The Kneser–Poulsen volume inequalities for $d + 4$ balls

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28 November 2026

hexagon:2611.01327v1

Abstract

We prove that for at most $d + 4$ balls in Euclidean d-space, $d \geq 3$, every expansion of the centers weakly increases the volume of the union and weakly decreases the volume of the common intersection. The radii may differ. This answers Open Problem 13 of Bezdek and Lángi and includes the case of eight balls in four dimensions. We show that the volume of a three-ball cut by two affine half-spaces is convex in its squared radius. A circular-arc overlap formula expresses this volume as a positive mixture of convex functions. Slicing gives the same convexity for the walls in a $(d + 4)$-dimensional power diagram. A second finite difference of the Csikós variation formula then recovers the two d-dimensional volume comparisons. The complete theorem is formally verified in Lean 4.

1 The comparison

Write $B_d(x, r)$ for the closed Euclidean ball of radius r. Given a finite, nonempty labeled configuration $x = (x_i)$ and $r = (r_i) \in [0, \infty)^N$, put

$$U_d(x, r) = \mathrm{vol}_d\left(\bigcup_{i=1}^{N} B_d(x_i, r_i)\right), \qquad I_d(x, r) = \mathrm{vol}_d\left(\bigcap_{i=1}^{N} B_d(x_i, r_i)\right).$$

We call q an expansion of p if $\|p_i - p_j\| \leq \|q_i - q_j\|$ for all i, j.

Theorem 1. *Let $d \geq 3$ and $1 \leq N \leq d+4$. For every expansion $p \to q$ in $\mathbb{R}^d$ and every $r \in [0, \infty)^N$,*

$$U_d(p, r) \leq U_d(q, r), \qquad I_d(p, r) \geq I_d(q, r).$$

Bezdek and Connelly proved both comparisons for $N \leq d + 3$ [3, Corollary 4], a range recalled in the survey by Bezdek, Lángi, and Naszódi [5, Corollary 2.9]. Bezdek singled out $N = d + 4$ as the next case in 2008 [1, §6]. Bezdek and Lángi asked for $N = d + 4$ in each $d \geq 3$ [4, §3.3, Open Problem 13]. Gorbovickis proved the union inequality for larger configurations under a hypothesis limiting the number of other balls meeting any pair intersection in the initial configuration [7, Theorem 2.2]. His argument uses higher derivatives in a common shift of the squared radii. For $N = d + 4$, all $d + 2$ other balls can meet a given pair intersection. Bezdek gives further volume comparisons for uniform contractions of congruent balls [2]. Our earlier seven-ball proof [9] obtained the $d = 3$ case by constructing a five-dimensional motion. Here we use four extra dimensions and compare second differences. The geometric input is a squared-radius convexity property of volumes of clipped power-diagram walls.

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2 Two cuts of a three-ball

For a measurable set $K \subset \mathbb{R}^m$, write

$$W_K(s) = \mathrm{vol}_m\{x \in K : \|x\|^2 \le s\}, \qquad s \in \mathbb{R}.$$

This definition gives zero for negative s. Put $t_+ = \max(t, 0)$. Exact formulas for volumes of spherical wedges also occur in sphere–mesh intersection algorithms [8]. For the sign of a second difference, a positive integral representation is convenient.

Lemma 2. *If $K \subset \mathbb{R}^3$ is an intersection of at most two closed affine half-spaces, W_K is convex on $\mathbb{R}$.*

Proof. Two nonconstant inequalities can be written $n_1 \cdot x \ge a$ and $n_2 \cdot x \ge b$ for unit inward normals with angle $\alpha \in [0, \pi]$. Rotate coordinates to put the normals in the horizontal plane. Denote the resulting volume by $W_\alpha(s)$. For $0 < \beta < \pi$ define

$$T(\beta) = \frac{a^2 + b^2 - 2ab\cos\beta}{\sin^2\beta}.$$

We will prove

$$W_\alpha(s) = W_\pi(s) + \frac{2}{3}\int_\alpha^\pi (s - T(\beta))_+^{3/2}\, d\beta. \tag{1}$$

The values of the integrand at $0, \pi$ are immaterial. For opposite normals,

$$W_\pi(s) = \pi \int_{[a,-b]} \left(s - z^2\right)_+ dz,$$

with integral zero if $a > -b$. Thus both terms in (1) are convex in s. For at most one inequality the same slice formula applies with a half-line or the whole line. Constant inequalities either leave the region unchanged or make it empty.

Fix a horizontal circle of radius $\rho > 0$ about the vertical axis. Each constraint cuts an arc centered in the direction of its normal. When $|a|, |b| < \rho$, let their half-widths be $u = \arccos(a/\rho)$ and $v = \arccos(b/\rho)$. Write $m = u + v$, $n = |u - v|$, and $f(t) = (m - t)_+ - (n - t)_+$ for $t \ge 0$. On the real line $f(t)$ is the overlap length of two intervals with these half-widths and center separation t. On the circle, counting the two possible translates, the angular overlap at separation $\alpha \in [0, \pi]$ is

$$L_\rho(\alpha) = f(\alpha) + f(2\pi - \alpha).$$

Indeed, with the first arc represented on $[-u, u]$, the second arc can meet it in positive length only from centers α and $\alpha - 2\pi$. Since $n \le \min(m, 2\pi - m)$, the piecewise-linear formula gives

$$L_\rho(\alpha) - L_\rho(\pi) = \int_\alpha^\pi \mathbf{1}_{\{n < \beta < \min(m, 2\pi - m)\}}\, d\beta. \tag{2}$$

Taking cosines, the condition under the integral is

$$ab - \sqrt{\rho^2 - a^2}\sqrt{\rho^2 - b^2} < \rho^2\cos\beta < ab + \sqrt{\rho^2 - a^2}\sqrt{\rho^2 - b^2}.$$

Squaring the equivalent absolute-value inequality and canceling gives $\rho^2 > T(\beta)$. If either arc is empty or a full circle the overlap is constant in α; the indicator of $\rho^2 > T(\beta)$ vanishes because

$T(\beta) \geq \max(a^2, b^2)$. This last bound follows from $T = b^2 + (b\cos\beta - a)^2/\sin^2\beta$ and its symmetric counterpart. The exceptional radii at which an arc degenerates are null for radial integration.

For $s > 0$ the ball gives each horizontal radius $\rho \leq \sqrt{s}$ a vertical height $2\sqrt{s-\rho^2}$. Multiply (2) by ρ times this height and integrate. Tonelli's theorem applies to the nonnegative indicator integrand. For each β the inner integral, interpreted as zero when $T(\beta) \geq s$, is

$$\int_{\sqrt{T(\beta)}}^{\sqrt{s}} 2\rho\sqrt{s-\rho^2}\,d\rho = \tfrac{2}{3}\,(s - T(\beta))_+^{3/2}.$$

This proves (1), which also holds for $s \leq 0$. □

Lemma 3. *If $m \geq 3$ and $K \subset \mathbb{R}^m$ is cut out by at most $m-1$ closed affine half-spaces, then W_K is convex. The same holds for a ball centered at any fixed point.*

Proof. Translate the center to the origin. Set aside at most two of the inequalities so that at most $m-3$ remain. Take a three-dimensional linear space E orthogonal to all the remaining normals. On a fiber $y + E$, $y \in E^\perp$, the remaining inequalities are constant. The fiber is empty or is cut out by the two inequalities set aside. Its volume inside the ball is a function from Lemma 2 evaluated at $s - \|y\|^2$. Integrating the convexity inequality over $E^\perp$ proves the claim. □

3 A comparison of second differences

We now prove Theorem 1. For $N = 1$ both quantities are translation invariant. Suppose $N \geq 2$ and first take positive radii. Let G_p, G_q be the Gram matrices of the $N-1$ differences from an anchor in each configuration. For $\varepsilon > 0$ set

$$G_\varepsilon(t) = (1-t)G_p + tG_q + \varepsilon I, \qquad 0 \leq t \leq 1.$$

This matrix is positive definite throughout the closed interval. A Cholesky factor $R(t)$ satisfying $R(t)^\mathsf{T} R(t) = G_\varepsilon(t)$ varies smoothly since all pivots are positive. Use its columns and an anchored zero as centers in $\mathbb{R}^{N-1} \subset \mathbb{R}^D$, where $D = d + 4$. The regularization adds ε to squared distances from the anchor and 2ε to squared distances between distinct nonanchor labels. These increments are independent of time. Every pairwise distance is therefore nondecreasing, and the centers are affinely independent even if some original centers coincide. The endpoints of this motion will tend to the prescribed distance data as $\varepsilon \downarrow 0$.

For $\sigma \geq 0$ replace each radius by $\sqrt{r_i^2 + \sigma}$. Write $F^\cup(t,\sigma)$ and $F^\cap(t,\sigma)$ for the resulting union and intersection volumes in $\mathbb{R}^D$. On every closed subinterval of $(0,1)$, the motion is smooth and the centers pairwise distinct. Write $d_{ij}(t) = \|c_i(t) - c_j(t)\|$. The Csikós first variation [6], in its union and intersection forms [3, Theorem 2], reads

$$\begin{aligned} \partial_t F^\cup(t,\sigma) &= \sum_{i<j} d'_{ij}(t) W_{ij}^{\mathrm{near}}(t,\sigma), \\ -\partial_t F^\cap(t,\sigma) &= \sum_{i<j} d'_{ij}(t) W_{ij}^{\mathrm{far}}(t,\sigma). \end{aligned} \tag{3}$$

Both W's measure $(D-1)$-dimensional volumes. To describe the near wall, put $\pi_i(x) = \|x - c_i(t)\|^2 - r_i^2$ and take the hyperplane $H_{ij} = \{\pi_i = \pi_j\}$. In this hyperplane impose $\pi_i \leq \pi_k$ for each $k \neq i, j$, and clip by $\pi_i \leq \sigma$. For the far wall reverse all the comparisons with the other sites, retaining the same clipping inequality. Subtracting a common σ from the powers preserves their

order. Affine independence guarantees that the comparison normals restricted to H_{ij} are linearly independent: a dependence of the restrictions would make a linear combination of $c_k - c_i$, $k \neq i, j$, a multiple of $c_j - c_i$. Thus the weak comparisons describe the closure of the genuine common face. Clipping with a ball at tangency can contribute zero wall volume.

For fixed t, set $s = \sigma + r_i^2 - \operatorname{dist}(c_i(t), H_{ij})^2$. The clip is empty for $s < 0$. For $s \geq 0$ it is the ball of squared radius s in H_{ij} centered at the projection of $c_i(t)$. Each wall in (3) consequently has the form in Lemma 3 up to a constant shift of its argument. The count needed there is

$$m = D - 1 = d + 3, \qquad N - 2 \leq d + 2 = m - 1.$$

For $h > 0$ put $\Delta_h^2 F(t) = F(t, 2h) - 2F(t, h) + F(t, 0)$. Apply (3) at these three fixed shifts. Convexity of every wall and nonnegativity of each speed d'_{ij} give

$$\frac{d}{dt}\Delta_h^2 F^{\cup}(t) \geq 0, \qquad \frac{d}{dt}\Delta_h^2 F^{\cap}(t) \leq 0.$$

The mean-value theorem compares any two interior times. Continuity of the volumes extends both comparisons to $t = 0, 1$.

4 Return to the endpoint dimension

First remove the Gram regularization at each fixed $h > 0$. For $b \in \{p, q\}$, the columns of $(G_b + \varepsilon I)^{1/2}$ and an anchored zero have the same labeled squared distances as the corresponding regularized endpoint. An ambient isometry therefore maps one configuration to the other. The positive semidefinite square root is continuous up to $\varepsilon = 0$. The limiting centers are congruent to the original centers of b embedded in $\mathbb{R}^D$. At each of the three fixed squared-radius shifts, the finite union and intersection volumes converge by dominated convergence off the finitely many limiting spheres, inside a common bounded ball. This proves the endpoint comparisons for the unregularized configurations at each h, before dividing by h^2.

At an endpoint b, use a rigid motion to place its centers in $\mathbb{R}^d \times \{0\} \subset \mathbb{R}^{d+4}$. Write $M_b(x)$ for the maximum over i of $r_i^2 - \|x - b_i\|^2$ in the union case and its minimum in the intersection case. At fixed $x \in \mathbb{R}^d$, the nonempty normal sections are four-balls with a common center. The fiber of the union or intersection has volume $\kappa_4 \left(M_b(x) + \sigma\right)_+^2$, where $\kappa_4 = \pi^2/2$. For every real u,

$$\frac{(u + 2h)_+^2 - 2\,(u + h)_+^2 + (u)_+^2}{h^2} \;\longrightarrow\; 2\mathbf{1}_{\{u \geq 0\}} \quad (h \downarrow 0).$$

The quotient belongs to $[0, 2]$, because the first derivative of $u \mapsto (u)_+^2$ is nondecreasing and 2-Lipschitz. For $0 < h \leq 1$ its spatial support is contained in the fixed union of balls of squared radii $r_i^2 + 2$. Fubini's theorem and dominated convergence show that $h^{-2}\Delta_h^2 F$ tends at each endpoint to $\pi^2 \operatorname{vol}_d\{x : M_b(x) \geq 0\}$. The two endpoint comparisons yield Theorem 1 for positive radii. For radii that can vanish, apply this result to $r_i + 1/(k+1)$, $k \geq 0$, and let $k \to \infty$. The balls decrease to closed balls of radii r_i. Their common intersections do likewise, and finiteness of the label set gives the same conclusion for their unions. Continuity from above applies because the unions at $k = 0$ have finite volume. This completes the proof. □

Formal verification

The accompanying Lean 4.19.0 development proves the statement of Theorem 1 for the standard Euclidean Lebesgue measure. In `DPlusFour.lean`, the declaration `volume_d_plus_four` takes

a pairwise distance comparison and nonnegative radii as inputs. The code also gives the four-dimensional eight-ball specialization. `WallTheorem.lean` proves clipped-ball convexity, including affine offsets, degeneracies and both signs of the power-cell comparisons.

The formal proof of the first variation works in the general position ensured by the Gram regularization. In coordinates centered at each site the ball stays fixed and its power cell is clipped by moving affine half-spaces. At a reference time, slice along the normal of one face. Outside the other cutting planes and the sphere rim, the clipped indicator is locally constant on the moving slice. Those exceptional sets have zero wall measure. Fubini's theorem and dominated convergence give the integral of the normal speed over the clipped face. Changing one cut at a time gives the derivative of the whole cell. Pairing the two contributions from a common wall cancels the terms involving the rotation of the normal and leaves the derivative of the center distance. Reversing the active inequality gives the far-cell sign. The declaration `csikos_power_variation` records this identity, with each ordered pair given half weight. The version 1.0.0 release contains the Lean sources, pinned dependencies and replay instructions. The original preprint and proof are archived at doi:10.5281/zenodo.24623358.

Author contributions and use of AI. Peiwei Wu and Qian Peng jointly directed the study, selected the four-dimensional eight-ball question and its generalization, and specified the scope of the volume comparisons and their formal verification. They used OpenAI Codex to search the literature, develop the reductions and proofs, implement and test the Lean formalization, and draft and edit the manuscript.

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