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Spin model unification of Hadamard matrix invariants

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Abstract

We unify the 5 types of invariants of the complex Hadamard matrices: (1) the algebraic invariants, coming from spin models, subfactors, planar algebras and quantum permutation groups, (2) the analytic invariants, coming from the glow of the Ising arrays and of the planar rotators, (3) the geometric invariants, such as the defect, (4) the combinatorial invariants, coming from the quantum isomorphisms and the closed diagrams in spiders, and (5) the arithmetic invariants, such as the trace, which live on the conjugacy class. The first 4 types are explicit functions of a single complete invariant, the partition function of the Jones spin model of the matrix on bicolored graphs, and the arithmetic invariants come from the same model on directed graphs. The meeting point of the algebraic and analytic invariants is the Jones transfer matrix T_2, whose top eigenspace is seen by the subfactor, while its traces twisted by a crossing, and squared, appear in the complex and real glow, at orders 8 and 16. The defect is the multiplicity of the eigenvalue 1 of the Berezin matrix, minus $2N-1$, and the flux harmonics are invariants of the strong quantum equivalence, the doubled rectangle R_2 being seen by the glow, at order 8, and by the defect of the McNulty-Weigert matrices.

To the memory of Vaughan Jones

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Introduction

A complex Hadamard matrix is a matrix $H \in M_N(\mathbb{T})$ whose rows are pairwise orthogonal, that is, $HH^* = N1_N$. The equivalence of such matrices consists of the permutations of the rows and columns, and of the multiplications of the rows and columns by numbers in $\mathbb{T}$.

The basic example is the Fourier matrix $F_N = (w^{ij})$, with $w = e^{2\pi i/N}$. The real ones go back to Sylvester [98] and Hadamard [53], and exist only in the orders $N = 1, 2$ and $N = 0(4)$, the converse being the Hadamard conjecture. We refer to [100] for the general theory.

There are several types of invariants for this equivalence, and for the finer relation of conjugacy, coming from quite different points of view, and which have been developed quite independently. Being a bit loose, as these invariants can mean many things, the central ones are as follows:

1. The algebraic invariants. As found by Jones [63], a Hadamard matrix gives a spin model, and so a commuting square, a subfactor and a planar algebra [67], whose k-boxes are the fixed points $Fix(u^{\otimes k})$ of a quantum permutation group $G_H \subset S_N^+$ [6, 11], the main question being whether $G_H = S_N^+$, a property called wildness.

2. The analytic invariants. The glow is the law of the sum of entries of D_aHD_b, with D_a, D_b random diagonal matrices, with signs or phases on the diagonal, and this is the density of states of a system of Ising spins, or of planar rotators, coupled through H [35, 90], with extensions to the almost Hadamard matrices and the block designs in [44].

3. The geometric invariants. These include the defect of Tadej and Zyczkowski [101], which bounds the dimension of the manifold of Hadamard matrices around H, and whose vanishing means that H is isolated [100], and more generally the local structure of this manifold, as described by the higher order deformations of H.

4. The combinatorial invariants, coming from the quantum symmetries of H and from the various notions of quantum equivalence, with quantum isomorphisms of graphs, closed diagrams in spiders and nonlocal games, as developed by Gromada [49], by Chan and Martin [32], and by Brannan, Gromada, Hernández Palomares and Priebe [26].

5. The arithmetic invariants, such as the trace, which for the Fourier matrix is the Gauss sum, and more generally the spectrum, which for the Butson matrices consists of Weil numbers [36]. These are not invariant under the equivalence, and live on the conjugacy class of H, being seen by the periodic chains having H as transfer matrix.

The purpose of this paper is to unify these invariants. The unifying object is the Jones spin model of H, with the N spin states being the indices of H, and with H itself as matrix of Boltzmann weights. For a bipartite multigraph Γ with signed edges, balanced in the sense that each vertex has as many positive as negative edges, the partition function of this model on Γ is

$$S_\Gamma(H) = \sum_{x,y} \prod_{e \in E_+} H_{x(e)y(e)} \prod_{e \in E_-} \bar{H}_{x(e)y(e)}$$

the sum being over the labelings x of the row vertices, and y of the column vertices, by $1, \ldots, N$. Our first result is as follows:

Theorem A. *The numbers $S_\Gamma(H)$ form a complete invariant for the equivalence of complex Hadamard matrices, and the algebraic, analytic, geometric and combinatorial invariants are explicit functions of them, as follows:*

1. *The dimensions of the planar algebra are the multiplicities of the eigenvalue 1 of the Jones transfer matrices T_k, and so are the limits of their power traces, which are partition functions.*

2. *The glow moments are sums over all bipartite graphs, weighted by the cumulants of the gauge group, real or complex.*

3. *The defect is the multiplicity of the eigenvalue 1 of the Berezin matrix Λ_H, minus $2N-1$, and the power traces of Λ_H are partition functions on necklaces of rectangles.*

4. *The closed diagrams in complementary spiders, which compute the combinatorial invariants, are the numbers $N^{-|E|/2}S_\Gamma(H)$.*

The completeness is a general fact from the invariant theory of compact groups, and the point is in the explicit formulas. The graphs in (1) are not planar in general, as for H real the planar graphs give universal values. The statement (3) is a reformulation of a result of Tadej, see [8], and makes the defect an eigenvalue multiplicity, exactly as the dimensions of the planar algebra. The second one comes from a simple dictionary: the glow is the law of $\Omega = \langle Hb, a\rangle$, which is the partition function of the Jones model on a single edge, computed in the gauge (a, b). Thus the arrays are the Jones model with its gauge made dynamical. Also, the Reidemeister II move, which is the Hadamard condition, becomes the maximal and uniform frustration of the arrays.

The algebraic invariants occupy the first half of the paper. Their central object is the sequence $c_k = \dim Fix(u^{\otimes k})$, which is the sequence of moments of the main character of G_H, and whose computation is notoriously difficult. For the deformations of the Fourier matrices the quantum group can be computed [10, 12], the numbers c_k being then given by the blowup of the spectral measure over a compact abelian group [9], and for the other matrices the general expectation is that $G_H = S_N^+$, that is, $c_k = Cat_k$ for any k, which means that the standard invariant of the subfactor is Temperley-Lieb. This was checked numerically in [12] for many matrices, up to $k = 9$ in the best cases, and proved up to $k = 18$ for the generic Karlsson matrices in [4]. The first proofs are due to Bisch and Nicoara [20, 21], as follows:

Theorem B. *The subfactor associated to a complex Hadamard matrix H of order $N > 4$ has Temperley-Lieb standard invariant, and so $G_H = S_N^+$, in the following cases:*

1. *H is real, and $N = 4(8)$.*
2. *H has entries in $\mathbb{Q}(i)$, and $N = 2(4)$.*
3. *H has cube roots of unity as entries, and $N = 3(9)$ or $N = 6(9)$.*
4. *H is a Petrescu matrix of order 7, outside a countable set of algebraic parameters.*

Observe that the conditions on N exclude the Diţă matrices, which are never wild, as explained in Section 3. For instance a real Diţă matrix of order N contains real Hadamard submatrices of orders M and N/M, with $1 < M < N$, and since the orders of the real Hadamard matrices are $1, 2$ and the multiples of 4, this is impossible at $N = 4(8)$, $N > 4$. In fact none of the matrices in Theorem B is of Diţă type, as shown in Section 5.

This is from [20, 21], the proof using a profile congruence, which forces the rotation to be trivial on the lowest weight spaces of the planar algebra, and the annular decomposition of Jones [64]. We extend this to the generic Karlsson matrices, and to the matrices having an affine profile modulo some prime, which gives the wildness of $H_9, BN_{10}, N_{11}, N'_{11}$, of the generic members of the family R_7^q, and of 2 new Butson matrices. For the circulant matrices, such as BF_6, wildness can be decided on the shift-invariant part of the planar algebra, and we find many new wild circulant matrices, including one of order 8, the first wild matrix of order 8, as far as we know, and the Björck matrices of simple index 2, at all the primes $p \geq 7$.

The algebraic and analytic invariants meet at the transfer matrix $T_2 \in M_{N^2}(\mathbb{R})$ of Jones, whose entries are the squared moduli of the profile entries of H. In the real case we have:

Theorem C. *For a real Hadamard matrix H of order N, $\dim P_2$ is the multiplicity of the eigenvalue 1 of T_2, and*

$$E(\Omega^{16}) = V(N) + \frac{4}{9} \cdot 16! \cdot N^6 \, Tr(T_2^2)$$

with V a universal polynomial, the lower moments being universal. At $N = 16$ we have $Tr(T_2^2) = 7 + 9 \cdot 2^{4-r}$, with r the 2-rank, and $\dim P_2 = 16, 7, 4, 3, 3$ for the 5 classes. At $N = 20$, the 3 classes have the same glow and Temperley-Lieb subfactors, but the spectra of their matrices T_2 are different.

Thus the sequence $m \to Tr(T_2^m)$ interpolates between the real glow, at $m = 2$, and the subfactor, at $m = \infty$, and at $N = 20$ both ends are blind, while the middle is not. In the complex case, the crossing F of the 2 legs of T_2 enters, already at order 8:

Theorem D. *For a complex Hadamard matrix H of order N, the moments of the complex glow are universal up to order 6, and*

$$E(|\Omega|^8) = V(N) + 18N^3\, Tr(T_2F)$$

with V a universal polynomial, where $N^3Tr(T_2F) = \sum_{jl} |\sum_k H_{jk}^2 \bar{H}_{lk}^2|^2$ is the sum over the rectangles of the squared flux phases. This number is between N^3 and N^4, the value N^4 being attained precisely by the matrices which are equivalent to real ones.

So the real Hadamard matrices, that is, the Ising arrays, are the most Ising-like rotator arrays, all their fluxes being 0 or π. As for the defect, it is universal on the real slice, as shown in [8], just like the low glow moments, and for the McNulty-Weigert matrices it is determined by the same number $Tr(T_2F)$:

Theorem E. *Every real Hadamard matrix of order N has defect $(N-1)(N-2)/2$. In the complex case, the defect is the multiplicity of the eigenvalue 1 of the Berezin matrix, minus $2N-1$, and for the McNulty-Weigert matrices of order $2p$, with outer matrix F_2, it is a function of $Tr(T_2F)$, and so of the moment $E(|\Omega|^8)$.*

The last assertion is a reformulation of a result from [8]. Regarding now the arithmetic invariants, these are not functions of the numbers $S_\Gamma(H)$, because the equivalence changes for instance the trace, and they live on the conjugacy class of H. However, they come from the same Jones model, on the directed graphs with a single type of vertices, where the row and column spins are identified. In terms of the arrays, this amounts in gluing the 2 layers of spins, the energy becoming $-\langle Ha, a\rangle/\sqrt{N}$, and we have:

Theorem F. *The partition functions $S_\Gamma(H)$ on the balanced directed graphs form a complete invariant for the conjugation of H by the monomial matrices, the bicolored graphs being particular cases, and the trace being the loop. For the glued arrays we have $E(\langle Ha, a\rangle) = Tr(H)$, and in the complex case $E(|\langle Ha, a\rangle|^2) = |Tr(H)|^2 + N^2 - N$.*

For the Fourier matrix F_N the trace is the quadratic Gauss sum, and for the Butson matrices the eigenvalues are Weil numbers, all their conjugates having modulus $\sqrt{N}$, which is where the arithmetic geometry of [36] enters. Together with the results of [26, 49], according to which all the real Hadamard matrices of a given size are quantum isomorphic, and all the complex ones are weakly quantum equivalent, this gives a hierarchy of blind regions for the 5 types of invariants, discussed at the end of the paper, the arithmetic invariants being the finest ones. We also show that the flux harmonics $R_k(H) = \|H^{\circ k}(H^{\circ k})^*\|_2^2$, with $H^{\circ k}$ being the entrywise power, which include the doubled rectangle R_2 of Theorem D, are invariants of the strong quantum equivalence of [26], so that in order 4 this is the usual equivalence, that for the Butson matrices the algebraic and geometric invariants are invariant under the Galois group, while the analytic and combinatorial ones are not, and that the numbers of points of the abelian varieties of [36] are supertraces of free fermion chains.

Motivation. The main motivation for the present work comes from the coupling matrices of high-energy physics, along the lines of [105], with our work here, based on [4, 8, 9, 12, 14, 34, 35, 36, 37, 44, 48, 90, 103, 104, 105], working out the mathematics for one of the several steps of the program mentioned in [105]. In fact, modulo a substantial amount of conformal field theory, this was also the motivation of Vaughan Jones for studying such questions, from the spin models of [63] to his later work on quantum spin chains and their continuum limits [65, 66], and we refer to [42, 69] for the operator algebra side of this story.

The paper is organized as follows. Sections 1-6 deal with the algebraic invariants: the spin models and subfactors, the quantum permutation groups, the deformed Fourier matrices, the Temperley-Lieb question, the work of Bisch and Nicoara, with Theorem B, and the spectral measures and their blowup. Sections 7-12 deal with the other invariants: the glow, the transfer matrices and fluxes, with Theorems C and D, the defect, with Theorem E, the combinatorial invariants, the arithmetic invariants, with Theorem F, and the unification, with Theorem A.

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1 Spin models and subfactors

The algebraic invariants of the Hadamard matrices come from the spin models of Jones, and from the commuting squares, subfactors, planar algebras and quantum groups associated to them. A spin model in the sense of Jones [63] consists of N spin states and of Boltzmann weights $w_\pm(a,b)$, and its partition function on a link diagram puts the spins on the shaded regions of the checkerboard coloring, with a weight at each crossing. The invariance under the 2nd Reidemeister move is guaranteed by the conditions

$$w_+(a,b)w_-(a,b)=1\quad,\quad \sum_x w_-(a,x)w_+(c,x)=N\delta_{ac}$$

which for $w_+=H$ and $w_-=\bar{H}$ are exactly the conditions $H\in M_N(\mathbb{T})$ and $HH^*=N1_N$. Thus, a Hadamard matrix is a spin model satisfying Reidemeister II. The 3rd move requires in addition the star-triangle relation, and the basic example is the Potts model, which for $n=2$ is the Ising model, and produces the Jones polynomial at $t=i$. Note that F_2 is the Ising weight $e^{K\sigma\sigma'}$ at the imaginary temperature $K=i\pi/4$, up to equivalence.

Without the star-triangle relation there is no knot invariant, but there is still a subfactor. As observed by Popa [93], a unitary $U\in U_N$ is of the form $U=H/\sqrt{N}$, with H Hadamard, precisely when the algebra Δ of diagonal matrices and its conjugate $U\Delta U^*$ are orthogonal, modulo the scalars, with respect to the trace. In other words, the Hadamard matrices are the commuting square case of the invariant of Connes for pairs of maximal abelian subalgebras, discussed in [105] in relation with the mixing matrices of the Standard Model. Iterating the basic construction on the commuting square formed by Δ and $U\Delta U^*$ gives a hyperfinite subfactor of index N [69, 86], whose planar algebra is the planar algebra P^H generated by H inside the spin planar algebra [67]. We recall that the possible indices of subfactors are the numbers $4\cos^2(\pi/n)$, and those in $[4,\infty)$ [62], and that the principal graphs below index 4 are of type ADE [47]. In terms of quantum groups, consider the vectors and projections

$$\xi_{ij}=\frac{1}{\sqrt{N}}\cdot\frac{H_i}{H_j}\quad,\quad P_{ij}=Proj(\xi_{ij})$$

with H_i being the rows of H, and the quotient being taken entrywise. The matrix $P=(P_{ij})$ is a magic unitary, in the sense of the quantum permutation group S_N^+ of Wang [108], a compact quantum group in the sense of Woronowicz [110], and we have:

Definition 1.1. *The quantum group $G_H\subset S_N^+$ of H is the Hopf image of the representation $u_{ij}\to P_{ij}$ of $C(S_N^+)$, and the Jones transfer matrices of H are the matrices $T_k\in M_{N^k}(\mathbb{C})$ given by*

$$(T_k)_{i_1\ldots i_k,j_1\ldots j_k}=tr(P_{i_1j_1}\ldots P_{i_kj_k})$$

with tr being the normalized trace of $M_N(\mathbb{C})$.

The planar algebra of H is then given by $P_k^H=Fix(u^{\otimes k})$, the fixed points of the k-th tensor power of the fundamental representation of G_H [6, 11]. The numbers $\dim P_k^H$ are the moments

of the main character of G_H, and we refer to [14] for such dimensions viewed as Hamburger moments, to [12] for their computation for many matrices, and to [4] for the generation questions, in the case of the Karlsson matrices. As observed by Jones in [67], P_k^H is the top eigenspace of a transfer matrix having the profile of H as Boltzmann weights, and in our normalization this matrix is T_k. The profile was recently used in [20] for computing the standard invariant of the subfactors associated to the Hadamard matrices of Petrescu of order 7, and other families, and in [21] for the real Hadamard matrices. Observe that

$$(T_2)_{i_1i_2,j_1j_2} = \frac{1}{N}|\langle\xi_{i_1j_1},\xi_{i_2j_2}\rangle|^2 = \frac{1}{N^3}\Big|\sum_x H_{i_1x}\bar{H}_{j_1x}\bar{H}_{i_2x}H_{j_2x}\Big|^2$$

which are the squared moduli of the profile entries of H. More generally, as noted in [12] and in [20, 21], we have $T_k = N^{-k-1}AA^*$, where $A \in M_{N^k}(\mathbb{C})$ is given by $A_{i,x} = \prod_s \bar{H}_{i_sx_s}H_{i_{s+1}x_s}$, with the indices taken modulo k, and this shows in particular that T_k is positive, and that the matrices $T_k(H^t) = N^{-k-1}A^*A$ associated to the transpose have the same spectrum. Equivalence does not change any of these objects, because the lines spanned by the vectors ξ_{ij} do not change, and so neither do the projections P_{ij}, up to a relabeling. Also, the power traces $Tr(T_k^m) = N^{-m(k+1)}Tr((AA^*)^m)$ are normalized partition functions $S_\Gamma(H)$, on the balanced bipartite graphs obtained by gluing cyclically m copies of A and m copies of A^*, which are not planar in general.

The algebra P_2^H is closely related to the Nomura algebra of H, which is the algebra of the matrices having all the vectors ξ_{ij} as eigenvectors [61], and in all our computations the dimensions coincide. In the real case, the real Hadamard matrices also produce genuine spin models, of size $4N$, satisfying the star-triangle relation, by a construction of Nomura [89], whose link invariant is computed in [60], and spin models also come from strongly regular graphs [59]. When the entries of H are roots of unity, G_H can be finite, and we are then in the setting of the finite-dimensional Hopf algebras, for which we refer to [1, 84]. The first relative commutants were computed for many matrices of small order by Nicoara [86], by means of the commuting square tower, and the angle operator of the subfactor, whose spectrum detects infinite depth, was studied for the Hadamard subfactors by Montgomery [82, 83]. We have:

Theorem 1.2. *For a complex Hadamard matrix H, the following hold:*

1. $\dim P_k^H$ *is the multiplicity of the eigenvalue* 1 *of* T_k*, and so* $\dim P_k^H = \lim_m Tr(T_k^m)$.
2. T_2 *is real, symmetric, doubly stochastic, and positive.*
3. $\dim P_2^H$ *is the number of connected components of the graph on the pairs of indices in which* (i_1,i_2) *and* (j_1,j_2) *are joined when* $\sum_x H_{i_1x}\bar{H}_{j_1x}\bar{H}_{i_2x}H_{j_2x} \neq 0$.

Proof. (1) The Haar state of G_H is the Cesàro limit of the convolution powers of $tr \circ \pi$, where $\pi : u_{ij} \to P_{ij}$ [109], and the matrices of these convolution powers, on the products $u_{i_1j_1} \ldots u_{i_kj_k}$, are the powers of T_k. Thus the Cesàro averages of T_k^r converge to the projection onto $Fix(u^{\otimes k})$. Also, $T_k = N^{-k-1}AA^*$ is positive, and is a contraction, being the slice $(id \otimes tr \circ \pi)(u^{\otimes k})$ of a unitary by a state, so its spectrum is in $[0,1]$, the Cesàro limit is the projection onto the eigenvalue 1, and $Tr(T_k^m)$ converges to its multiplicity.

(2) T_2 is the entrywise product of the Gram matrix of the vectors ξ_{ij} with its conjugate, so it is positive, by the Schur product theorem. It is doubly stochastic because the vectors ξ_{ij} with i fixed form an orthonormal basis.

(3) For a symmetric doubly stochastic matrix with entries ≥ 0 and positive spectrum, the multiplicity of the eigenvalue 1 is the number of connected components of its support, by Perron–Frobenius. □

2 Quantum permutation groups

Let us discuss now the quantum groups $G_H \subset S_N^+$ in more detail. Recall that $C(S_N^+)$ is the universal C^*-algebra generated by the entries of an $N \times N$ magic unitary, that is, a matrix of projections summing up to 1 on each row and each column [108]. Given a representation $\pi : C(S_N^+) \to B$, its Hopf image is the smallest quotient $C(S_N^+) \to C(G)$ through which π factorizes, and for the Hadamard models $\pi : u_{ij} \to P_{ij}$ this produces the quantum groups G_H. In terms of the main character $\chi = \sum_i u_{ii}$, with law μ_H with respect to the Haar integration, we have

$$c_k = \dim Fix(u^{\otimes k}) = \int_{G_H} \chi^k = \int_0^N x^k \, d\mu_H(x)$$

and so the sequence (c_k) is a moment sequence. In particular its Hankel matrices (c_{i+j}) are positive, and we refer to [14] for the general theory of such dimension sequences, and to [12] for the use of the Hankel inequalities as lower bounds, as for instance $c_5 \geq 206$ for one of the matrices of order 16. The basic examples are the Fourier matrices, for which the situation is as follows, see [11]:

Theorem 2.1. *For a complex Hadamard matrix H, the following are equivalent:*

1. *G_H is classical.*
2. *G_H is a group dual.*
3. *H is equivalent to a generalized Fourier matrix $F_G = F_{N_1} \otimes \ldots \otimes F_{N_k}$, with $G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$.*

In that case $G_H = G$, acting on itself by translations.

Proof. For $H = F_G$ we have $\xi_{ij} = \xi_{i-j,0}$, so the magic matrix is circulant over G, and its coefficients generate the algebra $C(G)$. The converse is from [11]. □

The same characterization holds at the level of commuting squares, in a quite different form, as recently shown by Hall and Nicoara [55]: the spin model commuting square of H satisfies their spanning relative commutant condition precisely when H is equivalent to the character table of a finite abelian group. In the opposite direction, the quantum groups G_H can be used as seeds for generating S_N^+, in relation with the maximality of $S_N \subset S_N^+$, the joint quantum group $\langle S_N, G_H \rangle$ being computed there numerically for the regular matrices of order $N \leq 7$, and shown to be S_N^+ for the generic deformations $F_A \otimes_Q F_2$.

An important feature of the construction $H \to G_H$ is its behavior under the symmetries of H. The conjugate $\bar{H}$ has the same quantum group, with the flipped comultiplication, and so $\mu_{\bar{H}} = \mu_H$. As for the transpose, the formula $T_k = N^{-k-1} AA^*$ from Section 1, and its version $T_k(H^t) = N^{-k-1} A^* A$, give the following result, from [12], refining the duality results of [7]:

Theorem 2.2. *For any complex Hadamard matrix H we have $T_k(H) \simeq T_k(H^t)$, for any k, and so $c_k(H) = c_k(H^t)$, and $\mu_H = \mu_{H^t}$. Moreover, if G_H is finite, then $G_{H^t} = \widehat{G}_H$.*

Proof. The first assertion follows from $AA^* \simeq A^* A$, and the multiplicities of the eigenvalue 1 are then the same, by Theorem 1.2. The last assertion is from [7]. □

Thus the transposition is invisible at the level of the moments, but not at the level of the quantum groups, as shown in [9] for the deformed Fourier matrices with $|X| \neq |Y|$. Another general principle, from [12], is an integrality statement. When G_H is finite, its Hopf algebra is finite dimensional, and the corresponding fusion data has integer entries, and this gives:

Proposition 2.3. *If G_H is finite, there is a monic polynomial $p \in \mathbb{Z}[x]$ such that $p(NT_k) = 0$, for any k. Thus, if some matrix NT_k has an eigenvalue which is not an algebraic integer, then G_H is infinite, and the subfactor has infinite depth.*

Proof. This is from [12]. With $\omega = Tr \circ \pi$ we have $NT_k = (id \otimes \omega)(u^{\otimes k})$, and $\varphi \to (id \otimes \varphi)(u^{\otimes k})$ is a unital morphism from the convolution algebra $C(G_H)^*$ to $M_{N^k}(\mathbb{C})$. When $C(G_H)$ is finite dimensional, the $\mathbb{Z}$-span of the traces of its irreducible representations is a unital ring, which is a finitely generated $\mathbb{Z}$-module, and ω belongs to it, so it is integral over $\mathbb{Z}$. □

This principle was used in [12] for proving infinite depth for several families of real matrices, as explained in Section 4 below, and recovers in particular the infinite depth results of Montgomery [82, 83], obtained with the angle operator. Let us also mention that the construction $H \to G_H$ extends to the partial Hadamard matrices, with the Hopf image being replaced by a bialgebra image, giving quantum partial permutation semigroups, and with the components of Theorem 1.2 becoming the finest invariant coloring of the ordered pairs of rows. Summarizing, the basic structure is as follows:

Theorem 2.4. *The quantum group $G_H \subset S_N^+$ is invariant under the equivalence, and satisfies $G_{\bar{H}} = G_H^\sigma$ and $\mu_{H^t} = \mu_H$. It is classical precisely in the generalized Fourier case, and when it is finite, the eigenvalues of all the matrices NT_k are algebraic integers, from a finite set.*

Proof. This follows by putting together the above results, from [7, 11, 12]. □

3 Deformed Fourier matrices

The main source of non-classical quantum groups G_H with computable structure are the deformations of the Fourier matrices, introduced by Diţă [40]. Given 2 complex Hadamard matrices $H \in M_M(\mathbb{T})$ and $K \in M_N(\mathbb{T})$, and a matrix of parameters $Q \in M_{M\times N}(\mathbb{T})$, the matrix $(H \otimes_Q K)_{ia,jb} = Q_{ib}H_{ij}K_{ab}$ is complex Hadamard, of order MN. For Q the all-one matrix we obtain $H \otimes K$, and the row and column phases of Q do not change the equivalence class, so we can assume that Q is dephased. The magic matrix of $H \otimes_Q K$ decomposes into blocks, and its quantum group sits inside a free wreath product [17], as follows, see [12]:

Proposition 3.1. *For any $H \in M_M(\mathbb{T})$, $K \in M_N(\mathbb{T})$ and $Q \in M_{M\times N}(\mathbb{T})$, the representation $u_{ia,jb} \to P_{ia,jb}^{H\otimes_Q K}$ factorizes through $C(S_N^+ \wr_* S_M^+)$, and more precisely we have $G_{H\otimes_Q K} \subset S_N^+ \wr_* G_H$.*

Proof. This is from [12], the factorization being given by the magic matrix of H, tensored with the identity, for the outer copy, and by the block sums $\sum_j P_{ia,jb}^{H\otimes_Q K}$ for the inner copies. □

For the deformed Fourier matrices $F_X \otimes_Q F_Y$, with X, Y finite abelian groups, the quantum group was computed for generic Q in [10], by using the group $\Gamma_{X,Y} = \mathbb{Z}^{(|X|-1)(|Y|-1)} \rtimes Y$, whose lattice part can be identified with $I_X \otimes I_Y$, the product of the augmentation ideals. The tools for the non-generic case were developed by Bichon [18], see also Burstein [27], and the general answer is as follows, from [12]:

Theorem 3.2. *For any X, Y and any dephased Q, encoded into $\theta \in (\mathbb{R}/\mathbb{Z})[X \times Y]$, the quantum group of $F_X \otimes_Q F_Y$ is given by $C(G) = C^*(\Gamma_{X,Y}/N_Q) \rtimes_\Phi C(X)$, where*

$$N_Q = \left\{ r \in I_X \otimes I_Y \middle| r\theta \in (\mathbb{R}/\mathbb{Z})^X + (\mathbb{R}/\mathbb{Z})^Y \right\}$$

and this quantum group is the generic one precisely when, for each Galois orbit of pairs of nontrivial characters of X, Y, the vector of character sums of the exponents of the parameters which are not roots of unity is nonzero. When Q has roots of unity as entries, the quantum group is finite.

Proof. This is from [12], by using the crossed coproduct description from [10], which always maps onto the Hopf image, and the tools from [18], the elements of N_Q being those of the lattice which act by scalars on all the blocks. □

In the generic case, the law of the main character is $\mu = (1-1/N)\delta_0 + (1/N)law(qq^*)$, where $q \in M_{M\times N}(\mathbb{T})$ is a Haar distributed matrix of phases, and $N = |Y|$, which is a free Poisson type law of a random matrix with independent phases [10]. The case of arbitrary Q is the blowup result of [9], discussed in Section 6 below. At order 6 the theorem gives explicit descriptions of the generic loci of $F_2 \otimes_Q F_3$ and $F_3 \otimes_Q F_2$, which have a nontrivial structure even away from the roots of unity, and the relation subgroups of $F_2 \otimes_Q F_3$ are the ideals of the Eisenstein integers [9]. In the real case, the theorem covers for instance the matrix $\text{had}_{16.1}$ from the catalogue of McKay [80], which is equivalent to $F_{\mathbb{Z}_2^2} \otimes_Q F_{\mathbb{Z}_2^2}$, with a single entry -1 in Q, and whose quantum group is finite, of dimension 256 [12].

For the Diţă matrices which are not of deformed Fourier type, the main examples studied in [12] are the regular families $F_{8,5a}, F_{8,5b}, F_{8,5c}$ of order 8, built from 2 copies of the matrices F_4^s. Here the quantum groups are contained in the free wreath products $S_4^+ \wr_* \mathbb{Z}_2$ and $H_2^+ \wr_* \mathbb{Z}_2$, the latter being the quantum automorphism group of the binary tree [41], with equality at all the levels computed. The equality reduces to a freeness question for 2 copies of S_4^+, which can be formulated in terms of Pauli representations, and in subfactor terms, the question is whether the planar algebra is a free composition in the sense of Bisch and Jones [19]. Let us also mention that the subfactors associated to the Diţă type matrices have intermediate subfactors, as shown by Nicoara [86], and so they are never Temperley-Lieb. In fact, we have the following result, which follows from the above, and from the definitions:

Theorem 3.3. *For a Diţă deformation $H = K \otimes_Q L$ of order $N = MN'$, with $M, N' \geq 2$, we have $c_2 \geq 3$, and so $G_H \neq S_N^+$. For the deformed Fourier matrices $F_X \otimes_Q F_Y$ the quantum group is described by Theorem 3.2, and it is infinite as soon as Q is generic.*

Proof. The partition of the indices into the M blocks is preserved by G_H, by Proposition 3.1, and its indicator is an extra fixed vector at level 2, on top of the 2 Temperley-Lieb ones. As for the second assertion, for generic Q the group $\Gamma_{X,Y}$ contains the free abelian group $\mathbb{Z}^{(|X|-1)(|Y|-1)}$. □

4 Temperley-Lieb and wildness

The quantum group G_H always contains, at the level of fixed points, the Temperley-Lieb algebra generated by the Jones projections, which at $N \geq 4$ has dimension Cat_k in degree k. The simplest possible answer is therefore $G_H = S_N^+$, and following [12] we use the following terminology:

Definition 4.1. *A complex Hadamard matrix $H \in M_N(\mathbb{T})$ is called wild when $G_H = S_N^+$, or equivalently, when $c_k = Cat_k$ for any k, or equivalently, for $N > 4$, when the standard invariant of its subfactor is Temperley-Lieb, with principal graph A_∞.*

The equivalences here come from Tannaka duality [111], the fixed points of S_N^+ being the Temperley-Lieb ones, and from the fact that the closed walks of length $2k$ on A_∞, based at the endpoint, are counted by Cat_k. The deformed Fourier matrices are not wild, by Theorem 3.3, and the conjecture, going back to [11], is that the matrices which are not of Diţă type, in a suitable general sense, are wild. The main computational tool is the following decomposition, from [12], based on the annular decomposition of Jones [64]:

Theorem 4.2. *For $H \in M_N(\mathbb{T})$ with $N \geq 5$, we have $T_k \simeq \bigoplus_s m_k(s)\Phi_s$, where $m_k(s)$ are the multiplicities of the irreducible corepresentations r_s of S_N^+ in $u^{\otimes k}$, and $\Phi_s \simeq \Psi_2 \oplus \ldots \oplus \Psi_s \oplus [s \text{ odd}] \cdot 0_{N-1}$ for $s \geq 1$, where Ψ_t is the restriction of T_t to the annular lowest weight space $\Lambda_t \subset (\mathbb{C}^N)^{\otimes t}$. In particular $G_H = S_N^+$ if and only if $\|\Psi_t\| < 1$ for all $t \geq 2$.*

Proof. This is from [12]. The first decomposition comes from the Jones-Wenzl decomposition of $u^{\otimes k}$ over S_N^+, with $\Phi_s = (id \otimes tr \circ \pi)(r_s)$, and the second one is the spectral version, for the tracial states ψ with $\psi(u_{ij}) = 1/N$, of the annular decomposition of Jones [64]. □

In practice, if $c_{k-1} = Cat_{k-1}$, then c_k is Cat_k plus the multiplicity of the eigenvalue 1 of Ψ_k, and this multiplicity is the annular multiplicity a_k in degree k, which will be a key object in Section 5. In this way, the equalities $c_k = Cat_k$ were checked in [12], in floating point, up to the levels k indicated below, for the matrices listed there, and here is the current status of these matrices, in view of the results in Section 5, the numerical range being indicated for the open cases:

N	matrices	status
6	T_6	wild, Theorem 5.6
6	$H_6^q,\ K_6,\ BN_6^q$	wild for generic parameters, Section 5
6	BF_6	open, $k \leq 9$
7	$P_7^q,\ R_7^q$	wild for generic q, Section 5
8	$F_{8,2}, F_{8,4}, K_{8,2}, M_{8,1}, I_{8,12}, K_{8,30}, L_{8,30}$	open, $k \leq 7$
$9-11$	$BN_9,\ H_9,\ BN_{10},\ N_{11},\ N'_{11}$	wild, Section 5
$12, 20$	$P_{12},\ P_{20}^1,\ P_{20}^2,\ \text{had}_{20.1}$	wild, Theorem 5.6
28	all the 487 classes	wild, Theorem 5.6
24	P_{24}^1, 45 classes of order 24	open, $k \leq 5,\ k \leq 4$
$32, 48$	$P_{32}^1,\ P_{48}^1$	open, $k \leq 4$
$36-108$	$P_{36}^2,\ P_{44}^1,\ P_{52}^2,\ P_{100}^2,\ P_{108}^1,\ P_{108}^2$	wild, Theorem 5.6
104	P_{104}^1	open, $k \leq 3$

Here P_{q+1}^1 and $P_{2(q+1)}^2$ are the Paley matrices of type I and II [91], H_6^q is the Diţă family through the Haagerup matrix, and K_6 is the 3-parameter family of Karlsson [71]. For the Karlsson matrices, consisting, up to equivalence, of the matrices which decompose into Hadamard blocks of size 2×2, the computations were pushed much further in [4], by using a limit argument at the boundary points of the family, where the matrix becomes a Diţă deformation of $F_2 \otimes F_3$. The fixed vectors at nearby parameters converge, along subsequences, to the fixed vectors of an overgroup of the quantum group at the boundary point, having 2 orbitals, and this leads to the following result, about quantum groups:

Theorem 4.3. *Let D be the quantum group of a generic Diţă deformation of $F_2 \otimes F_3$, and let $D \subset G \subset S_6^+$, with $c_2(G) = 2$. Then $c_k(G) = Cat_k$ for $k \leq 18$, and the first level k with $c_k(G) > Cat_k$, if any, satisfies $k = 3(4)$ with $k \geq 19$, or $k = 1(4)$ with $k \geq 37$.*

Proof. This is from [4]. A first extra tensor of such an overgroup would give a probability measure on the closed walks on the kagome lattice, with prescribed local distributions, read from a Jones-Wenzl projection, and this is excluded at the even levels by a straight walk argument, and at the odd levels by staircase tests, and a certificate at level 15. □

The quantum group D is contained in the free wreath product $S_3^+ \wr_* S_2^+$, the quantum symmetry group of 2 disjoint triangles, and so the quantum groups $\langle G_H, S_3^+ \wr_* S_2^+ \rangle$, with H of order 6 having $c_2 = 2$, are overgroups as above, having Catalan fixed space dimensions up to level 18 [4]. This is evidence for the maximality of $S_3^+ \wr_* S_2^+$ in S_6^+, which is the free analogue of the maximality of $S_3 \wr S_2$ in S_6, and the generic Karlsson matrices themselves are wild, at all levels, as explained in Section 5. Regarding the real matrices, the first step towards wildness is infinite depth, and here the integrality principle of Proposition 2.3 gives the following results, from [12], recovering and extending the results of Montgomery [82, 83]:

Theorem 4.4. *The matrices of order $2(q+1)$ built from symmetric conference matrices, including all the Paley II matrices, produce infinite depth subfactors, for any $q \geq 5$. The same holds for the Paley I matrices P^1_{q+1}, for all the prime powers $q = 3(4)$ with $11 \leq q < 10^5$.*

Proof. This is from [12]. In the conference case, an explicit eigenvector of NT_2 has eigenvalue $1 + (2q+1)(q-3)^2/(q+1)^2$, which is not an algebraic integer for $q \geq 5$, and for Paley I the matrices $X_\psi(x,y) = \psi(y-x)$, with ψ a nontrivial character of $\mathbb{F}_q^*$ of odd order, are eigenvectors of T_2, with eigenvalues given by character sums, which are not algebraic integers. □

For the Paley II matrices, and for the Paley I matrices with $q = 3(8)$, Theorem 4.4 is superseded by Theorem B, which gives wildness, and proves the conjecture of Montgomery [82]. For the Paley I matrices with $q = 7(8)$, of order $0(8)$, infinite depth is all that is known. Let us also mention that the classification of the regular complex Hadamard matrices at $N = 7, 8$, and the partial results at $N = 11$, as well as the characterization of the Fourier matrices of prime order from partial data, provide many more candidates for wildness, and that the Hadamard matrices of Petrescu [92], of prime order, form the first continuous families of prime order, which makes them particularly interesting.

5 Congruences of the profile

We explain here the work of Bisch and Nicoara [20, 21], which provides the first proofs of wildness, for many matrices. We use the cyclic coordinates from Section 1. For $n \geq 1$ consider the space $H_n = l^2(X^n)$, where $X = \{1, \ldots, N\}$ is the index set of H, the matrix T_n acting on it, and the cyclic shift $(\rho f)(a_1, \ldots, a_n) = f(a_n, a_1, \ldots, a_{n-1})$. Also, let $U_n \subset H_n$ be the space of functions f which vanish when 2 cyclically adjacent labels are equal, and whose sum over any coordinate is zero, when the other coordinates are fixed. We have then the following result, from [20, 21], building on [67, 82]:

Proposition 5.1. *For any $n \geq 1$, the relative commutant P_n is isomorphic to $F_n = \ker(1 - T_n)$, with the 2-click rotation of the planar algebra corresponding to the cyclic shift ρ, restricted to F_n. The common kernel $W_n \subset P_n$ of the cap maps corresponds to $F_n \cap U_n$, for $n \geq 2$, and $W_1 = 0$.*

Proof. The first assertion is the angle operator description of flatness, with $T_n = C_n C_n^*$, where C_n is the matrix of scalar products between the bases of the 2 algebras whose intersection is the relative commutant, in the commuting square tower. The rotation is computed in the spin planar algebra, where it acts on the labels by a cyclic shift, and the 2 types of caps either sum over a label, or identify 2 adjacent labels. □

The spaces W_n are the lowest weight spaces of the annular decomposition of Jones [64], and their dimensions $a_n = \dim W_n$ are the annular multiplicities, which are the numbers appearing in Theorem 4.2. With $b_n = \dim P_n$, and for an irreducible subfactor of index > 4, the annular decomposition gives

$$b_n = Cat_n + \sum_{k=2}^{n} \binom{2n}{n-k} a_k$$

and so the Temperley-Lieb property amounts in $a_n = 0$ for any n. The first ingredient of [20, 21] is a congruence for the profile. Consider the numbers

$$S(a,b,c,d) = \sum_y H_{ay} \bar{H}_{by} \bar{H}_{cy} H_{dy}$$

and call restricted the quadruples with $a \neq c$ and $b \neq d$. In the real case, we have the following congruence, recorded by Wallis [107], and used by Montgomery [83] for computing $\dim P_2$:

Lemma 5.2. *For a real Hadamard matrix of order N, and any 4 distinct rows a, b, c, d, we have $S(a, b, c, d) = -N(8)$. In particular, when $N = 4(8)$, the numbers $S/4$ satisfy*

$$\frac{S(a, b, c, d)}{4} = 1 + \delta_{ab} + \delta_{cd} + \delta_{ad} + \delta_{bc} \quad (2)$$

on all the restricted quadruples.

Proof. After multiplying the columns by the entries of row a, the other 3 rows become $1-2x, 1-2y, 1-2z$, with x, y, z being 0-1 vectors, and the orthogonality relations give $S = N - 8\sum_j x_j y_j z_j$. For the restricted quadruples which are not formed by 4 distinct rows, the orthogonality gives $S = 0$ or $S = N$, and this gives the second assertion. □

In the complex case, similar congruences hold modulo suitable primes, after dividing S by a suitable number α. For instance for Gaussian rational matrices of order $N = 2m$, with m odd, we have $S/2 = 1 + D$ modulo $1 - i$, where $D = \delta_{ab} + \delta_{cd} + \delta_{ad} + \delta_{bc}$, and for matrices with cube roots of unity as entries, of order $N = 3m$, with $3 \nmid m$, we have $S/3 = -m(1 - E)$ modulo $1 - e^{2\pi i/3}$, where $E = \delta_{ab} + \delta_{cd} - \delta_{ad} - \delta_{bc}$ [20]. The program `congruence.py` checks these congruences for the Paley matrices of orders 12 and 20, for the Haagerup and Tao matrices, and for the Petrescu matrix. The second ingredient is the following consequence, for the rotation:

Theorem 5.3. *Assume that H is real, of order $N = 4(8)$, with $N > 4$. Then the 2-click rotation ρ acts as the identity on the lowest weight spaces W_n, for any n odd.*

Proof. This is from [21]. On the words with no equal adjacent labels, the matrix $K_n = \prod_j S(a_j, b_j, a_{j+1}, b_{j+1})/4^n$ satisfies $K_n f = N(N/4)^n f$ for $f \in W_n$, by flatness. On the other hand, expanding the product of the congruences in Lemma 5.2, and using the cap equations, which kill the terms not involving all the input labels exactly once, gives $K_n f = (2 + \rho + \rho^{-1})f$ modulo 2, on the integer valued vectors $f \in U_n$. As N is even, we get $(\rho - 1)^2 f = 0$ modulo 2 on a ρ-stable integral lattice of W_n, and since $\rho^n = 1$ with n odd, this forces $\rho = 1$. □

The third ingredient is a uniform bound for the trace of the rotation, which holds for any complex Hadamard matrix, and uses the transfer matrix formula from [7]:

Proposition 5.4. *For any complex Hadamard matrix H of order N, and any $n \geq 1$, we have*

$$|Tr(\rho|P_n)| \leq N$$

where Tr is the usual trace, and ρ is the 2-click rotation.

Proof. This is from [20, 21]. With Q_{ab} the projection on ξ_{ab}, the block matrix $V = (Q_{ab}) \in M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$ is unitary, and with $V^{[r]} = V_{01}V_{02}\ldots V_{0r}$, the formula $T_n(a, b) = tr(Q_{a_1b_1}\ldots Q_{a_nb_n})$, and its version for T_n^r from [7], give $Tr(\rho T_n^r) = (Tr \otimes tr)((V^{[r]})^n)$, whose absolute value is at most N. Letting $r \to \infty$, we obtain the result. □

The bound is sharp, as shown by W_8 at $n = 2$, where $Tr(\rho|P_2) = 8$. Now since the rotation permutes the annular basis diagrams, and since for n odd the only rotation invariant Temperley-Lieb diagrams are the 2 alternating ones, and no lower weight diagram is invariant, we have $Tr(\rho|P_n) = 2 + Tr(\rho|W_n)$ for any $n \geq 3$ odd [21]. The program `congruence.py` computes the spaces F_n and W_n for small n, and checks this formula, and the bound in Proposition 5.4, for the above matrices, and for F_6 and W_8. Together with Theorem 5.3, this gives $a_n \leq N - 2$, for any n odd. Thus the annular multiplicities are bounded along the odd levels, and the point is that this is enough, together with $\dim P_2 = 2$, for having a Temperley-Lieb standard invariant, by the following general result, from [20, 21]:

Proposition 5.5. *Let P be the planar algebra of an irreducible subfactor of index $\delta^2 > 4$, with rooted law μ, the spectral measure at the root of the square of the adjacency operator of the principal graph. Then*

$$a_n = \int q_n(x)\, d\mu(x)$$

for any $n \geq 2$, where $q_0 = 2$, $q_1 = x - 2$ and $q_{n+1} = (x-2)q_n - q_{n-1}$. Thus, if the annular multiplicities are bounded along a subsequence, then μ is supported by $[0, 4]$, the principal graph has norm at most 2, and if moreover $\dim P_2 = 2$, the principal graph is A_∞.

Proof. The polynomials q_n are rescaled Chebyshev polynomials, $q_n(2 + 2\cos\theta) = 2\cos(n\theta)$, and the expansion $x^n = \binom{2n}{n} + \sum_{k\geq 1} \binom{2n}{n-k} q_k(x)$, integrated against μ, compared with the annular formula above, gives the formula for a_n. Since $q_n \geq -2$ on $[0, 4]$, and q_n grows exponentially on $(4, \infty)$, the boundedness forces μ to be supported by $[0, 4]$, and the norm bound follows by counting closed walks. Finally, the infinite graphs of norm ≤ 2 having a root of degree one are A_∞ and D_∞, and D_∞ is excluded by $\dim P_2 = 2$, as well as by the theorem of Popa on the graphs with an A_∞ tail [94]. □

Putting everything together, we obtain the following result, which is Theorem B in the introduction, and whose real case solves the conjecture of Montgomery [82] for the Paley II matrices:

Theorem 5.6. *The subfactor associated to a complex Hadamard matrix H of order $N > 4$ has Temperley-Lieb standard invariant, and so $G_H = S_N^+$, in the following cases:*

1. *H is real, and $N = 4(8)$.*
2. *H has entries in $\mathbb{Q}(i)$, and $N = 2(4)$.*
3. *H has cube roots of unity as entries, and $N = 3(9)$ or $N = 6(9)$.*
4. *H has an algebraic representative with unit entries at a prime of odd residue characteristic p, reducing to $J_N - 2I_N$, with $N = 4(p)$.*
5. *H is the Petrescu matrix of order 7, or a member of its family outside a countable set of algebraic parameters.*

Proof. This is from [20, 21]. In all cases the congruence of the profile forces $\rho = 1$ on W_n for the odd n prime to the residue characteristic, and so $a_n \leq N - 2$ along a subsequence, and the same congruence shows that the off-diagonal block of T_2 has connected support, so $\dim P_2 = 2$, and Proposition 5.5 applies. For the families, at a fixed degree n the dimension of P_n is given by the rank of a matrix with Laurent polynomial entries, which is maximal at the base point, and so outside a finite set of algebraic parameters. □

The hypotheses of Theorem 5.6 exclude the Diţă matrices, which have $c_2 \geq 3$ by Theorem 3.3, and this can be seen directly, as follows. We say that $H \in M_N(\mathbb{T})$ is of Diţă type when it is equivalent to a matrix of the form $D_{ia,jb} = K_{ij}(L_i)_{ab}$, with $K \in M_M(\mathbb{T})$ and $L_1, \ldots, L_M \in M_{N/M}(\mathbb{T})$ being complex Hadamard, and $1 < M < N$. This covers the deformations $K \otimes_Q L$ from Section 3, with $L_i = LQ_i$, where Q_i is the diagonal matrix of the i-th row of Q.

Proposition 5.7. *The matrices in Theorem 5.6 are not of Diţă type.*

Proof. Assume that H is equivalent to a matrix D as above, and set $N' = N/M$. After permuting the rows and columns of H, and indexing them as (i, a) and (j, b), we have $H_{ia,jb} = x_{ia}K_{ij}(L_i)_{ab}y_{jb}$, for some numbers $x_{ia}, y_{jb} \in \mathbb{T}$. We have then the following 3 observations, coming directly from this formula:

(a) For i, j fixed, $(H_{ia,jb})_{ab}$ is a complex Hadamard matrix of order N', equivalent to L_i.

(b) For a, b fixed, $(H_{ia,jb})_{ij}$ is a complex Hadamard matrix of order M, equivalent to K.

(c) For $j \neq j'$ and b fixed, the ratio $H_{ia,jb}/H_{ia,j'b} = K_{ij}y_{jb}/(K_{ij'}y_{j'b})$ does not depend on a.

(1) If H is real, (a) and (b) produce real Hadamard matrices of orders $M, N' \geq 2$, so these orders are 2 or multiples of 4. If both are multiples of 4 then $16|N$, if one of them is 2 then the other one is $N/2 = 2(4)$, with $N/2 > 2$, which is impossible, and if both are 2 then $N = 4$. Thus $N = 4(8)$ with $N > 4$ is excluded.

(2) If H has entries in $\mathbb{Q}(i)$, so do the matrices in (a) and (b), and since $N = 2(4)$, one of their orders M, N' is odd, and > 1. But there is no complex Hadamard matrix of odd order $n > 1$ with entries in $\mathbb{Q}(i)$. Indeed, such an entry is of the form $z = (a + ib)/c$, with $a, b, c \in \mathbb{Z}$ satisfying $a^2 + b^2 = c^2$ and having no common divisor, so c is odd, and exactly one of a, b is odd. Thus z belongs to the local ring of $\mathbb{Z}[i]$ at the prime $1 + i$, whose residue field is $\mathbb{F}_2$, and $z = 1$ modulo $1 + i$, because $i = 1$ modulo $1 + i$. It follows that the scalar product of 2 rows is $n = 1$ modulo $1 + i$, and so cannot vanish.

(3) If H has cube roots of unity as entries, so do the matrices in (a) and (b). For such a matrix of order n, the scalar product of 2 rows is $n_0 + n_1 w + n_2 w^2$, with $w = e^{2\pi i/3}$ and $n_0 + n_1 + n_2 = n$, which vanishes only for $n_0 = n_1 = n_2$, so $3|n$. Thus $3|M, N'$, so $9|N$, which is excluded.

(4) Assume that H has a representative with unit entries at a prime P of odd residue characteristic, reducing to $J_N - 2I_N$. This representative is of the above form too, with other numbers x_{ia}, y_{jb}, and by (c), for $j \neq j'$ and b fixed, the reductions of the columns (j, b) and (j', b) have a constant ratio on each row block $\{(i, a)|a\}$, of size $N' \geq 2$. On the other hand, each column of $J_N - 2I_N$ has exactly one entry -1, and 2 different columns have ratio -1, which is different from 1 in odd characteristic, exactly on the 2 rows where these entries -1 sit, and ratio 1 elsewhere. Thus the row block containing the entry -1 of the column (j, b) consists of this row, and of the row containing the entry -1 of the column (j', b). Since there are $M - 1$ choices for j', leading to different rows, we obtain $M = 2$ and $N' = 2$, so $N = 4$, which is excluded.

(5) For the Petrescu matrices, $N = 7$ is prime, so there is no decomposition $N = MN'$ with $M, N' \geq 2$. □

At $N = 0(8)$ the situation is different, the real Diţă matrices, such as the Walsh matrices, existing there, and the method of Theorem 5.6 cannot apply to them, since they are not wild.

By the same specialization argument, the Temperley-Lieb property holds at all the transcendental phases in the Diţă family through the Haagerup matrix, at the algebraically independent pairs of phases in a 2-parameter Karlsson family, and at many algebraic parameters in the affine families of orders 10, 12, 14 of [75], and in the families of Szöllősi through the real matrices of order 4(8) [20]. The method fails as such for the real matrices of order $N = 0(8)$, where no congruence of the above type holds, as shown for instance by the Paley matrix P_{24}^1, whose numbers $S/8$ take both parities on the quadruples of distinct rows, as computed by `congruence.py`, and in fact the numbers S themselves vanish somewhere, as explained at the end of this section.

The specialization argument applies as well to the 3-parameter Karlsson family. We use the parametrization of [4], where the 2×2 blocks are $H_{ij} = X_i \Omega^{ij} Y_j$, for $i, j \in \{0, 1, 2\}$. Here $\Omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}\Lambda$, where Λ is the reflection having first row $(\cos\theta, e^{i\varphi}\sin\theta)$, and X_i, Y_j depend on 4 phases z_1, z_2, z_3, z_4, with $z_1 = e^{i\psi}$, the other ones being determined by their squares, as $z_3^2 = M_A(z_1^2)$, $z_4^2 = M_B(z_1^2)$ and $z_2^2 = M_A^{-1}(z_4^2)$, where M_A, M_B are the Möbius maps of the unit circle associated to the matrices $A = F_2\Omega$ and $B = F_2\Omega^2$.

Theorem 5.8. *The generic Karlsson matrices are wild. That is, outside a countable union of nowhere dense real analytic subsets of the parameter space, the members H of the Karlsson family K_6 have $c_k = Cat_k$ for any k, and so $G_H = S_6^+$.*

Proof. Fix k, and set $m = 6^k - Cat_k$. Since $c_k \geq Cat_k$, we have $rank(1 - T_k) \leq m$, with equality precisely when $c_k = Cat_k$. Let U be the set of parameters where M_A, M_B are nondegenerate, and the above formulas are regular, whose complement is contained in a proper real algebraic subset. On U the entries of H are rational in the parameters and in z_2, z_3, z_4, and the sign choices for these square roots only permute rows or columns, or multiply them by -1, so they change T_k at most by a conjugation with a permutation matrix. Thus the sum Δ_k of the squared moduli of the minors of size m of $1 - T_k$ does not depend on these choices, and is a rational function of the parameters, whose numerator is a real analytic function on the connected parameter space, and which is nonzero precisely when $c_k = Cat_k$. Now the Diţă family through the Haagerup matrix consists of Karlsson matrices, at the angles $\theta = \arccos(1/\sqrt{3})$, $\varphi = \pi/4$, and at $\psi = 0$ we obtain the matrix in $BH(6, 4)$, which has entries in $\mathbb{Q}(i)$, with $N = 6 = 2(4)$, and so is wild, by Theorem 5.6 (2). At these angles M_A degenerates, but near this point the matrix is still given by regular formulas, with $z_2^2 = M_B^{-1}(z_3^2)$, as for the degenerate limits in [71, Appendix 1]. By lower semicontinuity of the rank, $\Delta_k \neq 0$ at the points of U close to it, so the zero set of the numerator of Δ_k, together with the complement of U, is contained in a nowhere dense real analytic subset, and the union over k gives the result. □

In particular the Catalan values up to level 18 for the generic Karlsson matrices, obtained in [4] by using Theorem 4.3, hold at all levels. The program `specialization.py` checks the ingredients of the proof, namely that the matrices given by both formulas are Hadamard, that the Diţă family through the Haagerup matrix is the above curve, with all the Haagerup invariants being 4th roots of unity at $\psi = 0$, and that the family is continuous at this point, and computes $c_2, c_3, c_4 = 2, 5, 14$ there, and at random parameters. Summarizing, the main open cases are the real matrices of order $0(8)$, the matrices with cube roots of unity of order divisible by 9, the Gaussian rational matrices of order $0(4)$, and some of the matrices in the table in Section 4, discussed at the end of this section.

The method of [20, 21] extends in fact to a general class of matrices, as follows. Assume that the entries of H belong to a number field K, and let P be a prime of K, with local ring R and residue field k, such that the entries of H are units of R. We say that the profile of H is affine at P when the numbers $\sigma(a, b, c, d) = S(a, b, c, d)/N$ belong to R, for all the restricted quadruples, and when there are $X, Y \in k$ such that

$$\sigma(a, b, c, d) = -1 + \delta_{ab} + \delta_{cd} + X\delta_{ad} + Y\delta_{bc} \ (P)$$

for all the restricted quadruples. By orthogonality, this holds automatically when $a = b$ or $c = d$, so the condition concerns the quadruples of distinct indices, where $\sigma = -1(P)$, and those with $a = d$ or $b = c$. The condition depends on the representative of H, the row phases multiplying $S(a, b, c, d)$ by $\lambda_a\bar{\lambda}_b\bar{\lambda}_c\lambda_d$, the column phases leaving it invariant. We have:

Lemma 5.9. *If the profile of H is affine at P, and $N \geq 5$, then $X + Y = N - 2$ and $XY = 1$ in k.*

Proof. By orthogonality we have

$$\sum_x S(a, b, c, x)S(c, x, e, f) = NS(a, b, e, f)$$

that is, $\sum_x \sigma(a, b, c, x)\sigma(c, x, e, f) = \sigma(a, b, e, f)$. Take a, b, c, e, f distinct. The terms with $x \notin \{a, b, c, e, f\}$ are products of 2 values at quadruples of distinct indices, so they are 1 modulo P, and there are $N - 5$ of them. The terms with $x = b, c, f$ vanish, by orthogonality, and those with $x = a, e$ are $-(X - 1)$ and $-(Y - 1)$ modulo P. Thus $N - 5 + 1 - X + 1 - Y = -1$ in k, which gives the first formula. Also, for a, b, c distinct, the vector $S(a, b, c, \cdot)$ is the image by H of the vector with entries $H_{ay}\bar{H}_{by}\bar{H}_{cy}$, of norm $\sqrt{N}$, so $\sum_x |S(a, b, c, x)|^2 = N^2$, and since $\bar{S}(a, b, c, x) = S(b, a, x, c)$, this reads $\sum_x \sigma(a, b, c, x)\sigma(b, a, x, c) = 1$. The $N - 3$ terms with

$x \notin \{a,b,c\}$ are 1 modulo P, those with $x = b, c$ vanish, and the term with $x = a$ is $(X-1)(Y-1)$ modulo P. Thus $N-3+(X-1)(Y-1) = 1$ in k, which with the first formula gives $XY = 1$. □

In the real case of Lemma 5.2 we have $X = Y = 1$, in the cube root case we have $X = Y = -1$, and for a reduction to $J_N - 2I_N$ we have $X = Y = 1$, and so the lemma explains the conditions on N in Theorem 5.6. Also, X and Y are the roots of $t^2 - (N-2)t + 1$, and in particular they are nonzero. In general, we have the following result, which contains the cases (1)-(4) of Theorem 5.6:

Theorem 5.10. *Assume that $N \geq 6$, and that some representative of H has an affine profile at a prime P. Then the subfactor associated to H has Temperley-Lieb standard invariant, and so $G_H = S_N^+$.*

Proof. Let $\Omega_n \subset X^n$ be the set of words with no equal cyclically adjacent labels. With $P_{ab} = \xi_{ab}\xi_{ab}^*$, the flatness equations for $f \in W_n$ involve $\bar{\sigma}$, and so the conjugate space $\bar{W}_n$ satisfies $\sum_b \prod_j \sigma(a_j, b_j, a_{j+1}, b_{j+1}) f(b) = Nf(a)$. Since the cap equations and the rotation are real, $\bar{W}_n$ satisfies the cap equations, and $\rho = 1$ on W_n if and only if $\rho = 1$ on $\bar{W}_n$. Let $L_n = \bar{W}_{n,K} \cap R^{\Omega_n}$, where $\bar{W}_{n,K}$ is the solution space over K of the equations defining $\bar{W}_n$. This is a free R-module, saturated in R^{Ω_n}, stable under ρ, and spanning $\bar{W}_n$. For $f \in L_n$, expand the product of the congruences. A term using fewer than n different labels b_j vanishes after summation, by the cap equations, and since each factor involves b_j or b_{j+1}, the terms using all of them correspond to the 2 cyclic matchings of the factors with the labels. For each matching, a change between the 2 local choices forces 2 cyclically adjacent labels b_j, b_{j+1} to coincide, which kills the term, and the 4 surviving terms give $2 + X^n\rho + Y^n\rho^{-1}$. With lifts $x, y \in R$ of X, Y we obtain $(2 - N + x^n\rho + y^n\rho^{-1})L_n \subset PL_n$, by saturation. Now take n a prime with $n = 1(|k|-1)$ and $n > |k|$, and there are infinitely many such primes, by Dirichlet. Then $X^n = X$ and $Y^n = Y$, so by Lemma 5.9 the reduction $\bar{\rho}$ of ρ to L_n/PL_n satisfies $(\bar{\rho} - 1)(X - Y\bar{\rho}^{-1}) = 0$. Since $\bar{\rho}^n = 1$ with n prime to p, $\bar{\rho}$ is diagonalizable, and an eigenvalue $\lambda \neq 1$ would satisfy $\lambda = Y/X \in k^*$, so its order would divide both n and $|k| - 1$, which are coprime, so $\lambda = 1$. Thus $\bar{\rho} = 1$, so the idempotent $e = (1/n)\sum_j \rho^j$ satisfies $(1-e)L_n \subset PL_n$, hence $(1-e)L_n \subset P^r L_n$ for any r, so $e = 1$, and $\rho = 1$ on $\bar{W}_n$, and on W_n. By Proposition 5.4 we conclude that $a_n \leq N - 2$ for these n. Finally, since $\sigma = -1(P)$ on the quadruples of distinct indices, the corresponding entries of T_2 are nonzero, and since $N \geq 6$, any 2 ordered pairs of distinct indices are both joined to a pair disjoint from their union, so the off-diagonal block of T_2 has connected support, $\dim P_2 = 2$, and Proposition 5.5 applies. □

As a first application, the case of Theorem 5.6 where the matrix reduces to $J_N - 2I_N$ extends to arbitrary diagonal values, as follows, with the program `affine.py` testing the affine condition at the primes not dividing N, for the Butson matrices, recovering the Petrescu matrix at $p = 3$:

Theorem 5.11. *Assume that $N \geq 6$, and that H has a representative whose entries are units at a prime P not dividing N, and which reduces modulo P to $J_N + (\lambda - 1)I_N$, for some $\lambda \in k^*$. Then $\lambda + \lambda^{-1} = 2 - N$ in k, and the subfactor associated to H has Temperley-Lieb standard invariant.*

Proof. With $M = J_N + (\lambda - 1)I_N$, the orthogonality of the rows modulo P gives $N - 2 + \lambda + \lambda^{-1} = 0$, and the profile of M is $N - 4 + 2\lambda + 2\lambda^{-1} = -N$ on the quadruples of distinct indices, $N - 3 + \lambda^2 + 2\lambda^{-1}$ when $a = d$, and $N - 3 + \lambda^{-2} + 2\lambda$ when $b = c$. Thus the profile of H is affine at P, with $NX = 2N - 3 + \lambda^2 + 2\lambda^{-1} = (\lambda - 1)^2$, by using $2N = 4 - 2\lambda - 2\lambda^{-1}$, and since $N = -(\lambda - 1)^2/\lambda$, this gives $X = -\lambda$, and similarly $Y = -\lambda^{-1}$. Thus Theorem 5.10 applies. □

For $\lambda = -1$ this is the reduction to $J_N - 2I_N$, with $N = 4(p)$, and for $p = 2$ and λ a primitive cube root of unity in $\mathbb{F}_4$, the condition $\lambda + \lambda^{-1} = 2 - N$ reads $N = 1(2)$, so this applies to the Butson matrices with sixth roots of unity, of odd order, whose reduction modulo 2 has this form.

The program `screen.py` tests the affine condition, at the primes dividing Nq, for the matrices of the catalogue of Butson matrices of Lampio, Östergård and Szöllősi [76], which is to be downloaded separately, from the Zenodo record 2585766, and the program `classify.py` classifies the results. Here $BH(N,q)$ denotes the set of $N\times N$ Hadamard matrices having as entries q-th roots of unity, and the catalogue contains 6815561 matrices with $N\geq 6$, including the large files $BH(12,12)$ and $BH(16,4)$, which are screened by the programs `conor.py` and `ronda.py`, with the same test.

Up to the embeddings $BH(N,q)\subset BH(N,qr)$, the matrices having an affine profile at a prime dividing Nq are the 844 representatives found in [20], which are real at $N=12,20$, with entries in $\mathbb{Q}(i)$ at $N=6,10,14$, with cube roots of unity at $N=6,12,21$, and with sixth roots of unity at $N=7$, plus 2 more matrices, in $BH(9,10)$ and $BH(10,6)$, whose reductions modulo the primes above 5 and 3 are equivalent to J_9-2I_9 and $J_{10}-2I_{10}$. These 2 matrices are thus wild, by Theorem 5.6 (4), and the first one, which is the only matrix in $BH(9,10)$ up to equivalence, is the matrix BN_9 from the table in Section 4, which is therefore a wild matrix of order 9. All the affine profiles found on the catalogue have $X=Y=\pm 1$, so there Theorem 5.10 gives nothing new, and its further applications have to come from other constructions, such as the algebraic families.

Let us go back now to the table in Section 4. The matrices H_9,BN_{10} there have entries in $\mathbb{Q}(\sqrt{-15})$, and N_{11},N_{11}' have entries in $\mathbb{Q}(\sqrt{-7})$, with the formulas from [12], and the family R_7^q passes through a Butson matrix. As for the family BN_6^q, which is defined on the arc formed by the numbers $q\in\mathbb{T}$ with $q+\bar{q}\leq\sqrt{3}-1$, and is real analytic in its interior, for a suitable choice of the square root in its formula, its generic members are wild by [20], by specialization at the interior point $q=i$, where the entries are $\pm 1,\pm i$, the paper [20] covering as well some algebraic parameters. We have the following result:

Theorem 5.12. *The matrices $H_9,BN_{10},N_{11},N_{11}'$ are wild, and so are the members of the family R_7^q, outside a countable set of parameters.*

Proof. For H_9 and BN_{10} the entries are units at the primes above 5 and 3, which ramify in $\mathbb{Q}(\sqrt{-15})$, with residue fields $\mathbb{F}_5$ and $\mathbb{F}_3$, and for N_{11},N_{11}' the same holds at the prime above 7, in $\mathbb{Q}(\sqrt{-7})$. The entries reduce to ± 1, and the reduced matrices are equivalent to J_N-2I_N, with $N=9,10,11,11$ satisfying $N=4(p)$, for $p=5,3,7,7$, so Theorem 5.6 (4) applies. The matrix R_7^1 is one of the 2 matrices of $BH(7,6)$, which have sixth roots of unity as entries, and modulo the prime above 3 of $\mathbb{Q}(e^{2\pi i/3})$ we have $e^{2\pi i/3}=1$, so the entries reduce to ± 1, and the reduced matrix is equivalent to J_7-2I_7, with $7=4(3)$, as noted in [20]. Thus R_7^1 is wild, and since the entries of R_7^q are Laurent polynomials in q, the specialization argument in the proof of Theorem 5.6 gives the result for R_7^q. □

The program `opencases.py` checks all this, in exact arithmetic, as well as the above facts about BN_6^q, the arc being checked numerically. As for the other open cases of the table, they are beyond the reach of Theorem 5.10, for a simple reason. Observe first that the numbers $|S(a,b,c,d)|$ do not depend on the representative of H, up to a relabeling of the indices, and that an affine profile forces $\sigma=-1(P)$ on the quadruples of distinct indices. Thus, we have:

Proposition 5.13. *If $S(a,b,c,d)=0$ for some quadruple of distinct indices, then no representative of H has an affine profile, at any prime. For a real Hadamard matrix of order $N=0(8)$, with $N\leq 56$, such a quadruple always exists.*

Proof. The first assertion is clear from the above. For the second one, the proof of Lemma 5.2 gives $S=N-8\sum_j x_jy_jz_j$, so $S\in 8\mathbb{Z}$ on the quadruples of distinct rows. On the other hand, as in the proof of Lemma 5.9, for a,b,c distinct we have $\sum_d S(a,b,c,d)^2=N^2$, and the terms with $d=b,c$ vanish, as does the term with $d=a$, which is $\sum_y H_{by}H_{cy}=0$. Thus the $N-3$ numbers $S(a,b,c,d)$ with $d\notin\{a,b,c\}$ are multiples of 8, whose squares sum up to N^2, and if they were

all nonzero, we would get $64(N-3) \leq N^2$. With $N = 8m$ this reads $m^2 - 8m + 3 \geq 0$, which fails for $1 \leq m \leq 7$. □

At $N = 4(8)$ the same proof gives $S = 4(8)$, so $S \neq 0$, in agreement with Theorem 5.6 (1). The program `opencases.py` computes the zeros of S on the quadruples of distinct indices, in exact arithmetic. For BF_6 there are 96 zeros, out of 360 quadruples, and for $L_{8,30}$ there are 384, out of 1680. For the families $F_{8,2}, F_{8,4}, K_{8,2}, M_{8,1}$ the numbers S vanish identically in the parameters, on respectively 288, 576, 384, 384 quadruples, so no member of these families can be treated, and the specialization argument is of no help. For P^1_{104} the program finds a quadruple of distinct rows with $S = 0$, and for $P^1_{24}, P^1_{32}, P^1_{48}$ and the 45 classes of order 24, Proposition 5.13 applies directly. The same holds for the transposes, which have zeros too.

The 2 remaining matrices, $I_{8,12}$ and $K_{8,30}$, have profiles with no zeros, but they fail a necessary condition, coming from the fact that the numbers $|\sigma(a,b,c,d)|^2 = \sigma(a,b,c,d)\sigma(b,a,d,c)$ do not depend on the row phases. Since these numbers must be 1 modulo P on the quadruples of distinct indices, the prime P must contain all the numbers $|S(a,b,c,d)|^2 - N^2$, and `opencases.py` finds that there is no such prime for $K_{8,30}$, and that for $I_{8,12}$ only the primes above 2 remain, where the full test, carried out by `screen.py` on $BH(8,12)$, fails. The program also sweeps over the residues of the parameters of $F_{8,2}, K_{8,2}, R_7^q$ in some small finite fields, finding only the affine members of R_7^q above 3, and checks the relations of Lemma 5.9 on them. Summarizing, after Theorem 5.12, the open cases of the table are BF_6, the matrices of order 8, and the real matrices of orders 24, 32, 48, 104, and Theorem 5.10 cannot apply to any of them.

Among these open cases, the circulant ones, of the form $H_{ij} = h(j-i)$, with $h : \mathbb{Z}_N \to \mathbb{T}$, have some extra structure. The simplest one is the Björck-Fröberg matrix BF_6, which is the circulant matrix with first row $(1, ia, -a, -i, -\bar{a}, i\bar{a})$, where $a = (1 - \sqrt{3} + i\sqrt{2\sqrt{3}})/2$, coming from the cyclic 6-roots [23]. By [22], a circulant matrix H is Hadamard precisely when its eigenvalues $\lambda_k = \sum_t h(t)w^{kt}$, with $w = e^{2\pi i/N}$, all have modulus $\sqrt{N}$, that is, when the Fourier transform of h is unimodular too, up to normalization, and we refer to [52] for the circulant matrices of prime order. The profile takes a simple form in terms of the eigenvalues:

Proposition 5.14. *For a circulant Hadamard matrix H, with eigenvalues λ_k, the profile is given by $S(a,b,c,d) = N^{-3}\sum_{k+n=l+m} \lambda_k\bar{\lambda}_l\bar{\lambda}_m\lambda_n w^{ka-lb-mc+nd}$, the sum being over $k,l,m,n \in \mathbb{Z}_N$. Thus the Fourier transform of the profile is supported by the quadruples with $k+n = l+m$, where it has constant modulus $1/N$.*

Proof. We have $H_{ay} = h(y-a)$, with $h(t) = \frac{1}{N}\sum_k \lambda_k w^{-kt}$, and inserting this in the definition of S, the sum over y gives N when $k+n = l+m$, and 0 otherwise, which gives the formula. The second assertion follows from $|\lambda_k|^2 = N$. □

Thus the zeros of the profile of BF_6, which block Theorem 5.10, are cancellations in sums of unimodular numbers, and disappear in the Fourier basis. The other feature of the circulant matrices is the symmetry under the cyclic shift $\sigma(i) = i+1$. Indeed, with U the cyclic shift of the coordinates, we have $\xi_{\sigma(i)\sigma(j)} = U\xi_{ij}$, so $P_{\sigma(i)\sigma(j)} = UP_{ij}U^*$, and by the trace property T_k commutes with $\sigma^{\otimes k}$, acting on $(\mathbb{C}^N)^{\otimes k}$. Thus P_k^H is invariant under $\sigma^{\otimes k}$, and we have:

Theorem 5.15. *Let H be circulant, with $N \geq 4$, and let $G' \subset S_N^+$ be the quantum group generated by G_H and by the cyclic group $\mathbb{Z}_N$, acting by cyclic shifts. Then $G_H = S_N^+$ if and only if $G' = S_N^+$, that is, if and only if the shift-invariant vectors of P_k^H are Temperley-Lieb, for any k.*

Proof. By Tannakian duality [111], the fixed vectors of G' are the shift-invariant vectors of P_k^H, which gives the second equivalence, and one implication in the first one is clear. Conversely, assume $G' = S_N^+$, and let $x \in P_k^H$ be an eigenvector of $\sigma^{\otimes k}$. The rank one operator $R = xx^*$ intertwines the representation $u^{\otimes k}$ of G_H with itself, because x is fixed, and commutes with

$\sigma^{\otimes k}$, so it belongs to $End_{G'}(u^{\otimes k})$, which is $End_{S_N^+}(u^{\otimes k})$, by the Frobenius isomorphism with $Fix(u^{\otimes 2k})$. Thus the line spanned by x is invariant under S_N^+, and so is a corepresentation of dimension 1. But for $N \geq 4$ the irreducible corepresentations of S_N^+ other than the trivial one have dimension at least $N-1$, so x is fixed by S_N^+, and so is Temperley-Lieb. Since P_k^H is spanned by such eigenvectors, we obtain $G_H = S_N^+$. □

The proof works more generally for any abelian group of permutations σ of the indices satisfying $P_{\sigma(i)\sigma(j)} = U_\sigma P_{ij} U_\sigma^*$, for some unitaries U_σ. The point is that the obstruction coming from the 1-dimensional corepresentations, which in the classical case comes for instance from $A_N \subset S_N$, is absent for S_N^+. In practice, the shift-invariant vectors are determined by their values on the words starting with 0, so they form a space of dimension N^{k-1}. More generally, T_k decomposes over the characters $t \to w^{ct}$ of $\mathbb{Z}_N$, into N blocks $T_k^{(c)}$ of size N^{k-1}, and so do the Jones-Wenzl pieces Φ_k from Theorem 4.2, which are the restrictions of T_k to the vectors whose leg sums vanish, and which vanish on the words with 2 equal adjacent letters [12]. By Theorem 5.15, a circulant matrix H is wild precisely when $\|\Phi_k^{(0)}\| < 1$ for any k. The program `circulant.py` implements this, checks Proposition 5.14, and computes the block norms for BF_6:

k	$c=0$	$c=1,5$	$c=2,4$	$c=3$
2	0.27897	0.63350	0.51414	0.30940
3	0.40192	0.63350	0.51414	0.50952
4	0.66084	0.66666	0.66619	0.59734
5	0.66084	0.66666	0.66619	0.59734
6	0.66516	0.66666	0.66619	0.66163

The maximum over the blocks is the norm $\|\Phi_k\|$ computed in [12], and the relevant block $c=0$ stays below the other ones, while approaching $2/3$ as well. The program `charles.py`, run on a laptop, extends these computations up to level 10, in dimension 6^9 for each block, and finds norms between 0.6660 and 0.6667 at the levels 7 to 10, these values being lower bounds, coming from 40 Lanczos steps, so this is only numerical evidence for $c_k = Cat_k$ with $k \leq 10$, extending [12] by one level. The same program computes the blocks for the 2 circulant matrices of order 7 discussed below, up to level 9, with norms below $4/7$. The question is then whether the Fourier form of the profile, which has no zeros, and the reduction to the block $c=0$, can be used for proving that these norms stay below 1, at all levels.

The circulant matrices can also be searched for directly. By [22], these are given by the vectors $h \in \mathbb{T}^N$ whose Fourier transform has constant modulus, which are the common points of 2 tori, and can be found by alternating projections between them, followed by Newton's method. The program `fury.py`, run on a laptop, does this from random starting points, for $N = 6, \ldots, 13$, and sorts the results by the spectrum of T_2, which is invariant under equivalence, transposition and conjugation, and the program `liston.py` does the same for $N = 14, \ldots, 20$. The numbers of spectra with $c_2 = 2$ found in this way, which are lower bounds for the numbers of classes, all these matrices being numerically isolated among the circulant ones, are as follows:

N	6	7	8	9	10	11	12	13
classes	1	2	3	2	8	12	19	23
N	14	15	16	17	18	19	20	
classes	50	69	93	121	365	371	821	

At $N=6$ this is BF_6, which is however not isolated among all the matrices of order 6, by [30], and at $N=7$ these are the circulant matrices coming from the cyclic 7-roots of simple index 2 and 3 [22, 52], denoted C_{7A} and C_{7C} in [100], the first one, which has 2 distinct entries, being the matrix of the Fano plane from [20]. For all these matrices with $N \leq 13$, `fury.py` computes the norms of the blocks $\Phi_k^{(c)}$ up to the level 7, 6 or 5, depending on N, and finds them all smaller

than 1, so the computed values support $c_k = Cat_k$ at these levels, and `liston.py` does the same up to level 4, for the matrices of order $N \geq 14$ whose profile has no zeros, and whose first row takes at most 8 values, up to the characters. In most cases the norms approach $4/N$, from below, which is the upper edge of the support of the free Poisson law, rescaled by $1/N$, in agreement with the main character being free Poisson, and the value $2/3$ for BF_6 is of this type, while some matrices have norms above $4/N$. Also, at the prime orders 7, 11, 13 all the matrices found have profiles with no zeros, and at the orders 8, 10, 12 some of them do too, so for them Theorem 5.10 can apply, and we have:

Theorem 5.16. *The circulant matrices C_8 and C_{10} having as first rows*

$$(1, z, z, -1, z, 1, 1, -1) \quad , \quad (1, a, 1, -i, ia, a, ia, a, ia, -i)$$

with $z = (-1 - 2i\sqrt{2})/3$ and $a = (\sqrt{5} + 2i)/3$ are complex Hadamard matrices, and are wild.

Proof. For C_8 the entries are in $\mathbb{Q}(i\sqrt{2})$, and $z = \bar{\pi}/\pi$, with $\pi = 1 + i\sqrt{2}$ of norm 3, so the entries are units at the prime $P = (i\sqrt{2})$ above 2, which has residue field $\mathbb{F}_2$, and with $v_P(8) = 6$. A direct computation shows that $v_P(S) = 6$ on the quadruples of distinct indices, and $v_P(S) \geq 7$ on the restricted quadruples with $a = d$ or $b = c$, but not both. Thus $\sigma = S/8$ belongs to the local ring, and is $1 = -1$ modulo P in the first case, and 0 modulo P in the second case, and 1 modulo P when $a = d$ and $b = c$, which is the affine condition with $X = Y = 1$. Thus Theorem 5.10 applies. For C_{10} the entries are in $\mathbb{Q}(i, \sqrt{5})$, whose ring of integers is $\mathbb{Z}[i, f]$, with $f = (1 + \sqrt{5})/2$, and they are units at the prime $P = (1 + i)$, the only prime above 2, with residue field $\mathbb{F}_4$, the reduction being given by $i \to 1$ and $f \to w$, with $w^2 = w + 1$. The computation shows that $\sigma = S/10$ belongs to the local ring, is 1 modulo P on the quadruples of distinct indices, 0 modulo P on the restricted quadruples with $a = d$ or $b = c$, but not both, and 1 modulo P when $a = d$ and $b = c$, so again $X = Y = 1$, and Theorem 5.10 applies. □

The program `circwild.py` checks all this, in exact arithmetic. Both congruences take place at the prime above 2, with $X = Y = 1$, as in the real case of Lemma 5.2, but at $N = 0(8)$ for C_8, an order where no real matrix can work, by Proposition 5.13. At the orders 14 and 15, the search finds circulant matrices with 3 distinct entries, of the form $1, z, z^2$, with z quadratic, and here the congruence takes place at a ramified prime dividing N, with $X = Y = -1$, as in the cube root case of Theorem 5.6:

Theorem 5.17. *The circulant matrices C_{14} and C_{15} having as first rows*

$$(1, 1, z, z^2, z, z^2, z^2, z, z, 1, z^2, 1, z^2, z^2) \quad , \quad (1, 1, 1, 1, y, y^2, y^2, 1, y^2, y, 1, 1, y^2, 1, y)$$

with $z = (-3 - i\sqrt{7})/4$ and $y = (-1 + i\sqrt{15})/4$ are complex Hadamard matrices, and are wild.

Proof. For C_{14} the entries are in $\mathbb{Q}(i\sqrt{7})$, and since they have modulus 1, and denominators powers of 2, they are units at the primes not above 2, and in particular at the prime $P = (i\sqrt{7})$ above 7, which is ramified, with local ring $\mathbf{Z}_7[i\sqrt{7}]$ and residue field $\mathbb{F}_7$. The computation shows that $\sigma = S/14$ belongs to the local ring, and that modulo P it is -1 on the quadruples of distinct indices, -2 on the restricted quadruples with $a = d$ or $b = c$, but not both, and -3 when $a = d$ and $b = c$, which is the affine condition with $X = Y = -1$. Thus Theorem 5.10 applies. For C_{15} the same holds at the prime above 5 of $\mathbb{Q}(i\sqrt{15})$, with the same residues $-1, -2, -3$. □

These matrices are checked by `circwild.py` too. Note that $z = (-3 - i\sqrt{7})/4$ is, up to conjugation, the entry of the 2-valued circulant matrix C_{7A} of order 7, and that the 2-valued circulant matrix of order 15 found by the search, which has entries $1, y^2$, is one of the matrices treated in [20]. The other circulant matrices with no zeros in the profile can be treated in the same way, once their entries are known exactly, and the simplest case is that of the matrix C_{7C}

of order 7, of simple index 3, which turns out to be accessible via Theorem 5.6 (4). Its first row is $(1, A, B, C, C, B, A)$, with $A, B, C \in \mathbb{T}$ subject to the 3 equations expressing the orthogonality of the rows, and we have:

Theorem 5.18. *The numbers $A + \bar{A}$, $B + \bar{B}$, $C + \bar{C}$ are roots of the polynomial*

$$f(u) = u^6 - 12u^5 + 18u^4 + 64u^3 - 96u^2 - 45u + 43$$

and C_{7C} reduces modulo any prime above 3 to a matrix equivalent to $J_7 - 2I_7$. Thus C_{7C} is wild, and so is its conjugate C_{7D}.

Proof. With $u = A + \bar{A}$ and $s = A - \bar{A}$, we have $s^2 = u^2 - 4$, and the numbers $B + \bar{B}$, $C + \bar{C}$, as well as the products of the imaginary parts of A with those of B and C, are polynomials in u, with rational coefficients, so the entries live in the field $\mathbb{Q}(u, s)$, where the orthogonality equations can be checked exactly. Since f is monic with integer coefficients, u is an algebraic integer, and so is A, which is a root of $x^2 - ux + 1$, and similarly for B, C, so the entries are algebraic integers, of modulus 1, and units. Now $f = (u - 1)^6$ modulo 3, so for any prime P above 3 we have $u = 1(P)$, and so $(A + 1)^2 = (u + 2)A = 0(P)$, that is, $A = -1(P)$, and similarly $B = C = -1(P)$. Thus C_{7C} reduces modulo P to the circulant matrix $2I_7 - J_7$, which is equivalent to $J_7 - 2I_7$, and since $7 = 4(3)$, Theorem 5.6 (4) applies. □

The polynomial f, and the expressions in terms of u used in the proof, were found by integer relations, from a solution computed with 120 digits, and the program `circwild.py` checks exactly that they define a solution of the orthogonality equations, matching the numerical one. The next candidate is the matrix of order 13 having 3 distinct entries, coming from the cyclic 13-roots of simple index 2, whose profile has no zeros as well, and which is wild by Theorem 5.20 below, and many more candidates come from the following searches.

Most of the matrices above are of multiplier type, in the sense that, up to equivalence, their first row is constant on the orbits of a subgroup U of the group of units of $\mathbb{Z}_N$, acting by multiplication. This is the case for C_{7A}, C_{7C}, C_8, C_{10}, C_{14} and C_{15}, and at a prime order p, with U of index d, these are the cyclic p-roots of simple index d of Björck [22]. The program `joshua.py`, run on a laptop, searches for such matrices, for $N = 6, \ldots, 32$, and for all the subgroups U generated by at most 2 elements, with 2 to 12 orbits, by solving the orthogonality equations for the phases on the orbits, by Gauss-Newton, from random starting points. At the singular points the solutions are only accurate to about 7 digits, and the program `blocks.py` snaps them to exact Butson matrices, and recomputes $\dim P_2$. The numbers of classes with $c_2 = 2$ found in this way are as follows:

N	6	7	8	9	10	11	12	13	14
classes	1	2	3	2	8	12	16	16	18
N	15	16	17	18	19	20	21	22	23
classes	43	26	33	28	43	41	60	29	19
N	24	25	26	27	28	29	30	31	32
classes	36	7	31	0	11	22	9	39	0

At the prime orders all these matrices have profiles with no zeros, and for all of them, the norms of the blocks $\Phi_k^{(0)}$ computed by `joshua.py`, at the levels $k \leq 4$ for $N \leq 20$ and $k \leq 3$ beyond, are smaller than 1, the largest one being 0.974, at $N = 24$. As for the matrices with $c_2 > 2$, their orbitals always produce block systems, as for the matrices of Diţă type, except for 4 matrices of orders 24 and 28, where the search for block systems is too long. Conversely, the program `dempsey.py` searches for the circulant matrices whose first row takes at most m values, up to the characters, for $m = 2, \ldots, 5$ and $N = 9, \ldots, 24$, with no symmetry assumed, by Gerchberg-Saxton iterations alternating with a quantization step, followed by Gauss-Newton on

the m phases. All the matrices with $c_2 = 2$ found in this way are of multiplier type, in an affine sense, the only other matrices being 51 classes at $N = 16$, with $c_2 = 3$, coming from a continuous family. Thus, the few-valued circulant candidates for wildness all have the multiplier structure, and so explicit algebraic entries.

At prime order the multiplier circulants can be followed further. The program `ali.py` computes them for the primes $5 \leq p \leq 127$ and the indices $d = 2, \ldots, 6$, and finds 250 classes, 246 of them with $c_2 = 2$, all having profiles with no zeros, and all being isolated, numerically, the defect being computed in floating point via the Berezin matrix of Section 9, and checked by a singular value computation in the doubtful cases, with an exact certificate for the 2-valued ones, explained below. The norms of the blocks $\Phi_k^{(0)}$ are all smaller than 1, at the levels up to $8, 7, 6, 5, 4, 3$, for p up to $7, 11, 13, 19, 47, 127$, the largest value at $p = 7$ and level 8 being 0.5712, close to $4/7$. Curiously, at $p = 71$ the linearized orthogonality equations of the 2-valued matrix have singular values of size 10^{-10}, instead of 10^{-2} at the neighboring primes. However, the defect is exactly 0 there, as certified by `isolated.py`, which reduces the Berezin matrix modulo a prime above 569, where $\sqrt{-71} \to 92$, and splits it, over the characters of the translations, into 71 blocks of size 71, whose nullities over $\mathbb{F}_{569}$ are $1, 2, \ldots, 2$, so that the nullity over $\mathbb{C}$ is at most $141 = 2p - 1$. The same holds for all the 2-valued matrices with $p = 3(4)$ and $7 \leq p \leq 127$. Among these matrices there are, for $p = 3(4)$, the 2-valued matrices coming from the quadratic residues, and these, and more generally those coming from the difference sets with parameters $(4n - 1, 2n - 1, n - 1)$, are wild by [20]. In our setting, this follows from Theorem 5.10, as follows:

Proposition 5.19. *Let D be a difference set with parameters $(v, k, \lambda) = (4n-1, 2n-1, n-1)$ in an abelian group G, and $h = 1+(z-1)\chi_D$, with $z = (1-2n+i\sqrt{v})/2n$. Then $H = (h(j-i))_{i,j\in G}$ is a complex Hadamard matrix, and for any prime q with $v_q(v)$ odd, the profile of H is affine at the prime above q of $\mathbb{Q}(i\sqrt{v})$, with $X = Y = -1$. Thus H is wild, for $v \geq 7$.*

Proof. Let $\varepsilon = z-1 = i\sqrt{v}(1+i\sqrt{v})/2n$. Since $|z| = 1$ we have $\varepsilon+\bar{\varepsilon} = -|\varepsilon|^2$, and $|\varepsilon|^2 = v/n$. For $s \neq 0$ the autocorrelation is $\sum_x h(x+s)\bar{h}(x) = v + k(\varepsilon+\bar{\varepsilon}) + \lambda|\varepsilon|^2 = v - (k-\lambda)v/n = 0$, so H is Hadamard. Now let P be the prime above q, which is ramified, and $e = v_q(v)$, so that $v_P(v) = 2e$ and $v_P(\varepsilon) = e$, the numbers $1 + i\sqrt{v}$ and $2n$ being units at P, because $4n = v + 1$. Expanding $S(a, b, c, d) = \sum_x h(x-a)\bar{h}(x-b)\bar{h}(x-c)h(x-d)$ in powers of $\varepsilon, \bar{\varepsilon}$, the terms of degree at least 3 vanish in $\sigma = S/v$ modulo P. Modulo P we have $n = 1/4$, $\varepsilon^2/v = \bar{\varepsilon}^2/v = -1/4n^2 = -4$, $\varepsilon\bar{\varepsilon}/v = 1/n = 4$ and $k(\varepsilon+\bar{\varepsilon})/v = -k/n = 2$, and with $A_{xy} = |(D + x) \cap (D + y)|$, which is $\lambda = -3/4$ for $x \neq y$ and $k = -1/2$ for $x = y$, we obtain

$$\sigma = 5 - 4A_{ad} - 4A_{bc} + 4(A_{ab} + A_{ac} + A_{bd} + A_{cd}) \ (P)$$

This is $5 + 8\lambda = -1$ on the quadruples of distinct indices, $5 - 4k + 12\lambda = -2$ when $a = d$ or $b = c$, but not both, and $5 - 8k + 16\lambda = -3$ when $a = d$ and $b = c$, which is the affine condition with $X = Y = -1$, and Theorem 5.10 applies. □

Since $v = 3(4)$ is not a square, there is always a prime q as above. This covers the quadratic residues modulo a prime $p = 3(4)$, which give the Björck matrices of simple index 2 at these primes, the Singer difference sets in $\mathbb{Z}_{2^m-1}$, and the twin prime difference sets in $\mathbb{Z}_p \times \mathbb{Z}_{p+2}$, and in particular the matrix C_{7A} of the Fano plane, and the 2-valued matrix of order 15, which comes from the cyclic $(15, 7, 3)$ difference set. The program `designs.py` checks the residues $-1, -2, -3$ in exact arithmetic, for the quadratic residues modulo $7, 11, 19, 23$, the Singer difference sets of orders 15 and 31, and the twin prime difference set of order 35, at all the relevant primes.

At the primes $p = 1(4)$ the Björck matrices of simple index 2 have 3 distinct entries, and the mechanism is a reduction to $2I_p - J_p$. The rows of $2I_p - J_p$ have scalar products $p - 4$, so such a reduction can only happen modulo the primes above the prime divisors q of $p - 4$, as needed for Theorem 5.6 (4). With $w = e^{2\pi i/p}$, the Gauss periods are the sums η_0, η_1 of the numbers w^x over the nonzero squares, and over the non-squares, and for $p = 1(4)$ these are $(-1 \pm \sqrt{p})/2$. We have:

Theorem 5.20. *Let $p = 1(4)$ be a prime, $s = \pm 1$, $c = 1/(1 - s\sqrt{p})$ and $a = c + i\sqrt{1 - c^2}$. The circulant matrix H of order p with first row $h(0) = 1$, $h(x) = a$ for the nonzero squares x, and $h(x) = \bar{a}$ for the non-squares x, is a complex Hadamard matrix. For $p \geq 13$, and any prime q dividing $p - 4$, the matrix H reduces modulo some prime above q to $2I_p - J_p$, and so H is wild.*

Proof. The eigenvalues of H are the numbers $\widehat{h}(k) = \sum_x h(x)w^{kx}$, and H is Hadamard when they all have modulus $\sqrt{p}$. We have $\widehat{h}(0) = 1 + (p-1)c$, and for $k \neq 0$ a square, $\widehat{h}(k) = 1 + a\eta_0 + \bar{a}\eta_1 = 1 - c + i\sqrt{p(1-c^2)}$, while for k a non-square we obtain the conjugate. Since $1 - c = -s\sqrt{p}\,c$ and $1 + (p-1)c = -s\sqrt{p}$, we get $|\widehat{h}(k)|^2 = pc^2 + p(1-c^2) = p$, and $|\widehat{h}(0)|^2 = p$. Also, $|c| < 1$, because $\sqrt{p} - 1 > 1$, so $|a| = 1$. For the second assertion, observe that $p - 4 \geq 9$ is odd, so it has odd prime divisors q, and since $(\sqrt{p} - 2s)(\sqrt{p} + 2s) = p - 4$, there is a prime P of $\mathbb{Q}(\sqrt{p})$ above q containing $\sqrt{p} - 2s$. Modulo P we have $\sqrt{p} = 2s$, so $1 - s\sqrt{p} = -1$ is a unit, and $2 - s\sqrt{p} = 0$, and so $1 + c = (2 - s\sqrt{p})/(1 - s\sqrt{p})$ is in the local ring, and vanishes modulo P. Now a is a root of $x^2 - 2cx + 1$, so it is a unit at any prime P' above P of the field of H, and from $(a+1)^2 = 2(1+c)a$ we obtain $a = -1(P')$, and so $\bar{a} = 2c - a = -1(P')$ too. Thus H reduces modulo P' to $2I_p - J_p$, which is equivalent to $J_p - 2I_p$, and since $p = 4(q)$, Theorem 5.6 (4) applies. □

At $p = 5$, where $p - 4 = 1$, we have $a = e^{4\pi i/5}$ for $s = 1$, and $a = e^{2\pi i/5}$ for $s = -1$, so $h(x) = w^{2x^2}$ and $h(x) = w^{x^2}$, and in both cases H is equivalent to F_5. The program `simpleindex.py` checks Theorem 5.20 in exact arithmetic, in the field $\mathbb{Q}(\sqrt{p}, a)$, for the primes $13 \leq p \leq 127$, and compares with `ali.py`, which finds exactly these 2 classes of simple index 2 at each prime $p = 1(4)$. Together with Proposition 5.19, this shows that the Björck matrices of simple index 2 are all wild, at the primes $p \geq 7$.

For the simple index 3, where $p = 1(3)$, the entries were found explicitly by Björck and Haagerup [24], in terms of the numbers A, B with $4p = A^2 + 27B^2$, the 12 unimodular solutions of their equations forming 2 orbits under the rotations of the cosets and the conjugation. Here we compute these matrices exactly, as for C_{7C}, and the same mechanism works, with $q = 3$ always available, because $3|p - 4$. The first row is $h(0) = 1$ and $h(x) = x_j$ on the j-th coset of the group of cubes, for $j = 0, 1, 2$, and we set $u_j = x_j + \bar{x}_j$. We have:

Theorem 5.21. *At the primes $p = 1(3)$ with $13 \leq p \leq 127$, the Björck matrices of simple index 3 found by `ali.py` form 2 classes, up to equivalence and conjugation, and all of them, except for one class at $p = 61$, reduce to $2I_p - J_p$ modulo some prime above a prime divisor of $p - 4$, and so are wild.*

Proof. For each class, the program `indexthree.py` refines the solution of `ali.py` to 600 digits, finds by integer relations the minimal polynomial f of u_0, which has degree 3 or 6, and the expressions of u_1, u_2, and of the products of the imaginary parts of x_0, x_1, x_2, as polynomials in u_0 with rational coefficients, and then checks in exact arithmetic, in the field $K = \mathbb{Q}(u_0, x_0)$, that they define a complex Hadamard matrix, and that f is irreducible, from its factorizations modulo small primes. It also checks that $u_1 = g(u_0)$, $u_2 = g(u_1)$ and $u_0 = g(u_2)$, for a polynomial g. Now if u_0, u_1, u_2 are integral and equal to -2 at a prime P of K above q, then $(x_j + 1)^2 = (u_j + 2)x_j$ gives $x_j = -1(P)$, so H reduces modulo P to $2I_p - J_p$, and Theorem 5.6 (4) applies, because $p = 4(q)$. As for the existence of P, consider the roots θ of f in an algebraic closure of $\mathbb{Q}_q$ which are integral and congruent to -2, whose number m is the multiplicity of -2 as a root of f modulo q. An embedding of K sending u_0 to such a root θ gives a prime P as above, as soon as $g(\theta)$ and $g(g(\theta))$ are among these m roots too. The program computes the number of roots θ having each of these 2 properties, from the Newton polygons of the characteristic polynomials of $g(u) + 2$ and $g(g(u)) + 2$ on the part of $\mathbb{Q}_q[u]/(f)$ corresponding to these m roots, which is cut by an idempotent, obtained by Hensel lifting. For all the classes but one, there is a prime q dividing $p - 4$ for which these numbers are both equal to m, so that all the m roots are suitable.

The q-adic computations are done with 90 digits, and the precision is certified, with a margin of at least 70 digits, by the Newton identities applied to integral elements. □

In each pair, one class has u_0, u_1, u_2 equal to the 3 roots of a cubic polynomial, and reduces modulo all the primes above the prime divisors $q \neq 3$ of $p-4$, and modulo those above 3 for $p = 13, 31$, where $p-4$ is a power of 3, and the other class has u_0, u_1, u_2 equal to 3 of the 6 roots of a sextic polynomial, and reduces modulo a prime above 3. In the notation of [24], the cubic class is the orbit of the solution $c^{(1)}$, and the sextic class is that of $c^{(2)}$, as checked numerically by `indexthree.py`, which also gives an interval isolating u_0 among the roots of f, and the sign of $Im(x_0)$, which determine the class. At $p = 7$ the 2 classes are F_7 and C_{7C}, and the program recovers Theorem 5.18. The exception is the sextic class at $p = 61$, where $f = 3u^6 - 158u^5 + 180u^4 + 4034u^3 - 1720u^2 - 9177u + 6109$, which has 4 roots congruent to -2, and 1 root which is not integral, at the primes above 3, and where only 2 of these 4 roots are sent by g to roots congruent to -2, so that, the roots being permuted by g in cycles of length 3, no prime above 3 is suitable. The prime 19, which is the other divisor of $p-4 = 57$, is not suitable either. As for a reduction to $J_p + (\lambda - 1)I_p$, as in Theorem 5.11, in the above normalization, this would require u_0, u_1, u_2 to be congruent to $2-p$. But $f(2-p) = 3^4 \cdot 61^2 \cdot 799091$, with 799091 prime, and $2-p$ is a simple root of f modulo 799091, so the 3 distinct roots u_0, u_1, u_2 cannot all be congruent to $2-p$ at a prime above it, and since $2-p = -2(3)$, the primes above 3 are excluded as well. Thus the wildness of this class remains open. For the indices $d \geq 4$ the fields are larger, the degree of u_0 being for instance above 24 at $p = 13$ and $d = 4$, and these matrices remain open.

Coming back to BF_6, whose profile has zeros, its profile on the Fourier side is given by the phase function $\Phi(k,l,m) = \lambda_k \bar{\lambda}_l \bar{\lambda}_m \lambda_{l+m-k}/N^2$ appearing in Proposition 5.14, and this function turns out to be very simple, arithmetically. The field of BF_6 is $K = \mathbb{Q}(i,a)$, of degree 8, and above 2 and 3 there is a single prime, with uniformizer $b = a+1$, which satisfies the Eisenstein equation $b^4 - 6b^3 + 12b^2 - 12b + 6 = 0$ at both primes, the residue fields being $\mathbb{F}_4$ and $\mathbb{F}_9$. We have:

Proposition 5.22. *For BF_6, the norms of the eigenvalues λ_k have the same valuations at 2 and 3 as $\sqrt{6}$, so the phase function Φ is a unit at all primes. Moreover, $\Phi = 1$ modulo the prime above 2, and $\Phi = (-1)^{(k-l)(k-m)}$ modulo the prime above 3.*

Proof. Since $\lambda_k \bar{\lambda}_k = 6$, the eigenvalues are units outside 2 and 3, and the valuations follow from the norms, there being a single prime above 2 and 3. The congruences are checked in exact arithmetic, by writing the elements of K in terms of i and b, and reducing modulo b. □

The program `fourierside.py` does these computations. Thus, modulo both primes dividing N, the profile of BF_6 on the Fourier side is congruent to phase functions of group type, as for the Fourier matrices, so its subfactor cannot be seen by congruences of order 0 at these primes. The deviations from these models have valuation 1 exactly when $k-l$ and $k-m$ are both nonzero modulo 3, at the prime above 2, and when $k-l$ and $k-m$ are nonzero, and not both even, at the prime above 3, the shadows of the 2 factors of $\mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$. Thus the wildness of BF_6, if true, should come from these first order terms, suggesting a first order version of Theorem 5.10.

6 Spectral measures and blowup

The rooted law μ of a subfactor, appearing in Proposition 5.5, is the spectral measure at the root of the square of the adjacency operator of the principal graph, whose moments are the dimensions $b_n = \dim P_n$. For a Hadamard subfactor, this is the law μ_H of the main character of G_H, from Section 2. For the subfactors of index ≤ 4 the rooted law has a circular lift, which is the image by $u \to (u + u^{-1})^2$ of a probability measure on the unit circle, see [2], and we refer

to Evans and Pugh [43] for the joint spectral measures of the $SU(3)$ nimreps. Above index 4 there is no circular lift in general, but the rooted law still determines the annular multiplicities, summed over the rotation eigenvalues, as explained by Arhancet [2], and this is precisely the formula in Proposition 5.5.

The rooted law does not determine the standard invariant, as shown for instance by the group subfactors of $\mathbb{Z}_4$ and $\mathbb{Z}_2^2$, which have the same principal graph, and different fusion rules. For the canonical Lagrangian inclusions associated to a unitary fusion category C, the rooted law was computed in [2], in terms of the formal codegrees f_E of Ostrik, the result being

$$\mu_C = \left(1 - \sum_E \frac{1}{f_E^2}\right)\delta_0 + \sum_E \frac{1}{f_E^2}\,\delta_{\dim(E)f_E}$$

with E ranging over the irreducible representations of the complexified fusion ring, and this explains why the extensions associated to the Haagerup categories have the same moments.

The paper [9] starts with 2 preliminary computations. First, at small index above 4 the rooted laws still come from finite graphs, and for the Haagerup, Asaeda-Haagerup, 3311 and 2221 subfactors [3, 51, 68] they can be computed from the moment recurrences, which give their atoms and their annular multiplicities, the scalar lift coming from the annular theory of Jones [64] having now in its support a reciprocal pair on the positive half-line. Second, the exceptional diagrams, whose ternary indices $2, 3, 5$ are the inverses of the Deligne parameters of E_6, E_7, E_8 [38], lead to the signed root measures of Deligne and de Man [39], whose atoms have a uniform description along the exceptional series, for the Cartan powers of the adjoint representation. On the general side, the numbers $c_k = \dim Fix(u^{\otimes k})$ are the moments of a probability measure, and as shown in [14], for a self-dual object in a complex tensor category the invariant tensor dimensions are Hamburger moments precisely when a Cauchy-Schwarz inequality holds for the sums of tensor powers. This is automatic in the unitary setting, and fails for some non-semisimple categories, with for instance a 6-dimensional representation whose doubled sequence is represented by a measure with 42 atoms up to degree 83, and is not a moment sequence at degree 84. The Hankel determinants of the sequence c_k, giving the Jacobi parameters of μ, are the natural tool for comparing 2 such sequences.

The question of spectral measure blowup is that of realizing the moments of μ on a space where the measure becomes simpler, as for the free Poisson law, which is the law of cc^*, with c a circular variable. For the deformed Fourier matrices, the following answer was found in [9], by extending the generic formula from Section 3 to arbitrary parameters, the blowup space being a compact abelian group:

Theorem 6.1. *For $H = F_X \otimes_Q F_Y$, with $|X| = M$, $|Y| = N$ and Q dephased, let $K_Q \subset \mathbb{T}^{X\times Y}$ be the closed subgroup generated by the dephasings of the matrices $W_{ij}^{k,e} = Q_{i+k,j+e}Q_{ij}/(Q_{i+k,j}Q_{i,j+e})$, with $k \in X$ and $e \in Y$. Then, with $Z \in K_Q$ Haar distributed, we have*

$$\mu_H = \left(1 - \frac{1}{N}\right)\delta_0 + \frac{1}{N}\,law(ZZ^*)$$

with respect to the normalized trace of $M_M(\mathbb{C})$, and so $\dim P_n = \frac{1}{MN}\int_{K_Q} Tr((ZZ^*)^n)dZ$, for any $n \geq 1$.

Proof. This is from [9], the proof using the crossed coproduct descriptions of [10, 12, 18], and the fact that the dephased second differences of Q are the fluxes of the rectangles. □

The group K_Q is the whole dephased torus for generic Q, and is finite when the entries of Q are roots of unity. Geometrically, the lattice $I_X \otimes I_Y$ is the first homology of the complete bipartite graph $K_{X,Y}$, and K_Q is the group of flat connections which are trivial on the relations, giving an abelian covering [9]. The generic moments are lower bounds for all the rooted moments, with

equality precisely in the generic case, and with $d = \dim K_Q$ we have $\dim P_n \sim C(MN)^n n^{-d/2}$, so the rooted law determines the dimension of K_Q. More precisely, the first strict inequality between the rooted moments and the generic ones occurs at the length of the shortest nontrivial relation between the crossed coproduct generators [9]. On the other hand, the rooted law loses information: transposition preserves it for any complex Hadamard matrix [12], while for generic parameters with $M \neq N$ it changes the isomorphism class of the quantum group [9]. At index 6 the relation subgroups of $F_2 \otimes_Q F_3$ are the ideals of the Eisenstein integers, the finite cases giving the coset graphs of the torus maps $\{6,3\}_{b,c}$, which include the cube, the Heawood graph and the Pappus graph, and the generic law is

$$\mu = \frac{2}{3}\,\delta_0 + \frac{1}{6}\,p_3(|x-3|)dx$$

with p_3 being the density of the length of the 3-step planar random walk. At index 8 the diamond lattice appears, with arbitrarily delayed exceptional moments, at index 9 the triangular lattice appears, and the circulant families at square index give planar random walk laws as well. Following [14], the first Jacobi defect can be located, and there are finite depth laws with arbitrarily long generic truncations [9]. In the generic case, the blowup can be reformulated in terms of lattice walks, as follows. Let Γ_{MN} be the maximal abelian cover of the complete bipartite graph $K_{M,N}$, which is the topological crystal of $K_{M,N}$ in the sense of Sunada [97], with deck group $\mathbb{Z}^{(M-1)(N-1)}$. At $M = 2$ this is the subdivision of the maximal abelian cover of the graph having 2 vertices and N edges between them, which is the honeycomb lattice at $N = 3$, and the diamond lattice at $N = 4$. The vertices of degree 2, sitting on the edges of these lattices, form their line graphs, which are the kagome lattice and the pyrochlore lattice.

Theorem 6.2. *For $H = F_X \otimes_Q F_Y$ with $|X| = M$, $|Y| = N$ and Q generic, and $k \geq 1$, $\dim P_k$ is $1/M$ times the number of closed walks of length $2k$ on Γ_{MN}, based at a vertex of degree M. In particular, at index 6, respectively 8, the number $\dim P_k$ is half the number of closed walks of length k on the kagome, respectively pyrochlore lattice, having 2 loops added at each vertex.*

Proof. For Q generic the group K_Q is the dephased torus, and $Z \in K_Q$ is the matrix having 1 on the first row and column, and free phases elsewhere. This is the Bloch form of the incidence matrix of Γ_{MN}, the free phases being the characters of the deck group, evaluated on the edges outside the spanning tree formed by the edges of the first row and column. Thus $\int_{K_Q} Tr((Z^*Z)^k)dZ$ is N times the number of closed walks of length $2k$ based at a vertex of degree M, all these vertices being equivalent, and since $Tr((ZZ^*)^k) = Tr((Z^*Z)^k)$, Theorem 6.1 gives the first assertion. At $M = 2$, the walks of length 2 between vertices of degree 2 are the moves to the $2(N-1)$ neighbors in the line graph, together with 2 returns, through the 2 endpoints, so Z^*Z is the Bloch form of the adjacency matrix of the line graph, plus 2, which gives the second assertion. □

For the kagome and pyrochlore lattices, the first values of these numbers are as follows, as computed by `crystal.py`, which checks them against the direct computation of $\dim P_k$ for $k \leq 4$, respectively $k \leq 3$:

$$1, 4, 18, 86, 426, 2164, 11208, \ldots \qquad\qquad 1, 5, 28, 167, 1036, 6608, 43072, \ldots$$

The flat bands of these 2 lattices, which make them frustrated, and the pyrochlore lattice the lattice of spin ice [25], contribute to the atom of μ at 0. Indeed, we have $\mu = (1-1/M)\delta_0 + (1/M)\nu$, where ν is the spectral measure of Z^*Z, which for $M \leq N$ has at least $N - M$ eigenvalues 0 at each point, with equality at the generic points, and at $M = 2$ these are the flat bands of the line graph at -2, namely 1 band for kagome, and 2 bands for pyrochlore. For non-generic Q the same holds with Γ_{MN} replaced by its quotient by the annihilator of K_Q, the parameters playing the role of twisted boundary conditions, and at roots of unity we obtain finite abelian covers of $K_{M,N}$, such as the kagome lattices on the torus maps $\{6,3\}_{b,c}$. The kagome lattice also appears

in [4], through the closed words in the generators of the generic Diţă quantum group at index 6, and Theorem 6.2 explains this, the moments of this quantum group being kagome walk counts.

By combining the blowup with Proposition 5.5, the annular multiplicities of the deformed Fourier matrices can be computed explicitly. Since $q_n(0) = 2(-1)^n$, we obtain, for $MN > 4$ and $n \geq 2$:

$$a_n = 2\left(1 - \frac{1}{N}\right)(-1)^n + \frac{1}{MN}\int_{K_Q} Tr(q_n(ZZ^*))dZ$$

In particular, for the generalized Fourier matrices, where K_Q is trivial and ZZ^* has eigenvalues MN and 0, we obtain the following formula, which is checked against the direct computation of the spaces W_n by `congruence.py`, for F_6 and W_8:

Theorem 6.3. *For a generalized Fourier matrix F_G, with $|G| = N > 4$, the annular multiplicities are given, for $n \geq 2$, by*

$$a_n = \frac{q_n(N) + 2(N-1)(-1)^n}{N}$$

and grow like λ^n/N, where $\lambda = (N - 2 + \sqrt{N(N-4)})/2$. For the deformed Fourier matrices $F_X \otimes_Q F_Y$, the annular multiplicities are given by the above integral over K_Q, and are unbounded.

Proof. For F_G the rooted law is $(1-1/N)\delta_0 + (1/N)\delta_N$, because $\dim P_n = N^{n-1}$ for $n \geq 1$, and Proposition 5.5 gives the formula, with the growth coming from $q_n(x) = \lambda(x)^n + \lambda(x)^{-n}$ for $x > 4$. In the deformed case, the formula is the one above, and since K_Q contains the all-one matrix, for which ZZ^* has the eigenvalue $MN > 4$, and the Haar measure has full support, μ_H charges $(4, \infty)$, and the proof of Proposition 5.5 shows that the numbers a_n grow exponentially. □

Summarizing, the rooted laws of the main classes of complex Hadamard matrices are as follows, with the wild matrices, where $c_k = Cat_k$, having as rooted law the free Poisson law of parameter 1, also called Marchenko-Pastur law, which is the law of the main character of S_N^+:

H	G_H	μ_H
$F_G,\ \|G\| = N$	G	$(1-1/N)\delta_0 + (1/N)\delta_N$
$F_X \otimes_Q F_Y$	crossed coproduct	$(1-1/N)\delta_0 + (1/N)law(ZZ^*),\ Z \in K_Q$
$F_2 \otimes_Q F_3,\ Q$ generic	generic Diţă	$(2/3)\delta_0 + (1/6)p_3(\|x-3\|)dx$
wild	S_N^+	$(2\pi)^{-1}\sqrt{4x^{-1}-1}\,dx$ on $[0,4]$

Thus the wild matrices are at the opposite end from the Fourier matrices, whose rooted law is supported by $\{0, N\}$, the deformed Fourier matrices interpolating between these 2 extremes.

7 Gauge fields and the glow

The analytic invariants come from the statistical mechanics of the arrays of spins coupled through H, studied in [35, 44, 90]. Consider 2 layers of N parallel wires, crossing each other, with a spin a_j on each wire of the first layer, and a spin b_k on each wire of the second layer, the junction at the crossing (j,k) coupling these 2 spins with strength $H_{jk}/\sqrt{N}$. With $\Omega = \langle Hb, a\rangle = \sum_{jk} H_{jk}\bar{a}_j b_k$, the energy of a configuration is $-\Omega/\sqrt{N}$, or its real part. The law of Ω, with a, b uniform, is the glow of H, which is the central analytic invariant of H, being the density of states of the array, the thermodynamic functions being functions of it. The glow is determined by its moments, which are sums of partition functions, as explained below, and in the real case Ω takes finitely many values, so its law is determined by finitely many moments. For the real matrices of order 16 and 20 the law itself can be compared directly, by enumerating the spins b, conditional on which Ω is a sum of independent symmetric signs, as in Section 8. There are 3 versions of this model, studied so far as follows:

1. The Ising arrays [90], with H real, and spins $a_j, b_k = \pm 1$. This is the switching game of Gale and Berlekamp [45] at finite temperature, and a deterministic instance of the random orthogonal model of Marinari, Parisi and Ritort [78, 79]. The moments are universal polynomials in N up to order 14, the first correction being given by the 4-profile, and at $N = 16$ the 5 densities of states form a 1-parameter family, the parameter being the number $2^{14-r} - 128$ of crystalline ground states, with r the 2-rank.

2. The planar rotators [35], with H complex, and spins $a_j, b_k \in \mathbb{T}$. This is an array of $2N$ superconducting wires, coupled by Josephson junctions, the matrix H being the pattern of fluxes, and the Fourier matrix giving the uniform field of Chandra, Ioffe and Sherrington [33]. The moments are computed through order 12, in terms of flux harmonics, and the ground states correspond to the bistochastic forms with positive line sums.

3. The design arrays [44], with $H = \sqrt{N}U$ being an almost Hadamard matrix, that is, with U being a local maximizer of the 1-norm on O_N, as for the matrices coming from the block designs. These arrays have crystalline states at any order, such as the hyperovals, the unitals and the Baer subplanes of the projective plane of order 4, and in the complex case there is a circulant matrix of order 16 contradicting the almost Hadamard conjecture, which is supported by numerical evidence at $N = 4, 5$ [34].

On the moving side, the matrix being now a particle on U_N, or on the phase torus, the geometry meets the defect, the Hessian of the trace norm at a Hadamard matrix being given by the Berezin matrix, as explained in Section 9. Back to the frozen arrays, the dictionary with the Jones model is as follows. The gauge transformations $H \to D_a H D_b$, with D_a, D_b diagonal, are part of the equivalence, and do not change the invariants of Section 1. But $\Omega(a, b) = \sum_{\sigma\sigma'} \bar{a}_\sigma H_{\sigma\sigma'} b_{\sigma'}$ is the partition function of the Jones model on the simplest open diagram, a single edge with free ends, computed in the gauge (a, b), and this quantity is not gauge invariant. Thus:

1. The glow is the law of the single edge partition function over the gauge group, and the arrays are the Jones model with the gauge made dynamical, with Boltzmann weight $e^{\beta\Omega/\sqrt{N}}$. The spins of the arrays are the gauge transformations of the Jones model, real for the Ising arrays, and complex for the planar rotators.

2. The crystalline states of the arrays are the gauges in which H is bistochastic. In the complex case such gauges always exist, by the Idel–Wolf theorem [58], and in the real case they exist precisely for the regular Hadamard matrices.

3. The gauge invariant quantities are the fluxes. For the rectangle on the rows a, b and the columns j, l, the flux phase is $e^{i\Phi} = H_{aj}\bar{H}_{bj}H_{bl}\bar{H}_{al}$, and in the real case the rectangle is frustrated in the sense of Toulouse [106] when $\Phi = \pi$.

The 3rd point gives the following interpretation of Reidemeister II, which in the real case is the result of [90] stating that a real Hadamard matrix has the maximal number $N^2/4$ of frustrated rectangles, on any 2 rows:

Proposition 7.1. *A matrix $H \in M_N(\mathbb{T})$ is Hadamard precisely when, for any 2 rows $a \neq b$, the flux phases of the N^2 rectangles on these rows sum up to 0, the degenerate rectangles included.*

Proof. With $d_j = H_{aj}\bar{H}_{bj}$, the sum of the flux phases is $\sum_{jl} d_j \bar{d}_l = |\sum_j d_j|^2 = |\langle H_a, H_b\rangle|^2$. □

Thus Reidemeister II for the Jones model is the same as maximal and uniform frustration for the arrays. Regarding now the moments, for a uniform sign u the cumulants are the numbers c_m given by $\log\cosh t = \sum_m c_m t^m/m!$, namely $c_2 = 1$, $c_4 = -2$, $c_6 = 16$, $c_8 = -272$. For a uniform phase u, the nonzero joint cumulants of $u, \bar{u}$ are those of k copies of u and k copies of $\bar{u}$, given by

$\log \sum_k x^k/k!^2 = \sum_k c'_k x^k/k!^2$, namely $c'_1 = 1$, $c'_2 = -1$, $c'_3 = 4$. Consider the $2p$ factors of Ω^{2p}, or of $\Omega^p \bar{\Omega}^p$, as $2p$ positions, and 2 partitions π, ρ of these positions, for the row and column indices. This gives a bipartite multigraph $D_{\pi\rho}$, whose vertices are the blocks of π and ρ, with an edge for each position, signed according to Ω or $\bar{\Omega}$. The gauge average glues the copies of the open edge into such graphs, planar or not, and only the gauge invariant graphs survive, namely the balanced ones in the complex case, and those with even degrees in the real case, which become balanced, with the same weight, after signing the edges alternately along an Euler circuit of each component. We have:

Theorem 7.2. *The moments of the glow are the following sums of Jones partition functions,*

$$E(\Omega^{2p}) = \sum_{\pi\rho} c_\pi c_\rho S_{D_{\pi\rho}}(H) \quad , \quad E(|\Omega|^{2p}) = \sum_{\pi\rho} c'_\pi c'_\rho S_{D_{\pi\rho}}(H)$$

in the real and complex case, where $c_\pi = \prod_{B\in\pi} c_{|B|}$, *and where* $c'_\pi = \prod_{B\in\pi} c'_{|B|/2}$, *the product being zero when a block is not balanced.*

Proof. For fixed labels, the moment-cumulant formula gives $E(a_{x_1} \ldots a_{x_{2p}}) = \sum_\pi \prod_{B\in\pi} c_{|B|}$, the sum being over the partitions π such that x is constant on the blocks, because the joint cumulants of independent variables vanish, unless all the variables coincide. Doing the same for b, and summing over the labels, gives the first formula, and the complex case is similar. □

8 Transfer matrices and fluxes

Let H be a real Hadamard matrix. The graphs of Theorem 7.2 can be reduced with the Jones relations: a vertex of degree 2 is removed by $HH^t = H^tH = N1_N$, which is Reidemeister II, and a pair of parallel edges is removed by $H_{jk}^2 = 1$. A graph which cannot be reduced has all degrees even and at least 4, and so at least 4 vertices on each side, and at least 16 edges, with equality only for $K_{4,4}$, which is not planar. In fact, the reductions preserve the planarity, and a simple planar bipartite graph has a vertex of degree at most 3, by the Euler formula, so the planar graphs reduce completely, and their values are universal. This gives in particular the universality of the moments up to order 14, and the formula $E(\Omega^{16}) = U_8(N) + \frac{32}{3} \cdot 16! \cdot P_4(H)$, where $P_4(H)$ is the sum over the sets I of 4 rows of $(\sum_k \prod_{j\in I} H_{jk})^4$, as explained in [90]. In terms of the transfer matrix T_2, this reads:

Proposition 8.1. *For a real Hadamard matrix* H *of order* N *we have* $Tr(T_2^2) = N^{-6} S_{K_{4,4}}(H)$, *and*

$$E(\Omega^{16}) = V(N) + \frac{4}{9} \cdot 16! \cdot N^6 \, Tr(T_2^2)$$

with V *a universal polynomial.*

Proof. We have $Tr(T_2^2) = N^{-6} \sum_{abcd} (\sum_x H_{ax} H_{bx} H_{cx} H_{dx})^4$, which is the contraction along $K_{4,4}$. The terms with a, b, c, d distinct give $4! P_4(H)$, and the other terms are universal, because a repeated row index can be removed with $H_{jk}^2 = 1$, and then the orthogonality applies. □

Thus the sequence $m \to Tr(T_2^m)$ goes from the real glow, at $m = 2$, to the subfactor, at $m = \infty$, because $Tr(T_2^m) \to \dim P_2^H$. The program `subfactor.py` computes all these objects for the matrices of order $N \leq 20$, taken from the catalogue of McKay [80], and converted by `convert.py`, the traces being computed exactly from the integer matrix $N^3 T_2$, and $\dim P_3$ exactly too, from the integer matrix $N^4(T_3 - 1)$, whose blocks have zero row sums, and a nonzero principal minor of size one less, checked modulo a prime. The spectra of T_2 are computed in floating point, and the program also computes the sizes of the components for the transposes,

which are used below. This gives the following table, where r is the 2-rank, and c_H the number of crystalline states, from [90]:

H	r	c_H	$Tr(T_2^2)$	spectrum of T_2	$\dim P_2$	$\dim P_3$
$\text{had}_{16.0}$	4	896	16	1^{16}	16	256
$\text{had}_{16.1}$	5	384	23/2	$1^7, (1/2)^{18}$	7	58
$\text{had}_{16.2}$	6	128	37/4	$1^4, (3/4)^2, (1/2)^{12}, (1/4)^{18}$	4	21
$\text{had}_{16.3}, \text{had}_{16.4}$	7	0	65/8	$1^3, (1/2 \pm \sqrt{2}/8)^6, (1/4)^{28}$	3	11
$\text{had}_{20.0}, \text{had}_{20.1}, \text{had}_{20.2}$	18	0	17356/3125	different	2	5

The traces of the minimal projections of P_2^H are proportional to the sizes of the components in Theorem 1.2, as in [86], and for $\text{had}_{16.3}$ these sizes are $128, 112, 16$, while for its transpose, as for $\text{had}_{16.4}$, they are $224, 16, 16$. Thus the subfactor, with its trace, separates $\text{had}_{16.3}$ from its transpose, while their glows and the moments of their subfactors coincide, as explained below. We should mention that [67] reports the dimensions $\dim P_2 = 16, 8, 5, 3, 3$ for the 5 classes of order 16, these relative commutants being used in [5] for distinguishing the 5 classes, and the subfactors of the real Hadamard matrices, and their symmetries, are studied in [21]. Our computations, done in 3 independent ways, with the eigenvalue 1 of T_2, with the Nomura algebra, and with the components of Theorem 1.2, give $16, 7, 4, 3, 3$. Summarizing, we have:

Theorem 8.2. *At $N = 16$ we have $Tr(T_2^2) = 7+9\cdot 2^{4-r}$, with r the 2-rank, and the dimensions of P_2, P_3 are as above. At $N = 20$, the 3 classes have the same glow, and Temperley-Lieb subfactors, but the spectra of their matrices T_2 are different, with $Tr(T_2^3) =$ 230644/78125, 1116356/390625, 219124/78125.*

Proof. The first formula is the formula of [90] for the 4-profile, rewritten with Proposition 8.1, and the rest is the computation done by `subfactor.py`, in exact arithmetic, together with Theorem 5.6 at $N = 20$, which is 4 modulo 8. As for the glow, conditional on b the variable Ω is a sum of independent symmetric signs, with coefficients $|(Hb)_j|$, and the program `isinglaw.py` finds that the histograms of the sorted lists of these numbers, over the 2^{19} vectors b with $b_1 = 1$, coincide for the 3 classes. □

In the complex case, the situation is different, the crossing of the 2 legs of T_2 entering already at order 8. Let now $H \in M_N(\mathbb{T})$ be a complex Hadamard matrix. In Theorem 7.2, a vertex with one positive and one negative edge is removed by $HH^* = H^*H = N1_N$, and a pair of parallel edges of opposite signs is removed by $|H_{jk}| = 1$. However, parallel edges of the same sign stay, since $H_{jk}^2 \neq 1$ in general. A graph which cannot be reduced has at least 2 edges of each sign at each vertex, and at least 2 vertices on each side, because a single row would produce a pair of parallel edges of opposite signs. So it has at least 8 edges, and with 8 edges the only one is the doubled rectangle, whose contraction is

$$R_2(H) = \sum_{jlkm} \left(H_{jk}\bar{H}_{jm}H_{lm}\bar{H}_{lk}\right)^2 = \sum_{jl} \left| \sum_k H_{jk}^2 \bar{H}_{lk}^2 \right|^2$$

which is the sum over the rectangles of the squared flux phases $e^{2i\Phi}$. In terms of the transfer matrix T_2, since $\sum_k H_{jk}^2\bar{H}_{lk}^2 = N\langle \xi_{jl}, \xi_{lj}\rangle$, we have $R_2(H) = N^3 Tr(T_2F)$, where F is the flip of the 2 legs of T_2, that is, a crossing. For H real we have $Tr(T_2F) = N$, so the crossing is invisible there.

Proposition 8.3. *The flux harmonics $R_k(H) = \|H^{\circ k}(H^{\circ k})^*\|_2^2$, with $H^{\circ k}$ being the entrywise power, satisfy*

$$N^3 \leq R_k(H) \leq N^4$$

with $R_1(H) = N^3$, and with equality at right precisely when H is equivalent to a Butson matrix with entries in the k-th roots of unity. Also, $R_k(H) = N^3$ precisely when $H^{\circ k}$ is Hadamard.

Proof. With $K = H^{\circ k}$, the matrix $KK^* \geq 0$ has diagonal entries N, so $R_k(H) \geq N \cdot N^2$, with equality when KK^* is diagonal, that is, K Hadamard, and $R_k(H) \leq (Tr\, KK^*)^2 = N^4$, with equality when K has rank 1, that is, $H^k_{jl} = x_j y_l$, which means that H has entries in the k-th roots of unity, up to equivalence [28]. □

The numbers $R_k(H)/N^4$ are the moments of the flux distribution of H, which is the law of the flux phase of a random ordered rectangle, the degenerate ones included. For the Fourier matrices we have $R_k(F_N) = N^3 gcd(k, N)$, as noted in [8], where the flux sums of the McNulty-Weigert matrices are computed as well, and these numbers will reappear in Section 10, as closed diagrams separating some Fourier matrices, up to quantum equivalence.

The program `glowmoments.py` computes the moments of the complex glow exactly, as $E(|\Omega|^{2m}) = \sum_{\alpha\beta} |C_m(\alpha, \beta)|^2$, where $C_m(\alpha, \beta)$ is the coefficient of $x^\alpha y^\beta$ in $(\sum_{ij} H_{ij} x_i y_j)^m$, computed by the recurrence $C_{m+1}(\alpha, \beta) = \sum_{ij} H_{ij} C_m(\alpha - e_i, \beta - e_j)$, in cyclotomic arithmetic. It finds the universal parts by exact interpolation on the Fourier matrices F_N with $N \leq 9$, and checks them at $N = 10, 11$, and for the matrices of Haagerup and Tao, and W_8. We obtain:

Theorem 8.4. *For a complex Hadamard matrix H of order N, the moments*

$$E(|\Omega|^2) = N^2 \quad , \quad E(|\Omega|^4) = 2N^4 - 2N^3 + N^2 \quad , \quad E(|\Omega|^6) = 6N^6 - 18N^5 + 33N^4 - 36N^3 + 16N^2$$

are universal, and $E(|\Omega|^8) = V(N) + 18R_2(H)$, where

$$V(N) = 24N^8 - 144N^7 + 528N^6 - 1368N^5 + 2434N^4 - 2580N^3 + 1089N^2$$

In particular, $E(|\Omega|^8) \leq V(N) + 18N^4$, with equality precisely when H is equivalent to a real Hadamard matrix.

Proof. The form of the moments is the above graph argument, with the coefficient 18 counting the positions of the 4 factors H and of the 4 factors $\bar{H}$ on the doubled rectangle, and the universal parts come from the exact interpolation. The last assertion follows from Proposition 8.3, with $k = 2$. □

So the real Hadamard matrices, that is, the Ising arrays, are the most Ising-like rotator arrays, maximizing the squared fluxes. Here are some examples, computed by `complexglow.py`, which does the above in floating point, and `defect.py`, with d denoting the defect, discussed in Section 9:

H	N	R_2	R_3	$\dim P_2$	$\dim P_3$	d
$F_2 \otimes_q F_2,\ q = e^{i\theta}$	4	$192 + 64\cos 2\theta$	64	$3 + \delta_{q^4,1}$	10 or 16	1 or 3
F_5	5	125	125	5	25	0
F_6	6	432	648	6	36	4
$F_3 \otimes_Q F_2,\ Q_{ad} = e^{0.4iad}$	6	432	648	4	18	4
Haagerup	6	336	216	2	5	4
Tao	6	216	1296	2	5	0

For the deformations $F_2 \otimes_q F_2$ the glow varies continuously with q, while the rooted law of the subfactor only depends on the order of q^4 [9]. More generally, for the deformations $F_X \otimes_Q F_Y$ of Diţă [40], the rooted law is a Gram law over the closed group generated by the second differences of Q, which are fluxes [9], while the glow sees the actual fluxes of H, through R_2 at order 8, the recovery of the whole flux distribution being open. Closure versus distribution, that is, arithmetic versus analysis. However, the glow can see the deformations quite late: for the curve $H(q)_{ab,cd} = w^{ac}(-1)^{bd} q^{ad}$ of Diţă deformations of $F_6 = H(1)$, with $a, c \in \mathbb{Z}_3$ and $b, d \in \mathbb{Z}_2$, the program `glowmoments.py` finds that the moments at $q = 1$ and $q = i$ coincide up to order 14, while at order 16 they are 3380149012050756 and 3380146268677956. Summarizing, we have:

Theorem 8.5. *Neither the glow nor the subfactor determines the other. More precisely, the matrices $F_2 \otimes_q F_2$ with $q = e^{\pi i/5}, e^{3\pi i/5}$ produce isomorphic subfactors, and the matrices of Haagerup [52] and Tao [102] are both wild, but in both cases the values of R_2, and so the glows, are different, while* $\text{had}_{16.3}$ *and its transpose have the same glow, real and complex, but different standard invariants, and the same rooted law.*

Proof. The Haagerup matrix has entries in $\mathbb{Q}(i)$, and the Tao matrix has cube roots of unity as entries, so both are wild, by Theorem 5.6, and their values of R_2 are $336, 216$, so their glows are different, by Theorem 8.4. For $F_2 \otimes_q F_2$, in the above normalization, the subfactor depends only on the smallest l with $q^{4l} = 1$, and on the sign of q^{2l}, by Burstein [27], and here $l = 5$ and $q^{2l} = 1$ in both cases, while $R_2 = 192 + 64\cos 2\theta = 176 \pm 16\sqrt{5}$. As for $H = \text{had}_{16.3}$, we have $\Omega_{H^t}(a, b) = \langle H^t b, a\rangle = \langle H\bar{a}, \bar{b}\rangle = \Omega_H(\bar{b}, \bar{a})$, and $(\bar{b}, \bar{a})$ has the same law as (a, b), so the glows of H, H^t coincide, in the real and complex case, and the traced algebras P_2 are different, the traces of their minimal projections being proportional to $128, 112, 16$ and $224, 16, 16$, as explained above. Finally, the rooted laws coincide, by Theorem 2.2. □

9 The defect, as deformations

The geometric invariants concern the manifold of Hadamard matrices around H, the basic one being the defect of Tadej and Zyczkowski [101]:

Definition 9.1. *The defect of a complex Hadamard matrix $H \in M_N(\mathbb{T})$ is the number $d(H) = \dim E_H - (2N - 1)$, where E_H is the space of real matrices $A \in M_N(\mathbb{R})$ satisfying*

$$\sum_k H_{ik}\bar{H}_{jk}(A_{ik} - A_{jk}) = 0$$

for any i, j.

The space E_H consists of the first order deformations $H_{ik} \to H_{ik}e^{itA_{ik}}$ preserving the Hadamard condition, and the matrices $A_{ik} = \alpha_i + \beta_k$, which are the infinitesimal gauge transformations, form a subspace of dimension $2N - 1$. Thus, in the language of Jones, the defect is the dimension of the space of first order deformations of the Boltzmann weights preserving Reidemeister II, modulo gauge. Equivalently, $d(H)$ is the dimension of the space of solutions A having 0 on the first row and column, and this shows that $d(H)$ is invariant under equivalence. The basic properties of the defect, as discussed in [8, 101], are as follows:

1. The local dimension of the manifold of dephased Hadamard matrices at H is at most $d(H)$, so $d(H) = 0$ implies that H is isolated, and the defect can jump, as for $F_2 \otimes_q F_2$, with defect 1 for $q \neq \pm 1$ and 3 for $q = \pm 1$, and local dimension 1 [8].

2. Every real Hadamard matrix of order $N \geq 4$ belongs to an affine family $H_{ik}e^{itA_{ik}}$, of dimension at least 1, by a construction in the spirit of Szöllősi [99].

3. For the Fourier matrix of a finite abelian group G we have $d(F_G) = \sum_g N/ord(g) - 2N + 1$, as found by Tadej and Zyczkowski for G cyclic [101], and by Tadej in general [8].

4. The defect has a differential interpretation, going back to Karabegov [70]: E_H identifies with the full kernel of the differential at $U = H/\sqrt{N}$ of the map $V \to (|V_{ik}|^2)_{ik}$ on U_N, and its quotient by the $2N - 1$ row and column phase directions has dimension $d(H)$.

Beyond the first order, a direction $A \in E_H$ extends to order 2 when there is a real matrix B such that $i\sum_k H_{ik}\bar{H}_{jk}(B_{ik} - B_{jk}) = \sum_k H_{ik}\bar{H}_{jk}(A_{ik} - A_{jk})^2$, for any $i \neq j$. For the Fourier matrices F_N, the analytic curves of Nicoara and White span the first order deformations [87],

every first order direction extends to order 2 [88], and when N is a prime power the defect is attained by an affine family [101]. However, when N is neither a prime power nor twice a prime, some first order directions cannot be realized by sequential deformations, as shown by Hopkins and Nicoara [57], so the local dimension at F_N is then strictly smaller than the defect, and numerically, an obstruction at order 11 was detected at F_{10} by Barros e Sá and Bengtsson [16], who also found evidence at F_{12} for 2 families of dimension 13 meeting along a family of dimension 9. In the other direction, isolation does not imply zero defect, as shown by Hall and Nicoara [54], who found a matrix in $BH(13,6)$ which is isolated, with defect 1, disproving the square case of the conjecture from [13], the proof using the second order condition of Nicoara for the directions of convergence of sequences of Hadamard matrices. A related finiteness result for the commuting squares is due to Nicoara [85].

In the language of Jones, the key object is the following matrix, which is the Berezin matrix of Karabegov [70], in the normalization of [8, 36], and whose entries are the flux phases of the rectangles, from Section 7:

Definition 9.2. *The Berezin matrix of $H \in M_N(\mathbb{T})$ is the matrix $\Lambda_H \in M_{N^2}(\mathbb{C})$, indexed by the pairs (row, column), given by*

$$(\Lambda_H)_{ij,kl} = \frac{1}{N} H_{ij}\bar{H}_{kj}H_{kl}\bar{H}_{il}$$

for any rows i, k and any columns j, l.

For a Hadamard matrix this matrix is unitary and symmetric, and the defect becomes the multiplicity of an eigenvalue, exactly as the dimensions of the planar algebra, with respect to the transfer matrices T_k. To be more precise, let us call necklace of r rectangles the balanced bipartite graph C_r obtained by gluing cyclically r rectangles, the s-th one being on the rows i_s, i_{s+1} and the columns j_s, j_{s+1}, with $s \in \mathbb{Z}_r$, so that C_2 is the doubled rectangle of Section 8. We have:

Theorem 9.3. *For a complex Hadamard matrix $H \in M_N(\mathbb{T})$, the Berezin matrix Λ_H is unitary and symmetric, and*

$$d(H) = \dim\ker(\Lambda_H - 1) - (2N-1)$$

Moreover, we have $Tr(\Lambda_H^r) = N^{-r}S_{C_r}(H)$ for any $r \geq 1$, and so the defect is a function of the partition functions of H on the necklaces C_r, with $r \leq N^2$.

Proof. Since $|H_{kl}| = 1$, we have $\sum_{kl}(\Lambda_H)_{ij,kl}(\bar{\Lambda}_H)_{ab,kl} = N^{-2}H_{ij}\bar{H}_{ab}\sum_k \bar{H}_{kj}H_{kb}\sum_l H_{al}\bar{H}_{il} = \delta_{ia}\delta_{jb}$, by orthogonality of the columns and of the rows, and the symmetry is clear. For A real, let $V = (H \circ A)H^*$, where $\circ$ is the entrywise product. The equations of E_H state that $V = V^*$, and we have $Tr(VV^*) = N\sum_{ij} A_{ij}^2$ and $Tr(V^2) = N\langle \Lambda_H A, A\rangle$, with A viewed as a vector indexed by the pairs, so that

$$\|V - V^*\|_2^2 = 2N\langle (1 - Re\Lambda_H)A, A\rangle$$

Now Λ_H being unitary and symmetric, we have $\bar{\Lambda}_H = \Lambda_H^* = \Lambda_H^{-1}$, so $Re\Lambda_H = (\Lambda_H + \Lambda_H^*)/2$ is a real symmetric matrix, whose eigenvalues are the numbers $\cos\theta$, with $e^{i\theta}$ being the eigenvalues of Λ_H. Thus $1 - Re\Lambda_H$ is positive, and E_H is its real kernel, whose dimension is the multiplicity of the eigenvalue 1 of Λ_H. As for the traces, $Tr(\Lambda_H^r)$ is the sum over the pairs (i_s, j_s), with $s \in \mathbb{Z}_r$, of the products of the flux phases of the rectangles on the rows i_s, i_{s+1} and the columns j_s, j_{s+1}, divided by N^r, and this is the partition function on C_r. Finally, Λ_H being diagonalizable, the multiplicity of the eigenvalue 1 is determined by the characteristic polynomial, and so by the traces $Tr(\Lambda_H^r)$ with $r \leq N^2$. □

The formula for the defect is due to Tadej, see [8], and the program `berezin.py` checks it, together with the necklace formula for $r \leq 3$, for the Fourier matrices F_N with $N \leq 12$, the

matrices of Haagerup and Tao, the deformations $F_2 \otimes_q F_2$ and $F_3 \otimes_Q F_2$, and the real matrices of orders 2, 4, 8, 12. Observe that $Tr(\Lambda_H^2) = R_2(H)/N^2$ is the doubled rectangle, which appears in the complex glow, by Theorem 8.4. Also, as shown in [8], the Hessian at H of the trace norm $S \to Tr|S|$ on the torus $M_N(\mathbb{T})$, which is maximal at the Hadamard matrices, with value $N\sqrt{N}$, and which is the energy of the dual model in [44], is $(Re\Lambda_H - 1)/2\sqrt{N}$, in the phase coordinates. Thus the defect counts the flat directions of this energy, modulo gauge, and the higher traces see more: the McNulty-Weigert matrices $K_{11}(3), K_{11}(5)$ of order 22 from [8] have the same flux distribution and the same defect 20, but different values of $Tr(\Lambda_H^3)$, and so different Hessians. For the real matrices, the defect is always the same, as follows:

Theorem 9.4. *Every real Hadamard matrix of order N has defect*

$$d(H) = \frac{(N-1)(N-2)}{2}$$

so the defect is universal on the real slice, just like the moments of the real glow up to order 14.

Proof. This is from [8], and here is a short proof. For H real the coefficients are real, so E_H is the kernel of $N(N-1)/2$ real equations, one for each pair $i < j$, and we must prove that they are linearly independent. Assume that $\sum_{i<j} \lambda_{ij} E_{ij} = 0$, with E_{ij} being the equation for (i, j), and let μ be the antisymmetric matrix with $\mu_{ij} = \lambda_{ij}$ for $i < j$. The coefficient of A_{ik} in the combination is $\sum_j \mu_{ij} H_{ik} H_{jk} = H_{ik}(\mu H)_{ik}$, so we must have $\mu H = 0$, and so $\mu = 0$, because H is invertible. Thus $\dim E_H = N^2 - N(N-1)/2$, and $d(H) = N^2 - N(N-1)/2 - 2N + 1 = (N-1)(N-2)/2$. □

At small orders, the manifold of complex Hadamard matrices is now well understood. In order 6, the classification has been completed by Cárdenes Wuttig and Tindall [29], who proved a strong form of the conjecture of Szöllősi, stating that every matrix can be reconstructed from a dephased 3×3 corner, given by 4 phases. In a second paper [30], they showed that all the complex Hadamard matrices of order 6, except for the isolated matrix of Tao, form a connected space, and deduced from this that there are no 4 mutually unbiased bases in $\mathbb{C}^6$, every triple being either in a 2-parameter family, or the isolated triple coming from the Fourier matrix. In prime order p, the Fourier matrix can be recognized from small pieces of it, as shown in [104], the starting point being the fact that for a regular or Butson matrix of order p, the entrywise p-th power is either Hadamard or of rank 1, which in terms of Proposition 8.3 means $R_p(H) \in \{p^3, p^4\}$. For the regular matrices, whose scalar products between rows decompose as sums of rotated regular polygons, the classification at $N = 7$ gives F_7, the Petrescu family P_7^q and the family R_7^q, with a complete classification at $N = 8$ too [48], and at $N = 11$ the homogeneous partial matrices with 3 rows are classified in [103], with maximal dimensions $0, 3, 4, 3, 4, 6$ for the 6 possible cycle profiles of the row quotients.

For the McNulty-Weigert matrices [81], built from 2 product bases of $\mathbb{C}^N \otimes \mathbb{C}^p$, with p an odd prime, and an outer Hadamard matrix of order N, the isolation in order $2p$ is controlled by a cross-ratio of the labels of the bases, which must not be a square modulo p [36]. The structure of the deformations was worked out in [8]: when the cross-ratio is a square, there are $p^2 + 2$ analytic families of dimension $p - 1$ through the matrix, with distinct tangent spaces, every first order direction extends through order 2, and in order 10 the defect is 8, while the local dimension is 4. Also from [8], with outer matrix F_3 and 6 distinct labels the matrices are isolated for any prime $5 \leq p \leq 31$, every Diţă deformation of $F_N \otimes F_M$ with N, M coprime has the defect of F_{NM}, and a McNulty-Weigert matrix is equivalent to a master matrix, of the form $(e^{2\pi i x_i y_j})$, precisely when both label lists are constant, and the outer matrix is equivalent to a Hadamard submatrix of a Fourier matrix of order prime to p. The connection with the glow comes from the following result:

Theorem 9.5. *For the McNulty-Weigert matrices H of order $N = 2p$, with outer matrix F_2, we have $d(H) = (N-2)(Tr(T_2 F) - 1)$, and so the defect is determined by the moment $E(|\Omega|^8)$, via Theorem 8.4. In particular H is isolated precisely when its entrywise square is Hadamard.*

Proof. This is from [8], where the formula is written as $d(H) = (N-2)(R_2(H)/N^3 - 1)$, and we have $R_2(H) = N^3 Tr(T_2 F)$, as explained in Section 8. The last assertion follows from Proposition 8.3, with $k = 2$, since $Tr(T_2 F) = 1$ precisely when $R_2(H) = N^3$, together with the fact that $d(H) = 0$ implies isolation, while for $d(H) > 0$ the matrix lies on the analytic families of positive dimension constructed in [8]. □

10 Quantum groups and categories

The classical symmetries of H are the pairs (P, Q) of monomial matrices with $PHQ = H$, and these are exactly the symmetries of the arrays of Section 7, mapping a configuration (a, b) to $(P^* a, Qb)$, which preserves $\langle Hb, a\rangle$, as used in [90] for computing the ground states. With G being the group of monomial matrices, this group is $G \cap HGH^{-1}$, and its quantum versions, for G being a quantum reflection group, are the quantum automorphism groups of H, introduced by Gromada in the real case [49], and by Brannan, Gromada, Hernández Palomares and Priebe in general [26]. Thus, the combinatorial invariants concern the quantum symmetries of the arrays. The quantum reflection groups involved are the free wreath products $H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$ of Bichon [17], with $s \in \{2, 3, \ldots, \infty\}$ and $\mathbb{Z}_\infty = \mathbb{T}$, and the free complexification K_N^+ of $H_N^+ = H_N^{2+}$, and we refer to [37] for a recent look at the quantum reflection groups between H_N and H_N^+. We have:

Definition 10.1. *Let $G \subset U_N^+$ be a quantum reflection group, with $C(G)$ given by relations on the entries of its fundamental corepresentation. Two complex Hadamard matrices H, K of order N are called G-quantum equivalent when the universal algebra generated by the entries of a matrix p, such that p and $K^{-1}pH$ both satisfy the relations of $C(G)$, is nonzero. For $G = H_N^{s+}, H_N^{\infty+}, K_N^+$ we speak of s-quantum, strong quantum, and weak quantum equivalence.*

At $K = H$ the above algebra is the algebra of functions on the quantum automorphism group $G \cap HGH^{-1}$, and in general it is the linking algebra between the quantum automorphism groups of H and K. As shown in [26], s-quantum equivalence implies strong quantum equivalence, which implies weak quantum equivalence, and these notions are of combinatorial nature, being related to the quantum isomorphisms of graphs, and to the nonlocal games of quantum information, as well as to the quantum association schemes of Gromada [50]. In the real case, the main results of Gromada [49] are as follows:

1. The quantum automorphism group $H_N^+ \cap H^{-1} H_N^+ H$ has as representation category the image of the category of the planar bipartite graphs with even degrees, generated by 2 spiders and a cup, whose dimensions are the squares $Cat_{(k+l)/2}^2$ of the Catalan numbers. This category is isomorphic, but not monoidally, to a twisted product of 2 Temperley-Lieb categories, at $\delta = \sqrt{N}$.

2. For $N \geq 4$ there are no negligible morphisms, and the quantum automorphism group is not classical, even when the classical automorphism group is $\mathbb{Z}_2$, and for $N = 4$ it is the anticommutative twist SO_4^{-1}.

3. All the real Hadamard matrices of a given order are quantum isomorphic, and the same holds for the Hadamard graphs, as obtained independently in [32]. There are also quantum Hadamard matrices acting on finite quantum spaces, the transposition of $M_n(\mathbb{C})$ being an example.

Observe that (3) is in agreement with the reduction argument of Section 8, according to which the planar graphs, which are those of the category in (1), give universal values on the real slice. In the complex case, the main results of [26], obtained with the graphical calculus of complementary spiders, are as follows:

1. All the complex Hadamard matrices of a given order are weakly quantum equivalent, the corresponding category being pure, in the sense that all its closed diagrams are universal.

2. For $N = mp^2$ the Fourier matrices F_N and $F_{mp} \otimes F_p$ are not strongly quantum equivalent, and nor are some Butson matrices of order 14, with entries in the 7-th roots of unity, taken from [76], with computer evidence for the Butson matrices of order $2p$ coming from the construction of Butson, at the primes $7 \leq p \leq 47$, and the conjecture that for any $s > 2$ there are Butson matrices with entries in the s-th roots of unity which are not s-quantum equivalent.

3. The s-quantum equivalence is modeled by a synchronous nonlocal game, the perfect quantum commuting strategies corresponding to quantum s-equivalence, and the quantum automorphism group of a Butson matrix embeds into that of its Hadamard graph, the inclusion being strict in general.

4. The finite quantum groups have only classical symmetries, as found by Kasprzak, Soltan and Woronowicz [72], and all the finite quantum groups of a given dimension are quantum affinely equivalent.

The closed diagrams in these categories are built from complementary spiders, as follows:

Definition 10.2. *The spiders of H are the tensors $\delta_{i_1 \ldots i_m}$ on $\mathbb{C}^N$ in the standard basis $\{e_i\}$, and the same tensors in the orthonormal basis $\{f_k\}$ given by $(f_k)_i = \bar{H}_{ik}/\sqrt{N}$. A closed diagram is a graph whose vertices are spiders of the 2 types, with each edge joining 2 spiders of different types, and oriented.*

Here we use the fact that 2 spiders of the same type, joined by an edge, fuse into a single spider, so we can assume that the edges join spiders of different types. The 2 bases are complementary, or mutually unbiased, which is the Hadamard condition. We have $\langle e_i, f_k \rangle = H_{ik}/\sqrt{N}$, which gives:

Proposition 10.3. *A closed diagram in the spiders of H evaluates to $N^{-|E|/2} S_\Gamma(H)$, with Γ being the underlying bicolored graph, with the edges signed by their orientation.*

Proof. Expanding each spider in its basis, the diagram becomes a sum over the labels of the spiders, with each edge contributing a factor $\langle e_i, f_k \rangle$ or its conjugate. □

Thus the combinatorial invariants are again partition functions of the Jones model, with the class of graphs being dictated by the category. For the representation categories of quantum groups the diagrams are planar, and we obtain:

Theorem 10.4. *If a unitary monoidal equivalence maps the generating spiders of H to those of K, then $S_\Gamma(H) = S_\Gamma(K)$, for any graph Γ appearing as a closed diagram in the category. In particular, for the weak quantum equivalence, which relates all the Hadamard matrices of a given size, the closed diagrams of the corresponding category are universal.*

Proof. A monoidal equivalence preserves compositions and tensor products, and acts as the identity on the scalars, so it preserves the values of the closed diagrams, and Proposition 10.3 gives the first assertion. The second one follows from the first one, and from [26]. □

The closed diagram used in [26] for separating the Fourier matrices F_N and $F_{mp} \otimes F_p$ is the doubled rectangle with p-fold edges, that is, the flux harmonic R_p of Section 8, and its values there, $N^3 p$ and $N^3 p^2$, are those given by the formula $R_k(F_N) = N^3 gcd(k, N)$ and by multiplicativity. In general, we have:

Theorem 10.5. *The flux harmonics $R_k(H)$, with $k \geq 1$, and so the flux distribution of H, are invariants of the strong quantum equivalence, and so of the s-quantum equivalence, for any s. In particular:*

1. *In order 4, the strong quantum equivalence coincides with the equivalence.*
2. *The matrices F_6, Haagerup and Tao are pairwise not strongly quantum equivalent.*
3. *The moment $E(|\Omega|^8)$ of the complex glow is an invariant of the strong quantum equivalence.*

Proof. Consider the closed diagram having 2 spiders i, i' of the first type, and 2 spiders j, j' of the second type, with k parallel edges joining i to each of j, j', and i' to each of j, j', with the orientations chosen such that the corresponding product is $H^k_{ij}\bar{H}^k_{i'j}H^k_{i'j'}\bar{H}^k_{ij'}$. This diagram is planar, and each spider has k legs of each orientation, placed consecutively, so it belongs to the category of $H_N^{\infty+}$, whose intertwiners are given by the noncrossing partitions having in each block as many legs of each color, this being the case for the spiders of both types. By Proposition 10.3 its value is $N^{-2k}R_k(H)$, and Theorem 10.4 gives the invariance of R_k, and so of the flux distribution, which is determined by its moments $R_k(H)/N^4$, with $R_{-k} = \bar{R}_k = R_k$. In order 4 the matrices are the deformations $F_2 \otimes_q F_2$ with $q = e^{i\theta}$ and $\theta \in [0, \pi/2]$, up to equivalence [100], and $R_2 = 192 + 64\cos 2\theta$ separates them, which gives (1). For (2) we use the values $432, 336, 216$ of R_2, and (3) follows from Theorem 8.4. □

The program `berezin.py` checks the formula $R_k(F_N) = N^3 gcd(k, N)$ for $N \leq 12$ and $k \leq 6$, the values N^3p and N^3p^2 of [26] for $(m, p) = (2, 2), (3, 2), (2, 3)$, and the formula for $R_2(F_2 \otimes_q F_2)$, along with the equivalence of $F_2 \otimes_q F_2$ with its versions at $\bar{q}$ and $-q$, used in the proof. On the real slice we have $R_k(H) = N^4$ for k even, and $R_k(H) = N^3$ for k odd, in agreement with the fact that all the real Hadamard matrices of a given order are quantum isomorphic [49], and the first non-universal moment of the real glow, at order 16, comes from the graph $K_{4,4}$, which is not planar. As for the weak quantum equivalence, since R_2 is not universal, the doubled rectangle is not a closed diagram there. We will see in Section 11 that R_2 is not invariant under the Galois group either, so the Galois conjugation of the Butson matrices, which is part of the Hadamard equivalence considered in [26], does not preserve the strong quantum equivalence classes.

11 Traces and conjugacy classes

The arithmetic invariants of a complex Hadamard matrix H are those of its conjugacy class, under the monomial matrices, or even under the whole unitary group, the basic ones being the trace, and more generally the spectrum. These are not invariant under the equivalence. For instance, consider the Fourier matrix $F_N = (w^{jk})$, with $w = e^{2\pi i/N}$ and $j, k = 0, \ldots, N-1$. Its trace is the quadratic Gauss sum, and the computation of Gauss gives $Tr(F_N) = \sum_k w^{k^2} = (1+i)(1+i^{-N})\sqrt{N}/2$, while with $b_k = w^{-k^2}$ we have $Tr(F_N D_b) = N$. In terms of the Jones model, the trace is the partition function of the model on a loop, with the row and column spins being identified, and this suggests the following definition:

Definition 11.1. *A balanced directed graph is a directed multigraph Γ with signed edges, such that at each vertex, the number of positive outgoing edges plus negative incoming edges equals the number of positive incoming edges plus negative outgoing edges. The partition function of the Jones model on Γ is*

$$S_\Gamma(H) = \sum_x \prod_{e \in E_+} H_{x(s(e))x(t(e))} \prod_{e \in E_-} \bar{H}_{x(s(e))x(t(e))}$$

with the sum over the labelings x of the vertices by $1, \ldots, N$, and with $s(e), t(e)$ being the source and target of e.

The bicolored graphs of the introduction are particular cases, with all the edges going from the row vertices to the column vertices. The trace is the loop, and the power traces $Tr(H^n)$ are the oriented cycles, that is, the partition functions of the periodic chains of length n, with H as transfer matrix. As a first arithmetic fact, we have, from [36]:

Proposition 11.2. *Let $H \in M_N(\mathbb{Z}[w])$ be a Butson matrix, with $w = e^{2\pi i/l}$. Then the eigenvalues of H are Weil numbers, in the sense that they are algebraic integers, all of whose conjugates have modulus $\sqrt{N}$.*

Proof. The eigenvalues are algebraic integers, being roots of a monic polynomial with coefficients in $\mathbb{Z}[w]$. If τ is an automorphism of the normal closure, and σ is its restriction to $\mathbb{Q}(w)$, then $\tau(\lambda)$ is an eigenvalue of $\sigma(H)$, which is Hadamard, because σ commutes with the complex conjugation, $\mathbb{Q}(w)$ being abelian, and so $|\tau(\lambda)| = \sqrt{N}$. □

The arithmetic behind this was developed in [36], and the main findings there are as follows. For a Butson matrix H of prime order p, with entries in $\mathbb{Z}[w]$, $w = e^{2\pi i/l}$, $l \geq 3$, and not having $\pm\sqrt{p}$ as eigenvalues, there is an abelian variety A_H over $\mathbb{F}_p$, of dimension $p\varphi(l)/2$, with multiplication by $\mathbb{Z}[w]$, whose lattice in the sense of Centeleghe and Stix [31] is $\mathbb{Z}[w]^p$, with Frobenius map H and Verschiebung H^*. This variety depends only on the conjugacy class of H, and its number of points over $\mathbb{F}_{p^n}$ is $|N_{K/\mathbb{Q}} \det(1 - H^n)|$, with $K = \mathbb{Q}(w)$. In order 2 this gives, as simple factors, all the simple abelian varieties over $\mathbb{F}_2$ having a single rational point, as classified by Madan and Pal [77], and Robinson [95]. In general, the Hodge polygons, coming from the Smith normal form, are invariant under equivalence, while the Newton polygons, coming from the valuations of the eigenvalues, are invariant only under conjugation, and lie above the Hodge ones, and a Butson matrix has finite order, in the sense that a power of $H/\sqrt{N}$ is the identity, precisely when its Newton polygons are straight lines.

In the equivalence class of F_p there are the Clifford matrices, which have finite order, the matrix $G_p = (w^{i/j})$, whose eigenvalues are $\pm\sqrt{p}$ and the nontrivial Gauss sums, with Newton slopes given by the Stickelberger theorem, and exotic matrices of finite order, with a criterion analogous to that of Shioda and Katsura for the Fermat varieties [96]. For the Maiorana-McFarland matrices $(w^{if(j)+g(j)})$, with f a permutation polynomial of degree $2 \leq d \leq p-2$, the eigenvalues of $H/\sqrt{p}$ become equidistributed, as $p \to \infty$, by the bounds of Deligne [36].

In physical terms, H is here a transfer matrix, acting on a single set of spins. The Clifford matrices are the quantizations of the linear maps of the torus, or quantum cat maps, of Hannay and Berry [56], whose power traces are given exactly by sums over the periodic orbits [73], the equidistribution questions for them being studied by Kurlberg and Rudnick [74]. Thus, the finite order matrices correspond to chains with perfect revivals, while the Maiorana-McFarland matrices have equidistributed spectra. The relation between the chains and the abelian varieties of [36] is as follows:

Theorem 11.3. *Let H be a Butson matrix of prime order p, with entries in $\mathbb{Z}[w]$, $w = e^{2\pi i/l}$, $l \geq 3$, not having $\pm\sqrt{p}$ as eigenvalues, and let A_H be its abelian variety over $\mathbb{F}_p$. Then, with σ ranging over the Galois group of $\mathbb{Q}(w)$:*

1. *The traces of the powers of the Frobenius map of A_H, on its first cohomology, are the sums $\sum_\sigma Tr(\sigma(H)^n)$ of partition functions of periodic chains, with transfer matrices $\sigma(H)$.*
2. *The numbers of points are $|A_H(\mathbb{F}_{p^n})| = |\prod_\sigma Str(\Gamma(\sigma(H))^n)|$, where $\Gamma(X) = \oplus_k \Lambda^k X$ is the free fermion transfer matrix on the Fock space $\Lambda\mathbb{C}^p$, and Str is the supertrace, with sign $(-1)^k$ on $\Lambda^k\mathbb{C}^p$.*
3. *The Riemann hypothesis of Weil for A_H, stating that the eigenvalues of the Frobenius map have modulus $\sqrt{p}$, comes from the unitarity of the transfer matrices $\sigma(H)/\sqrt{p}$, which is the Hadamard condition, that is, Reidemeister II for the Jones models of the matrices $\sigma(H)$.*

Proof. By [36], the lattice of A_H is $\mathbb{Z}[w]^p$, with Frobenius map H, so the Frobenius map on the rational lattice K^p, which is a vector space over $\mathbb{Q}$ of dimension $p\varphi(l) = 2\dim A_H$, has as eigenvalues the numbers $\sigma(\lambda)$, with λ being the eigenvalues of H, and this gives (1). For (2), we have $|A_H(\mathbb{F}_{p^n})| = |N_{K/\mathbb{Q}}\det(1-H^n)| = |\prod_\sigma \det(1-\sigma(H)^n)|$, and $\det(1-X) = \sum_k (-1)^k Tr(\Lambda^k X) = Str(\Gamma(X))$, with $\Gamma(X^n) = \Gamma(X)^n$. As for (3), the matrices $\sigma(H)$ are Hadamard, as in the proof of Proposition 11.2. □

In other words, the characteristic polynomial of the Frobenius map is $\prod_\sigma \det(t-\sigma(H))$, the points of A_H are counted by fermionic chains, and the Verschiebung $H^* = pH^{-1}$ is the transfer matrix of the time-reversed chain. In the language of spin models, the trace is well-defined for the spin models of Jones in the strict sense, satisfying the star-triangle relation, being N times the modulus of the model. For the cyclic models $W = (tw^{c(i-j)^2})$ of Goldschmidt and Jones [46], see also [15], with p an odd prime, $w = e^{2\pi i/p}$ and $c \in \mathbb{F}_p^*$, the matrix W is Hadamard, with trace pt, and it is a spin model precisely when $t^2 = \bar{G}(c)/\sqrt{p}$, where $G(c)$ is the quadratic Gauss sum, so that the modulus of the model encodes the sign of the Gauss sum [36]. Also, the modular invariance equation $(PT)^3$ scalar of Bannai and Bannai [15], with $P = F_N$ and T diagonal, is a finite order condition, placing the corresponding matrices in the supersingular case [36].

As noted in [36], a Hadamard matrix has 2 possible physical roles: coupling matrix between 2 sets of spins, as for the arrays, where only its equivalence class matters, and transfer matrix acting on a single set of spins, where its conjugacy class matters. In terms of the arrays, identifying the row and column spins amounts in gluing the 2 layers, with energy $-\langle Ha, a\rangle/\sqrt{N}$. The law of $\langle Ha, a\rangle$ is invariant under the conjugation by the monomial matrices, because $\langle QHQ^*a, a\rangle = \langle HQ^*a, Q^*a\rangle$, and Q^*a has the same law as a, with Q being a signed permutation matrix in the real case. The program `arithmetic.py` checks the formulas below, and the Gauss formula, for F_N with $N \leq 12$, and for the Haagerup and Tao matrices. We have:

Theorem 11.4. *The numbers $S_\Gamma(H)$, over the balanced directed graphs Γ, form a complete invariant for the conjugation of complex Hadamard matrices by the monomial matrices, and the arithmetic invariants are functions of them. For the glued arrays we have $E(\langle Ha, a\rangle) = Tr(H)$, with a uniform on $\mathbb{T}^N$, or on $\{\pm 1\}^N$, as well as $E(|\langle Ha, a\rangle|^2) = |Tr(H)|^2 + N^2 - N$, with a uniform on $\mathbb{T}^N$, and, for H real and a uniform on $\{\pm 1\}^N$, $E(\langle Ha, a\rangle^2) = Tr(H)^2 + Tr(H^2) + N^2 - 2N$.*

Proof. The completeness is as in the proof of Theorem 12.1, with the compact group $S_N \ltimes \mathbb{T}^N$ acting by conjugation: the averaging over $\mathbb{T}^N$ kills the monomials in $H, \bar{H}$ which are not balanced, in the above sense, the averaging over S_N produces sums over the injective labelings, and the Möbius inversion gives the numbers $S_\Gamma(H)$. The spectrum is determined by the power traces $Tr(H^k)$ with $k \leq N$, which are cycles. As for the moments, in the complex case we have $\langle Ha, a\rangle = \sum_{jk} H_{jk} a_k \bar{a}_j$, and $E(a_k \bar{a}_j) = \delta_{jk}$ gives the first formula. For the second one, $E(a_k a_l \bar{a}_j \bar{a}_m)$ is 1 when $\{k, l\} = \{j, m\}$, and 0 otherwise, and the 2 pairings give $|Tr(H)|^2$ and $\sum_{jk} |H_{jk}|^2 = N^2$, the N terms with $j = k = l = m$ being counted twice. In the real case there are 3 pairings, giving $Tr(H)^2$, $\sum_{jk} H_{jk}^2 = N^2$ and $Tr(H^2)$, and the N diagonal terms, with $H_{jj}^2 = 1$, are counted 3 times. □

More generally, the moments of the glued arrays are sums of partition functions on balanced directed graphs, weighted by cumulants, exactly as in Theorem 7.2, with a single type of vertices. Some arithmetic invariants are however functions of the equivalence class only, such as the Hodge polygons, and by Theorem A they are then functions of the numbers $S_\Gamma(H)$ on the bicolored graphs. A remarkable instance of this is the 2-rank, which is a piece of the Hodge polygon [36], and which determines the density of states of the Ising arrays at $N = 16$ [90]. Also, the crystalline states of the array of $F_p \otimes F_p$ include the vectors $(t_i w^{ij})$, with $t \in \mathbb{T}^p$ arbitrary, and those having p-th roots of unity as entries are the vectors $(w^{f(i,j)})$ with f a bent function, and the matrices $F_p P D$ of the equivalence class of F_p, with P a permutation matrix and D diagonal

with p-th roots of unity as entries, are such crystalline states [36]. As for the Galois group, its action produces a clear separation between the various types of invariants, as follows:

Theorem 11.5. *Let H be a Butson matrix, with entries in $\mathbb{Q}(w)$, $w = e^{2\pi i/l}$, and σ be in the Galois group of $\mathbb{Q}(w)$. Then $\sigma(H)$ is a Butson matrix, with $S_\Gamma(\sigma(H)) = \sigma(S_\Gamma(H))$ for any Γ, and:*

1. *The algebraic and geometric invariants are Galois invariant, in the sense that $\dim P_k^{\sigma(H)} = \dim P_k^H$ for any k, and $d(\sigma(H)) = d(H)$.*
2. *The analytic and combinatorial invariants are not: for $q = e^{\pi i/5}$ the matrices $F_2 \otimes_q F_2$ and $F_2 \otimes_{q^3} F_2$ are Galois conjugate, and have $\dim P_2 = 3$ and defect 1, but have $R_2 = 176 \pm 16\sqrt{5}$, and so different complex glows, and are not strongly quantum equivalent.*

Proof. The Galois group of $\mathbb{Q}(w)$ being abelian, σ commutes with the complex conjugation, and this gives the first assertions. (1) We have $T_k = N^{-k-1}AA^*$, with the entries of A being products of entries of H and $\bar{H}$, so $T_k(\sigma(H)) = \sigma(T_k(H))$, entrywise. Now $\dim P_k^H$ is the multiplicity of the eigenvalue 1 of the Hermitian matrix $T_k(H)$, which is $N^k - rank(T_k(H) - 1)$, and the rank is preserved by σ. The same argument works for the Berezin matrix, with $\Lambda_{\sigma(H)} = \sigma(\Lambda_H)$, by using Theorem 9.3. (2) Here $\sigma : w \to w^3$, with $w = q$, maps $F_2 \otimes_q F_2$ to $F_2 \otimes_{q^3} F_2$, and the values $\dim P_2 = 3$ and $d = 1$ come from the table in Section 8, while $R_2 = 192 + 64\cos 2\theta$, with $\theta = \pi/5, 3\pi/5$, gives $176 \pm 16\sqrt{5}$. Theorem 8.4 and Theorem 10.5 give the last assertions. □

The program `berezin.py` checks these values, and in particular, the 2 matrices in (2) are not equivalent, while being Galois conjugate, and so equivalent in the sense of the Hadamard equivalence of [26]. Observe that the values of R_2 there are Galois conjugate, in agreement with $R_2(\sigma(H)) = \sigma(R_2(H))$. Thus, the algebraic and geometric invariants only see the Galois orbit of H, while the glow and the strong quantum equivalence see the matrix itself.

12 Unification of the invariants

Let us summarize. The raw object is the Jones spin model of H, that is, the collection of its partition functions $S_\Gamma(H)$, or equivalently the model $\pi : u_{ij} \to P_{ij}$ and its transfer matrices T_k. Each type of invariants uses a class of graphs, and an operation:

invariants	graphs	operation	blind region
algebraic	gluings of A, A^*	eigenvalue 1 of T_k	wild matrices, as at $N = 20$
analytic	all bipartite	gauge average	orders ≤ 14, or ≤ 6
geometric	necklaces of rectangles	eigenvalue 1 of Λ_H	the real slice
combinatorial	planar, spiders	monoidal equivalence	weak equivalence, real slice
arithmetic	directed	traces, Galois	none, finer

The first place where the theories meet is the transfer matrix T_2, through $Tr(T_2^\infty)$ for the subfactor, and through $Tr(T_2F)$ and $Tr(T_2^2)$ for the complex and real glow. The second one is the doubled rectangle $R_2 = N^3Tr(T_2F) = N^2Tr(\Lambda_H^2)$, which is seen by the complex glow, at order 8, and by the defect of the McNulty-Weigert matrices, and more generally the flux harmonics R_k, which are invariants of the strong quantum equivalence. In fact, there are 3 transfer matrices around: the matrices T_k, whose eigenvalue 1 gives the algebraic invariants, the Berezin matrix Λ_H, whose eigenvalue 1 gives the defect, and H itself, as transfer matrix of the chains, whose eigenvalues are Weil numbers in the Butson case, where the multiplicities of the eigenvalue 1 of the first 2 only depend on the Galois orbit of H, while the matrices themselves do not. The arithmetic invariants are of a different nature, being finer than the equivalence, and they are seen by the glued arrays, whose first moment is the trace. As for the old question of Jones

on the statistical mechanical meaning of the Hadamard matrices [62, 63], a Hadamard matrix has 2 dual meanings: N-state Boltzmann weights satisfying Reidemeister II, and couplings of a magnet made of 2-state spins, or of planar rotators, which are the gauge transformations, and Reidemeister II becomes maximal frustration.

For the circulant matrices of Section 5, $H_{ij} = h(j-i)$, with eigenvalues $\lambda_m = \sum_t h(t)w^{mt}$, all these objects take a simple Fourier form. The transfer matrices T_k and the Berezin matrix Λ_H commute with the cyclic shifts, and decompose into N blocks, the subfactor being seen by the shift-invariant block, by Theorem 5.15. The entrywise powers $H^{\circ k}$ are circulant too, so $R_k(H) = \sum_m |\widehat{h^k}(m)|^4$, with $\widehat{h^k}$ being the Fourier transform of h^k, the glow is the law of $\Omega = \frac{1}{N}\sum_m \lambda_{-m}\widehat{\bar{a}}(m)\widehat{b}(m)$, and the spectrum consists of the numbers λ_m, all of modulus $\sqrt{N}$, with $Tr(H) = Nh(0)$.

As for the dichotomy between the Diţă type and the wildness, from Section 4, the matrices which are not wild should be of Fourier or Diţă type, and in particular their quantum group should preserve a partition of the indices into blocks, these blocks being unions of orbitals, in the sense of Theorem 1.2. At prime order there are no such partitions, so the matrices which are not wild should be equivalent to the Fourier matrix. The program `royce.py`, run on a laptop, tests this at the primes $p = 7, 11, 13, 17$, by finding complex Hadamard matrices from random starting points, and computing their defect, $\dim P_2$ and $\dim P_3$, and finds 7265 classes, all having $\dim P_2 = 2$ and $\dim P_3 = 5$, except for F_7. The program `wladimir.py` does the same at the orders $N = 6, 8, 9, 10, 12$, with 33116 classes, the matrices having $\dim P_2 = 2$ always having $\dim P_3 = 5$, and those with $\dim P_2 > 2$ always having block systems, the only exceptions being near-degenerate points of continuous families. There are many questions left, and regarding the subfactors, we have for instance:

1. Is there a version of the method of Section 5 allowing zeros in the profile, as needed for BF_6, $L_{8,30}$, the families of order 8, and the real matrices of order 0(8)? Can a finer congruence deal with $I_{8,12}$ and $K_{8,30}$, and a Fourier one with BF_6, using Proposition 5.22? Which other circulants, such as the Björck matrices of simple index $d \geq 4$, or the one of order 61 in Theorem 5.21, have an affine profile, and which of the matrices of [24] are wild?

2. Is the free wreath product $S_3^+ \wr_* S_2^+$ maximal in S_6^+, and can the kagome walks of Theorem 4.3 be pushed beyond level 18?

As for the relations between the invariants, and the dichotomy, we have:

1. Do the traces $Tr(T_k^m)$ appear in the glow at higher orders, as $Tr(T_2^3)$ at order 24, and does the rooted law determine the real glow, or the converse?

2. Does R_k appear in the complex glow at order $4k$, with coefficient $((2k)!/k!^2)^2(c_k')^4/2$, which is 18 at $k = 2$, and does the complex glow determine the flux distribution?

3. What is the glow of the spin models of Nomura [89], and of the spin models in general [60]?

4. What is the law of $\langle Ha, a\rangle$ for the Fourier matrices, whose first moment is the Gauss sum, and do the higher moments of the glued arrays see the Weil numbers of [36]?

5. Is the flux distribution a complete invariant for the strong quantum equivalence in order 6, as in order 4, in view of [29], and are the traces $Tr(\Lambda_H^r)$ seen by the glow, as $Tr(\Lambda_H^2)$ is?

6. Is a complex Hadamard matrix whose quantum group preserves no partition of the indices into blocks either equivalent to F_p, at prime order p, or wild, as supported by Section 5 and by `royce.py`, `wladimir.py`?

7. Is the Hodge polygon determined by the glow, as the 2-rank is at $N = 16$, and is there a physical meaning for the Newton strata of [36]?

Finally, here is the unification statement, which includes Theorem A:

Theorem 12.1. *Two complex Hadamard matrices H, K of the same order are equivalent precisely when $S_\Gamma(H) = S_\Gamma(K)$ for any balanced bipartite multigraph Γ with signed edges. Moreover, the algebraic, analytic, geometric and combinatorial invariants are the explicit functions of the numbers $S_\Gamma(H)$ given by Theorems 1.2, 7.2, 9.3 and 10.4, and the arithmetic invariants are functions of the numbers $S_\Gamma(H)$ on the balanced directed graphs, which separate the conjugacy classes, by Theorem 11.4.*

Proof. The equivalence is the action of the compact group $(S_N \ltimes \mathbb{T}^N) \times (S_N \ltimes \mathbb{T}^N)$ on $M_N(\mathbb{C})$, by left and right multiplication with monomial matrices, whose orbits are separated by the invariant polynomials. The averaging over $\mathbb{T}^N \times \mathbb{T}^N$ kills the unbalanced monomials in $H, \bar{H}$, and the averaging over $S_N \times S_N$ turns a balanced monomial into a sum over the injective labelings of a graph Γ. By Möbius inversion, these sums are combinations of the numbers $S_{\Gamma'}(H)$, with Γ' obtained from Γ by identifying vertices of the same type, giving the first assertion. The rest follows from Sections 1, 7, 9, 10, 11. □

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