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A FIVE-NINTHS LOWER BOUND FOR WASTE IN PACKING A SQUARE WITH UNIT SQUARES

Abstract. Let $W(L)$ be the minimum area left uncovered by a packing of independently rotated unit squares in a square of side L. For each fixed $0 < \delta \le 1/2$, we prove that $W(L) \gg_\delta L^{5/9}$ whenever L is sufficiently large and $\delta \le L - \lfloor L \rfloor \le 1 - \delta$. In particular $W(L)/\sqrt{L}$ tends to infinity along the half-integers, ruling out the uniform square-root bound proposed by Erdős and Graham. The proof tracks two axial characters of the tile centers and their product. The boundary phases force appreciable integrated squared inclination, while coherence of the axial characters and a separate estimate for their product force the same integral to be small on a set of rows. The per-packing theorem and its extremal and half-integer consequences have been formally verified in Lean.

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1. Statement and mechanism

Place closed unit squares in $[0, L]^2$, allowing each square its own center and orientation and requiring pairwise disjoint interiors. Write W for the Lebesgue area of the uncovered set and $W(L)$ for its minimum over packings. Every finite subfamily of N tiles has area $N \le L^2$, since tile boundaries have measure zero. This excludes infinite families, gives $W = L^2 - N$, and shows that the minimum is attained because the empty packing is admissible and there are only finitely many attainable counts. Write $\{L\} = L - \lfloor L \rfloor$.

Theorem 1.1. *For every $0 < \delta \le 1/2$ there are constants $c_\delta, L_\delta > 0$ such that, for every $L \ge L_\delta$ with $\delta \le \{L\} \le 1 - \delta$, every packing as above has*

$$W \ge c_\delta L^{5/9}.$$

Corollary 1.2. *For every $C > 0$, one has $W(k + 1/2) > C\sqrt{k + 1/2}$ for all sufficiently large integers k.*

Proof. Apply Theorem 1.1 with $\delta = 1/2$ and divide by $\sqrt{L}$. The resulting lower bound is $c_{1/2}L^{1/18}$. □

Erdős and Graham constructed packings with $W(L) = O(L^{7/11})$ and suggested that the correct upper bound might be $O(L^{1/2})$ [5]. Roth and Vaughan proved $W(L) \gg \sqrt{L \operatorname{dist}(L, \mathbb{Z})}$ when $L\{L\} > 1/6$, and expressed doubt about the proposed upper bound [9]. Corollary 1.2 answers the

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uniform upper-bound question negatively. Recent constructions by Bui and McClenagan give $W(L) = O(L^{3/5})$ [3, 7]. An earlier claim of the same upper exponent by Chung and Graham [4] rested on an angle calculation subsequently corrected by Arslanov and Bui [1]. On half-integer sides the two exponents now differ by $3/5 - 5/9 = 2/45$. At integer sides the grid gives zero waste. The theorem leaves open both the optimal exponent on half-integers and the behavior as $\{L\}$ approaches an endpoint at a rate depending on L.

The proof uses three observations of the *same* population of tiles. For a center c_i, take the two unit complex phases associated with its coordinates modulo one and also their product. When two averages of unit phases have almost full modulus, the average of their product must also have almost full modulus, provided all three use the same weights. On the other hand, the three Fourier integrals over a nearly aligned tile have quite different first moments. This gives two transport equations for the axial phases and an algebraic constraint on their product. Integrating these relations on rows where the inclination is controlled exposes a conflict with the phase change across the container.

Integration of exponential characters over tiles has classical precedents, including the rectangle-tiling test in Robins [8, Ch. 1]. Here the information comes from the quadratic terms near Fourier zeros and from the interaction of the three characters. The comparison with previous work concerns this packing problem. A bounded bibliographic search cannot establish absolute priority.

2. Controlling inclination by the original void

Identify the physical plane with $\mathbb{C}$. For a tile Q_i let d_i be any unit edge direction, and put

$$\alpha_i = \tfrac{1}{4}|d_i^4 - 1| = |\Re(d_i)\Im(d_i)|.$$

The value is independent of the chosen edge and satisfies $0 \le \alpha_i \le 1/2$. We use physical coordinates x, macroscopic coordinates $X = x/L$, and the notation

$$\Omega = [0,1]^2, \qquad S = \sqrt{L}, \qquad w = 1 + W/S.$$

In estimates with $\lesssim$, constants are independent of L, the tile count and the packing. Once a smoothing profile is fixed they may depend on that profile.

Lemma 2.1 (Localized inclination)**.** *Let V_p be the original void inside the axis-parallel square of side eight about the center of Q_p. For centers at distance at most three, $|d_p^4 - d_q^4| \lesssim |V_p|$. If the center of Q_p is at distance at most $5/2$ from a wall, then $|d_p^4 - 1| \lesssim |V_p|$.*

Proof. We use the rectangle, vertex-free strip and chain of facing edges of Roth and Vaughan [9, §3, pp. 172–173]. Their Fundamental Lemma uses distance at most one between the tile *sets* and a different neighborhood for the waste. We derive the stated center and wall estimates.

For distinct tiles, take the segment joining their centers as longitudinal axis. At each end the transverse segment of length one is in the tile's inscribed disk. The rectangle between the end segments lies in the container by convexity, and in the radius-four disk about the first center. Every tile meeting that disk lies in the concentric square of side twelve. There are thus at most $m = 144$ relevant tiles. Divide the transverse unit interval into $4m + 1$ equal bins. Since the bin interiors are disjoint, at least one contains no projected vertex in its interior. Its closed middle half has width $h = 1/[2(4m + 1)]$ and avoids every projected vertex.

Throughout this strip each active tile has a longitudinal section of positive length whose two endpoints are affine functions of the transverse coordinate. Their order among tiles is fixed because an exchange of two section midpoints would force interior overlap. Between successive sections in the chain from the first designated tile to the second, the gap is nonnegative and affine. If the facing slopes are s, t and the end gaps g_0, g_1, then $h|s - t| \le g_0 + g_1$. In an orthonormal coordinate frame with the transverse axis first, the fourth harmonic of an edge of slope s is $P(s) = ((1+\mathrm{i}s)/(1-\mathrm{i}s))^2$. Since $|P'(s)| \le 4$, the gap area $h(g_0+g_1)/2$ is at least $h^2|P(s)-P(t)|/8$. The fourth harmonics of both edges of each tile agree, and changing frame preserves their differences in modulus. The gaps lie between the end caps, are uncovered, and belong to the measuring neighborhood. Summing along the chain proves the first assertion.

For a wall, append an aligned unit square outside the container whose center lies half a unit beyond the wall on the normal through the given center. The enlarged family has disjoint interiors and the two centers are at distance at most three. Each longitudinal fiber starts inside the original container. Its transverse coordinate is constant, so it cannot pass outside an adjacent wall, even near a corner. Its portion beyond the chosen wall is in the appended square. The same gap sum thus charges only the original void. □

Divide each side into $\lceil 4L \rceil$ equal intervals. The physical cell side $s = L/\lceil 4L \rceil$ lies in $[1/5, 1/4]$ for $L \ge 1$. For each cell C set α_C equal to the maximum inclination of tiles meeting its *closure*, with value zero for an empty cell. Give exterior cells value zero. For every pointwise evaluation use the unique half-open owner $[js, (j+1)s) \times [ks, (k+1)s)$ on the extended lattice. Thus the upper and right container edges have exterior owners. The same conventions apply after rescaling.

Lemma 2.2 (Mesh estimates)**.** *Counting nearest-neighbor faces once, including jumps to exterior zero, one has*

$$\sum_{C \sim C'} |\alpha_C - \alpha_{C'}| + \sum_{C} \sum_{i:\, |C \cap Q_i| > 0} |\alpha_i - \alpha_C| \lesssim W, \qquad \sum_i \alpha_i^2 \lesssim W^2. \quad (2.1)$$

For the macroscopic field $a(X) = S\alpha_C$ evaluated at the owner of LX, the variation $\mathrm{Var}(a)$ *and the integral in y of* $\sup_x a(x, y)$ *are $O(w)$. Here variation is the sum of (macroscopic face length) times (absolute jump).*

Proof. Centers of tiles meeting adjacent closed cells are at distance at most $2 + 3s < 3$. A tile meeting a wall cell has center within $1 + s$ of the wall. Apply Lemma 2.1 and the reverse triangle inequality. An empty cell pays for incident jumps by its void area s^2. Only boundedly many tiles meet one cell, by the area bound in a fixed surrounding square. Each local void charge lies in a side-sixteen square about its assigned cell midpoint. These squares have bounded overlap since $s \geq 1/5$. This proves the first inequality, including the within-cell comparisons.

Each row maximum of the unscaled representatives is bounded by the sum of horizontal jumps along that row starting from exterior zero. The sum of row maxima and, by the same argument, the sum of column maxima are $O(W)$. A tile center is in the interior of its tile and in the closure of its owner, so its inclination is bounded by that owner's representative. If r_k, t_j are the respective row and column maxima, a center in cell (j, k) contributes at most $r_k t_j$ to the energy. Centers have bounded multiplicity per cell. Summation gives the second inequality in (2.1). For the two macroscopic estimates multiply the face-jump sum or the row-maximum sum by sS/L. □

Proposition 2.3 (Editing the packing)**.** *For $D \geq 1$ with $2D/S \leq 1/16$, there exist an edited packing $(Q'_i)_{i \in I'}$ of waste W', an owned-cell field β supported on Ω, and measurable $G \subset [1/4, 3/4]$ with $|G| \geq 1/4$, such that*

$$W'/L \lesssim w/D, \qquad \alpha'_i \leq 2D/S \quad (i \in I'), \qquad \sum_i (\alpha'_i)^2 \lesssim Lw^2, \tag{2.2}$$

$$0 \leq \beta \leq 4D^2, \quad \mathrm{Var}(\beta) \lesssim Dw, \qquad \sup_{x \in \mathbb{R},\, y \in G} \beta(x, y) \lesssim w^2. \tag{2.3}$$

Writing A' for the edited occupied set in macroscopic coordinates and α' for its tile inclinations,

$$\int_{A'} |L(\alpha')^2 - \beta| \, dX \lesssim w/S. \tag{2.4}$$

Proof. First select rows in the *original* packing. Lemma 2.2 and Markov's inequality give a set G of the required measure on which $\sup_x a(x, y) \lesssim w$. One may discard the finitely many horizontal cell-boundary ordinates.

For $D < t \leq 2D$ mark a cell when $a_C = S\alpha_C \geq t$. Let H_t be its pointwise indicator by ownership, P_t its macroscopic perimeter counting faces between opposite colors, and $V = \mathrm{Var}(a)$. Summing the lengths of threshold intervals crossed by each face gives $\int_D^{2D} P_t \, dt \leq V$, including the wall faces. Choose t with $P_t \leq V/D$. At that same threshold put

$$\beta = b_t(a), \qquad b_t(v) = v^2 \mathbf{1}_{v<t} = \min(v, t)^2 - t^2 \mathbf{1}_{v \geq t}.$$

The first term is $2t$-Lipschitz on $[0, \infty)$, so $\mathrm{Var}(\beta) \leq 2tV + t^2 P_t \leq 8DV$. The global and good-row caps follow from $b_t(v) \leq \min(t^2, v^2)$.

Keep an old *closed* tile precisely when every point of it has mask zero. Insert an integer-grid closed unit square precisely when every point of it has mask one (in physical coordinates use $H_t(x/L)$). These tests use the owner of *every* point. The closed union of marked cell closures describes

their location but can include faces of mask zero. Substituting that closed union in the tile tests could change the edit. The tests above give contained tiles with pairwise disjoint interiors. The center of any old tile of scaled inclination at least t lies in a marked owner because its closure meets that tile. Thus such a tile is removed. Inserted tiles have zero inclination, giving the tilt and energy estimates in (2.2).

Consider a formerly occupied point lost by the edit. If its mask is zero, its old square contains a point of mask one. If its mask is one, an integer-grid square containing it must contain a point of mask zero. In either case the two points are within one tile diameter. The horizontal-then-vertical lattice crossing between their owners encounters opposite labels on adjacent cells, with a separating face midpoint at sup-norm distance at most four in physical units. This counts exterior faces as well. The union of the side-eight squares about these midpoints has area at most $64LP_t/s$. Consequently $W' \le W + O(LP_t) \lesssim Lw/D$, using $s \ge 1/5$ and $D \le S$.

On refill tiles both densities in (2.4) vanish. On a retained tile the coefficient in each owned cell is $L\alpha_C^2$. Since $\alpha_i + \alpha_C \le 1$, the physical cellwise discrepancy after rescaling is at most $L^{-1}|Q_i \cap C|\,|\alpha_i - \alpha_C|$. Summing (2.1) proves (2.4). Tile and cell edges have area zero in this integral. Their pointwise ownership was needed earlier for the edit. □

Cutting away large inclinations and refilling with grid squares also appears in Bui's good-square construction [2]. The path estimate there is Proposition 19 in the proceedings (Proposition 21 in the v1 preprint PDF). Here the threshold varies with L. The bound on the variation of the retained coefficient in (2.3) still uses the *original* waste after the edit has possibly increased it.

3. Three Fourier measurements

Use the edited packing for the rest of the proof. In this section α_i denotes the inclination of edited tile i. For the physical center c_i, write

$$X_i = c_i/L, \qquad z_i = c_i + (1/2, 1/2) \pmod{\mathbb{Z}^2}, \qquad \omega = 2\pi.$$

For $p = e_1, e_2, e_1 + e_2$, give tile i the phase $\zeta_{i,p} = \exp(\mathrm{i}\omega p \cdot z_i)$. Let m_p take this value on the interior of each macroscopic tile and vanish elsewhere. Thus the diagonal tile phase is the product of the axial tile phases.

3.1. Moments of a rotated tile.

For a physical tile Q translated to have center zero, define the signed mass and vector first moment

$$(T_p, B_p) = (-1)^{p_1+p_2} \int_Q (1, v)\mathrm{e}^{\mathrm{i}\omega p \cdot v}\, dv.$$

The sign accounts for the shift by $(1/2, 1/2)$ in z_i.

Lemma 3.1. *For either axial mode, $T_{e_j} = -\alpha^2/2 + O(\alpha^3)$ and $B_{e_j} = e_j/(\mathrm{i}\omega) + O(\alpha)$. For the diagonal, $T_{e_1+e_2} = -\alpha^2 + O(\alpha^3)$ and $B_{e_1+e_2} = O(\alpha)$, uniformly for $\alpha \le 1/16$.*

Proof. Choose orthogonal edge directions $u = (c, s_0)$, $u^\perp = (-s_0, c)$, with $c \geq |s_0|$. Then $\alpha = c|s_0|$, $|s_0| \leq 2\alpha$ and $c^2 + s_0^2 = 1$. Before multiplication by the sign the mass equals $\operatorname{sinc}(\pi p \cdot u)\operatorname{sinc}(\pi p \cdot u^\perp)$, where $\operatorname{sinc}(t) = \sin(t)/t$ with value one at the origin. Use $\operatorname{sinc}(\pi(1+h)) = -h + O(h^2)$ and $\operatorname{sinc}(\pi h) = 1 + O(h^2)$. For either axis, after the sign the answer is $-(1-c) + O(\alpha^4)$, and $2(1-c) - \alpha^2 = (1-c)^2(c^2 + 2c + 2)$. For the diagonal the two linear factors give $(c - 1 + s_0)(c - 1 - s_0) = (1-c)^2 - s_0^2 = -\alpha^2 + O(\alpha^4)$. The product of the linearizations has error $O(\alpha^3)$.

At alignment the integrals of $\exp(\mathrm{i}\omega x)$ and of $x\exp(\mathrm{i}\omega x)$ on $[-1/2, 1/2]$ are zero and $-1/(\mathrm{i}\omega)$, respectively. Multiplying the one-dimensional integrals gives the stated first moments after the signs. Pulling each integral back to the fixed unit square gives a uniformly bounded angular derivative. The canonical angle has magnitude $O(\alpha)$. □

3.2. **Smoothing and exterior data.** For the one-dimensional smooth bump b, take the real Mathlib `ContDiffBump` at 0 with inner radius $1/2$ and outer radius 1. For $0 < \eta \leq 1/4$ set

$$k = b/\int_{\mathbb{R}} b, \qquad k_\eta(x) = \eta^{-1}k(x/\eta), \qquad K_\eta(X) = k_\eta(X_1)k_\eta(X_2).$$

The profile is nonnegative, has unit mass and vanishes outside $[-1, 1]$. For the exterior completion put $\lambda = \mathrm{i}\omega L$ and

$$I_\nu[g](x) = \int_0^1 \mathrm{e}^{\nu s} g(x - s)\, ds, \qquad F(x) = \int_{-\infty}^{x} k_\eta(t)\, dt.$$

Define

$$E_1(X) = \big[1 - F(X_1) + \mathrm{e}^{\lambda} F(X_1 - 1)\big] I_0[k_\eta](X_2), \quad E_2(X) = E_1(X_2, X_1), \tag{3.1}$$

$$u_j = K_\eta * m_{e_j} + E_j \quad (j = 1, 2), \qquad u_{12} = K_\eta * m_{e_1+e_2}.$$

The bracket supplies the wall phases on the two exterior half-strips. Its derivative in X_1 is $\mathrm{e}^{\lambda} k_\eta(X_1 - 1) - k_\eta(X_1)$. We use $\|E_j\|_\infty \leq 3$, hence $\|u_j\|_\infty \leq 4$. On $\Omega_\eta = [\eta, 1 - \eta]^2$ both exterior contributions vanish, and the occupied fields have modulus at most one by positivity and unit mass.

Proposition 3.2 (Global residuals for the fixed profile)**.** *Write $R_j = \partial_j u_j + \mathrm{i}\pi\beta u_j$ and $M = \beta u_{12}$. All three residuals are integrable on $\mathbb{R}^2$, and*

$$\|R_1\|_1 + \|R_2\|_1 + \|M\|_1 \lesssim \mathcal{E}, \tag{3.2}$$

$$\mathcal{E} = \frac{w}{D} + \frac{Dw^2 + w}{S} + \frac{w/S + w^2/L}{\eta} + \frac{1}{L\eta^2} + \eta D w. \tag{3.3}$$

The completed axial fields themselves can be nondecaying along the exterior strips.

Proof. For each frequency measure the whole physical container by

$$\frac{1}{L}\int_{[0,L]^2} \phi(x/L)\mathrm{e}^{\mathrm{i}\omega p\cdot x}\, dx, \qquad \phi(X) = K_\eta(Z - X).$$

All error norms below integrate in Z over the whole plane. Taylor expansion of ϕ on tile i about X_i gives

$$\frac{\zeta_{i,p}}{L}\left[T_p\phi(X_i) + L^{-1}B_p \cdot \nabla\phi(X_i)\right]$$

up to its second-order remainder. Spread each center evaluation uniformly over its own macroscopic tile. The mass and gradient then acquire the true tile phase there. Substitute Lemma 3.1 and the coefficient comparison (2.4). Up to the errors below, the axial and diagonal measurements become, respectively,

$$-\tfrac{1}{2}\int \beta m_{e_j}\phi + (\mathrm{i}\omega)^{-1}\int m_{e_j}\partial_j\phi, \qquad -\int \beta m_{e_1+e_2}\phi. \tag{3.4}$$

The derivative here acts on X.

For clarity, the contributions are displayed before and after insertion of the geometric bounds. In the table N is the edited count and $K = K_\eta$.

Source of error	Bound up to a constant	Scale
True void	$W'\|K\|_1/L$	w/D
Mass expansion	$L^{-1}\sum_i \alpha_i^3$	Dw^2/S
First-moment expansion	$L^{-2}\|\nabla K\|_1 \sum_i \alpha_i$	$w/(S\eta)$
Spreading the quadratic mass	$L^{-2}\|\nabla K\|_1 \sum_i \alpha_i^2$	$w^2/(L\eta)$
Taylor remainder and gradient spread	$NL^{-3}\|\nabla^2 K\|_1$	$1/(L\eta^2)$
Replacing energy density	Left side of (2.4)	w/S
Commuting and completing	$O(\eta)\,\mathrm{Var}(\beta)$	ηDw

Indeed, $N \le L^2$, $\sum_i \alpha_i^2 \lesssim Lw^2$, and $\sum_i \alpha_i \le (N\sum_i \alpha_i^2)^{1/2}$. Each edited tile satisfies $\alpha_i \le 2D/S$, so summation gives $\sum_i \alpha_i^3 \le (2D/S)\sum_i \alpha_i^2$. Both sums are zero when the edit leaves no tiles. Each displacement from a rescaled center is at most $1/L$. Integrating translated derivatives of the kernel in Z gives $\|\nabla^j K\|_1 = O(\eta^{-j})$ for $j = 0, 1, 2$, also for the integral Taylor remainder. Integrating the omitted void first in Z gives the first line. These estimates bound integrals of absolute values.

It remains to evaluate the container side of (3.4) and justify the last line of the table. Put $g = k_\eta$ and $B_\lambda[g](x) = \mathrm{e}^\lambda g(x-1) - g(x)$. Integration by parts gives

$$\lambda I_\lambda[g] = B_\lambda[g] + I_\lambda[g'], \qquad \|I_\lambda[g]\|_1 \le \|g\|_1, \quad \|B_\lambda[g]\|_1 \le 2\|g\|_1.$$

In direction 1 the whole-container measurement is $LI_\lambda[g](X_1)I_0[g](X_2)$. Multiply it by $\mathrm{i}\omega$, subtract $B_\lambda[g](X_1)I_0[g](X_2)$ and integrate by parts once more. The remainder has norm at most $(2\|g'\|_1+\|g''\|_1)/|\lambda|$, since $\|I_0[g]\|_1 = 1$. The subtracted term is exactly $\partial_1 E_1$. In direction 2 the same formulas hold with coordinates exchanged. The diagonal measurement is $LI_\lambda[g](X_1)I_\lambda[g](X_2)$. One integration in each variable gives norm at most $L(2\|g\|_1 + \|g'\|_1)^2/|\lambda|^2$. All three remainders are $O(1/(L\eta^2))$.

For a bounded occupied phase m, the difference $K * (\beta m) - \beta(K * m)$ has norm bounded by

$$\int K(h)\|\beta(\cdot - h) - \beta\|_1 \, dh \le 2\eta \, \mathrm{Var}(\beta).$$

For the mesh function the translation inequality follows by summing the lengths swept out by each face, first horizontally and then vertically, with exterior faces included. The support of K has Euclidean radius at most 2η. In Ω, E_j can be nonzero only within distance η of the two j-ends. Translating each such end strip outwards by 2η gives coefficient zero. The same translation bound and $\|E_j\|_\infty \le 3$ therefore pay for $\|\beta E_j\|_1$ at cost $O(\eta \, \mathrm{Var}(\beta))$.

Finally $\partial_{X_j} K(Z - X) = -\partial_{Z_j} K(Z - X)$. Equating the axial expression in (3.4) times $\mathrm{i}\omega$ with the boundary derivative moves the latter to the left as $+\partial_j E_j$, giving the sign $+\mathrm{i}\pi\beta u_j$ in the residual. The diagonal expression has no leading gradient. This proves (3.2). □

Scope of the smoothing estimate. The same calculations work for any *fixed* nonnegative C^∞ probability density supported in $[-1, 1]$ and its rescaled tensor products. They use positivity, support, unit mass, the first absolute spatial moment and the indicated L^1 derivative norms. The constants may depend on these data. This generality follows from the analytic argument. The machine-checked proposition uses the particular normalized bump specified above.

4. Coherence on selected rows

The imaginary potential in the axial equations changes phase without contributing to the derivative of squared modulus. We use this along finite segments with known endpoint phases. Put $J = [-1, 2]$ and define

$$r_1(y) = \int_J |R_1(x, y)| \, dx, \qquad r_2(x) = \int_J |R_2(x, y)| \, dy.$$

For $y \in [\eta, 1 - \eta]$, equation (3.1) and the kernel support give $u_1(-1, y) = 1$ and $u_1(2, y) = \mathrm{e}^{\mathrm{i}\omega L}$. The same endpoints hold for u_2 on a column with $x \in [\eta, 1 - \eta]$.

Lemma 4.1 (Propagation and the joint phase)**.** *On each such row, throughout J,*

$$1 - |u_1(x, y)| \le 8 r_1(y), \qquad |\mathrm{e}^{\mathrm{i}\omega L} - 1| \le r_1(y) + 4\pi \int_J \beta(x, y) \, dx. \tag{4.1}$$

The modulus estimate holds on the corresponding columns with indices exchanged. Throughout Ω_η,

$$1 - |u_{12}| \le 2(1 - |u_1|) + 2(1 - |u_2|). \tag{4.2}$$

Proof. Along a row, $(|u_1|^2)' = 2\Re(\overline{u_1} R_1)$ pointwise because β is real. The field is smooth along the line. The residual is integrable on J since the coefficient is bounded and measurable there. Integration from -1 gives

$|1-|u_1|^2| \le 8r_1(y)$, using the global bound four. Dividing by $1+|u_1|$ gives the first estimate even at zeros or at coefficient jumps. Integrating $\partial_1 u_1 = R_1 - \mathrm{i}\pi\beta u_1$ between the endpoints proves the second. Apply the same argument vertically.

In the trimmed square all three fields come from the same positive subprobability measure, obtained by restricting the density $K_\eta(X-\cdot)$ to occupied tiles. Write z, v for the two unit phases on that space, $m \le 1$ for its mass, and rotate the two phases separately so their integrals are nonnegative real numbers $|u_1|, |u_2|$. The magnitude of the product integral stays $|u_{12}|$. Now $|1-zv|^2 \le 2|1-z|^2 + 2|1-v|^2$. Integrating and using $\Re(z) = 1 - |1-z|^2/2$ yields

$$m - |u_{12}| \le 2(m-|u_1|) + 2(m-|u_2|).$$

Adding the missing mass and using $m \le 1$ proves (4.2). □

Proposition 4.2 (Boundary obstruction)**.** *For the parameters of Proposition 2.3 and $0 < \eta \le 1/4$,*

$$|\mathrm{e}^{2\pi\mathrm{i}L} - 1| \lesssim w^2(\mathcal{E} + \eta). \tag{4.3}$$

Proof. Take the rows G selected before the edit and let $B \lesssim w^2$ be their pointwise cap on β. Integrating (4.1) on G gives

$$|G|\,|\mathrm{e}^{\mathrm{i}\omega L} - 1| \le \|R_1\|_1 + 4\pi \int_G \int_J \beta(x,y)\,dx\,dy.$$

Outside $[0,1]$ the coefficient vanishes. The two end strips of width η cost at most $2\eta B|G|$. On the remaining slab write $\beta = \beta|u_{12}| + \beta(1-|u_{12}|)$. The integral of the first term is at most $\|M\|_1$. The second term is at most $16B(r_1(y) + r_2(x))$ pointwise by Lemma 4.1. Integrating in $x \in [\eta, 1-\eta]$ costs at most $|J| r_1(y)$ for the horizontal term, with $|J| = 3$. Fubini bounds the vertical term integrated on the slab by $|G|\|R_2\|_1$. Thus no selection of columns is required. Use (3.2), $|G| \ge 1/4$ and $w \ge 1$. In particular the whole-plane commutator estimate was taken *before* restricting to rows, so inclinations just across a row boundary are already accounted for. □

5. The exponent and the scope of verification

Proof of Theorem 1.1. On the specified fractional parts $|\mathrm{e}^{2\pi\mathrm{i}L} - 1| \ge b_\delta := 2\sin(\pi\delta) > 0$. If the theorem failed for some fixed δ, we could choose a sequence of admissible sides $L \to \infty$ and packings with $W/L^{5/9} \to 0$, hence $w = o(L^{1/18})$. In (4.3) take

$$D = Kw^3, \qquad \eta = \frac{\varepsilon}{Dw^3}.$$

Choose $K \ge 1$ large and $0 < \varepsilon < 1/4$ small in terms of b_δ and the absolute constant in (4.3). The cutoff condition $2D/S \le 1/16$ holds eventually. After multiplying by w^2, the terms w/D, $\eta D w$, and η contribute $1/K, \varepsilon, \varepsilon/(Kw^4)$.

Choose the constants so their combined contribution to the right side of (4.3) is less than $b_\delta/2$. The remaining terms are bounded by

$$C_{K,\varepsilon}\left(\frac{w^7+w^3+w^9}{\sqrt{L}}+\frac{w^{10}+w^{14}}{L}\right)=o(1).$$

This contradicts the endpoint mismatch. The constants may degenerate as $\delta\downarrow 0$. □

The particular budget (3.3) explains the exponent. For power scales $w=L^\gamma,\ D=L^d,\ \eta=L^{-e}$, boundedness of the costs w^3/D, $\eta D w^3$ and $w^3/(S\eta)$ requires $d\ge 3\gamma,\ e\ge d+3\gamma,\ e\le 1/2-3\gamma$. Thus $\gamma\le 1/18$ for this balance. Determining the sharp growth rate of $W(k+1/2)$ remains open.

Formal verification. The accompanying Lean 4.24.0 project uses a revision-pinned Mathlib [6, 10]. In the namespace `PackingPhaseIII`, the theorem `unrestricted_five_ninths` quantifies over geometric packings of every finite cardinality. The minimum-waste and half-integer statements also have proof objects. An independent AI-based audit checked the interpretation using arbitrary index types, topological interiors and the area outside the union of the closed tile images. The audit replayed the union of the three transitive dependency sets in a Lean kernel initially containing no modules. All 45,452 reachable declarations were replayed and the three theorem types recovered. The only named axioms in the chains were propositional extensionality, classical choice and quotient soundness. The audit report, pinned revisions, replay program and manuscript-to-Lean index accompany the paper. Kernel verification concerns the formal statements and their proofs. Bibliographic priority and academic peer review are separate questions.

Research provenance. The investigator reports selecting the classical packing problem and proposing control theory, including PID control, as a starting point. The investigator directed the subsequent research, auditing, Lean formalization and publication preparation with AI-assisted instructions. AI systems produced the mathematical discoveries, proofs, computational implementations, formalization and manuscript. Separate AI sessions conducted the mathematical and formal verification audits. The investigator understands the central ideas and overall proof structure and has not independently verified every technical step. The public contribution record gives the evidence by stage.

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