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A COMPUTER-ASSISTED PROOF OF THE LIH–WANG PERMANENT CONVEXITY CONJECTURE FOR ORDER FIVE

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Abstract. Let J_5 be the 5×5 matrix with every entry $1/5$. For every 5×5 doubly stochastic matrix A and every $\alpha \in [1/2, 1]$, we prove that the permanent of $\alpha J_5 + (1 - \alpha)A$ does not exceed $\alpha \operatorname{per} J_5 + (1 - \alpha) \operatorname{per} A$. Equality holds precisely when $A = J_5$ or $\alpha = 1$. We reduce the parameter interval to its midpoint using cubic Bernstein coefficients. Radial Bernstein expansions then reduce the midpoint inequality to lower-dimensional Birkhoff faces. The proof uses exact rational sum-of-squares certificates and an exhaustive face classification.

1. Introduction

Let Ω_n denote the polytope of real nonnegative $n \times n$ matrices with all row and column sums equal to 1. Put $J_n = \mathbf{1}\mathbf{1}^{\mathsf{T}}/n$ and $\operatorname{per} A = \sum_{\pi\in S_n} \prod_{i=1}^{n} a_{i,\pi(i)}$. The Lih–Wang conjecture asks whether

$$\operatorname{per}\big(\alpha J_n + (1-\alpha)A\big) \le \alpha \operatorname{per} J_n + (1-\alpha) \operatorname{per} A \qquad (A \in \Omega_n, \quad 1/2 \le \alpha \le 1).$$

Lih and Wang proved the order-three case [11]. Minc catalogued the conjecture, and Cheon and Wanless listed it as Conjecture 34 in their update [12, 2]. Foregger proved an order-four parameter range [5]. Udayan and Somasundaram reported an order-four result and a restricted order-six result [13]. Lavi's research record gives the full order-four interval and the order-five case for mixtures of up to four permutation matrices [9].

Theorem 1. *The Lih–Wang inequality holds for every $A \in \Omega_5$ and every $\alpha \in [1/2, 1]$. Equality holds exactly when $A = J_5$ or $\alpha = 1$.*

The general-order conjecture remains open. The order-five proof combines inequalities on the centered subpermanents, exact polynomial identities on individual faces, and a finite induction over supports.

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2. Preliminaries and the midpoint reduction

Write $J = J_5$, $t = 1 - \alpha$, $X = A - J$ and $r^2 = \|X\|_F^2$. For $k \geq 1$, let $\sigma_k(M)$ be the sum of all order-k subpermanents, with $\sigma_0 = 1$. By expanding the permutation products,

$$\operatorname{per}(J_n + zX) = \sum_{k=0}^{n} \frac{(n-k)!}{n^{n-k}} z^k \sigma_k(X). \tag{1}$$

The centered matrix X has zero row and column sums. In this case $\sigma_1(X) = 0$, $\sigma_2(X) = r^2/2$ and $\sigma_3(X) = \frac{2}{3}\sum_{ij} x_{ij}^3$. With $R_i = \sum_j x_{ij}^2$ and $C_j = \sum_i x_{ij}^2$, one also has

$$\sigma_4(X) = \frac{r^4}{8} + \frac{3}{2}\sum_{ij} x_{ij}^4 - \frac{3}{4}\Big(\sum_i R_i^2 + \sum_j C_j^2\Big) + \frac{1}{4}\operatorname{tr}\big((X^{\mathsf{T}}X)^2\big). \tag{2}$$

For these identities, sum over ordered tuples of distinct rows and columns, then divide by $k!$. Apply partition-lattice inclusion–exclusion to distinctness, using a block of size j with weight $(-1)^{j-1}(j-1)!$. Singleton blocks vanish by the margins. For $k = 2, 3$ only the one-block partition remains. For $k = 4$, the three pairings and the single four-element block survive. Equal pairings contribute $3r^4$, distinct pairings contribute $6\operatorname{tr}((X^{\mathsf{T}}X)^2)$, mixed terms contribute $-18(\sum_i R_i^2 + \sum_j C_j^2)$, and the two four-element blocks contribute $36\sum_{ij} x_{ij}^4$. In particular,

$$\sigma_4(X) \geq \frac{3}{2}\sum_{ij} x_{ij}^4 - \frac{9}{16}\Big(\sum_i R_i^2 + \sum_j C_j^2\Big). \tag{3}$$

Indeed, both r^4 and the trace term are at least the mean of $\sum_i R_i^2$ and $\sum_j C_j^2$.

Set

$$F(A,t) = t\operatorname{per} A + (1-t)\operatorname{per} J - \operatorname{per}(J + tX), \qquad Q(A,t) = F(A,t)/[t(1-t)].$$

The quotient extends polynomially. Formula (1) gives

$$Q(A,t) = \frac{3}{125}r^2 + \frac{2(1+t)}{25}\sigma_3(X) + \frac{1+t+t^2}{5}\sigma_4(X) + (1+t+t^2+t^3)\sigma_5(X).$$

Let B_j be its cubic Bernstein coefficients in $2t \in [0,1]$. The endpoint coefficient $H = B_3$ is

$$H(A) = 2\operatorname{per} A + 2\operatorname{per} J - 4\operatorname{per}\big((A+J)/2\big). \tag{4}$$

We have $B_0 = \operatorname{per} A - \operatorname{per} J$ and $B_1 = \frac{7}{6}B_0 - \frac{1}{250}r^2$. We use the quantitative permanent bound

$$\operatorname{per} A \geq \frac{n!}{n^n}\exp\left[\left(\frac{1}{2n} + \frac{1}{3n^2}\right)\|A - J_n\|_F^2\right] \qquad (A \in \Omega_n,\ n \geq 2). \tag{5}$$

from Lavi [10, Theorem 1.2], based on Gurvits's capacity inequality [6, 8]. Thus $B_0 > 0$ and $B_1 \geq (101/93750)r^2 > 0$ if $A \neq J$. The ordinary lower bound also follows from the van der Waerden theorem [3, 4].

Lemma 2. *For $A \in \Omega_5$, $B_2(A) - \frac{34}{45}H(A) \geq \frac{61}{24000}\|A - J\|_F^2$. Consequently, strict positivity of H on $\Omega_5 \setminus \{J\}$ implies Theorem 1.*

Proof. The left side is $\frac{11}{1875}r^2 + \frac{2}{125}\sigma_3(X) + \frac{17}{900}\sigma_4(X)$. By (3) it is at least the sum, over five rows and five columns of A, of the function

$$\Psi(p) = \frac{11}{3750}R + \frac{2}{375}\sum_i x_i^3 + \frac{17}{1200}\sum_i x_i^4 - \frac{17}{1600}R^2.$$

Here $x_i = p_i - 1/5$ and $R = \sum_i x_i^2$. The second identity of Appendix A yields $\Psi(p) \geq \frac{61}{48000}R$ on the probability simplex. This proves the bound. If $A \neq J$ and $H(A) > 0$, all four Bernstein coefficients are positive. Multiplying by $t(1-t)$ gives the strict inequality for $0 < t \leq 1/2$. At $t = 0$ or $A = J$, the difference vanishes directly. □

3. Radial reduction

First take a ray $A = J + s(Y - J)$ with Y on the boundary of Ω_5 and $0 < s \leq 1$. Such a ray exists because J has all entries positive and Ω_5 is bounded. For $X = Y - J$, the four cubic Bernstein coefficients in s of $H(A)/s^2$ are

$$\frac{3}{125}r^2, \quad \frac{3}{125}r^2 + \frac{1}{25}\sigma_3(X), \quad \frac{3}{125}r^2 + \frac{2}{25}\sigma_3(X) + \frac{7}{60}\sigma_4(X), \quad H(Y).$$

The degree-three Schur inequality gives $\sigma_3(X) \geq -r^2/10$. More generally, for $n \geq 3$, nonnegative p_i of sum S, and $x_i = p_i - S/n$, expansion gives

$$\sum_i x_i^3 + \frac{n-2}{n(n-1)}S\sum_i x_i^2 = \frac{2}{n(n-1)}\sum_{i<j<k}\sum_{\text{cyc}} p_i(p_i - p_j)(p_i - p_k).$$

For ordered entries $a \geq b \geq c \geq 0$, the triple sum is $(a-b)^2(a+b-c) + c(a-c)(b-c) \geq 0$. Apply the identity to the rows when $n = 5$ and multiply by $2/3$. For the third radial coefficient use (3) and the first identity of Appendix A on every row and column. The result is a lower bound $\frac{53}{24000}r^2$. Thus all three coefficients preceding $H(Y)$ are positive for $Y \neq J$. The midpoint inequality therefore reduces to the boundary.

For a proper face P, take a rational $M \in \operatorname{relint} P$. For $Y \in P$ write

$$d_i(Y) = [s^i]H(M + s(Y - M)), \qquad c_k(Y) = \sum_{i=0}^{k} \frac{\binom{k}{i}}{\binom{5}{i}} d_i(Y). \tag{6}$$

The c_k are the degree-five Bernstein coefficients. We have $c_5(Y) = H(Y)$. If $c_0, \ldots, c_4 > 0$ on P and $H > 0$ on its relative boundary, then $H > 0$ on P. In fact, for an interior point other than M the radial endpoint Y is on the relative boundary and the Bernstein expansion has nonnegative basis factors, with a positive $c_0(1-s)^5$ term. At M, the value is c_0. This

argument applies as well when the boundary endpoint belongs to a face of codimension greater than one.

4. Exact certificates on faces

For a support $E \subseteq [5] \times [5]$, let P_E consist of the matrices in Ω_5 that vanish off E. Suppose that the bipartite graph of E is connected and that E is the union of its perfect matchings. The margins have rank nine on E. One can choose the nine edges of a spanning tree as dependent coordinates, since the oriented incidence matrix on a tree has rank nine and deleting one vertex equation gives an invertible system. The supplied charts solve the nine margin equations for nine entries with an invertible rational matrix. The fifth column margin then follows from the sum of the row margins. The remaining $d = |E| - 9$ entries give a chart $Y(z)$ of the entire affine hull. Its domain in the face is exactly $Y_e(z) \geq 0$ for $e \in E$. Rescaling or translating the free coordinates preserves this chart.

For the certificates on faces with at least three zeros, the constant $c_0 = H(M)$ and the minimum of the affine polynomial c_1 over the permutation vertices are verified to be positive. Let $v_j(z)$ be the vector of monomials of total degree at most j, including the constant. Each certificate for c_k, $k = 2, 3, 4$, has the form

$$c_k(Y(z)) = v_2(z)^{\mathsf{T}} Q_0 v_2(z) + \sum_{w \in \mathcal{W}} w(z) v_1(z)^{\mathsf{T}} Q_w v_1(z). \tag{7}$$

The list $\mathcal{W}$ contains Y_e for $e \in E$ and $Y_e Y_f$ for unordered pairs of edges, including repetitions. All matrices are symmetric rational and positive definite. The constant entry of v_2 guarantees strict positivity on P_E. The check constructs c_k from (4) and (6) and compares every monomial of degree at most four. Interpolation in s uses six distinct rational arguments and recovers the coefficients of a polynomial of degree at most five. After clearing positive denominators, the matrix check finds a nonsingular integer matrix U and verifies strict diagonal dominance with positive diagonal of $U^{\mathsf{T}} Q U$, using integer arithmetic. A floating-point search proposes U. Positive definiteness follows from the checked congruence. Similarly, the rational coefficients of a computed polynomial residual are assigned to a correction of Q_0, and the corrected matrix is required to pass the same check.

4.1. **The one- and two-zero faces.** By independent row and column permutations every one-zero face is equivalent to $a_{11} = 0$. For this face take M with $m_{11} = 0$, $m_{1j} = m_{i1} = 1/4$ for $i, j > 1$, and $m_{ij} = 3/16$ otherwise. For $Z = Y - M = \left(\begin{smallmatrix} 0 & u^{\mathsf{T}} \\ v & W \end{smallmatrix}\right)$, put $D = \|u\|_2^2 + \|v\|_2^2$ and $\lambda_b = 7611/128000$, $\lambda_i = 6017/128000$. Expansion in the tangent space gives

$$d_0 = \frac{8049}{8192000}, \quad d_1 = 0, \quad d_2 = \frac{\lambda_i}{2}\|W\|_F^2 + \frac{5\lambda_b - \lambda_i}{8} D,$$

$$d_3 = \frac{729}{6400}\sum_j (u_j^3 + v_j^3) + \frac{3}{50}\sum_{ij} w_{ij}^3 - \frac{69}{1280}\sum_{ij}(u_j + v_i)w_{ij}^2 - \frac{97}{1024}v^{\mathsf{T}}Wu.$$

These identities can be checked by substituting the tangent margins and expanding the permutation products. It follows that $c_0, c_1, c_2 > 0$. For $c_3 = d_0 + \frac{3}{10}d_2 + \frac{1}{10}d_3$, the last bilinear term has $|v^{\mathsf{T}}Wu| \le D/2$, since the inner block of Y is doubly substochastic and u, v have zero sums. Split the remaining inner terms equally between four rows and four columns. For each line with border entry $b \in [-1/4, 3/4]$ and shifted inner entries $w_j = y_j - 3/16$, set $s_0 = 3/4 - b$ and $D_2 = \sum_j (y_j - s_0/4)^2$. The four-variable Schur bound makes its contribution at least $A(b) + B(b)D_2$, where

$$A(b) = \frac{b^2(201840b + 113279)}{20480000}, \quad B(b) = \frac{16131}{5120000} - \frac{914}{128000}b.$$

The allowed interval is $0 \le D_2 \le 3s_0^2/4$. The lower endpoint gives $A(b) \ge 0$. At the upper endpoint the resulting cubic has positive second derivative on the interval and its tangent at $b = 19/100$ is bounded below by $695916071/1024000000000 > 0$ throughout the interval. Thus $c_3 \ge d_0 > 0$.

The certificate for c_4 on this face uses the same positivity principle as (7). Its terms are averages over the support stabilizer of $wv_1^{\mathsf{T}}Q_w v_1$, with w an allowed entry or a product of two allowed entries, together with positive multiples of averages of products of four allowed entries. The matrices are positive definite. Permuting rows, permuting columns, and transposing preserve H and the feasible face when they belong to its stabilizer. Every summand is nonnegative. Each single-entry orbit is represented, and each feasible matrix has positive entries, so a single-entry term gives strict positivity.

Two zeros lie either on a common row or column, or in different rows and columns. For the representative with $a_{11} = a_{12} = 0$, take the reference point whose first row has three entries equal to $1/3$ outside the prescribed zeros. In the remaining rows, the first two entries equal $1/4$ and the last three equal $1/6$. The linear term vanishes and the quadratic term is positive semidefinite, so $c_0 = c_1 = 17/6000$ and $c_2 \ge c_0$. For the representative with $a_{11} = a_{22} = 0$, put $a = 29/100$. Take a 2×2 leading block with zero diagonal and off-diagonal entries a, cross blocks with entries $(1-a)/3$, and a 3×3 trailing block with entries $(1+2a)/9$. Here $c_0 = 914717153/486000000000$, and direct minimization of the affine c_1 gives $c_1 \ge 546254137/303750000000$. The averaged certificates cover c_3, c_4 in both cases and c_2 in the second. Their identities are checked in the whole affine chart. Both reference points have unit margins and are positive in every allowed position.

5. Support enumeration and terminal cases

Use the Birkhoff–von Neumann description of Ω_5 as the convex hull of permutation matrices [1]. The vertices of P_E are precisely the permutation matrices supported by E. Thus P_E is empty if there is no perfect matching.

Otherwise its actual support E' is the union of all its perfect matchings, and $P_E = P_{E'}$. The average of these vertices is strictly positive on E', so further zero positions describe the relative boundary of $P_{E'}$. Independent permutations of rows and columns, together with transposition, preserve H and give equivalent face problems.

The connected actual supports are precisely the fully indecomposable supports. For if a proper nonempty set of rows has the same number of neighbors, every perfect matching uses those columns for that set. Every edge belongs to a perfect matching, so the complementary rows have no edges to those columns, and the support is disconnected. Conversely, every connected component of an actual support has equally many rows and columns, since its vertices are matched internally. This gives the block forms used below when the support is disconnected.

For three prescribed zeros, the four classes represent a three-edge star, a three-edge path, a two-edge path and an isolated edge, and three disjoint edges in the graph of zeros. Each is a connected actual support after taking the complement. For four prescribed zeros, the verifier enumerates all $\binom{25}{4}$ choices and obtains nine connected actual supports up to symmetry. Larger levels can be found by deleting one allowed edge, taking the union of the remaining perfect matchings, and canonicalizing. Completeness can also be seen by adding edges to any connected actual support until there are 21 allowed edges. Each added edge (i, j) extends to a perfect matching because deleting row i and column j from a fully indecomposable support leaves a perfect matching. This follows from the strict Hall inequality on each nonempty proper set of rows and Hall's matching theorem [7]. The supports along this chain are connected and actual. The verifier additionally checks the reduced support of every one-edge deletion, including deletions that force additional zeros.

Zeros	4	5	6	7	8	9	10	11	12	13	14	15
Connected	9	17	31	52	75	93	89	68	36	15	3	1
Terminal	0	0	0	0	0	0	0	0	0	1	1	1

TABLE 1. Support classes up to row and column permutations and transposition.

There are 486 nonterminal classes in Table 1. Every one of their 7,762 one-edge deletions reduces to a deeper nonterminal class (5,394 deletions), a terminal class (11), or a partly decomposable support (2,357). Here a terminal connected class has no proper connected actual subface. The four three-zero classes and the 486 nonterminal classes in the table use the certificates of (7).

5.1. **Partly decomposable matrices.** After permuting rows and columns, such a matrix is either $1 \oplus C$ with $C \in \Omega_4$, or $D \oplus E$ with $D \in \Omega_2$ and

$E \in \Omega_3$. A component of size one gives the first form, and otherwise the only partition of five with smaller components is $2+3$. Write $u_k = \sigma_k(C)$, $p = \operatorname{per} C$. Expanding (4) using (1) or the subpermanent convolution for the two blocks gives

$$H(1 \oplus C) = \frac{37}{20}p - \frac{2}{125}u_2 - \frac{7}{200}u_3 + \frac{3}{125}.$$

With $S = \|C - J_4\|_F^2 \le 3$, the centered identities give $u_2 = 9/2 + S/2$ and $u_3 = 3/2 + S/2 + \frac{2}{3}\sum(c_{ij} - 1/4)^3 \le 3/2 + S$, since $c_{ij} - 1/4 \le 3/4$. Apply (5) with $n = 4$ and $e^x \ge 1 + x$. This yields

$$H(1 \oplus C) \ge \frac{1167}{16000} - \frac{4533}{256000}S \ge \frac{5073}{256000} > 0.$$

For $D \oplus E$, put $p_2 = \operatorname{per} D$, $p_3 = \operatorname{per} E$ and $v_2 = \sigma_2(E)$. The same convolution gives

$$H(D \oplus E) = \frac{15}{8}p_2p_3 - \frac{1}{40}p_2v_2 - \frac{9}{250}p_2 - \frac{3}{50}p_3 - \frac{13}{500}v_2 + \frac{3}{250}.$$

The centered quadratic identity gives $v_2 \le 3$, and (5) gives $p_2 \ge 1/2$ and $p_3 \ge 2/9$. Substituting $v_2 = 3$ decreases the expression. The two partial derivatives of the resulting bilinear function are positive for these bounds, so the minimum lower bound occurs at $(1/2, 2/9)$ and is $147/2000 > 0$.

5.2. **The three terminal connected classes.** The class with ten edges has representative $I_5 + C_5$, with C_5 the cyclic permutation matrix. Its vertices are I_5, C_5. For $Y = (1-s)I_5/2 + (1+s)C_5/2$ the expansion of $H(Y)$ is even with positive coefficients $89/8000$, $839/800$, and $37/64$ in degrees $0, 2, 4$, respectively. The other two classes have three and four permutation vertices, respectively. Their archived coefficients are in barycentric weights $w_1, \dots, w_k$. If $z_2(w)$ contains every homogeneous quadratic monomial in the weights, the certificates read

$$H\Big(\sum_{i=1}^{k} w_i P_i\Big) = \sum_{i=1}^{k} w_i \, z_2(w)^{\mathsf{T}} Q_i z_2(w) \qquad \big(w_i \ge 0, \ \sum_i w_i = 1\big).$$

The left side is homogenized to degree five using $\sum w_i$ before comparing coefficients. The verifier checks all 21 or 56 coefficients and uses exact rational LDL^{T} to certify the matrices. At least one w_i is positive and $z_2(w) \ne 0$, so the right side is positive.

Completion of the proof of Theorem 1. On proper faces, use strong induction on the size of the actual support. For the induction step a boundary point lies in the reduced support obtained from at least one one-edge deletion. That support has fewer edges. The disconnected and terminal cases have just been proved. Every other support has the radial coefficient checks in Section 4; table and enumeration ensure they cover the boundary. Thus $H > 0$ on all proper faces. The ray from J gives strict positivity away from J, and Lemma 2 completes the proof. □

6. Computational verification and reproducibility

The verification uses Python's standard library and includes a support enumeration based on permutation matching, an independent coefficient check for the standard face, and the rational Gram checks. For the 486 non-terminal supports in Table 1, the three certified polynomials per support give 1,458 identities, 617,727 coefficient comparisons, and 225,366 positive-definite matrix checks. The four three-zero supports have 12 additional identities. Their certificates use the same scheme as (7). Certificate discovery used numerical projections. All polynomial identities and all successful definiteness decisions are established with exact arithmetic. The supplement specifies the data layout, file hashes and a single verification command.

Code and data availability. The ancillary files provide the proof inputs, certificates, and verification programs. Download the complete `anc` directory. From the enclosing directory run

`python -I anc/unpack_proof_inputs.py proof-inputs`.

From `proof-inputs`, execute

`python -I -u round11-final-review/reproducibility/verify.py`.

The reconstruction checks SHA-256 digests of the contents. Gzip containers are regenerated from the exact certificate JSON text. The source provenance is the fixed project revision

`59bb4c208c268fa688431921a122163e48a75c6d`.

Author contributions. Jiayuan Wang commissioned the final audit and coordinated the preparation of the manuscript for review. Both authors approved this version and its deposit. Human authors have checked and understood some parts of the contribution.

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Appendix A. Simplex identities for the global reductions

For $p \in \mathbb{R}^5_{\geq 0}$ define $S = \sum_i p_i$, $x_i = p_i - S/5$, $R = \sum_i x_i^2$, and

$$W_{\pm} = \sum_i p_i \left(10p_i^2 - 7Sp_i + S^2 \ \pm 5\Big(\sum_j p_j^2 - Sp_i\Big) \right)^2 .$$

Define also

$$W_1 = S\sum_{i<j} p_ip_j(p_i-p_j)^2, \qquad W_2 = \sum_{i<j} p_ip_j(p_i+p_j)(p_i-p_j)^2,$$

$$W_4 = S\sum_{\substack{i<j,\ k<l\\ \{i,j\}\cap\{k,l\}=\varnothing}} p_ip_j(p_k-p_l)^2, \qquad W_5 = \sum_{i<j<k} p_ip_jp_k(p_l-p_m)^2,$$

$$W_6 = \sum_{i<j<k} p_ip_jp_k(p_i+p_j+p_k-3S/5)^2.$$

In W_5, $\{l,m\}$ is the complement of $\{i,j,k\}$. For the cubic radial coefficient at the midpoint, write

$$\widehat{\Phi} = \frac{3}{250}S^2R + \frac{2}{75}S\sum_i x_i^3 + \frac{7}{80}\sum_i x_i^4 - \frac{21}{320}R^2.$$

For the midpoint-to-interval step put

$$\widehat{\Psi} = \frac{11}{3750}S^2R + \frac{2}{375}S\sum_i x_i^3 + \frac{17}{1200}\sum_i x_i^4 - \frac{17}{1600}R^2.$$

The homogeneous polynomial identities, each checkable by expansion in the 126 monomials of degree five, are

$$\begin{aligned} S\left(\widehat{\Phi} - \frac{53}{48000}S^2R\right) &= \frac{443}{1612800}W_+ + \frac{5807}{8064000}W_- + \frac{1031}{26880}W_2 \\ &\quad + \frac{29}{512}W_5 + \frac{1411}{23040}W_6, \\ S\left(\widehat{\Psi} - \frac{61}{48000}S^2R\right) &= \frac{179}{11520000}W_+ + \frac{1969}{11520000}W_- + \frac{859}{192000}W_1 \\ &\quad + \frac{679}{1440000}W_4 + \frac{179}{48000}W_5. \end{aligned}$$

These imply the desired bounds at $S=1$, since all generators are nonnegative.

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