $-m$ is infeasible. The square contributions already fixed and the lower bounds for all placed rows count disjoint unordered pairs. The column-side version of (1) gives another lower bound: the overlap of a pair with at least one full column is fixed, and an overlap of at least two between unfilled columns can only increase. Both bounds are necessary inequalities. A run omitting the whole column-side bound serves as a check of this pruning.

At each complete matrix the implementation takes a representative of its bipartite isomorphism class, allowing the two colors to be swapped by transposition. Representatives are obtained by actual permutations and possible transposition, so equality of representatives cannot merge two distinct classes. Both enumerations traverse the whole search space just described, irrespective of the choice of canonical ordering. An independent coloring-and-individualization routine tests within-list distinctness and finds a class-wise bijection between the two lists. In this routine a color refinement compares multisets of neighbor colors, and a nonsingleton color cell is resolved by trying all possible images of one vertex. A discrete coloring is checked by adjacency. Thus an isomorphism cannot be lost by refinement, and a reported match is checked directly.

For a matrix T with rows indexed from zero, let $F(S)$ count the ways to assign the first $|S|$ rows distinct allowed columns with used set S. Start from $F(\varnothing) = 1$ and for each unused column j with $T_{|S|,j} = 1$ add $F(S)$ to $F(S \cup \{j\})$. The permanent is $F(\{0, \dots, 12\})$. An independent arbitrary-precision evaluation of this recurrence checks both permanents for every representative and the standard design. The census results are shown in Table 1. The original search visited 110,221,478 nodes. The search without the column-side bound visited 110,307,638 nodes. Each reached 1,970,760 complete matrices before isomorphism reduction. □

3. Rook roots and their moments

Let m_j be the number of j-matchings of the incidence graph of A, with $m_0 = 1$, and let $R(t) = \sum_{j=0}^{13} (-1)^j m_j t^{13-j}$. Inclusion–exclusion on the

TABLE 1. The low four-cycle census for $A \in \Lambda_{13}^4$.

C	Classes	Minimum of $\operatorname{per}(J-A)$
0	1	64,803,969
1, 2, 3, 5	0 each	—
4	2	64,839,709
6	11	64,857,234
7	41	64,865,578

forbidden entries in a permutation matrix, followed by the factorial integral, gives

$$\operatorname{per}(J-A) = \sum_{j=0}^{13} (-1)^j m_j (13-j)! = \int_0^\infty e^{-t} R(t)\, dt. \tag{2}$$

The matching polynomial of the graph is $R(z^2)$, so real-rootedness [5] makes all roots of R nonnegative. The graph has a perfect matching by Hall's theorem, since the four edges incident with each vertex on one side go to a set of neighbors of aggregate capacity four per vertex. Hence $m_{13} > 0$. Denote the positive roots, with multiplicities, by λ_i and put $p_j = \sum_{i=1}^{13} \lambda_i^j$. In particular, $\max_i \lambda_i \geq p_1/13 = 4$.

Let $D = AA^{\mathsf{T}} - 3I - J$ and $D' = A^{\mathsf{T}}A - 3I - J$. Both have zero diagonal and zero row sums. Let K_r count the subgraphs with two row and three column vertices forming a complete bipartite graph, and let K_c count those with three row and two column vertices.

Lemma 3. *The first seven moments are*

$$\begin{aligned}(p_0, p_1, p_2, p_3) &= (13, 52, 364, 3016),\\ p_4 &= 27196 - 4C,\\ p_5 &= 258232 - 120C,\\ p_6 &= 2538172 - 2340C - \operatorname{tr}(D^3) + 24(K_r + K_c).\end{aligned}$$

Proof. We give the finite coefficient calculation used in the proof. For j labeled edges, let π, ρ range over the partitions of $[j]$ on the two sides of an ordered list of edges. Form a bipartite multigraph $H(\pi, \rho)$ with vertices the blocks of these partitions and one edge for each label, joining the two blocks containing it. Write $\mu(\pi) = \prod_{U \in \pi} (-1)^{|U|-1}(|U|-1)!$. Partition-lattice inversion for the injective assignments of row and column endpoints gives $j! m_j = \sum_{\pi,\rho} \mu(\pi)\mu(\rho) \hom(H(\pi,\rho), A)$, where homomorphisms preserve the two sides. The factors in the sum multiply over connected components. The logarithm of the exponential generating function therefore retains only connected H. Since $\sum m_j z^j = \prod_i (1 + \lambda_i z)$, the moment p_j is the weighted connected sum multiplied by $(-1)^{j-1}/(j-1)!$.

Erase repeated edges, and successively erase degree-one vertices, accumulating a factor four for each erased vertex when the core is nonempty. A connected tree on q vertices has $13 \cdot 4^{q-1}$ homomorphisms. With at most six edges the nonempty cores are the following four. The table lists their homomorphism counts before leaf factors.

Core	Count
C_4	$364 + 4C$
C_6	$4420 + 36C + \operatorname{tr}(D^3)$
$K_{2,3}$ with two rows	$988 + 12C + 12K_r$
$K_{2,3}$ with two columns	$988 + 12C + 12K_c$

The cycle homomorphisms are the second and third traces of AA^{T}, using $D\mathbf{1} = 0$ and $\operatorname{tr}(D^2) = 4C$. For the bicliques sum the cubed overlaps over ordered pairs on the side with two vertices. The diagonal contributes $13 \cdot 4^3$. The off-diagonal terms use $c^3 = c + 6\binom{c}{2} + 6\binom{c}{3}$. There are no other cores because the minimum degree is two, and a connected bipartite simple graph of minimum degree two with at most six edges has at most six vertices. On four vertices it is a square, on five it is a complete 2 by 3 bipartite graph, and on six all degrees are two.

For clarity, the summation can be reproduced by starting the partition list with the empty tuple and appending to each tuple of length ℓ each integer between zero and one plus its maximum, taking the maximum of the empty tuple as -1. Each pair of resulting tuples is converted to its graph. Disconnected pairs contribute zero. Sum the appropriate table row multiplied by the factors above, componentwise in the basis $1, C, \operatorname{tr}(D^3), K_r, K_c$. The six rows after multiplying by $(-1)^{j-1}/(j-1)!$ are

j	1	C	$\operatorname{tr}(D^3)$	K_r	K_c
1	52	0	0	0	0
2	364	0	0	0	0
3	3016	0	0	0	0
4	27196	-4	0	0	0
5	258232	-120	0	0	0
6	2538172	-2340	-1	24	24

The supplied exact script performs this sum with integer and rational arithmetic. □

4. Upper bounds on the sixth moment

Write $p_6 = 2538172 - 2340C + \Delta$, and put $T = -\operatorname{tr}(D^3)$. The restrictions of D and D' to $\mathbf{1}^{\perp}$ have the same spectrum, by the equality of the spectra of AA^{T} and $A^{\mathsf{T}}A$. The twelve eigenvalues μ_i of either restriction satisfy $\mu_i \geq -3$, $\sum \mu_i = 0$ and $\sum \mu_i^2 = 4C$. The diagonal of D is zero, so its cubic trace is a multiple of six.

Lemma 4. *For $C \geq 14$ use*

$$\Delta \leq 6 \left\lfloor \frac{108C - 4C^2}{54} \right\rfloor + 48 \left\lfloor \frac{2C}{3} \right\rfloor.$$

For $C = 8, 9, 10, 11, 12, 13$, upper bounds on Δ are $102, 156, 264, 270, 324, 330$, respectively.

Proof. For any real a, let $g_a(x) = (x+3)(x-a)^2$. Expanding and summing yields

$$T = 36a^2 + 4C(3 - 2a) - \sum_i g_a(\mu_i). \tag{3}$$

All the subtracted terms are nonnegative. Use $a = C/9$ and round down to a multiple of six. On each side, $\binom{1+d}{3} \leq (d^2+d)/3$ for $d = -1, 0, 1, 2, 3$. Equation (1) and the zero defect sum bound each of K_r, K_c by $\lfloor 2C/3 \rfloor$. This proves the general bound.

On the row side, let P_3, P_2, E_1 count the pairs with defects $3, 2, 1$. The analogous column-side counts carry primes. For each side,

$$6P_3 + 3P_2 + E_1 = C, \qquad K_r = 4P_3 + P_2. \tag{4}$$

The first equality follows by adding the zero sum of defects to their squared sum. A defect three is a pair of identical rows. For any subset S of rows the sum of defects over unordered pairs inside S equals the sum inside S^c, by zero row sums. If S is one identical pair, the positive defects inside S^c sum to at least three. By (4) the pair and its complement cost at least nine. If two identical pairs meet, three identical pairs already cost eighteen. If they are disjoint with cross overlap t, their four rows have internal defect sum $2 + 4t$. The two pairs and the positive defects of the complement cost at least $12 + 2 + 4t$. Thus $P_3 = P_3' = 0$ for $C = 8$ and each is at most one for $C \leq 13$.

For an eigenvalue lower bound $\mu_1 \geq L \geq a \geq 0$, the derivative $g_a'(x) = (x-a)(3x+6-a)$ gives $g_a(\mu_1) \geq g_a(L)$. Rayleigh quotients give the following values of L. For one defect two or three, use $e_1 + e_2 - 2\mathbf{1}/13$, giving $26/11$ or $39/11$. For two incident defects two, use $3e_1 + 2e_2 + 2e_3 - 7\mathbf{1}/13$, giving $130/43$ since the third defect is at least -1. For a triangle of defects two, use $e_1 + e_2 + e_3 - 3\mathbf{1}/13$, giving $26/5$.

Any column-side $K_{2,3}$ determines a triangle in the graph of positive row defects: its three rows each share at least two columns with the others. When $C = 8$ and neither side has two defects two, their combined biclique count is at most two. If there is a defect two, use $L = 26/11$ and $a = 3/5$ in (3) to get $T < 54$ and hence $\Delta \leq 48 + 48$. If both counts vanish the general cubic bound suffices. If one side has two defects two, assume it is the row side. Its positive graph has only four edges by (4), hence at most one triangle. Two distinct column pairs supported on that triangle would force the three rows to have at least three common columns, giving three defects at least two, an impossibility. Thus $K_r + K_c \leq 3$. For incident row

TABLE 2. Rational substitutions for the bound on $T = -\operatorname{tr}(D^3)$.

	$L = 26/11$		$L = 39/11$	
C	a	$T <$	a	$T <$
10	7/10	72	1/2	30
11	4/5	78	3/5	36
12	9/10	84	7/10	42
13	9/10	84	4/5	48

defects use $L = 130/43$, $a = 1/2$, giving $T < 36$. For disjoint row defects relabel their pairs as 12 and 34 and put $s = d_{13} + d_{14} + d_{23} + d_{24}$. The cross defects are in $\{-1, 0, 1\}$ and at most two are positive, so $-4 \le s \le 2$. On the orthonormal vectors

$$\tfrac{1}{2}(e_1 + e_2 - e_3 - e_4), \qquad \tfrac{1}{\sqrt{468}}\big(13(e_1 + e_2 + e_3 + e_4) - 4\mathbf{1}\big)$$

the compression of D is diagonal with entries $2 - s/2$ and $26/9 + 13s/18$. The off-diagonal entry vanishes by $D\mathbf{1} = 0$ and $d_{12} = d_{34}$. Interlacing bounds the two largest eigenvalues from below by the ordered compression entries. In (3), take $a = 2/5$ and subtract g_a of each entry after taking its maximum with a. The result is less than 36 for each integer $-4 \le s \le 2$, by the indicated rational calculation. Hence $\Delta \le 30 + 72$ in either case.

For $C = 9$ with an identical row pair, the row positive graph has either two edges ($P_2 = 1$) or four ($P_2 = 0$). In the former case it has no triangle and $K_r = 5$. In the latter it has at most one triangle, and its three rows cannot share three columns, since that would give three row defects at least two. Thus $K_r + K_c \le 5$ in either case. With $L = 39/11$, $a = 1/2$, we have $T < 24$. Transposition treats an identical column pair. With no identical pair, if one side has three defects two its positive graph has three edges. For a triangle use $L = 26/5$, $a = 1/5$ to get $T < 0$ and $K_r + K_c \le 6$. Without a triangle the other side has no biclique. For all remaining cases with a defect two use $L = 26/11$, $a = 7/10$ to get $T < 66$. Then $K_r + K_c \le 4$, giving $\Delta \le 60 + 96$. If no defects two or three occur, the cubic bound alone suffices.

For $10 \le C \le 13$, if neither side has an identical pair, each biclique count is at most $\lfloor C/3 \rfloor$. If either has one, each count is at most $4 + \lfloor (C - 6)/3 \rfloor$. The Rayleigh substitutions used in these two cases are given in Table 2; the case without defects two or three uses just the general cubic estimate. Each table inequality follows by substituting the displayed rationals into (3) and dropping the unspecified nonnegative terms. Round T down to the next multiple of six strictly below the indicated bound and add the biclique contribution. This gives the stated bounds. □

5. Exact quadrature and continuous certificates

For each C in $[8, 117]$, relax the incidence problem to all multisets of thirteen positive real numbers with moments $p_0, \dots, p_5$ as in Lemma 3 and p_6 at most the bound in Lemma 4. Each root retains its unit mass. A proof tree consists of branches covering these multisets and leaves which give a contradiction or bound the value of (2) from below. We give the coverage and leaf conditions here to specify why a successful replay proves the inequality.

5.1. **Quadrature.** Use the Laguerre polynomials

$$L_n(t) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} t^k/k!, \qquad Q = L_8 + bL_7 + cL_6.$$

Each of L_6, L_7, L_8 is orthogonal to degrees at most five for $e^{-t}\,dt$ on $[0, \infty)$, as follows by substituting the factorial integrals in the polynomial formula. Suppose Q has eight distinct real roots ξ_i. For any polynomial of degree at most thirteen, division by Q gives an orthogonal multiple and a remainder of degree at most seven. Interpolate that remainder at the ξ_i to get

$$\begin{aligned} \int_0^\infty e^{-t} R(t)\,dt &= \sum_{i=1}^{8} w_i R(\xi_i), \\ w_i &= \frac{\displaystyle\int_0^\infty e^{-t} \frac{Q(t)}{t - \xi_i}\,dt}{Q'(\xi_i)}. \end{aligned} \tag{5}$$

All quadrature weights used in the proof are verified to be positive. The verifier constructs Q exactly from dyadic b, c, isolates eight distinct roots using rational interval endpoints and signs, and uses interval synthetic division and factorial integration to enclose the weights. Approximate root locations only supply starting points for the isolation.

5.2. **Coverage of branches.** At each node there are intervals for K located root copies. The unlocated roots lie in $(0, L]$ and avoid a list of open intervals. Bounds on L use the known largest-root interval when available or the moment upper bounds applied to a fixed polynomial $P(t) = \prod_{\ell=1}^{3} (t - a_\ell)^2$. The latter is nonnegative, and increasing to the right of the largest a_ℓ. Subtract lower bounds for located-root contributions to bound its sum over unlocated roots. The upper endpoint can then be trimmed when the omitted tail is covered by open intervals, respecting whether endpoints are included.

The first located root is a largest root, starting from a closed interval with lower endpoint four. A split of a located interval is the union of two closed intervals. For the other kind of split choose an open interval $H = (h_-, h_+)$ and an anchor ξ. One branch adds H to the list of root-free intervals. In every other branch there is some unlocated root in H. Select one copy r_* there whose distance from ξ is smallest among those in H. The closed

subintervals declared at the node cover $[h_-, h_+]$. Their intersections with the current allowed support provide the intervals for the newly located root, including allowed singleton intersections in H. The intervals may include their endpoints as a relaxation.

On a branch where the new root is in $[a, b]$, if $b < \xi$ then no remaining root lies in $H \cap (b, 2\xi - b)$. Indeed each point in this intersection has strictly smaller distance than r_* could have on that branch. If $a > \xi$, the excluded set is $H \cap (2\xi - a, a)$. The reflected endpoint is rounded inward, and the original H is retained when the root's interval is refined. An interval meeting the anchor gives no such exclusion. Open endpoints preserve equal distances and repeated root values. Trimming the reflected interval to H is essential because the choice of a nearest root was conditional on membership in H. For example, with $H = (1, 4)$, anchor 5 and roots 3 and $9/2$, the selected root is 3, but excluding the full interval $(3, 7)$ would erroneously remove $9/2$.

5.3. **Leaves and certified inequalities.** Let U be the allowed unlocated support and $s = t/L$ when $L > 0$. A degree-six polynomial $q(s) = \sum_{j=0}^{6} q_j s^j$ with $q_6 \geq 0$ has a rigorous upper bound on its total sum over the roots from the moments, using the upper bound for p_6. This sum can be turned into a bound on the unlocated sum by subtracting lower bounds on the located boxes. For a gap in U, the indicator of its left component (all support to the left of the gap) sums over unlocated roots to an integer m. Majorants of both signs of that indicator can prove a contradiction or determine m. Alternatively, the verifier checks every possible $m \in \{0, \ldots, 13 - K\}$ in a distinct subcase, using majorants of the form $q + \alpha \mathbf{1}_{\text{left}}$ whose indicator term contributes the exact quantity αm. A nonnegative majorant on U with a negative bound for its sum rules out a state or subcase. Other infeasibility leaves use an empty support with unlocated roots, impossible power sums of the unlocated roots, or a negative bound on the sum of a square.

At a quadrature node above all roots, or in a gap where the verified number of roots to its right is even, the contribution in (5) is nonnegative. It may then be assigned lower bound zero. This also holds if a root lies exactly at the node. For a node certified strictly above all roots, a polynomial majorant of $-\log(\xi_i - t)$ and the moment bound give a useful positive lower bound by exponentiation, times a lower bound on w_i. For the lower bound on a possible negative contribution, majorize $\log|t - \xi_i|$ on the unlocated support. Use the upper envelope over the interval enclosing ξ_i, with $\log 0 = -\infty$ when needed. A located root box separated from that node is handled by subtracting a lower bound for the polynomial minus the log function on the box. An overlapping located box may instead be removed from the moment calculation by subtracting a lower bound on its polynomial value, and then handled by multiplying by an upper bound on its absolute distance. For a positive node one may either use the polynomial-minus-log bound or account for the located roots separately after subtracting their polynomial

bounds. An upper bound on w_i times the resulting upper bound for the absolute product gives a lower bound with the sign reversed.

The univariate inequalities are checked on subintervals covering the required sets. Interval Horner evaluation and the mean-value theorem bound polynomial and polynomial-minus-log values using derivative enclosures, with each derivative coefficient enclosed separately. For log absolute values near a possible singularity, a maximum-distance bound is available without differentiation. The verifier uses correctly rounded Decimal logarithm and exponential values and takes adjacent Decimal numbers as enclosing endpoints. Their conversion to binary64 is checked by exact comparison and done outward. Basic binary64 operations are enclosed using directed enlargement with `nextafter`. The numerical assumptions are round-to-nearest binary64 with gradual underflow and the CPython Decimal contract [6]. Nonfinite or unrepresentable certificate fields and invalid or incomplete branch records are rejected.

Proposition 5. *For every integer* $8 \le C \le 117$*, every feasible root multiset gives* $\int_0^\infty e^{-t} R(t)\, dt > 64{,}803{,}969$.

Proof. The accompanying file contains one proof tree per integer C in this range. Its 61,114 nodes include 35,745 leaves. A replay checks all 110 distinct values and every branch and subcase, using the preceding coverage and leaf conditions. The smallest certified lower-bound margin over 64,803,969 is at $C = 8$. Its exact binary64 representation is `0x1.4112f929fffffp+5`, which exceeds 40.134264. At $C = 9$ the certified margin exceeds 374.532520. Here a margin is the verifier's lower bound for the excess on the accepted leaves that bound the integral. The other leaves eliminate their states. The certificate supplies all leaves needed by the coverage argument. The generator uses numerical linear programming to propose witnesses. Replay checks their feasibility inequalities directly, with no linear-programming solver. □

Proof of Theorem 1. The moment relaxation contains the rook roots of every feasible A. Proposition 5 and (2) treat $C \ge 8$. Proposition 2 treats all $1 \le C \le 7$ and gives strictness there. Section 2 gives the equality value and the isomorphism characterization at $C = 0$. □

Code and data availability

The public reproduction archive at a fixed Git revision supplies the complete proof trees, exact moment check, low-four-cycle enumerator, independent class and permanent checker, and a SHA-256 manifest. The proof trees, Python checks, and low-cycle census results are also included in the ancillary files. Run `python -B anc/replay.py` from the expanded source archive. The full proof-tree replay used CPython 3.12.10. The finite census was run with GCC 16.2.0 and OpenMP. A separate Lean 4.36.0 development checks Gram-defect identities, a four-cycle gap for general parameters, and

an integer cubic-moment inequality. The real-root certificate and census are validated by the computations described above.

AI and human involvement

The human project set the research scope, commissioned the verification, and approved deposit. AI research agents developed the candidate approach and computational tools. OpenAI Codex audited and repaired the interval and branch logic, replayed the computations, and drafted the manuscript. The contributors report understanding some parts of the work. The documented independent checks are mathematical and computational checks rather than an external peer-review report.

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