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A PROOF OF THE CAO–KOOL–MONAVARI CREPANT RESOLUTION CONJECTURE FOR CALABI–YAU FOURFOLDS

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Abstract. Cao, Kool and Monavari conjectured that the Donaldson–Thomas series of points on certain quotient fourfolds factors into a point series and two stable-pair series on a crepant resolution. We prove their Conjecture 5.16 for the cyclic and Klein-four actions. The identity holds in equivariant K-theory at generic weights and mass, in every degree and curve class. On the orbifold, factorization over disjoint supports bounds the poles of the plethystic logarithm. The cyclic formula then determines the contribution along each of the three Klein-four branches. On the resolution, two framings give a recurrence in the Euler characteristic of a stable pair. Its boundary values follow from the derived McKay correspondence. We specify the global orientations and give the cohomological products for both actions.

Introduction

A point of a quotient stack carries information about its stabilizer. For a finite subscheme, this information enters the count through the representation on its functions. How can the same count be recovered on a resolution of the coarse space? The crepant resolution conjecture of Cao, Kool and Monavari (CKM) answers this question using the curves contracted to points. It expresses the orbifold point series as the product of the point series on the resolution and two stable-pair series on its contracted curves [2, Conj. 5.16].

The two geometries. Work over $\mathbb{C}$ and write $\mathbb{A}^d$ for affine d-space. CKM consider a finite abelian group $G \subset \mathrm{SU}(4)$ all of whose elements have age at most one. For eigenvalues $\exp(2\pi\sqrt{-1}\,a_j)$, $0 \le a_j < 1$, the age is $\sum_j a_j$. By [2, Prop. 2.1], such a group acts through $\mathrm{SU}(2)$ on two coordinates or through $\mathrm{SO}(3)$ on three. A finite abelian subgroup of $\mathrm{SU}(2)$ is cyclic. Commuting rotations in $\mathrm{SO}(3)$ either have a common axis or belong to the group of half-turns about three perpendicular axes. A cyclic rotation has eigenvalues $1, \zeta, \zeta^{-1}$ on the complexified three-space [2, Rem. 4.11]. Thus the nontrivial actions in the conjecture are, up to conjugacy,

$$\begin{array}{ll} G = \mu_r,\ r \ge 2 & (x_1, x_2, x_3, x_4) \text{ of weights } (1, -1, 0, 0) \pmod r, \\ G = (\mathbb{Z}/2)^2 & x_1, x_2, x_3 \text{ of distinct nontrivial characters, } x_4 \text{ invariant.} \end{array}$$

Here μ_r denotes the group of complex rth roots of unity. Put $\mathcal{X} = [\mathbb{A}^4/G]$ and let $S = \operatorname{Spec} \mathbb{C}[x_1, x_2, x_3, x_4]^G$ be its coarse space. A G-cluster is a G-invariant finite subscheme whose coordinate ring is the regular representation. The Nakamura resolution $X = G\text{-}\operatorname{Hilb}(\mathbb{A}^4)$ parametrizes such clusters. Its universal cluster $\mathcal{Z} \subset X \times \mathbb{A}^4$ fits into the following commutative diagram.

$$\begin{array}{ccccc} & & [\mathcal{Z}/G] & & \\ & \swarrow^{\mathrm{pr}_X} & & \searrow^{\mathrm{pr}_{\mathcal{X}}} & \\ X & & & & \mathcal{X} \\ & \searrow_{f} & & \swarrow_{\pi} & \\ & & S & & \end{array} \tag{1}$$

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Here f sends a cluster to the image of its support and π is the coarse quotient map. In a cluster algebra the invariant summand is spanned by the unit. Thus x_4 acts by a scalar specifying a point of $\mathbb{A}^1$, also in families, and $X = Y \times \mathbb{A}^1$ for $Y = G\text{-}\operatorname{Hilb}(\mathbb{A}^3)$. The three-dimensional action preserves the standard volume form. The McKay theorem gives the crepant resolution in dimension three and the equivalence with kernel the universal cluster [1, Thm. 1.2]. Taking its product with the fourth line gives the correspondence in (1). The two upper arrows define the Fourier–Mukai functor

$$\Phi = \mathbf{R}(\mathrm{pr}_{\mathcal{X}})_* \, \mathbf{L}(\mathrm{pr}_X)^* \colon D^b \operatorname{Coh}(X) \xrightarrow{\sim} D^b \operatorname{Coh}(\mathcal{X}) = D^b_G \operatorname{Coh}(\mathbb{A}^4).$$

The category on the right consists of bounded complexes of G-equivariant coherent sheaves, up to quasi-isomorphism. Every proper curve in X is contracted by f since S is affine. Write $[C_1], \dots, [C_h]$ for the McKay basis of exceptional curve classes, indexed by the nontrivial characters of G, with $h = |G| - 1$ [2, (2.1), §4.2].

Three equivariant indices. Let T be the diagonal torus preserving the volume form. Its coordinate function characters t_1, t_2, t_3, t_4 satisfy $\prod_j t_j = 1$. Write $w = t_4$. The mass y is the character of a further torus $\mathbb{C}^*_y$ acting trivially on the spaces. We take the cover $\mathbb{T}$ of $T \times \mathbb{C}^*_y$ obtained by adjoining half-characters and write $\mathbb{F} = \operatorname{Frac} R_{\mathbb{Q}}(\mathbb{T})$ for the fraction field of its rational representation ring. For a character z and a virtual character $\sum_z m_z z$ set

$$[z] = z^{1/2} - z^{-1/2}, \qquad \left[\sum_z m_z z\right] = \prod_z [z]^{m_z}.$$

We use brackets of bundles by the splitting principle whenever the indicated denominators are invertible.

On a Calabi–Yau fourfold, the Serre pairing makes the trace-free deformation complex of an ideal or a stable pair self-dual up to shift. An orientation chooses a trivialization of its determinant line compatible with the induced quadratic form. This fixes the sign of the square-root Euler class in virtual localization [11, §4.3, Thm. 7.3]. If M is one of the moduli spaces below, write $\widehat{\mathcal{O}}^{\mathrm{vir}}_{M,o}$ for its twisted virtual structure sheaf with orientation o. For its tautological complex V_M put

$$\mathsf{I}(M, V_M) = \chi^{\mathrm{loc}}_{\mathbb{T}}\big(M, \widehat{\mathcal{O}}^{\mathrm{vir}}_{M,o} \otimes [y V_M^\vee]\big) \ \in \mathbb{F}.$$

Here χ^{loc} denotes pushforward from the proper fixed locus with the virtual normal factor of [11, Thm. 7.3]. This is the trivial-line insertion of [2, Defs. 3.4, 5.3]. We use $G_0^{\mathbb{T}}(M)$ for the Grothendieck group of equivariant coherent sheaves and $K^0_{\mathbb{T}}(M)$ for the group of equivariant perfect complexes, which acts on it by tensor product.

Let $\mathbf{q} = (q_0, \dots, q_h)$ record the character multiplicities, with 0 indexing the trivial character. The three moduli problems are as follows.

- $M_v = \operatorname{Hilb}^v(\mathcal{X})$ parametrizes G-equivariant quotients $\mathbb{C}[x_1, x_2, x_3, x_4] \twoheadrightarrow B$ with character multiplicities $v = (v_0, \dots, v_h)$. The tautological bundle V_{orb} has fibre B^G, of rank v_0.
- $\operatorname{Hilb}^n(X)$ parametrizes length-n subschemes $Z \subset X$. Its tautological bundle V_X has fibre $H^0(Z, \mathcal{O}_Z)$.
- $P_n(X, \beta)$ parametrizes stable pairs $\mathcal{O}_X \xrightarrow{s} F$ with F a pure one-dimensional sheaf of proper support, $[F] = \beta$, $\chi(F) = n$, and $\operatorname{coker} s$ zero-dimensional. The tautological complex V_{PT} is the derived pushforward of the universal sheaf, with fibre $\mathbf{R}\Gamma(X, F)$.

For $\operatorname{Hilb}^n(X)$ every fixed zero-cycle is supported on the finitely many fixed points of X, and Hilbert–Chow is proper. For M_v the fixed locus is the finite set of coloured monomial ideals [2, Lem. 5.5 and §5.6]. Properness of the stable-pair fixed locus for $\beta \neq 0$ follows from [2, Prop. 4.2], applied to the volume torus of either abelian action.

Give the empty object weight one. With $\mathbf{q}^v = \prod_{i=0}^{h} q_i^{v_i}$ and $Q^\beta = \prod_{i=1}^{h} Q_i^{\beta_i}$ for $\beta = \sum_i \beta_i [C_i]$, the definitions are

$$Z_{\mathrm{orb}}(\mathbf{q}) = 1 + \sum_{v \neq 0} \mathsf{I}(M_v, V_{\mathrm{orb}})\, \mathbf{q}^v,$$

$$Z_0(q) = 1 + \sum_{n \geq 1} \mathsf{I}(\mathrm{Hilb}^n(X), V_X)\, q^n,$$

$$Z_{\mathrm{PT}}(q, Q) = 1 + \sum_{\substack{\beta > 0 \\ n \in \mathbb{Z}}} \mathsf{I}(P_n(X, \beta), V_{\mathrm{PT}})\, q^n Q^\beta.$$

Here $\beta > 0$ means a nonzero effective class. The t_j and y are common to all three series and are suppressed in their arguments.

The conjecture and its proof. One copy of the regular representation has multiplicity one in every colour. Use its variable for the point degree on X and measure the remaining multiplicities relative to the trivial colour. Explicitly,

$$q = \prod_{j=0}^{h} q_j, \qquad Q_i = q_i \ (i > 0), \qquad \mathbf{q}^v = q^n Q^\gamma, \quad n = v_0, \quad \gamma_i = v_i - n. \tag{2}$$

A regular colour vector has $\gamma = 0$. In order two, for example, the vectors $(1, 0)$ and $(1, 2)$ contribute to qQ_1^{-1} and qQ_1, respectively.

Crepant resolution correspondence [2, Conj. 5.16]**.** *For each of the two actions there are orientations for which, after* (2)*,*

$$Z_{\mathrm{orb}}(q, Q) = Z_0(q)\, Z_{\mathrm{PT}}(q, Q)\, Z_{\mathrm{PT}}(q, Q^{-1}). \tag{3}$$

Here $Q^{-1} = (Q_1^{-1}, \ldots, Q_h^{-1})$. The two sides lie in $\mathcal{R} = \mathbb{F}[Q_1^{\pm 1}, \ldots, Q_h^{\pm 1}][[q]]$. Indeed each nontrivial colour of the polynomial ring is generated over the invariant ring by x_1^i, x_2^{r-i} in cyclic type or by $x_i, x_j x_k$ in Klein type, with i, j, k distinct. Thus $v_i \leq 2v_0$. On the resolution Lemma I.5 gives $0 \leq \beta_i \leq n$. Every coefficient of q^n is a finite Laurent polynomial in the curve variables, so inversion and multiplication are defined coefficientwise.

Theorems I.1 and II.1 establish (3) at generic weights and mass. CKM proved that their proposed closed forms imply (3) [2, Thm. 5.17]. Our task is to identify the series of geometric indices. We use the fourth coordinate to fix orientations on whole moduli spaces. Projection from $Y \times \mathbb{A}^1$ presents a compactly supported sheaf by a sheaf on Y and an endomorphism. The same description holds equivariantly on the affine cover of $\mathcal{X}$. Scale this endomorphism and the section by a common parameter. At zero the quadratic complex splits into a half and its dual. Evaluating their determinant lines gives an orientation. The endpoint is taken in the stack of complexes, which permits the section to vanish. We transport the orientation along the family and multiply it by $(-1)^n$ in degree n. Lemmas I.2 and I.7 give the comparisons on the orbifold and the pair space. For points on X use the chart-compatible fourth-direction orientation of [9, Prop. 3.6] with the same degree sign.

To describe the dependence on curve class in (3), let ψ^k be the Adams operation on $\mathcal{R}$ raising all characters and counting variables to their kth powers. For $f \in q\mathcal{R}$ put

$$\mathrm{PE}(f) = \exp\left(\sum_{k \geq 1} \frac{\psi^k(f)}{k} \right).$$

Its inverse on $1 + q\mathcal{R}$ is the plethystic logarithm PL. We call a coefficient of $\mathrm{PL}\, Z$ a *primitive coefficient* of Z. The common dependence on mass is encoded by $H(q, y) = \sum_{n \geq 1} [y^n] q^n$. We use $[q^n Q^\gamma] Z$ for coefficient extraction.

The proof has an orbifold calculation and a stable-pair calculation. On the cyclic orbifold, splitting at disjoint supports bounds the denominator of a primitive. The cohomological point and orbifold series computed by Liu [9, Thms. A, B] then fix the remaining scalar. The order-two case of this *equivariant K-theoretic* formula is used on each open Klein-four branch in Part II. Once these three contributions are removed, a growth estimate and a threefold point count determine the primitive on their intersection. On both resolutions a second frame, given by a tautological line, relates the pair index at fixed curve class to one at a smaller Euler characteristic. Iteration either leaves the support or reaches $n = \max_i \beta_i$, where the McKay transform identifies the pair complex with an orbifold ideal. These initial values determine the interior, including classes of arbitrary multiplicity.

Part I. The cyclic group

Let $G = \mu_r$ act with weights $(1, -1, 0, 0)$, $r \geq 2$. Then $X = S_r \times \mathbb{A}^2$, where S_r is the minimal resolution of $\mathbb{A}^2/\mu_r$ and its exceptional curves form an A_{r-1} chain [2, §2.2]. The ordinary affine four-space point series is known in K-theory [6, Thm. 1.1]. Liu computed both cyclic point series, on $\mathcal{X}$ and on X, in equivariant cohomology [9, Thms. A, B]. The task here is to recover the full character of the orbifold index and to determine the series of stable pairs with all multiplicities along the chain.

Write $a = t_3$ for the first translation weight on $\mathbb{A}^2$, so the second is $w = t_4$. Set $K = [aw]/([a][w])$ and $Q_{ij} = Q_i \cdots Q_j$ for an interval $[i, j]$ of exceptional curves.

Theorem I.1. *For the cyclic action the global orientations above give*

$$\mathrm{PL}\, Z_{\mathrm{PT}} = KH(q, y) \sum_{1 \leq i \leq j < r} Q_{ij}, \tag{4}$$

$$\mathrm{PL}(Z_{\mathrm{orb}}/Z_0) = KH(q, y) \sum_{1 \leq i \leq j < r} (Q_{ij} + Q_{ij}^{-1}). \tag{5}$$

We first prove (5). Away from the singular plane the quotient is smooth, where its point contribution cancels that of Z_0. Factorization makes this cancellation a bound on poles. Symplectic vanishing and the growth in the remaining characters leave a single scalar for each degree, which the cohomological calculation determines. We then prove (4) by lowering the Euler characteristic at fixed curve class until we reach the saturated face or fall below the support bound.

I.1. The orbifold vertex and its mass dependence.

For either group we will first determine the mass dependence of each orbifold primitive. This requires two descriptions of the same index. The intrinsic obstruction theory specifies the orientation on M_v. A model by matrices gives coherent sheaves on M_v that we can split when their supports separate. We compare these descriptions at the T-fixed points.

Let $\widehat{G}$ be the group of characters of G. For a graded space $V = \bigoplus_{g \in \widehat{G}} V_g$ set $v_g = \dim V_g$, $n = \dim V_0$, and $N = \dim V$. Here 0 denotes the trivial colour. Let ρ_j be the colour of x_j. We write colours additively as indices and multiplicatively as characters. In $R(T \times G)$ put

$$D_j = \mathbb{C}_{t_j^{-1}} \otimes \rho_j^{-1}, \qquad D' = \bigoplus_{j=1}^{3} D_j, \qquad D = D' \oplus D_4.$$

These lines record the tangent characters, dual to the coordinate functions. For a bundle or representation B write $\lambda_{-1} B = \sum_k (-1)^k \wedge^k B$.

Lemma I.2. *For either action each coefficient of* $\mathrm{PL}\, Z_{\mathrm{orb}}$ *of invariant rank $n > 0$ is $[y^n]$ times a rational function of spatial weights. The same holds for* $\mathrm{PL}\, Z_0$.

Proof. The coherent sheaf. Let A_v be the space of four operators X_j of colour ρ_j on V and a vector in V_0 generating V under these operators, modulo $\prod_g \mathrm{GL}(V_g)$. The generating condition is open and makes the change-of-basis action free. The smooth quotient A_v carries the tautological graded bundle, also denoted V. The Hilbert scheme M_v is the zero scheme of the six commutators $[X_i, X_j]$. They form a section s of the symmetric bundle $E_v = (\wedge^2 D \otimes \operatorname{End} V)^G$. Wedging the two exterior factors and taking the matrix trace defines its pairing, since $\det D$ is trivial. The Jacobi identity gives $(s, s) = 0$. A maximal isotropic subbundle, or polarization, is $P_V = ((D_4 \wedge D') \otimes \operatorname{End} V)^G$.

The action of s by Clifford multiplication on $\wedge^\bullet P_V$ squares to zero. Its even and odd cohomology form a coherent $\mathbb{Z}/2$-graded sheaf $\mathcal{H}_V$ on M_v [6, Lem. 2.16]. We retain this grading, taking even minus odd when passing to $G_0^{\mathbb{T}}(M_v)$. Put $d_0 = \dim(\operatorname{End} V)^G$ and $d_j = \dim(D_j \otimes \operatorname{End} V)^G$ for $1 \le j \le 3$. The sheaf whose index we will use is

$$\mathcal{F}_v = c_v\, \mathcal{H}_V[N] \otimes \wedge^\bullet(yV_0^\vee), \qquad c_v = y^{-n/2} w^{(d_0 + \sum d_j)/2} \prod_{j=1}^{3} t_j^{d_j}. \tag{6}$$

Here $[N]$ changes the parity by N mod 2. Indeed c_v is the inverse square root of the product of the three determinants of the ambient tangent, P_V and $yV_0^\vee$. Opposite-colour Hom factors pair by trace in these determinants. The character cover makes c_v a character of $\mathbb{T}$, so $\mathcal{F}_v$ exists before localization.

Comparison with the oriented index. A solid partition π is a finite order ideal in $\mathbb{Z}^4_{\ge 0}$. Its quotient has character $V = \sum_{b \in \pi} t^b \rho(b)$, where $t^b = \prod_j t_j^{b_j}$ and $\rho(b) = \prod_j \rho_j^{b_j}$. Since $G \subset T$ is finite abelian, these monomial quotients of colour ranks v are exactly the reduced isolated points of M_v^T [2, Lem. 5.5 and §5.6]. Their virtual tangent characters have zero T-fixed part by [2, Lem. 5.9]. We claim that $\mathcal{F}_v$ and the intrinsic oriented Hilbert scheme with its insertion give the same fixed-point contribution, given by

$$(-1)^{n+\mu_\pi} \frac{[yV_0^\vee]}{[\mathsf{H}(V)]}, \qquad \mathsf{H}(V) = V_0 - (\lambda_{-1} D'\, V^\vee V)^G, \qquad \mu_\pi = \#\{(k,k,k,l) \in \pi : k < l\}. \tag{7}$$

$\mathsf{H}(V)$ is a half of the virtual tangent character, meaning that its sum with its dual gives that character. Indeed the ideal triangle supplies the framing term V_0, and the equivariant Koszul resolution of the diagonal in the first three directions supplies $-(\lambda_{-1} D'\, V^\vee V)^G$. The fourth Koszul direction supplies the dual half.

For the sign, project the commutator derivative to $P_V^\vee$, the three-coordinate equation space. Its fixed cokernel controls the orientation sign [6, Prop. 2.30]. The fixed part of the three-coordinate Koszul complex has vanishing Ext^2 and has Ext^3 of dimension μ_π [6, Prop. 4.5, Lem. 4.6]. Its last term is $w \operatorname{End} V$. Exactness at the preceding term therefore gives fixed cokernel dimension $\dim(((w \operatorname{End} V)^G)^T) - \mu_\pi$ for the projected derivative. Indeed T-weight zero is already G-invariant since $G \subset T$. Replacing the complementary matrix half by $\mathsf{H}(V)$ exchanges the moving part of $w \operatorname{End}(V)^G$ with its dual, adding parity $d_0 - \dim(((w \operatorname{End} V)^G)^T)$. The parity shift in (6) cancels d_0 since $d_0 \equiv N$ mod 2, and the exterior mass gives $(-1)^n$. To compare with intrinsic determinant evaluation, scale the fourth operator and the map from $\mathcal{O}$ by a parameter. The trace-free endomorphism complex on the quotient stack is the invariant summand of the ordinary family. The ideal triangle removes the scalar endomorphisms on both sides and invariants are exact. Choose quotient volume by descent, so the Serre traces agree up to the factor $1/|G|$ in integration. Transport evaluation along the positive real rescaling between these trace forms. At the split endpoint we use the plus choice in [3, Rem. 4.4] on the sheaf half. Order the frame and its Serre dual by the two triangles in (4.4) of that paper. At zero frame map these triangles split, so their determinant isomorphism is evaluation of the frame against its dual. This is the transfer used for ordinary points in [9, Lem. 3.5].

Here is the parity comparison for the two groups. Put $p = \operatorname{rk} P_V = d_1 + d_2 + d_3$, $d_4 = d_0$. Orient the matrix presentation by the ambient tangent and the positive subbundle P_V, as in [9, eqs. (18)–(20)]. Its normalized Clifford module has even vacuum in $\wedge^\bullet P_V^\vee$. Exterior duality gives the grading $[p]$ on $\wedge^\bullet P_V$ [6, Ex. 2.15]. In the diagonal Koszul resolution first take the isotropic quotient with terms in degrees $0, 1, 2$ given by $\operatorname{End} V$, $D \otimes \operatorname{End} V$, and $\wedge^2 D' \otimes \operatorname{End} V$, taking invariants throughout. Relative to the matrix orientation this exchanges the degree-zero and degree-one terms with their trace duals. Their ranks give the sign $(-1)^{d_0+p+d_4}$. Passing to the three-coordinate quotient for the spectral orientation exchanges $D_4 \otimes \operatorname{End} V$ with its third exterior dual, of invariant rank d_4. The two determinant comparisons of [9, Lemmas 4.14–4.15] apply here because the invariant projection commutes with the wedge-trace dualities and with the ideal triangles. Each exchange of a rank-e summand gives $(-1)^e$ by [3, Prop. B.2]. Including exterior duality, the parity on $\wedge^\bullet P_V$ is

$$\underbrace{p}_{\text{exterior duality}} + \underbrace{(d_0 + p + d_4)}_{\text{first exchange}} + \underbrace{d_4}_{\text{second exchange}} \equiv d_0 = \sum_g v_g^2 \equiv N \pmod 2.$$

This recovers the cyclic sign of [9, Cor. 4.16]. For the Klein trivial-colour box $p = 0$, $d_0 = 1$, so the matrix and spectral orientations have opposite signs. With the degree sign the intrinsic weight is $-[y]/[w]$, as in (7). Combined with the fixed Koszul calculation, the comparison identifies the two indices by localization on the actual Hilbert fixed points.

Two framings. Write $f_v(y) = [\prod_g q_g^{v_g}] \operatorname{PL} Z_{\mathrm{orb}}$. To separate the dependence on the mass we prove the identity

$$[y_2^n] f_v(y_1) = [y_1^n] f_v(y_2). \tag{8}$$

Replace the single generator by two colour-zero generators of framing-torus characters ξ_1, ξ_2, with independent masses y_1, y_2. The ambient matrix construction now parametrizes jointly generating vectors, and its commuting locus is the G-graded Quot scheme for $\mathcal{O}_{\mathbb{A}^4}^{\oplus 2}$ of graded rank v. Use the same Clifford construction, replacing the mass factor by $\prod_{j=1}^2 y_j^{-n/2} \wedge^\bullet (\xi_j y_j V_0^\vee)$. The frame determinants cancel as in (6). The index is Laurent in $l^{1/2}$, $l = \xi_2/\xi_1$, by the following properness argument as in [6, Props. 5.5–5.6]. The graded scheme is open and closed in the G-fixed locus of the ordinary Quot scheme, whose support morphism to $\operatorname{Sym}^N \mathbb{A}^4$ is proper. This affine base maps finitely to the elementary symmetric functions of the N values of each coordinate separately. Proper pushforward followed by a graded free resolution on that affine space leaves only spatial weights in the denominator. The framing torus acts trivially on the base, and its common scalar acts trivially after normalization.

A framing-fixed quotient splits by ξ_1, ξ_2 into two rank-one quotients with graded characters V, W, after removing their framing characters. The cross terms between the two blocks will determine the limits of the index as $l \to 0, \infty$. For the moving cross terms set $C = (\lambda_{-1} D'\, V^\vee W)^G$, $D_0' = (\lambda_{-1} D'\, W^\vee V)^G$. These have rank zero. In cyclic type one spatial colour is trivial. In Klein type $(1-\rho_1)(1-\rho_2)(1-\rho_3) = 0$ in $R(G)$, as $\rho_3 = \rho_1\rho_2$ and $\rho_i^2 = 1$. Since $(D_0')^\vee = -w^{-1} C$, their determinants agree. The two diagonal indices in (7) are multiplied by

$$(-1)^{n_1+n_2} \frac{[l y_2 V_0^\vee][l^{-1} y_1 W_0^\vee]}{[l(W_0 - C)][l^{-1}(V_0 - D_0')]}, \qquad n_1 = \dim V_0, \quad n_2 = \dim W_0.$$

Indeed the fixed projected differentials split by framing weight, the off-diagonal ambient gauge ranks occur in opposite ordered pairs, and the two extra exterior masses give the displayed parity. For any virtual character B, the leading coefficients of $[lB]$ at $\infty, 0$ are $(\det B)^{1/2}$ and $(-1)^{\operatorname{rk} B} (\det B)^{-1/2}$ after removing the powers of l. The cross factor thus has limits $y_2^{n_1/2} y_1^{-n_2/2}$ and its inverse. A Laurent polynomial bounded at both ends is constant. The two limits of the generating series are products of the ordinary one-frame series with these monomial rescalings.

Taking PL, which commutes with these monomial rescalings, gives (8). Thus $f_v(y)/[y^n]$ is independent of y.

For ordinary points, distinct toric vertices of X have disjoint support. The intrinsic orientations multiply on these vertices [9, Prop. 3.6]. Applying the rank-one case of [6, Thm. 1.1] on each smooth affine fourfold chart therefore gives

$$\mathrm{PL}\, Z_0 = H(q,y) \sum_{p \in Y^T} -\frac{[uv][vz][zu]}{[u][v][z][w]}, \tag{9}$$

where u, v, z are the local coordinate characters and $uvzw = 1$. □

In particular, if $g_v(y)$ denotes a primitive coefficient for either the orbifold or its normalization by Z_0, then at generic spatial weights

$$g_v(y) = \frac{[y^n]}{[w^n]}\, g_v(w).$$

This identity follows from the two framings and (9).

I.2. **Factorization over the coarse space.** The individual weights in (7) have many poles. To bound the poles left after summing, we will realize each primitive as the index of a coherent class on $S = \mathbb{A}^4/G$. If this class is supported on a smaller locus, a finite pushforward of that locus to affine space bounds the denominator by the weights of the target.

Here and below the Euler character of an affine coherent sheaf is its localized pushforward. If $A \simeq \mathbb{A}^m$ is a representation with nontrivial coordinate function characters b_j and origin i_0, concentration and the Koszul resolution give

$$\chi^{\mathrm{loc}}_{\mathbb{T}}(A, E) = \frac{\sum_j (-1)^j [H^j(\mathbf{L} i_0^* E)]}{\prod_{j=1}^m (1 - b_j)}.$$

The numerator is the class of the derived fibre in $R_{\mathbb{Q}}(\mathbb{T})$. We may first push a class properly to A and apply this formula. It works even when individual weight spaces of global sections are infinite dimensional, as happens for weight zero on the volume torus.

For a quotient algebra B in M_v, its invariant algebra B^G is a length-n quotient of $\mathcal{O}_S$. Define

$$h_v : M_v \longrightarrow \mathrm{Hilb}^n S \longrightarrow \mathrm{Sym}^n S, \qquad B \longmapsto \mathrm{Spec}\, B^G \longmapsto [\mathrm{Spec}\, B^G]_{\mathrm{cycle}}.$$

Write $\mathcal{S}_v$ for the pushforward of $\mathcal{F}_v$ along h_v, taking all higher direct images with their total parity. For $n_1 + n_2 = n$, let $U_{n_1,n_2} \subset \mathrm{Sym}^{n_1} S \times \mathrm{Sym}^{n_2} S$ be the open of cycles with disjoint support, and let a_{12} add them. The factorization needed here is an isomorphism of $\mathbb{T}$-equivariant graded coherent sheaves on this open,

$$a_{12}^* \mathcal{S}_v \;\cong \bigoplus_{\substack{v'+v''=v \\ v'_0=n_1,\ v''_0=n_2}} (\mathcal{S}_{v'} \boxtimes \mathcal{S}_{v''})|_{U_{n_1,n_2}}. \tag{10}$$

The maps for three disjoint cycles must agree in the two orders of splitting. Interchanging the two cycles must act by the graded switch, which contributes a minus sign when two odd factors cross.

Lemma I.3. *For both group actions, the sheaves $\mathcal{S}_v$ admit compatible factorizations* (10)*. There are classes $\mathcal{G}_v \in G_0^{\mathbb{T}}(S)$ whose localized Euler characters satisfy*

$$f_v(y) = \chi^{\mathrm{loc}}_{\mathbb{T}}(S, \mathcal{G}_v).$$

These classes restrict compatibly on opens where the normalized sheaves and their factorization maps agree, for the indices occurring on the open.

Proof. The factor for two ordered blocks. Given two quotient modules V, W with disjoint coarse supports, the subscheme they define together is their disjoint union. To lift this splitting to their normalized sheaves, we compute the extra Clifford factor of the off-diagonal matrices. Let $C^k_{VW} = (\wedge^k D \otimes \mathrm{Hom}(V,W))^G$ and $P_{VW} = ((D_4 \wedge D) \otimes \mathrm{Hom}(V,W))^G$. The off-diagonal Koszul complex $C^\bullet_{VW}$ computes the G-invariant derived Hom between the two modules over $\mathbb{C}[x_1,x_2,x_3,x_4]$. Its differential is the alternating sum of $f \mapsto X_{j,W} f - f X_{j,V}$ in the four directions. The supports upstairs are disjoint, so the complex is exact. Thus $N_{VW} = C^1_{VW}/C^0_{VW}$ injects into C^2_{VW} by the commutator derivative, with annihilator N_{WV} in its trace dual. The two ordered N's are the normal to the direct-sum map of matrix spaces, since off-diagonal gauge variations remove C^0 and frame variations split. On linear slices transverse to the gauge orbits the diagonal commutators have no linear terms in the off-diagonal entries. The Clifford comparison of [6, Assumption 2.22, Lemmas 2.24–2.27] therefore applies. In the polarization $C^2_{VW} = P_{VW} \oplus P^\vee_{WV}$ the additional factor is the spinor line killed by Clifford multiplication by the normal image. It is

$$L_{VW} = \det N_{VW} \otimes \det P_{WV}.$$

Indeed, wedge by N_{VW} requires its top exterior power and contraction by its annihilator removes the complement. The resulting parity is $\mathrm{rk}\, N_{VW} + \mathrm{rk}\, P_{WV}$ modulo two. The ranks are equal, since D_4 is colour-neutral and the three spatial colours are preserved as a multiset under inversion. Thus this line is even.

The switch and the three-block compatibility. The determinant gives a line for each ordered pair. To obtain (10) with its required symmetries we must choose generators of these lines compatibly. Put $H_j = (D_j \otimes \mathrm{Hom}(V,W))^G$ with the spatial weight suppressed, $j = 1,2,3$, and $H_0 = \mathrm{Hom}_G(V,W)$. Taking the quotient determinant in N_{VW} cancels the underlying Hom lines against the trace-dual lines of P_{WV}. The remaining character is $c_v c_{v'}/c_{v+v'}$ by (6). On separated fourth spectra let $s_j(f) = X_{4,W} f - f X_{4,V}$ act on H_j, including $j = 0$. In the determinant coordinates just described, the projection induced by $N_{VW} \hookrightarrow C^2_{VW}$ has

$$\det(N_{VW} \longrightarrow P_{VW}) = \frac{\det s_1 \, \det s_2 \, \det s_3}{\det s_0}, \tag{11}$$

up to the fixed ordering of the determinant factors. Indeed eliminating the fourth gauge arrow gives the denominator and the other three arrows give their fourth commutators. For example, in cyclic order three let the two blocks have bases $\{1\}$ and $\{1, x_1\}$, with fourth coordinates s, t. The third and gauge determinants cancel, and exactly one directed arrow connects the first block to a nontrivial colour of the second. The resulting factor is $t - s$. Thus retaining the order of the Hom spaces already matters for these two quotients. Equation (11) fixes the determinant generator using interlaced top forms $\bigwedge_\nu (f_\nu \wedge f^\vee_\nu)$ for the trace-dual Hom spaces. The generator itself, defined in the quotient determinant, extends over the whole disjoint locus. Upon interchange the Sylvester operators become negative transposes, with the inverse colours permuted. The interlaced volume contributes $(-1)^{\sum_{j=1}^3 \dim H_j}$. The combined sign is $(-1)^{\dim H_0}$.

The determinant generators multiply under further splitting since both their projection determinants and the interlaced even pairs multiply. Translating the fourth operators of the individual blocks by independent variables extends the identity from separated spectra to arbitrary disjoint blocks, also over nonreduced bases, since the separation resultants are monic up to sign in the translation differences. The shifts in (6) contribute $(-1)^{\dim V \dim W}$ on interchange. Fix an ordering of the colours and put $b(v,v') = \sum_{g<h} v_g v'_h$, where v, v' are the ranks of V, W. Since

$$b(v,v') + b(v',v) = (\dim V)(\dim W) - \sum_g v_g v'_g, \qquad \sum_g v_g v'_g = \dim H_0,$$

multiplication by $(-1)^{b(v,v')}$ in the ordered isomorphism cancels the remaining switch sign. Bilinearity of b preserves associativity, and exterior powers of the mass factor already respect the graded switch.

Passage to the symmetric product. The invariant quotient algebra is a length-v_0 quotient of $\mathcal{O}_S$. The full graded quotient is a quotient of the finite cover algebra tensored with this invariant quotient. Ideal kernels cut out a closed locus in a relative Grassmannian over $\mathrm{Hilb}^{v_0} S$. Composing with the projective Hilbert–Chow morphism proves that h_v is proper. On the ordered disjoint open of $\mathrm{Sym}^{n_1} S \times \mathrm{Sym}^{n_2} S$ the coarse idempotents split the quotient. The two graded ranks are locally constant. Thus the base change of the support morphism is the disjoint union over $v' + v'' = v$, $v'_0 = n_1$, $v''_0 = n_2$, of the two-block support morphisms. Flat base change and Künneth identify the pullback of the pushed-forward super-sheaf with the direct sum of the external products over these splittings, taking higher direct images in total parity. This gives (10), the isomorphism required in [14, Def. 3.5, (10)].

The allowed indices are those with $M_v \neq \varnothing$ and $v_0 > 0$. They form a factorization index set by [14, Lem. 3.3] and are closed under addition because one can translate two quotients apart along x_4. Thimm's factorization theorem and its Euler-characteristic form [14, Thm. 3.22, Cor. 3.25] now give the classes $\mathcal{G}_v$. To see the Euler-characteristic formula on the volume torus, take a finite equivariant map to affine space (use the invariant coordinate powers and x_4). A k-cycle permuting external factors has graded trace ψ^k on the derived fibre numerator. On each coordinate it replaces $1 - b_j$ in the Koszul denominator by $1 - b_j^k$, since the eigenvalues of the cycle are the kth roots of unity. Averaging over permutations gives the plethystic exponential with these localized Euler characters.

For the open comparison we use the restriction argument of [14, proof of Thm. 3.57] with the graded ranks as explicit indices. The complex there is indexed by partitions of a total degree into allowed degrees. A term with an absent degree vanishes on the open. The isomorphisms of the two coherent systems identify the surviving external products and their splitting maps. Formation of this complex and passage to its supported class on the small diagonal commute with open restriction. This yields compatible choices of primitive classes for the comparison opens below. We make the comparison before taking Euler characters. □

I.3. **Comparing the sheaves on the free charts.** On an open where a subgroup acts freely, its quotient has fewer colours. Restriction of quotient algebras alone gives the correct classical Hilbert scheme there. To compare the primitive classes of Lemma I.3 we must also identify the sheaves and their disjoint-support maps. We do this first for a free μ_r-action. Keeping a residual grading in the calculation gives the chart with stabilizer of order two used in Part II.

Lemma I.4 (Removing a free cyclic factor)**.** *On $x_1^r \neq 0$ for the cyclic action, the normalized Clifford system, with its disjoint-support maps, restricts to the ordinary system in coordinates $(x_1^r, x_2 x_1^{1-r}, x_3, x_4)$. The same statement holds on $x_2^r \neq 0$. On $x_i^2 \neq 0$ for the Klein action the comparison is with the residual order-two system in coordinates $(x_i^2, x_k/x_i, x_j, x_4)$, with $\{i, j, k\} = \{1, 2, 3\}$ and the middle two coordinates odd. The systems on the right are restricted to their first coordinate being invertible.*

Proof. First take $G = \mu_r$. The operator for x_1 maps each multiplicity space to the next. When it is invertible the common rank is some m. On this open of the smooth matrix space use the first $r - 1$ forward arrows to identify the multiplicity bundles. Write U, R for the forward and reverse arrows on the closing edge. The remaining gauge group is GL_m acting by simultaneous conjugation. The four ordinary quotient coordinates will be represented by U, R, A, B, where A and B are the averages of the third and fourth loops. To exhibit the normal directions, put $\lambda_s = 1 - (s+1)/r$, the internal reverse arrows and the loops have coordinates

$$\lambda_s UR + (1 - \lambda_s) RU + F_s, \qquad A + G_s, \quad B + J_s, qquad \sum G_s = \sum J_s = 0.$$

The internal edges are $0 \le s < r-1$, and the loops include the last vertex. Here F_s, G_s, J_s are deviations from the displayed expressions with all deviations zero. On the commuting locus the successive 12 equations give the common reverse matrix $UR = RU$, and the $13, 14$ equations give equality of the loops. Thus $F = G = J = 0$. Along this locus the framing is cyclic exactly when it is cyclic for U, R, A, B, since the closing forward arrow is invertible.

Comparison of the isotropic sections. We need an isometric inclusion of the ordinary equation bundle in the coloured bundle such that projection of the coloured section agrees with the ordinary section modulo the square of the normal ideal. This is the formal comparison required in [6, Assumption 2.22]. Write D_{ab} for an entry in the ordinary commutator equation space, indexed by $1 \le a < b \le 4$. Embed the ordinary equation bundle in the coloured one by repeating $D_{12}/r, D_{34}$ on all vertices. For $l = 3, 4$ place D_{1l}, D_{2l} on the closing edges, zero on internal forward edges, and

$$\lambda_s(UD_{2l} + D_{1l}R) + (1-\lambda_s)(D_{2l}U + RD_{1l})$$

on internal reverse edges. This preserves the fourth polarization and the wedge-trace form, since complementary edge entries pair only on the closing edges. The image of the ordinary section is the coloured section at $F = G = J = 0$. Put $\Delta G_s = G_s - G_{s+1}$. The adjoint projection cancels the additional terms $UG_{r-1} - G_0U$ and $RG_0 - G_{r-1}R$ in the closing $13, 23$ commutators using $\sum \lambda_s \Delta G_s = G_0$ and $\sum (1-\lambda_s)\Delta G_s = -G_{r-1}$, and the same holds for J. In particular, the only quadratic correction needed in the diagonal equations is visible in

$$(\mathrm{pr}\, s)_{12} = [U, R], \qquad (\mathrm{pr}\, s)_{34} = [A, B] + \frac{1}{r}\sum_s [G_s, J_s],$$

where pr is the trace adjoint of the inclusion. Thus the sections agree modulo $(F, G, J)^2$ on the ambient space. The transverse derivative split-injects by the successive differences of F with zero end values and by $\Delta G, \Delta J$. This verifies the formal-coordinate and normal hypotheses of [6, Assumption 2.22 and Lemmas 2.24–2.27].

The transverse determinant. For the coherent sheaves to agree, the extra spinor line must cancel the ratio of normalization characters in (6). In the complementary equation bundle the normal image meets the fourth polarization precisely in the J-directions. Its Clifford annihilator line is their determinant. After factoring their top wedge, the other directions are a graph over the dual polarization and have invertible degree-zero projection. Thus the line has character $w^{-(r-1)m^2}$ and parity $(r-1)m^2$. Equivalently one shifts the complementary spin module by this parity before using the positive-normal comparison of [6, Lem. 2.25]. The normalization ratio, coloured divided by ordinary, is $w^{(r-1)m^2}$, and the difference of parity shifts is $(r-1)m$. This identifies the normalized super-sheaves.

Compatibility on disjoint supports. Consider the two maps obtained by splitting a disjoint pair of modules before or after eliminating the redundant arrows. The two comparisons compose by the Clifford composition rule. In the relative off-diagonal variables the nonclosing gauge eliminations are successive differences, the reverse-arrow substitutions are triangular with unit diagonal, and loop averaging has constant determinant. After cancelling the J determinants one pairs (F, G) in one ordered Hom space with the remaining fourth polarization in the opposite one. Representatives there are the sum-zero 34 entries and the internal reverse 24 entries with closing entry zero. In the first pairing the matrix is $F \mapsto (F_{s-1} - F_s)_{s=0}^{r-1}$ with both end values zero. Dropping the last output coordinate gives a triangular matrix of determinant ± 1. In the second pairing Δ sends $(G_0, \dots, G_{r-1})$, with sum zero, to its $r-1$ forward differences. Its determinant in the first $r-1$ input coordinates is r. Tensor these matrices with the ordered Hom spaces. The closing operator U occurs only in shears of the other coordinates. The two compositions therefore differ by a nonzero scalar depending only on the ranks. Associativity and the corrected block switch make this a symmetric multiplicative two-cocycle. It can be removed by rescaling

each degree, since the associated commutative extension of the rank monoid by $\mathbb{C}^*$ is cancellative and its group completion splits by lifting a basis of the generated lattice. This makes the chart isomorphism compatible with factorization.

The residual order-two action. On $x_i^2 \neq 0$ in the Klein case let $G_{\mathrm{f}} = \ker \rho_j$ and $G_{\mathrm{res}} = \ker \rho_i$. Then $G = G_{\mathrm{f}} \times G_{\mathrm{res}}$. The first involution acts freely, and the quotient by it retains the G_{res}-grading. Apply the preceding calculation with $r = 2$ and coordinate order (x_i, x_k, x_j, x_4). If necessary, multiply one of the first three coordinates by -1 to make this permutation volume-preserving. This leaves the torus characters and the fourth-axis polarization unchanged. In this order U, B are even, R, A are odd, and F, G, J have parities odd, odd, even. Apply the Clifford comparison to the invariant equation and normal bundles themselves. The trace pairing on them is nondegenerate by the character decomposition. The displayed differences and pivots are homogeneous, so the projected section and split normal calculation apply on each ordered graded Hom space. In residual ranks u_0, u_1 put $d = u_0^2 + u_1^2$, $e = 2u_0u_1$. For the full grading $(d_0, d_i, d_j, d_k) = (2d, 2d, 2e, 2e)$. With c_{full} and c_{res} the two normalization characters, substitution into (6) gives

$$c_{\mathrm{res}} = y^{-u_0/2} w^{d+e} (t_i^2)^d (t_j t_k / t_i)^e, \qquad \frac{c_{\mathrm{full}}}{c_{\mathrm{res}}} = w^d, \qquad \det J = w^{-d}.$$

The characters cancel. The normal parity is d, which equals $u_0 + u_1$, the difference of grading shifts, modulo two. The scalar argument for disjoint supports applies to the residual rank monoid in $\mathbb{Z}^2$. Thus the branch comparison also preserves the factorization maps, as required to use it in Part II. □

I.4. **Determination of the cyclic orbifold series.**

Proof of (5). Write $F_{n,\gamma} = [q^n Q^\gamma] \operatorname{PL}(Z_{\mathrm{orb}}/Z_0)$. The argument has three steps. The free charts bound its denominator, a vanishing and a boundedness calculation force a single spatial factor, and Liu's cohomological calculation determines its scalar.

The denominator. We show that $(1-a)(1-w)F_{n,\gamma}$ belongs to the Laurent character ring $R_{\mathbb{Q}}(\mathbb{T})$. On the free locus the colour vector is regular, $v = n\delta$, where $\delta = (1, \dots, 1)$. In regular degree subtract from $\mathcal{G}_v$ the proper pushforward from X of

$$[y^n]\, w\, \lambda_{-1}(\wedge^2 \Omega^1_Y) \tag{12}$$

pulled back along $X \to Y$. Its character is the point primitive by (9). On $x_1^r \neq 0$ the two classes agree by Lemmas I.3 and I.4. On the full ordinary affine comparison chart before inverting x_1^r, equality with (12) follows from equality of characters, since its equivariant coherent K-group is the representation ring. For a nonregular index the primitive has zero restriction there by the support assertion in [14, Rem. 3.24], applied to the regular subsemigroup. Indeed on the open every quotient has regular rank, every $m\delta$ occurs there, and the difference of two regular indices, when admissible, is regular. This also verifies the index condition for the open comparison of Lemma I.3. Localization for coherent K-groups therefore represents the difference by a class supported on $x_1^r = 0$. The invariant algebra there is finite over x_2^r, x_3, x_4. Coherent localization and finite pushforward thus give possible denominator $(1 - t_2^r)(1-a)(1-w)$. Interchanging the axes gives the same bound with $1 - t_1^r$. The extra factors are coprime.

The spatial factor. We next prove that $[a][w]F_{n,\gamma}/[aw]$ is independent of the spatial weights. On the symplectic subtorus $t_1 t_2 = aw = 1$, every nonempty solid partition has negative trivial multiplicity in the tangent half, as in [6, proof of Prop. 5.1]. A weight restricting trivially there has equal first and second exponents, hence has trivial cyclic colour. Invariant projection preserves this multiplicity, giving a zero in (7), on either square-root lift. The point numerator contains $[t_1 t_2]$ on every chart and the remaining chart denominators are generically nonzero there. The vanishing survives Adams operations. Consequently $J = [a][w]F_{n,\gamma}/[aw]$ is Laurent.

Keep a, t_2 fixed and vary t_1. At zero or infinity the growth exponent of a bracket is half the absolute slope of its weight. The slopes of the mass cancel those of the frame in (7). The self-Hom term has neutral factor $1 - a^{-1}$ and hence zero signed sum of absolute slopes. Each summand is bounded at both ends. On a point chart the spatial weights are u, v, a with $uv = t_1 t_2$, so numerator and denominator in (9) have equal slope sums. These estimates persist under products and Adams operations. Since $[a][w]/[aw]$ also has finite nonzero limits, the Laurent polynomial J is independent of t_1, and likewise of t_2. A determinant-one interchange of the two translation coordinates preserves the global orientation since it lies in the connected group SL_2 acting on that plane. Thus J is symmetric in a, w, and is independent of both since $w = (t_1 t_2 a)^{-1}$. By Lemma I.2, $F_{n,\gamma} = K[y^n] c_{n,\gamma}$ for a scalar $c_{n,\gamma}$.

The scalar. Write $t_j = e^{\epsilon \lambda_j}$, $y = e^{\epsilon m}$ with $\sum_j \lambda_j = 0$, and let $\epsilon \to 0$. The factor $K[y^n]$ tends to $nm(\lambda_3^{-1} + \lambda_4^{-1})$, also after every Adams operation. Equivariant virtual Riemann–Roch and localization [11, Thms. 6.1, 7.1, 7.3] give the cohomological index in this limit. Equivalently in the localized vertex one takes the leading factors $[e^{\epsilon u}] \sim \epsilon u$. Here the fixed loci are proper and reduced, and the insertion rank equals the rank of the half. Liu's cohomological point and cyclic orbifold formulas [9, Cor. 3.10, Thm. 5.14], in the same degree sign, have precisely this primitive for $\gamma = \pm \alpha_{[i,j]}$, and zero otherwise, where α_I denotes an indicator vector. Taking the plethystic logarithm in the counting variables therefore gives $c_{n,\gamma} = 1$ on the signed interval classes and zero on all other classes, including $\gamma = 0$. This proves (5). In particular the transverse $r = 2$ series on a Klein-four branch is now known at generic weights. □

I.5. **Initial values on the resolution.** We now turn from quotients on the orbifold to stable pairs on X. Their invariants will be determined by a recurrence in $n = \chi(F)$. For a fixed nonzero class $\beta = \sum_j \beta_j [C_j]$, the recurrence can stop at $n = \max_j \beta_j$. The initial value there comes from an isomorphism of moduli spaces. We prove it for both groups.

Let R_j denote the character-j line bundle in the pushforward of the universal cluster algebra on Y, and also its pullback to X. In particular $R_0 = \mathcal{O}_X$. Polynomials of colour j generate R_j. For a cyclic group the positive-dimensional fibres of $X \to S$ are reduced chains of projective lines. Indeed, for $a = x_1^r$, $b = x_2^r$, $c = x_1 x_2$ the minimal resolution of $ab = c^r$ has charts (u, v) indexed by $0 \le k < r$, with $(a, b, c) = (u^{k+1} v^k, u^{r-k-1} v^{r-k}, uv)$. The fibre ideal is (uv) on the inner charts and (u) or (v) on the two end charts. The surface McKay intersection pairing gives $\deg_{C_i} R_j = \delta_{ij}$ [4, §6, following Prop. 6.4].

For the Klein group, first fix the fourth coordinate and work on Y. On the cluster chart with basis $1, x_1, x_2, x_3$, the quotient $u_i = x_j x_k / x_i$ is the ratio of two sections of the same character line, where $\{i, j, k\} = \{1, 2, 3\}$. Multiplication gives $x_i^2 = u_j u_k$ and $x_1 x_2 x_3 = u_1 u_2 u_3$. These four invariant monomials generate the invariant algebra. The ideal of the central fibre is therefore $(u_1 u_2, u_2 u_3, u_3 u_1)$. The outer chart in which $x_j x_k$ replaces x_i has coordinates x_j^2, x_k^2, u_i^{-1}. These charts cover the cluster space. If x_i vanishes in a cluster algebra, $x_j x_k$ must span its character line, so x_j and x_k are both nonzero. Otherwise the three x_i themselves give a basis together with 1. The outer chart completes the u_i-axis to $C_i \simeq \mathbb{P}^1$. On the overlap the ith frame changes by $x_j x_k = u_i x_i$, while the other two frames agree. This proves $\deg_{C_i} R_j = \delta_{ij}$ also in this case. Above a nonzero point of the x_i^2-axis the fibre is a line of class $[C_j] + [C_k]$, since on $u_i = 0$ it specializes by $u_j u_k = x_i^2$ to $C_j \cup C_k$. Write $\alpha_I = \sum_{j \in I} [C_j]$. The connected reduced subcurves of the fibres thus have indicator classes α_I, indexed by intervals in the cyclic chain and by nonempty subsets of the three Klein curves.

Lemma I.5 (The saturated face)**.** *For a rank-one stable pair, $0 \le \beta_i \le n$. For each i, retain in Z_{PT} just the terms with $\beta_i = n$ and in Z_{orb} just those with $\gamma_i = n$. These restrictions agree in the orientations of the introduction. Every orbifold exponent retained in this way is effective. The same assertions hold for their plethystic logarithms.*

Proof. Which pairs can occur on the face? Write $\mathcal{O}_Z$ for the image of the section $\mathcal{O}_X \to F$. The image of $\mathcal{O}(S) \to H^0(Z, \mathcal{O}_Z)$ is a finite-dimensional algebra since Z is proper. Multiplication by a composition series of ideals filters $\mathcal{O}_Z$. Each nonzero quotient is generated by a coarse function and annihilated by a coarse maximal ideal, so is $\mathcal{O}_D$ for a subscheme D of a single scheme-theoretic fibre. That fibre is reduced and has dimension at most one. Removing the finite-length part of $\mathcal{O}_D$ gives the structure sheaf of a union of reduced curves. Each connected component has Euler characteristic one, as the normalization sequence shows even at the central triple point, where the three values have to agree. Each component contributes at most one to β_i. Additivity of χ and of the curve cycle, followed by adding the length of $\operatorname{coker}(\mathcal{O}_X \to F)$, gives the bound. Equality forces the section to be surjective. Every nonzero piece is then $\mathcal{O}_D$ for a connected reduced subcurve with i in its indicator class.

Which quotients can occur on the same face? For a G-graded quotient algebra A, multiply A by a composition series of ideals of A^G. A successive quotient is a cyclic module with its specified generator, which we call a pointed quotient. Its invariants have dimension one, since $(JA)^G = J$ for an ideal $J \subseteq A^G$. The condition $\gamma_i = n$ means $\dim A_i = 2n$. Every pointed quotient has dimension at most two in colour i, so all n pieces must have dimension two there.

At a fixed coarse position each such piece is uniquely specified by the set I of its doubled colours. In cyclic type the nonconstant powers of x_1 and x_2 over the singular point occupy the intervals $[1, p]$ and $[r-b, r-1]$, respectively, since $x_1x_2 = x_1^r = x_2^r = 0$. A proportionality between two nonzero powers must relate the two terminal elements, as one sees by multiplying by x_1 and x_2. In that case the intervals meet only at that colour, which has dimension one. Thus if a colour is doubled, the two lists of powers are independent, their intersection I is a nonempty interval, and $p = \max I,\ b = r - \min I$ determine the quotient. At a smooth coarse position the pointed quotient is regular.

For Klein type the two generators for colour k are x_k and x_ix_j. If k is doubled they are independent, so the two other colours occur as well. The squares and triple product act by scalars. The equality $x_k^2(x_ix_j) = (x_ix_jx_k)x_k$ forces both scalars in this equality to vanish. The resulting pointed algebras are as follows, up to permutation, with x_4 fixed at its coarse value.

I	pointed quotient with generator 1
$\{3\}$	$\mathbb{C}[x_1, x_2, x_3]/(x_1^2, x_2^2, x_3^2, x_1x_3, x_2x_3)$
$\{2,3\}$	$\mathbb{C}[x_1]/(x_1^2 - s) \otimes \mathbb{C}[x_2, x_3]/(x_2, x_3)^2$
$\{1,2,3\}$	$\mathbb{C}[x_1, x_2, x_3]/(x_1^2, x_2^2, x_3^2, x_1x_2x_3)$.

Here s is the value of x_1^2. For instance, in the first row any nonzero relation $x_1x_3 = cx_2$ would make $x_1x_2 = c^{-1}x_1^2x_3$ proportional to x_3. Hence $x_1x_3 = x_2x_3 = 0$, and multiplying these vanishings by x_1 and x_2, respectively, shows $x_1^2 = x_2^2 = 0$. In the second row, $x_2x_3 = cx_1$ implies $cx_1x_3 = 0$, so $c = 0$. These quotients occur at exactly the coarse positions of the corresponding connected subcurves. In both actions each pointed quotient with a doubled colour has deviation $+\alpha_I$. In particular the total deviation on the face is effective.

McKay equivalence on the face. Let $\Phi_3 : D^b \operatorname{Coh}(Y) \to D^b_G \operatorname{Coh}(\mathbb{A}^3)$ be the transform with kernel the universal cluster. It is an equivalence by [1, Thm. 1.2], since $\mathbb{A}^3$ is smooth of dimension three and $G \subset \mathrm{SL}_3$. Taking the exterior product of the kernel with the diagonal on the invariant fourth line gives an equivalence Φ from X to $\mathcal{X}$. The cluster kernel and the diagonal have their natural T-linearizations, so this construction also acts on the spatially equivariant categories. The projection formula identifies its jth space of derived sections with $\mathbf{R}\Gamma(X, R_j \otimes F)$ for $F \in D^b \operatorname{Coh}(X)$. In particular $\mathbf{R}\mathrm{Hom}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}, \Phi(F)) = \mathbf{R}\Gamma(X, F)$ since $R_0 = \mathcal{O}_X$. By adjunction and Yoneda, $\Phi(\mathcal{O}_X) = \mathcal{O}_{\mathcal{X}}$, the polynomial ring with invariant generator 1. Thus the map below sends a quotient on the saturated pair face to a pointed quotient on the saturated orbifold face.

$$(\mathcal{O}_X \twoheadrightarrow F) \ \longmapsto\ (\mathcal{O}_{\mathbb{A}^4} \longrightarrow \Phi(F)).$$

To see surjectivity first take $F = \mathcal{O}_D$ for a single connected subcurve. The globally generated line $R_j|_D$ has $H^1 = 0$ and one or two sections according to whether its degree is zero or one. In degree one the polynomial sections restrict to a basepoint-free system on the distinguished $\mathbb{P}^1$ and hence span its two sections. The image is the unique pointed quotient of the prescribed deviation. The assertion for arbitrary objects follows by induction on the filtrations. Extensions of sheaves in degree zero remain in degree zero, and the generators of the graded pieces are coarse functions times the original section. The transform commutes with these functions. The inverse transform gives extensions of pure curve sheaves, which are again pure. The transform and its inverse commute with change of the parameter scheme, by base change for the two kernels. Fibrewise concentration in degree zero gives flat families of sheaves, and Nakayama's lemma gives surjectivity of the frame in families. This proves the isomorphism of the two moduli functors.

For equality of the virtual indices we also need the obstruction maps, the Serre form and the orientation. The kernel commutes with derived base change, so identifies lifts of a pair complex over square-zero extensions in both directions, including their identifications on the base. This identifies the obstruction maps. Put $R = \mathbb{C}[x_1, x_2, x_3, x_4]$. On the tilting generator $\bigoplus_j R_j^\vee$, relative duality over R^G identifies the skew group algebra $R \rtimes G$ with its relative bimodule dual. The transform gives the same bimodule on the quotient stack. The ratio of the two trace identifications is a unit of the centre R^G. On the free locus the transform is descent and the two volumes agree, so the unit is one. The unit splits off the scalar endomorphisms R^G in degree zero on both sides. Their cofibres are the trace-free quadratic theories. The trivial colour of $\Phi(F)$ is $\mathbf{R}\Gamma(F)$, so the tautological insertions also agree. Finally, the product form of the kernel implies that Φ commutes with scaling the fourth operator and the framing map to zero. The two endpoint hyperbolic evaluations correspond under Φ and carry the same degree sign $(-1)^n$.

On each side restriction to the face keeps the zero set of a nonnegative additive function, namely $n - \beta_i$ or $n - \gamma_i$. Only terms on the face can contribute to a product on the face, and the functions scale by k under ψ^k. Restriction therefore commutes with PL in the completed series ring. □

I.6. **An equivariant lift on the isotropic flag.** To move away from the face, we will attach two framing lines to the same target. The proof compares two limits of the resulting index in the relative framing character. Localization initially produces a rational function, and equality of those limits requires regularity. The following lemma gives a criterion for an oriented virtual sheaf with a determinant insertion to lift to equivariant coherent K-theory before inverting characters.

For a torus U acting on M, let $G_0^U(M)$ denote the Grothendieck group of U-equivariant coherent sheaves. We compare classes after extension to the character field, adjoining half-characters when necessary. An orientation fixes a square root of the determinant pairing of a quadratic complex. Its spin gerbe records the further obstruction, in $H^2([M/U], \mu_2)$, to lifting the oriented complex to a spin structure. For an equivariant line N the Kummer boundary ∂N is the class of its gerbe of square roots.

Lemma I.6 (Cancellation on the isotropic flag)**.** *Let a complex torus U act on a quasi-projective scheme M with a U-linearized ample line and an oriented equivariant Oh–Thomas obstruction theory of even virtual rank. Suppose that the spin gerbe of the quadratic complex on $[M/U]$ has class ∂N for a U-linearized line N. Then $\sqrt{N} \star \widehat{\mathcal{O}}_M^{\mathrm{vir}}$ is the localization of a class in the ordinary equivariant coherent group $G_0^U(M)$. On a torus-fixed component the operator $\sqrt{N}\star$ multiplies by the ordinary rank-one square root of the underlying line and the half-power of its character* [11, eq. (128)]. *Tensoring by $\lambda_{-1}(B)$ for an equivariant bundle B preserves this conclusion.*

Proof. Choose an equivariant self-dual resolution $[P^\vee \to W \to P]$ and the cone $C \subset W$ of its two-term truncation. Isotropy of this cone is the consequence of the shifted symplectic structure used in [11, Prop. 4.3]. The linearized ample line allows equivariant resolutions by twisted vector

bundles in the self-dual construction of [11, Prop. 4.1, eq. (112)]. One may first stabilize the resolution by an acyclic hyperbolic summand. Let $\rho : \mathcal{F} \to M$ be the proper isotropic flag bundle of [11, eq. (19)], with positive maximal isotropic bundle $\Lambda \subset \rho^* W$ of rank m. Write τ for the tautological section over $\rho^* C$. Form the class

$$K = (-1)^m c_m(\Lambda^\vee, \tau)[\mathcal{O}_{\rho^* C}] \quad \text{in } G_0^U(\mathcal{F}).$$

The operator in this expression is specialization followed by the Kiem–Li cosection-localized Gysin map in [11, eq. (98)]. Both act on unlocalized coherent K-groups equivariantly [11, §7, following eq. (132)]. The sign in (99) of that paper is $(-1)^{m+|\Lambda|}$, and $|\Lambda| = 0$ on the positive flag for the chosen orientation on each component.

On $[\mathcal{F}/U]$ the spin obstruction of the resolution is $\partial(\rho^* \det P \otimes \det \Lambda^\vee)$. To see this, a spin structure on the three-term resolution amounts to a Clifford module for $\rho^* W$ with prescribed norm line $\rho^* \det P$. This is the Spin^{c} description. The exterior module from the isotropic bundle has norm $\det \Lambda$, and tensoring the module by a line multiplies its norm by the square of that line [7, Prop. 5.9 and proof of Thm. 5.22]. By the hypothesis and the Kummer exact sequence there is consequently a U-linearized line D on $\mathcal{F}$ with

$$D^2 = \rho^*(N \otimes \det P) \otimes \det \Lambda^\vee.$$

On a connected component of $\mathcal{F}^U$ each line is an ordinary line times a character. Formula (128) takes the product of the rank-one square roots in ordinary K-theory and the character half-powers. The former are unique in rank one after inverting two by [11, Lem. 5.1, Rem. 5.2]. Compute the individual half-powers after adjoining half-characters of U. Their product is already the character of D, by its square and torsion-freeness of the lattice. Thus the product of the three root operators for $\rho^* N, \rho^* \det P, \det \Lambda^\vee$ acts by tensoring with D on the fixed locus, and hence on localized $G_0^U(\mathcal{F})$ by concentration. The operators commute with proper pushforward by the fixed-locus definition and the projection formula. Now [11, eqs. (99), (103), (133), Defs. 5.7–5.8] give

$$\sqrt{N} \star \widehat{\mathcal{O}}_M^{\mathrm{vir}} = \rho_*(D \otimes K)$$

after localization. The right side is a proper pushforward of an ordinary equivariant coherent class. This gives the asserted lift. □

I.7. **Translation by a tautological line.** Let $f_{n,\beta}(y) = [q^n Q^\beta] \operatorname{PL} Z_{\mathrm{PT}}$ be the stable-pair primitive with the orientation fixed in the introduction. Give it value zero off the positive effective support, including $n = 0$. The following identity will determine these coefficients for either group. Throughout, square brackets around a character denote the symmetrized factor $[z] = z^{1/2} - z^{-1/2}$.

Lemma I.7 (Translation of the Euler characteristic)**.** *Let $L = \mathcal{O}_X$ or $L = R_i$, put $h = c_1(L) \cdot \beta$, and let y_1, y_2 be independent mass characters. For every $n > 0$,*

$$[y_2^{n-h}] f_{n,\beta}(y_1) = [y_1^n] f_{n-h,\beta}(y_2). \tag{13}$$

In particular there are mass-independent spatial coefficients $c_{n,\beta}$ such that

$$f_{n,\beta}(y) = [y^n] c_{n,\beta}, \qquad c_{n,\beta} = c_{n-\beta_i,\beta} \quad (n > \beta_i).$$

Proof. The source of the shift h is tensoring one of two ordinary pairs by L. Its curve class stays fixed and its Euler characteristic changes by $c_1(L)$ paired with that class. To compare the twisted and untwisted contributions, we realize the two ordinary pairs as the fixed locus of a single auxiliary problem.

The auxiliary moduli and its support map. Let the framing torus $T_f = (\mathbb{C}^*)^2$ act trivially on X and with characters ξ_1, ξ_2 on the two summands of $E = \xi_1 \mathcal{O}_X \oplus \xi_2 L$. At fixed (n, β) write M_E for the moduli of maps $s : E \to F$ to a properly supported pure curve sheaf with zero-dimensional cokernel. Include the empty object when both indices are zero. We use the cover of $T \times T_f$ times the two mass tori, on which the original characters have square roots. The diagonal framing

subtorus acts trivially on the problem, since its scalar on E can also act on F. Put $l = \xi_2/\xi_1$. For the universal target let

$$V_1 = \xi_1^{-1}\mathbf{R}\Gamma_X(F), \qquad V_2 = \xi_2^{-1}\mathbf{R}\Gamma_X(L^{-1} \otimes F).$$

Here and below relative global sections and Hom complexes are pushed to the parameter space of the objects in question.

Compactify Y in a smooth projective toric threefold $\bar{Y}$. Form $W = \mathbb{P}_{\bar{Y}}(\mathcal{O} \oplus K_{\bar{Y}})$ by adjoining the section at infinity D to the total space of $K_{\bar{Y}}$. The projection formula for the canonical bundle gives $K_W = \mathcal{O}_W(-2D)$. Over Y a canonical form has character $t_1t_2t_3 = w^{-1}$, giving the required fourth coordinate of weight w on its total space. Extend L by closing a toric divisor and pull it back to W. For a fixed Hilbert polynomial on this smooth polarized projective fourfold, choose Lin's stability polynomial of degree at least one. His criterion for one-dimensional targets is purity and generic generation, and nondegenerate semistable pairs are stable [8, Lemmas 2.8, 2.10]. The projective moduli and universal pair then exist by [8, Thm. 1.1]. The support condition in X cuts out an open, on which the prescribed numerical class is locally constant. Restricting the GIT polarization gives a linearized ample line on M_E.

For properness over the support coordinates we need $\mathbf{R}\Gamma_X(F)$ to be a vector bundle. Filter the image of s by its two successive source summands. The quotients are $\mathcal{O}$ or L restricted to subschemes of the support. Apply the coarse-ideal filtration in Lemma I.5 to these quotients, with either twist $\mathcal{O}$ or L^{-1}. On the whole reduced fibre each resulting line has degree zero except for at most one component of degree 1 or -1. Such a line has $H^1 = 0$. On a chain remove degree-zero leaves using surjectivity of evaluation at the attaching node. At the triple point the constants on the other two branches supply the two matching conditions. On any quotient of the line the same vanishing follows by the cohomological dimension bound one for the fibre. The cokernel of s has dimension zero. It follows by base change that V_1, V_2 are bundles and $\operatorname{rk} V_1 = n$.

The action of $\mathcal{O}(S)$ on the rank-n space of sections defines a support map

$$M_E \longrightarrow \operatorname{Sym}^n S$$

by its determinant polynomial law, equivalently by its joint generalized eigenvalues with multiplicities on geometric fibres. Every coarse support position occurs since the source lines are globally generated and their image generates F generically. To see that this map is proper, extend a generic pair over a trait in Lin's projective moduli, keeping its support cycle in $\operatorname{Sym}^n S$. The incidence of that cycle has finite support over the trait. Its inverse image under the projective map $X \to S$ is proper over the trait, hence closed also in W over the trait. Flatness keeps the limiting target on this support. The limit therefore belongs to X and retains its numerical class, which can be measured by the extended tautological lines.

The quadratic theory of the framed pair. A member of M_E gives a complex $I = [E \xrightarrow{s} F]$ in degrees $0, 1$, together with its identification with E along D. Consider the derived stack

$$\operatorname{Perf}(W) \times^{\mathbf{R}}_{\operatorname{Perf}(D)} \{E|_D\},$$

the fibre of restriction at this framed object. The target of the mapping stacks, the absolute derived stack Perf of perfect complexes, has a 2-shifted symplectic form [12, Thm. 2.12]. The schemes $W, D, 2D$ are projective, so their stacks of perfect complexes are locally geometric [10, Thm. 3.1, Rem. 3.2], locally of finite presentation since perfect complexes descend through filtered colimits on a finite affine cover. We may restrict to bounded-amplitude opens around the given objects. In [13, Thm. 3.3(1)], the additional effective divisor is zero because $2D$ is anticanonical. The required restriction from $D + 0$ to D is the identity. The theorem therefore supplies a $(2 - \dim W) = -2$ shifted symplectic form on this fibre.

The pair scheme is open in its classical truncation. Indeed, $\mathbf{R}\operatorname{Hom}_W(E, E(-D)) = 0$ since $\mathcal{O}(-D)$ has degree -1 on every projective-line fibre of $W \to \bar{Y}$. Also $\operatorname{Ext}^{\leq 3}_W(F, E(-D)) = 0$ by

codimension three on the smooth fourfold. The pair triangle now gives $\operatorname{Ext}^{\leq 1}_W(I, E(-D)) = 0$. Restricting the Hom complex of I into E to D thus gives a unique lift $I \to E$ of the framing, locally in a classical family. The cone is a flat sheaf on the open where its derived fibres are sheaves. Purity, generic generation and avoidance of the boundary are open there. This recovers the original pair from the framed complex. Generic generation and purity give trivial framed stabilizers. The only possible negative self-extension maps the zero-dimensional cokernel of the section to a subsheaf of the frame, so vanishes. Serre duality now bounds the centred quadratic complex in degrees $[-1, 1]$. It is

$$\mathbf{R}\mathrm{Hom}_W(I, I(-D))[2].$$

Composition takes two morphisms vanishing on D to a morphism twisted by $\mathcal{O}(-2D) = K_W$, where the Serre trace supplies the form. Oh–Thomas's isotropy result applies to such quasi-projective (-2)-shifted symplectic schemes [11, Prop. 4.3 and §4.3].

A global orientation and a Laurent index. Let $p : X \to Y$ forget x_4. On the relative support of a family of targets this map is affine and proper, hence finite. Writing $B = p_*F$, a target is a sheaf on Y with an endomorphism X_4 describing multiplication by x_4. Replace (X_4, s) by (tX_4, ts) for $t \in \mathbb{A}^1$. The map s here is specified by the images of its generators over Y. The underlying flat B remains fixed. On the smooth W this gives a perfect family I_t, and proper pushforward gives a family of quadratic perfect complexes even where I_0 leaves the pair moduli. The spin gerbe of such a complex is defined without a three-term amplitude assumption [7, §4]. At zero, F_0 is supported on the zero section and the frame map vanishes. The zero-section Koszul resolution gives a hyperbolic complex with half

$$J = \mathbf{R}\mathrm{Hom}_Y(B, B)[2] \oplus (V_1 \oplus V_2)[1].$$

Indeed, the self-Hom of the sheaf has two trace-dual normal degrees, the two opposite frame-to-sheaf terms are trace dual, and the frame-to-frame term vanishes by the fibre calculation above. Hyperbolic evaluation gives an orientation and its spin obstruction is $\partial(\det J)$. For the equivariant parameter stack, pullback along product with $\mathbb{A}^1$ induces isomorphisms on μ_2-cohomology in degrees $0, 1, 2$. Over $\mathbb{C}$ this follows from finite-coefficient comparison and contraction of the affine line on a simplicial atlas. The orientation at zero therefore extends uniquely and gives the stated spin class at one.

It remains to identify the line in this obstruction. Apply Φ_3 in families to B and let P_ρ be the perfect complex of sections of its ρ-isotypic summand over the parameter space. Its support on the affine side is finite over the parameter space. In the diagonal Koszul resolution the term for $J' \subseteq \{1, 2, 3\}$ is a geometric character times

$$P_\rho^\vee \otimes P_{\rho + \sum_{j \in J'} \rho_j} \quad \text{with sign } (-1)^{|J'|},$$

where ρ_j is the colour of x_j. For cyclic G, toggle the neutral third coordinate. For Klein G, take the complementary subset, which has the same colour sum since all colours have order two and $\rho_1 + \rho_2 + \rho_3 = 0$. In each case the two perfect bundle factors coincide and the signs are opposite. Thus in $\det \mathbf{R}\mathrm{Hom}_Y(B, B)$ only a geometric character survives. Its square root is defined on our character cover. We have proved $\partial(\det J) = \partial N$ with $N = (\det V_1 \det V_2)^{-1}$. In particular the rank of J is $-(\operatorname{rk} V_1 + \operatorname{rk} V_2)$, since the same cancellation of the alternating Koszul terms gives zero rank for the threefold self-Hom. The quadratic theory has even virtual rank, as required by Lemma I.6.

Apply that lemma with the equivariant ample line already constructed on M_E, and tensor the lift by $\prod_j y_j^{\operatorname{rk} V_j/2} \lambda_{-1}(y_j^{-1} V_j)$. The splitting principle identifies the localized result with the twisted virtual class with the two insertions $[y_1 V_1^\vee][y_2 V_2^\vee]$. Its proper support pushforward can

be followed by the finite map

$$\operatorname{Sym}^n S \;\longrightarrow\; \prod_{j=1}^{4} \operatorname{Sym}^n \mathbb{A}^1$$

whose four components take the elementary symmetric functions separately in $x_1^{m_1}, x_2^{m_2}, x_3^{m_3}, x_4$, where m_j is the order of the colour of x_j. Finiteness follows because each labelled coordinate is integral over its elementary symmetric functions and $\mathcal{O}(S)$ is finite over the four displayed coordinates. Every weight of the target affine space is a nontrivial spatial character. Localization on it only inverts spatial Euler factors. Consequently the index in each degree is Laurent in $l^{1/2}$ over the field of spatial and mass characters. This regularity is the reason for constructing the lift before taking the framing limits.

Localization in the framing torus. Since T_f acts trivially on W, on its fixed locus we have $F = \xi_1 F_1 \oplus \xi_2 F_2$. Generic generation by E and purity exclude any other weight. Writing $F_2 = L \otimes G_2$, the fixed sections are ordinary rank-one pairs with targets F_1 and G_2, either possibly empty. The same decomposition holds in derived deformations. Weight complexes split and any further weight with zero classical fibre vanishes by derived Nakayama.

On D a deformation of a line is a deformation of its determinant. Moreover $\mathbf{R}\Gamma_W \mathcal{O}(-D)[1] = 0$ is the relative tangent complex of $\operatorname{Pic}(W) \to \operatorname{Pic}(D)$ at the chosen line. Thus framing a rank-one complex at infinity formally fixes its determinant. Its trace-free complex is supported on the compact support of the target. Restriction to X identifies this complex and its Serre pairing with the ordinary pair theory. Tensoring the second block by L^{-1} gives the theory of G_2. At $t = 0$ the evaluation of J is the product of the diagonal evaluations and the evaluation on the off-diagonal summands. Placing each exterior basis element beside its trace dual groups the factors in even pairs. This fixes the relative sign on the entire fixed locus. The weight-zero cone is the product of the two diagonal cones. On fully spatially fixed components the oriented square-root Gysin map takes the exterior product as well. Indeed the two tautological isotropic sections lie in separate orthogonal summands. Apply the localized Whitney sum argument of [5, Lem. 4.8], using its K-theoretic isotropic reduction (10.8) in [5, §10.2]. On the successive blowups used in that argument the two section lines split off, and their determinant square roots multiply. The remaining ordinary Koszul Gysin maps compose. Thus the fixed virtual class is the product of the two ordinary pair classes, including on obstructed components.

For the moving part define perfect complexes on this product by

$$A = \mathbf{R}\Gamma_X(L^{-1} \otimes F_1), \qquad B' = \mathbf{R}\Gamma_X(F_2),$$
$$C_{12} = \mathbf{R}\mathrm{Hom}_Y(p_* F_1, p_* F_2), \quad C_{21} = \mathbf{R}\mathrm{Hom}_Y(p_* F_2, p_* F_1).$$

Both threefold Hom complexes have rank zero by Riemann–Roch, since the inputs have dimension at most one. Serre duality gives $C_{21}^\vee = -w^{-1} C_{12}$ in equivariant K-theory. Hence $\det C_{12} = \det C_{21}$, since twisting a rank-zero complex by a character preserves its determinant. The off-diagonal part of J is $l(C_{12} - B') + l^{-1}(C_{21} - A)$. Applying the Oh–Thomas K-theoretic localization formula [11, Thm. 7.3] therefore multiplies the product fixed class by

$$\frac{[l y_2 A^\vee][l^{-1} y_1 (B')^\vee]}{[l(B' - C_{12})][l^{-1}(A - C_{21})]}. \tag{14}$$

The formula applies on fixed components of positive dimension as well. We can first localize in spatial weights and then take the two limits of this factor before pushing forward.

Put $n_1 = \chi(F_1)$, $n_2 = \chi(G_2)$, $h_1 = c_1(L) \cdot [F_1]$ and $h_2 = c_1(L) \cdot [G_2]$. Riemann–Roch in dimension one gives $a = \operatorname{rk} A = n_1 - h_1$ and $b = \operatorname{rk} B' = n_2 + h_2$. At infinity the first bracket in the numerator of (14) contributes the leading determinant $\det(y_2 A^\vee)^{1/2}$, and the second contributes $(-1)^b \det(y_1 (B')^\vee)^{-1/2}$, after removing the powers of l. The denominator contributes $\det(B' - C_{12})^{1/2}$ and $(-1)^a \det(A - C_{21})^{-1/2}$. The ranks cancel the powers of l, and the equality of

the cross-Hom determinants cancels the remaining bundle factors. Applying the same computation at zero gives

	$l \longrightarrow \infty$	$l \longrightarrow 0$
(14)	$(-1)^{a+b} y_2^{a/2} y_1^{-b/2}$	$(-1)^{a+b} y_2^{-a/2} y_1^{b/2}$.

The leading determinants are units on the fixed base by the splitting principle. Multiply the total index by the numerical sign $(-1)^{\mathrm{rk}\, V_1 + \mathrm{rk}\, V_2} = (-1)^{n_1+n_2+a+b}$. The two diagonal factors in both limits now carry their ordinary signs $(-1)^{n_1}$ and $(-1)^{n_2}$.

The primitive identity. The Laurent polynomial in each total degree is bounded at both ends and hence constant. Its two limits give the same product of rank-one indices, with the mass rescalings displayed above. To write this as a series identity, for $\varepsilon = \pm 1$ define two monomial substitutions on the completed series ring with the two masses. With $h_\alpha = c_1(L) \cdot \alpha$, they are

$$S_1^\varepsilon(q^k Q^\alpha) = q^k Q^\alpha\, y_2^{\varepsilon(k-h_\alpha)/2},$$
$$S_2^\varepsilon(q^k Q^\alpha) = q^{k+h_\alpha} Q^\alpha\, y_1^{-\varepsilon(k+h_\alpha)/2}.$$

Both fix the coefficient characters. The index shift in S_2 is $\chi(F_2) - \chi(G_2) = h_2$. In particular these substitutions are continuous on the effective pair subring, since $h_\alpha \geq 0$. Writing $Z_{\mathrm{PT}}(y_j)$ for the ordinary pair series with mass y_j, the equality of the limits is

$$S_1^+ Z_{\mathrm{PT}}(y_1)\, S_2^+ Z_{\mathrm{PT}}(y_2) = S_1^- Z_{\mathrm{PT}}(y_1)\, S_2^- Z_{\mathrm{PT}}(y_2).$$

Both substitutions commute with the Adams operations by additivity in (k, α). Applying PL and comparing $q^n Q^\beta$ gives (13). For $L = \mathcal{O}$ this says that $f_{n,\beta}(y_1)/[y_1^n] = f_{n,\beta}(y_2)/[y_2^n]$. Independence of the two masses gives $c_{n,\beta}$. For $L = R_i$ one has $h = \beta_i$. If $n > \beta_i$ both brackets are nonzero and cancellation gives the translation. At $n = \beta_i$ the identity reads $0 = 0$. □

Completion of the proof of Theorem I.1. The same uniqueness argument will be used in Part II. For an effective $\beta \neq 0$ put $d = \max_j \beta_j$ and choose any i attaining this maximum. Each use of Lemma I.7 at $n > d$ lowers the index by d. If the terminal index lies in $0 < n < d$, the support bound of Lemma I.5 forces its primitive to be zero. Otherwise the terminal index is d. This proves

$$c_{n,\beta} = \begin{cases} c_{d,\beta}, & d \mid n, \\ 0, & d \nmid n \end{cases} \qquad (n > 0). \tag{15}$$

At $n = d$ the saturated face lemma identifies $[y^d] c_{d,\beta}$ with the orbifold primitive in the same degree and class. The point primitive has curve class zero. Formula (5) thus gives $c_{d,\beta} = K$ precisely for $d = 1$ and β the indicator of an interval, with all other initial values zero. There is no nonempty pair of class zero. Substituting in (15) proves (4), and plethystic exponentiation of (4) and (5) gives (3). □

In the additive limit $t_j = e^{\epsilon \lambda_j}$, $y = e^{\epsilon m}$, define $\theta = m(\lambda_3^{-1} + \lambda_4^{-1})$. For the resolution, [2, Thm. 6.4] identifies this limit of each K-theoretic index with the cohomological invariant in the same orientation. We apply it to $L = \mathcal{O}_X$ since the torus preserves the volume form and the fixed loci of the resolved moduli are proper as verified in the introduction. On the orbifold the same limit follows directly from (7), by replacing each moving bracket with its leading linear Euler weight. Thus the cyclic stable-pair series is

$$Z_{\mathrm{PT}}^{\mathrm{coh}}(q, Q) = \prod_{1 \leq i \leq j < r} \prod_{\ell \geq 1} (1 - q^\ell Q_{ij})^{-\ell\theta}. \tag{16}$$

Indeed $\psi^k(K[y^\ell]) \to \ell\theta$ for every $k \geq 1$. Only finitely many terms contribute to each counting degree, so the limit can be taken term by term in the exponential. The normalized cohomological orbifold series is the product of (16) with its $Q \mapsto Q^{-1}$ transform. For example, when $r = 2$

the coefficient of q^nQ^d in the ordinary logarithm of the pair series is $n\theta/d^2$ if $d \mid n$ and is zero otherwise $(n, d > 0)$.

Part II. The Klein four group

For $G = (\mathbb{Z}/2)^2$ the singular locus of $S = \mathbb{A}^4/G$ consists of three planes meeting along the fourth coordinate line. At a general point of each plane the stabilizer has order two. Thus Part I already determines the transverse orbifold theory along all three branches. A different input, Young's coloured plane-partition formula and Bryan's numerical DT sign [16, Thms. 1.5, A.3], determines a threefold specialization at their intersection. The problem is to recover the fourfold series there with its full dependence on the equivariant weights. We do this by bounding the poles and the growth of each primitive. On the resolution the result gives the initial values for the two-framing recurrence, which then determines the stable-pair series in the interior.

Denote the three nontrivial colours by ρ_1, ρ_2, ρ_3, with x_i of colour ρ_i. For $\varnothing \neq I \subseteq \{1, 2, 3\}$ let $\alpha_I = \sum_{i\in I}[C_i]$ and $Q_I = \prod_{i\in I} Q_i$. Write $\widehat{i} = \{1, 2, 3\} \setminus \{i\}$. The two types of spatial coefficient in this case are

$$B_i = A_{\widehat{i}} = \frac{[t_i^2 w]}{[t_i^2][w]}, \qquad A_I = -\frac{1}{[w]} \quad (|I| = 1, 3).$$

Theorem II.1. *In the orientations of the introduction,*

$$\mathrm{PL}\, Z_{\mathrm{PT}} = H(q, y) \sum_{\varnothing\neq I\subseteq\{1,2,3\}} A_I Q_I, \tag{17}$$

$$\mathrm{PL}(Z_{\mathrm{orb}}/Z_0) = H(q, y) \sum_{\varnothing\neq I\subseteq\{1,2,3\}} A_I (Q_I + Q_I^{-1}). \tag{18}$$

II.1. **Removing the contributions of the three branches.** Put $U_i = x_i^2$ and $b = x_1x_2x_3$. A monomial is G-invariant precisely when its first three exponents have the same parity. Hence

$$S = \operatorname{Spec} \mathbb{C}[U_1, U_2, U_3, b, x_4]/(b^2 - U_1U_2U_3).$$

The ith branch plane Π_i has coordinates U_i, x_4 and equations $b = U_j = U_k = 0$, where $\{i, j, k\} = \{1, 2, 3\}$. Its equivariant Euler character as a structure sheaf is $1/((1 - t_i^2)(1 - w))$. Let $v = (n, n+\gamma_1, n+\gamma_2, n+\gamma_3)$ be a colour vector of invariant rank $n > 0$. We will prove that subtracting the three transverse cyclic primitives leaves only a possible pole in the common line direction. More precisely, define

$$R_{n,\gamma} = [q^nQ^\gamma]\, \mathrm{PL}(Z_{\mathrm{orb}}/Z_0) - [y^n] \sum_{\{i:\, \gamma=\pm\alpha_{\widehat{i}}\}} B_i. \tag{19}$$

Write $z_j = t_j^{1/2}$ for $j = 1, 2, 3$. The rational character ring of the covering volume and mass tori before localization is $R_{\mathrm{ch}} = \mathbb{Q}[z_1^{\pm1}, z_2^{\pm1}, z_3^{\pm1}, y^{\pm1/2}]$, with $w^{1/2} = (z_1z_2z_3)^{-1}$. In the following comparison $G_0(Z)$ denotes the rational group $G_0^{\mathbb{T}}(Z)$ of equivariant coherent sheaves. The pole bound to be proved is

$$(1 - w)R_{n,\gamma} \in R_{\mathrm{ch}}. \tag{20}$$

By Lemma I.3 the orbifold primitive is the Euler character of a class on S. The morphism $S \to \mathbb{A}^4$ given by (U_1, U_2, U_3, x_4) is finite. Pushforward followed by a graded free resolution on $\mathbb{A}^4$ gives denominator dividing $(1 - w)\prod_i(1 - t_i^2)$. The subtractions in (19) also come from $G_0(S)$. For $\gamma = 0$ subtract the proper pushforward of (12) from X to S. For $\gamma = \pm\alpha_{\widehat{i}}$ subtract $[y^n](t_i^2w - 1)\mathcal{O}_{\Pi_i}$. Its Euler character is $[y^n]B_i$. We show that these subtractions remove each factor $1 - t_i^2$.

Fix i and choose an ordering j,k of $\widehat{i}$. On $U_i \neq 0$ write $G = K_i \times H_i$ with $K_i = \ker \rho_j$ and $H_i = \ker \rho_i$. The first factor acts freely. Taking its quotient gives coordinates

$$(U_i, P, A, x_4) = (x_i^2, x_k/x_i, x_j, x_4), \qquad (U_i, P, A, x_4) \longmapsto (U_i, -P, -A, x_4)$$

for the residual involution. The resulting coarse open embeds in the full cyclic quotient

$$S_i = \mathbb{A}^1_{U_i} \times (\mathbb{A}^2_{P,A}/H_i) \times \mathbb{A}^1_{x_4}.$$

In particular we have the two open immersions $S \supset \{U_i \neq 0\} \cong S_i \cap \{U_i \neq 0\} \subset S_i$. Regular clusters descend on this open through the finite étale K_i-cover. Pullback and descent are inverse for the equivariant ideals, and the pullback of a regular H_i-cluster has the regular G-representation. This also identifies the two resolutions and their point classes over the common open.

The identification for the orbifold primitive requires the disjoint factorization maps. First observe that invertible multiplication by x_i forces $v_g = v_{g+\rho_i}$ in additive colour notation. The possible indices are precisely

$$\gamma = h\alpha_{\widehat{i}}, \qquad (v_0^{\mathrm{res}}, v_1^{\mathrm{res}}) = (n, n+h), \quad |h| \leq n.$$

Indeed P, A generate the odd part over the invariant algebra, so its rank lies between zero and $2n$. At fixed $U_i \neq 0$, take blocks with bases $\{1\}, \{1, P\}, \{1, P, A\}$, in the last two cases setting all quadratic odd products to zero. Placing n such blocks at distinct fourth coordinates realizes every rank between the bounds. Thus the indices on the open form the intersection of the ambient index semigroup with a sublattice. If a sum and one summand belong to this intersection, so does the other summand whenever it is an ambient index. Lemma I.4 identifies the oriented coherent systems and their disjoint-support isomorphisms on the common open. These identifications apply to every surviving indexed partition in the open-restriction argument of Lemma I.3, giving equality of the corresponding primitive classes. For indices absent on the open the restriction is zero by [14, Rem. 3.24].

The cyclic formula (5) at $r = 2$ computes a character on the full residual affine quotient S_i. Its translation weights are t_i^2, w. It gives the branch character $[y^n]B_i$ exactly for $h = \pm 1$, the point character for $h = 0$, and zero for all other h. Indeed the independent residual half-characters pull back to $z_i^2, z_k/z_i, z_j$. Adjoining z_i makes R_{ch} free of rank two over the residual Laurent ring, with the same mass variable. Thus the cyclic formula pulls back along the corresponding isogeny of tori.

To use this character identity on an open it must first be an equality of classes at the prime in question. Let $\mathfrak{p}$ be a height-one prime of R_{ch} over $1 - z_i^4 = 1 - t_i^2$. The concentration subgroup of the covering torus is cut out by the characters χ with $1 - \chi \in \mathfrak{p}$. It fixes U_i and contains the subtorus $z_i = 1$ with z_j, z_k varying. Each of P^2, A^2, PA, x_4 has a nontrivial weight on that subtorus, so the reduced fixed locus in S_i is the U_i-axis. Thomason's concentration theorem [15, Thm. 2.1] applies to this diagonalizable action on the affine scheme S_i. Pushforward and equivariant homotopy invariance give isomorphisms

$$(R_{\mathrm{ch}})_{\mathfrak{p}} \;\cong\; G_0(\mathbb{A}^1_{U_i})_{\mathfrak{p}} \;\xrightarrow{\sim}\; G_0(S_i)_{\mathfrak{p}}.$$

After passing to the fraction field, concentration at the origin detects classes by their Euler character. The module at $\mathfrak{p}$ is torsion-free, so the cyclic primitive class equals the stated subtraction already there. The punctured axis alone has coherent group $R_{\mathrm{ch}}/(1 - t_i^2)$ by the exact sequence for its origin. Its character over the fraction field carries no information at this prime.

Restricting the equality of classes to $U_i \neq 0$ and using the factorization comparison now shows that the difference on S is represented at $\mathfrak{p}$ by a class supported on $U_i = 0$. The other branch planes already restrict to zero on this open. The reduced complement is finite over the remaining two squares and x_4. By coherent localization and finite pushforward its character is regular at $\mathfrak{p}$. This works separately for each prime and each i, proving (20).

II.2. **The primitive on the common line.** Lemma I.2 gives $R_{n,\gamma} = [y^n]r_{n,\gamma}(t)$ with $r_{n,\gamma}$ independent of the mass. Extracting the coefficient of $y^{n/2}$ in (20) shows $(1-w)r_{n,\gamma}$ is a Laurent polynomial in the spatial half-characters. We shall show that there are constants a, b, c depending on the index for which

$$r_{n,\gamma} = \begin{cases} c/[w], & \gamma \neq 0, \\ (a+bw)/(1-w), & \gamma = 0. \end{cases} \tag{21}$$

First vary the weights along any one-parameter subgroup fixing w, keeping the mass and the constant multipliers of the spatial weights generic. A bracket of a character of slope s has growth exponent $|s|/2$ at either end of the parameter. For this calculation use the half of the tangent character in (7) obtained by omitting a spatial coordinate instead of x_4. Both halves have the same symmetrization. Their difference is a sum of opposite pairs $\chi - \chi^{-1}$, which change a bracket only by a sign. In this half the self-Hom character has a factor $1 - w^{-1}$ outside the invariant part. Its two translates have the same slopes and opposite multiplicities. The invariant framing space and the insertion cancel in the absolute-slope sum as well. The orbifold vertex is therefore bounded at both ends. The point kernel (9) is bounded, because the three spatial slopes in each chart sum to zero, and the branch kernels B_i are bounded. Products, finite sums in fixed degree and Adams operations preserve these bounds. A Laurent polynomial bounded at both ends of every cocharacter fixing w has all exponents on the line spanned by the character $w^{1/2}$. It follows that $(1-w)r_{n,\gamma}$ is Laurent in $w^{1/2}$ alone.

Next vary t_j alone among t_1, t_2, t_3, so w has slope -1. We claim that for a nonregular vector v one can choose j for which the remainder decays at both ends. Let g_j be the nonidentity element fixing x_j and let $\{k, l\} = \widehat{j}$. For a solid partition π write $\rho(p)$ for the colour of the box $p = (p_1, p_2, p_3, p_4)$ and put

$$D_j(s) = \sum_{p \in \pi} \rho(p)(g_j) s^{p_j - p_4}, \qquad b_j(v) = D_j(1) = v_0 + v_{\rho_j} - v_{\rho_k} - v_{\rho_l}.$$

Here s records the slope. After cancellation of the insertion and the framing, the remaining character for the slope calculation is $(\lambda_{-1} D' V^\vee V)^G$, with D' the three spatial directions as in (7). Project onto invariants by averaging the trace over the four group elements. The factor $(1-\rho_k)(1-\rho_l)$ vanishes except at g_j, where its value is four. The result is $(1 - s^{-1})D_j(s)D_j(s^{-1})$. If d_a is the coefficient of s^a in $D_j(s)D_j(s^{-1})$, then $d_a = d_{-a}$, and the growth exponent at either end is

$$\frac{1}{2}\sum_a d_a(|a| - |a-1|) = -\frac{1}{2}d_0 = -\frac{1}{2}\sum_h \big([s^h]D_j(s)\big)^2.$$

Thus every summand is bounded, with strict decay if $b_j(v) \neq 0$. That conclusion also holds for the coefficient of the plethystic logarithm. Indeed b_j is additive on products and scales under Adams operations, so a nonzero total b_j requires a strictly decaying factor.

The three b_j are the nonidentity rows of the character table of G. They all vanish precisely at $v = (n, n, n, n)$, or $\gamma = 0$. For $\gamma = \pm\alpha_{\widehat{j}}$ choose its complementary j. Then $b_j \neq 0$ and the slopes in the subtracted B_j are $1, 2, -1$ for $t_j^2 w, t_j^2, w$, giving strict decay there too. Thus for $\gamma \neq 0$, $r_{n,\gamma}(w)$ decays at zero and infinity. The only half-integral exponent strictly between 0 and 1 in its numerator $(1-w)r_{n,\gamma}$ is $1/2$. This proves the first line of (21).

For $\gamma = 0$ the point term is bounded in this variation. The three spatial slopes on the central chart are $(-1, 1, 1)$ up to permutation, and the three outer charts have slopes $(0, 0, 1)$ or $(2, 0, -1)$. These follow from the central coordinates $x_k x_l / x_j$ and the outer coordinates described before Lemma I.5. Substitution in (9) gives a nonpositive growth exponent on each chart. In (6) the only spatial half-power is $w^{N^2/2}$ with $N = \sum_g v_g$, since the four endomorphism colour ranks sum to N^2. Its parity is N modulo two, which is additive in the degree and scales under Adams operations. The regular degree has $N = 4n$ and the point class (12) has integral spatial powers.

In this degree the Laurent numerator therefore has integral exponents in w. Boundedness at both ends gives the second line of (21).

II.3. **Evaluation at the threefold specialization.** It remains to determine the constants in (21). First set $y = w$ in the fourfold vertex with generic spatial weights. A solid partition extending in the fourth direction contains the box of character w and trivial colour. Its insertion vanishes. On a surviving plane partition $\mu_\pi = 0$, and the ideal triangle and threefold Serre duality give virtual tangent $\mathsf{H}(V) - wV_0^\vee$ on $[\mathbb{A}^3/G]$. The ratio of brackets in (7) is thus its symmetrized threefold DT weight, multiplied by our degree sign $(-1)^n$.

Now take $w^{1/2} \to 1$, keeping two spatial parameters generic. The plane-partition Zariski tangents have no invariant under the threefold volume torus. Taking G-invariants retains this property, as used in [16, proof of Thm. A.3]. Thus each threefold fixed contribution has a limit on this lift. Serre duality pairs a tangent weight with its inverse obstruction weight, giving the numerical DT sign. Bryan's sign changes the three nontrivial colour variables to their negatives. The factor $(-1)^n$ changes q_0 to $-q_0$, so $q = q_0q_1q_2q_3$ is unchanged. To see what the limit measures, write $M(z,q) = \prod_{\ell\geq 1}(1-zq^\ell)^{-\ell}$ and $\widetilde{M}(z,q) = M(z,q)M(z^{-1},q)$. Young's product [16, Thm. 1.5] for these colours, with Bryan's sign substitution [16, Thm. A.3] and our degree sign, gives

$$Z_{\mathrm{orb}}\big|_{y=w,\ w^{1/2}\to 1} = M(1,q)^4 \frac{\prod_{|I|=2}\widetilde{M}(Q_I,q)}{\prod_{|I|\in\{1,3\}}\widetilde{M}(Q_I,q)}.$$

The coefficient of $q^n z$ in the primitive for $M(z,q)$ is n. Consequently the specialized primitive has coefficient $4n$ in regular degree, $+n$ on the signed doubletons, and $-n$ on the signed singletons and triple. These specializations commute with Adams operations on the specified lift. Each of the four point charts in (9) contributes n in the limit, and $[w^n]B_i \to n$. Subtracting these values in (19) leaves exactly the signed singleton and triple terms. Since $[w^n]/[w] \to n$, the first line of (21) has $c = -1$ on those terms and $c = 0$ in all other nonregular degrees.

In regular degree the vanishing specialization yields $a+b = 0$, so $r_{n,0}$ is a constant. Simultaneous inversion of all spatial and mass characters fixes (7), since its insertion and half have equal rank. It also fixes the point contribution by (9). Since inversion negates $[y^n]$, it must negate $r_{n,0}$. The constant is zero. This proves (18) at generic characters and mass.

II.4. **From the boundary to the stable-pair interior.** To finish the proof of Theorem II.1, fix an effective curve class $\beta \neq 0$. Write $f_{n,\beta}(y) = [q^nQ^\beta]\,\mathrm{PL}\,Z_{\mathrm{PT}}$. The two-framing construction of Lemma I.7 expresses $f_{n,\beta}(y) = [y^n]c_{n,\beta}$ and gives

$$c_{n,\beta} = c_{n-d,\beta} \qquad (n > d), \qquad d = \max_i \beta_i.$$

Indeed we take for the second source a tautological line R_i with $\beta_i = d$. There are two possible stopping points for this descent. If $d \nmid n$, the last positive index is strictly less than d and its coefficient vanishes by the support bound in Lemma I.5. If $d \mid n$, the last positive index is $n = d$. At that index the two-mass identity (13) reads $0 = 0$. This is the second stopping case in (15). Its initial value comes from the oriented McKay comparison of Lemma I.5.

More explicitly, on $n = \beta_i$ all orbifold classes are effective and the primitive agrees with the pair primitive. The face can be taken before or after PL because $n - \beta_i$ is nonnegative on the support, additive on products and homogeneous under Adams operations. The point term has curve class zero. Formula (18) therefore gives the complete initial data

$$c_{d,\beta} = \begin{cases} A_I, & d = 1, \quad \beta = \alpha_I, \quad i \in I, \\ 0, & \text{otherwise.} \end{cases}$$

This determines the primitive at every interior index by induction. In particular only the seven indicator classes have nonzero pair primitives, and in those classes $d = 1$ so their values propagate

to every $n \geq 1$. Higher curve multiplicities enter the pair partition function through the plethystic exponential. This proves (17). Applying PE to (18) gives its product with the inverse-class transform, proving the correspondence (3).

II.5. **Cohomology and coefficient extraction.** For additive weights $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$ and mass m, set

$$\theta_I = \begin{cases} m((2\lambda_i)^{-1} + \lambda_4^{-1}), & I = \widehat{i}, \\ -m/\lambda_4, & |I| = 1 \text{ or } 3. \end{cases}$$

With $t_j = e^{\epsilon\lambda_j}$, $y = e^{\epsilon m}$ one has $\psi^k([y^\ell]A_I) \to \ell\theta_I$ for every $k \geq 1$. Indeed $[e^{\epsilon x}] \sim \epsilon x$, and replacing ϵ by $k\epsilon$ leaves this limit unchanged. Each fixed counting degree receives finitely many terms. The same cohomological limit used in (16) therefore gives

$$Z_{\mathrm{PT}}^{\mathrm{coh}}(q,Q) = \prod_{\varnothing \neq I \subseteq \{1,2,3\}} \prod_{\ell \geq 1} (1 - q^\ell Q_I)^{-\ell\theta_I},$$
$$Z_{\mathrm{orb}}^{\mathrm{coh}}/Z_0^{\mathrm{coh}} = Z_{\mathrm{PT}}^{\mathrm{coh}}(q,Q)\, Z_{\mathrm{PT}}^{\mathrm{coh}}(q,Q^{-1}).$$

The two group formulas also give a recursion for individual coefficients at the original generic weights. Let $\mathcal{I}$ be the nonempty intervals of the cyclic chain or the nonempty subsets of the three Klein colours, respectively. Use $A_I = K$ of Theorem I.1 in the first case and A_I of Theorem II.1 in the second. Set $N_{n,\gamma} = [q^n Q^\gamma](Z_{\mathrm{orb}}/Z_0)$. The ordinary logarithmic derivative $q\,\partial_q \log(Z_{\mathrm{orb}}/Z_0)$ multiplied by Z_{orb}/Z_0 yields

$$\begin{aligned} nN_{n,\gamma} &= \sum_{u=1}^{n} \sum_{k|u} \frac{u}{k}\, [y^u] \sum_{I \in \mathcal{I}} \psi^k(A_I)\big(N_{n-u,\gamma-k\alpha_I} + N_{n-u,\gamma+k\alpha_I}\big), \\ N_{0,\gamma} &= \mathbf{1}_{\gamma=0}. \end{aligned} \tag{22}$$

Here $u = k\ell$ is the total counting degree of the kth Adams operation on an index-ℓ primitive. Every term on the right has smaller remaining degree, so these initial values determine all coefficients uniquely. Keeping only the shifts $\gamma - k\alpha_I$ gives the recurrence for Z_{PT}. For example,

$$[q^2 Q_1 Q_2 Q_3] Z_{\mathrm{PT}} = -\frac{[y^2]}{[w]} - \frac{[y]^2}{[w]}(B_1 + B_2 + B_3).$$

The first term is the triple-class primitive. The second adds the three ways to split this class into a single curve and its complementary pair.

There are two direct checks in the Klein-four interior. At $n = 2$ start with the empty solid partition and successively adjoin a box whenever all its predecessors are present. For each $\gamma = \pm\alpha_I$ discard diagrams exceeding the target $(2, 2+\gamma_1, 2+\gamma_2, 2+\gamma_3)$ in any colour. The resulting lists have 14, 12 and 13 diagrams per sign and subset of size one, two and three, respectively. Clearing bracket denominators in (7) gives, for all fourteen exponents, the same Laurent numerator as (22) multiplied by the point series (9).

On the resolution, fixed pairs on the reduced triple curve also test the contribution of positive-dimensional fixed loci. Purity identifies the framed sheaf with a fractional ideal on the three branches. At the central point only the constants have a weight in common. When no branch has a pole there, a two-plane of constants in $\mathbb{C}^3$ containing the diagonal varies in a $\mathbb{P}^1$. The independent localization sum in this class agrees with the one-sided recurrence through $n = 10$ at two choices of characters. At $n = 10$ it includes 1,207 points and 45 projective lines. Writing $c_i = t_j t_k / t_i$ for the central curve weights, the total index modulo 1000000007 at $(c_1^{1/2}, c_2^{1/2}, c_3^{1/2}, y^{1/2}) = (2, 3, 5, 7)$ and $(3, 5, 7, 11)$ is respectively 587040426 and 796482852.

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