Cauchy completions and normalized length

Abstract

We give an overview of the construction of Faltings' normalized length through extensions of premeasures. For a small abelian category $\mathcal{A}$ with a finite premeasure ν, we construct its unique continuous extension κ to $\mathrm{Ind}(\mathcal{A})$. For every $M \in \mathrm{Ind}(\mathcal{A})$ of finite value, we construct a Cauchy sequence (F_n) in $\mathcal{A}$ and a map $\mathrm{colim}_n F_n \to M$ with kernel and cokernel of value zero. For a presentable stable category with a right complete t-structure compatible with filtered colimits and heart $\mathrm{Ind}(\mathcal{A})$, we prove that every Cauchy sequence in the quotient of $\{E : \sum_i \kappa(\pi_i E) < +\infty\}$ by $\{E : \kappa(\pi_i E) = 0 \text{ for all } i\}$ admits a colimit and converges to it. We also prove uniform continuity of finitely presented continuous additive linear endofunctors over discrete coherent commutative rings for continuous premeasures finite on the ring. We apply the extension theorem to finite flat towers of Noetherian local rings. The second part treats regular parameter towers with parameters $t_1, \dots, t_r$ over A and computes their ordered Grothendieck groups. For a finite generically separable extension $A \subseteq B$ of the base, set

$$P = \Big(\operatorname*{colim}_n A[t_1^{1/p^n}, \dots, t_r^{1/p^n}]\Big)^{\wedge}_p, \qquad D = (B \otimes_A P)_{\mathrm{perfd}},$$

and $d = [\mathrm{Frac}(B) : \mathrm{Frac}(A)]$. We prove

$$\kappa_A\big(\pi_i(D \otimes_A V)\big) = \begin{cases} d\,\mathrm{length}_A(V)[P/\mathfrak{m}_A P], & i = 0, \\ 0, & i \neq 0, \end{cases}$$

for every finite-length discrete A-module V. We make no claim to originality in this second expository companion article, which we wrote while studying OpenAI's proof of the positivity of Serre's intersection multiplicity.

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Part I
Measures on Abelian and stable categories

1 Premeasures on abelian categories

Throughout the paper L is an ordered real line, equipped with its order topology, and $L_{\geq 0}$ is its nonnegative cone. We write $L_{\geq 0} \cup \{+\infty\}$ for $L_{\geq 0}$ with a new largest element $+\infty$, equipped with the order topology. Addition and multiplication by positive real numbers extend by $x + (+\infty) = (+\infty) + x = +\infty$ and $t(+\infty) = +\infty$ for $x \in L_{\geq 0} \cup \{+\infty\}$ and $t > 0$.

Definition (Directed union)**.** Let $M : I \to \mathcal{A}$ be a functor from a small directed poset to an abelian category. A cocone $(a_i : M_i \to X)_{i \in I}$ on M is a *directed union* if it is a colimit cocone in $\mathcal{A}$ and every a_i is a monomorphism. We then write $X = \bigcup_{i \in I} M_i$.

Definition 1.1. A *premeasure* on an abelian category $\mathcal{A}$ is a function

$$\nu : \mathcal{A}^{\cong} \longrightarrow L_{\geq 0} \cup \{+\infty\}$$

with $\nu(0) = 0$ and $\nu(M) = \nu(M') + \nu(M'')$ for every short exact sequence $0 \to M' \to M \to M'' \to 0$. It is *finite* if all its values are finite, and *faithful* if $\nu(M) = 0$ implies $M = 0$. A *measure* is a faithful premeasure.

The premeasure ν *preserves directed unions* if $\nu(U) = \sup_{i \in I} \nu(M_i)$ for every directed union $(u_i : M_i \to U)_{i \in I}$.

The premeasure ν is *continuous* if the following conditions hold for every functor $M_\bullet : I \to \mathcal{A}$ from a small directed poset with monomorphic transitions and $\sup_i \nu(M_i) < +\infty$.

1. The diagram admits a cocone $(u_i : M_i \to M)_{i\in I}$ that is a directed union.
2. $\nu(M) = \sup_i \nu(M_i)$.
3. For every subobject $N \subseteq M$, the canonical map in (1.1)

$$\operatorname{colim}_i(N \cap M_i) \longrightarrow N \tag{1.1}$$

is an isomorphism.

Lemma 1.2. *For a measure, conditions (1) and (2) in Definition 1.1 imply condition (3).*

Proof. Suppose ν is faithful and satisfies Definition 1.1(1)–(2). Let $M = \bigcup_i M_i$ have finite value, and let $N \subseteq M$. Additivity gives

$$\nu(N \cap M_i) = \nu(N) + \nu(M_i) - \nu(N + M_i).$$

Since $\nu(M_i) \le \nu(N + M_i) \le \nu(M)$ and $\nu(M_i) \to \nu(M)$, both values tend to $\nu(M)$. It follows that $\nu(N \cap M_i) \to \nu(N)$. Put $V = \operatorname{colim}_i(N \cap M_i)$, and let $v : V \to N$ be the canonical map. Its kernel meets each canonical subobject $N \cap M_i$ trivially, so

$$\nu(\ker v) \le \nu(V) - \nu(N \cap M_i) \longrightarrow 0.$$

Faithfulness makes v monomorphic. Definition 1.1(2) gives $\nu(V) = \nu(N)$, so $\operatorname{coker} v$ also has value zero and vanishes. This proves Definition 1.1(3). □

A continuous premeasure κ on $\mathcal{A}$ restricts to a finite continuous premeasure on the Serre subcategory $\mathcal{A}_{<\infty} = \{M \in \mathcal{A} : \kappa(M) < +\infty\}$.

On a Grothendieck abelian category, continuity is equivalent to preservation of directed unions: the existence and exactness of these unions are automatic, and the equality when the supremum is $+\infty$ follows from monotonicity. For a continuous premeasure κ on a Grothendieck abelian category $\mathcal{A}$, the subcategory $\mathcal{A}_0 = \{M \in \mathcal{A} : \kappa(M) = 0\}$ is localizing, and the quotient $q : \mathcal{A} \to \mathcal{A}/\mathcal{A}_0$ has a fully faithful right adjoint [15, Appendix A.2].

2 Extension of premeasures to $\operatorname{Ind}(\mathcal{A})$

Theorem 2.1 is the abelian-category form of Virili's extension argument [9, Main Theorem and p. 70].

Theorem 2.1. *Let $\mathcal{A}$ be a small abelian category, and let ν be a finite premeasure on $\mathcal{A}$. For a finitely generated object $M \in \operatorname{Ind}(\mathcal{A})$, set*

$$\kappa_{\mathrm{fg}}(M) = \inf_{Q \twoheadrightarrow M} \nu(Q), \qquad Q \in \mathcal{A}. \tag{2.1}$$

For arbitrary $M \in \operatorname{Ind}(\mathcal{A})$, set

$$\kappa(M) = \sup_{M' \subseteq M} \kappa_{\mathrm{fg}}(M'), \qquad M' \text{ finitely generated.} \tag{2.2}$$

Then κ is the unique continuous premeasure on $\operatorname{Ind}(\mathcal{A})$ extending ν. For finitely generated M, one has $\kappa(M) = \kappa_{\mathrm{fg}}(M) < +\infty$. Here an object is finitely generated if every directed union of subobjects equal to it contains the whole object as one member.

Proof. The category $\mathrm{Ind}(\mathcal{A})$ is abelian, its filtered colimits are exact, and its compact objects are the objects of $\mathcal{A}$. Every object is a directed union of images of objects of $\mathcal{A}$, hence of finitely generated subobjects. In particular, every finitely generated object is a quotient of an object of $\mathcal{A}$. A finitely generated subobject H of $Q \in \mathcal{A}$ is therefore the image of a map $Q' \to Q$ in $\mathcal{A}$, so $H \in \mathcal{A}$.

If $Q \in \mathcal{A}$ and $H \subseteq Q$ is an arbitrary subobject, put

$$v_Q(H) = \sup_{H_0 \subseteq H} \nu(H_0), \qquad H_0 \text{ finitely generated.}$$

Both H_0 and Q/H_0 belong to $\mathcal{A}$, so positivity and additivity imply

$$0 \leq v_Q(H) \leq \nu(Q) < +\infty.$$

First, for any presentation $M = Q/K$ with $Q \in \mathcal{A}$, we claim

$$\kappa_{\mathrm{fg}}(M) = \nu(Q) - v_Q(K). \tag{2.3}$$

The quotients Q/K_0, for finitely generated $K_0 \subseteq K$, are objects of $\mathcal{A}$ and surject onto M. The infimum of their values is the right side of (2.3). Conversely, let $Q' \twoheadrightarrow M$ with $Q' \in \mathcal{A}$. Since

$$M = \operatorname*{colim}_{K_0 \subseteq K,\ K_0 \text{ finitely generated}} Q/K_0,$$

compactness of Q' makes its map to M factor through one Q/K_0. The cokernels of these factorizations are compact and form a filtered system of quotients with colimit zero. Hence, after enlarging K_0, the map $Q' \to Q/K_0$ is surjective. Positivity now gives $\nu(Q/K_0) \leq \nu(Q')$. This proves (2.3). In particular, taking $K = 0$ proves $\kappa_{\mathrm{fg}}(Q) = \nu(Q)$.

Second, for arbitrary subobjects $K \subseteq H \subseteq Q$, we claim

$$\kappa(H/K) = v_Q(H) - v_Q(K). \tag{2.4}$$

Every finitely generated subobject of H/K is the image of a finitely generated $H_0 \subseteq H$: its inverse image is a directed union of finitely generated subobjects, and one of their images already equals the given subobject. By (2.3),

$$\kappa_{\mathrm{fg}}\big(H_0/(K \cap H_0)\big) = \nu(H_0) - v_Q(K \cap H_0).$$

For any finitely generated $K_0 \subseteq K$, both $H_0 \cap K_0$ and $H_0 + K_0$ belong to $\mathcal{A}$. Their short exact sequence gives

$$\nu(H_0) - \nu(H_0 \cap K_0) = \nu(H_0 + K_0) - \nu(K_0) \leq v_Q(H) - \nu(K_0). \tag{2.5}$$

As K_0 varies, the supremum of $\nu(H_0 \cap K_0)$ is $v_Q(K \cap H_0)$: every finitely generated subobject of $K \cap H_0$ can itself be chosen as K_0. Taking infima over K_0 in (2.5) therefore gives

$$\nu(H_0) - v_Q(K \cap H_0) \leq v_Q(H) - v_Q(K).$$

The reverse bound after taking the supremum over H_0 follows from $v_Q(K \cap H_0) \leq v_Q(K)$. This proves (2.4). Taking $H = Q$ also proves that (2.2) agrees with (2.1) on finitely generated objects.

Now consider $0 \to M' \to M \to M'' \to 0$ with M finitely generated. Choose a surjection $Q \twoheadrightarrow M$ with $Q \in \mathcal{A}$, let K be its kernel, and let H be the inverse image of M'. Then

$$M = Q/K, \qquad M' = H/K, \qquad M'' = Q/H.$$

Equations (2.3) and (2.4) give additivity by cancellation of finite elements of L. They also give monotonicity for subobjects and quotients of a finitely generated object. Monotonicity for arbitrary subobjects follows from (2.2).

For a general short exact sequence, let M_0 range over finitely generated subobjects of M. Additivity for $0 \to M' \cap M_0 \to M_0 \to \operatorname{im}(M_0 \to M'') \to 0$ gives

$$\kappa(M_0) = \kappa(M' \cap M_0) + \kappa\big(\operatorname{im}(M_0 \to M'')\big). \tag{2.6}$$

The images in M'' form a directed union equal to M''. For any finitely generated $N'' \subseteq M''$, the intersections $N'' \cap \operatorname{im}(M_0 \to M'')$ form a directed union equal to N'', so one of them equals N''. Thus the images are cofinal among finitely generated subobjects of M'', and

$$\sup_{M_0} \kappa\big(\operatorname{im}(M_0 \to M'')\big) = \kappa(M'').$$

For M', monotonicity gives

$$\sup_{M_0} \kappa(M' \cap M_0) \le \kappa(M').$$

Conversely, every finitely generated $N' \subseteq M'$ can itself be chosen as $M_0 \subseteq M$, so

$$\kappa(N') = \kappa(M' \cap N') \le \sup_{M_0} \kappa(M' \cap M_0).$$

Taking the supremum over N' gives the reverse inequality. The values $\kappa(M' \cap M_0)$ and $\kappa(\operatorname{im}(M_0 \to M''))$ increase with M_0, and the finitely generated subobjects $M_0 \subseteq M$ form a directed poset. Taking suprema over M_0 in (2.6) therefore yields $\kappa(M) = \kappa(M') + \kappa(M'')$, including infinite values.

For a directed union, every finitely generated subobject of the union lies in one member. Equation (2.2) gives the asserted continuity. Finally, any continuous premeasure extending ν must have value $v_Q(K)$ on a subobject K of $Q \in \mathcal{A}$. The exact sequence $0 \to K \to Q \to Q/K \to 0$ forces (2.3); directed unions then force (2.2). This proves uniqueness. □

3 Continuous premeasures and filtered colimits

Proposition 3.1. *Let $\mathcal{B}$ be a Grothendieck abelian category with a continuous premeasure κ.*

1. *Let $M : I \to \mathcal{B}$ be a functor from a small directed poset with epimorphic transitions. If $\kappa(M_{\alpha_0}) < +\infty$ for some $\alpha_0 \in I$, then*

$$\kappa(\operatorname{colim}_\alpha M_\alpha) = \inf_\alpha \kappa(M_\alpha).$$

2. *Direct sums, sums of subobjects, and directed colimits of objects of value zero have value zero.*

3. *Let $M = \operatorname{colim}_n M_n$, with canonical maps $u_n : M_n \to M$. There is a subsequence $n_1 < n_2 < \cdots$ such that*

$$\kappa(M) = \lim_n \kappa(\operatorname{im} u_n) \le \liminf_n \kappa(M_n) = \lim_j \kappa(M_{n_j}). \tag{3.1}$$

In particular, $\liminf_n \kappa(M_n) = 0$ implies $\kappa(M) = 0$. If $\kappa(M) < +\infty$, a subsequence of the maps u_n satisfies $\kappa(\ker u_{n_j}) + \kappa(\operatorname{coker} u_{n_j}) \to 0$ if and only if $\kappa(M) = \liminf_n \kappa(M_n)$.

Proof. For (1), restrict to the cofinal indices $\alpha \geq \alpha_0$. The kernels $K_\alpha = \ker(M_{\alpha_0} \to M_\alpha)$ increase with union the kernel of $M_{\alpha_0} \to \operatorname{colim}_\alpha M_\alpha$. All their values are finite, and

$$\begin{aligned}\kappa(\operatorname{colim}_\alpha M_\alpha) &= \kappa(M_{\alpha_0}) - \sup_{\alpha \geq \alpha_0} \kappa(K_\alpha) \\ &= \inf_{\alpha \geq \alpha_0} \kappa(M_\alpha) = \inf_\alpha \kappa(M_\alpha).\end{aligned}$$

For (2), direct sums are directed unions of finite direct sums, while sums and directed colimits are quotients of direct sums. For (3), $M = \bigcup_n \operatorname{im} u_n$ gives (3.1). If $\kappa(M) < +\infty$, then $\kappa(\operatorname{coker} u_n) = \kappa(M) - \kappa(\operatorname{im} u_n) \to 0$. Since $\kappa(M_n) = \kappa(\ker u_n) + \kappa(\operatorname{im} u_n)$, the last assertion follows. □

For a map $f : X \to Y$ in an abelian category with a premeasure κ, define its error by

$$\delta_\kappa(f) = \kappa(\ker f) + \kappa(\operatorname{coker} f) \in L_{\geq 0} \cup \{+\infty\}. \tag{3.2}$$

If $\kappa(X), \kappa(Y) < +\infty$, then $\delta_\kappa(f) < +\infty$ and

$$\delta_\kappa(f) = \kappa(X) - \kappa(Y) + 2\kappa(\operatorname{coker} f).$$

Lemma 3.2. *For composable maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, one has*

$$\begin{aligned}\delta_\kappa(gf) &\leq \delta_\kappa(f) + \delta_\kappa(g), \\ \delta_\kappa(f) &\leq \delta_\kappa(gf) + \delta_\kappa(g), \\ \delta_\kappa(g) &\leq \delta_\kappa(gf) + \delta_\kappa(f).\end{aligned} \tag{3.3}$$

For a commutative square with horizontal maps $f : X \to Y$, $f' : X' \to Y'$ and vertical maps $a : X \to X'$, $b : Y \to Y'$, one has

$$\begin{aligned}\delta_\kappa(f) &\leq \delta_\kappa(f') + \delta_\kappa(a) + \delta_\kappa(b), \\ \delta_\kappa(f') &\leq \delta_\kappa(f) + \delta_\kappa(a) + \delta_\kappa(b).\end{aligned} \tag{3.4}$$

Proof. Use the exact sequence

$$0 \to \ker f \to \ker(gf) \to \ker g \to \operatorname{coker} f \to \operatorname{coker}(gf) \to \operatorname{coker} g \to 0.$$

The bound on $\delta_\kappa(gf)$ in (3.3) follows by bounding $\kappa(\ker(gf))$ and $\kappa(\operatorname{coker}(gf))$ separately. For the bound on $\delta_\kappa(f)$, use $\kappa(\ker f) \leq \kappa(\ker(gf))$ and $\kappa(\operatorname{coker} f) \leq \kappa(\ker g) + \kappa(\operatorname{coker}(gf))$. The bound on $\delta_\kappa(g)$ follows from $\kappa(\ker g) \leq \kappa(\ker(gf)) + \kappa(\operatorname{coker} f)$ and $\kappa(\operatorname{coker} g) \leq \kappa(\operatorname{coker}(gf))$. Apply (3.3) to $bf = f'a$ to obtain (3.4). □

Lemma 3.3. *Let ν be a continuous premeasure on an abelian category $\mathcal{A}$. Let $(M_i \hookrightarrow X)_{i \in I}$ be a family of subobjects indexed by a small directed poset, with $M_i \subseteq M_j$ for $i \leq j$ and $\sup_i \nu(M_i) < +\infty$. The diagram admits a directed union $(u_i : M_i \to U)_{i \in I}$ in $\mathcal{A}$. The map $u : U \to X$ induced by the inclusions is a monomorphism. For every morphism $f : U \to Y$ in $\mathcal{A}$,*

$$\operatorname{im} f = \bigcup_{i \in I} \operatorname{im}(f \circ u_i) \qquad \text{as subobjects of } Y.$$

Proof. Definition 1.1(1) gives the directed union $(u_i : M_i \to U)_{i \in I}$. Each composite $u \circ u_i$ is the inclusion $M_i \hookrightarrow X$. Applying (1.1) to $\ker u \subseteq U$ gives

$$\ker u \cong \operatorname{colim}_i \big((\ker u) \cap M_i\big) \cong \operatorname{colim}_i \ker(u \circ u_i) = 0.$$

Since $\nu(\operatorname{im}(f \circ u_i)) \leq \nu(M_i)$, the first assertion gives a directed union $V = \bigcup_i \operatorname{im}(f \circ u_i) \hookrightarrow Y$. Each $f \circ u_i$ factors through V. These factorizations are compatible because $V \to Y$ is monomorphic, so f factors through V by the colimit property of U. Thus $\operatorname{im} f \subseteq V$. Conversely, the inclusions $\operatorname{im}(f \circ u_i) \subseteq \operatorname{im} f$ induce a factorization of $V \hookrightarrow Y$ through $\operatorname{im} f$, giving $V \subseteq \operatorname{im} f$. □

Lemma 3.4. *Let ν be a continuous premeasure on an abelian category $\mathcal{A}$, and let $f : X \to Y$ satisfy $\nu(X) < +\infty$. For every small directed family of subobjects $Y_i \subseteq Y$ with $\sup_i \nu(Y_i) < +\infty$, one has*

$$\bigcup_i f^{-1}(Y_i) = f^{-1}\Big(\bigcup_i Y_i\Big) \quad \text{in } X.$$

Proof. By Lemma 3.3, the maps $\operatorname{colim}_i Y_i \to Y$ and $\operatorname{colim}_i f^{-1}(Y_i) \to X$ are monomorphisms. Write $Y' = \bigcup_i Y_i$ and let $\Gamma_f \subseteq X \oplus Y$ be the graph of f. Since $X \oplus Y' = \operatorname{colim}_i (X \oplus Y_i)$ and $\sup_i \nu(X \oplus Y_i) < +\infty$, Definition 1.1(3) gives

$$\Gamma_f \cap (X \oplus Y') = \bigcup_i (\Gamma_f \cap (X \oplus Y_i)).$$

Projection to X identifies $\Gamma_f \cap (X \oplus T)$ with $f^{-1}(T)$ for each $T \subseteq Y$. □

Lemma 3.5. *Let I be a small directed poset and $(X_i, f_{ij} : X_i \to X_j)_{i \le j}$ a diagram in an abelian category $\mathcal{A}$ with continuous premeasure ν. If $\sup_{i \in I} \nu(X_i) < +\infty$, then the diagram admits a colimit $X_\infty \cong \operatorname{colim}_{j \in I} X_j$ whose structure maps $u_i : X_i \to X_\infty$ satisfy*

$$\nu(\operatorname{im} u_i) = \inf_{j \ge i} \nu(\operatorname{im} f_{ij}), \tag{3.5}$$

$$\nu(X_\infty) = \liminf_{\substack{(i,j) \in I^2 \\ i \le j}} \nu(\operatorname{im} f_{ij}) \le \liminf_{i \in I} \nu(X_i). \tag{3.6}$$

Proof. For each i, Lemma 3.3 gives a subobject $K_i = \bigcup_{k \ge i} \ker f_{ik} \hookrightarrow X_i$. Fix $i \le j$. The indices $k \ge j$ are cofinal among the $k \ge i$. Applying Lemma 3.3 to $f_{ij}|_{K_i} : K_i \to X_j$ gives the first equality in

$$\begin{aligned} f_{ij}(K_i) &= \bigcup_{k \ge j} f_{ij}(\ker f_{ik}) \\ &= \bigcup_{k \ge j} (\operatorname{im} f_{ij} \cap \ker f_{jk}) = \operatorname{im} f_{ij} \cap K_j. \end{aligned}$$

The second equality uses $f_{ik} = f_{jk} \circ f_{ij}$. The last is (1.1) applied to $\operatorname{im} f_{ij} \cap K_j \subseteq K_j$. Since $\ker f_{ij} \subseteq K_i$, we obtain $f_{ij}^{-1}(K_j) = K_i$. The quotients X_i/K_i therefore have monomorphic transitions and uniformly bounded values. Their colimit X_∞ exists by continuity. Every cocone on (X_i, f_{ij}) kills each K_i, so $X_\infty \cong \operatorname{colim}_{i \in I} X_i$ and $\operatorname{im} u_i \cong X_i/K_i$.

Additivity and continuity give $\nu(X_i/K_i) = \nu(X_i) - \sup_{j \ge i} \nu(\ker f_{ij})$, proving (3.5). Taking $\sup_i$ in (3.5) and using continuity on $X_\infty \cong \bigcup_i (X_i/K_i)$ gives

$$\nu(X_\infty) = \sup_i \inf_{j \ge i} \nu(\operatorname{im} f_{ij}) \le \liminf_{j \in I} \nu(X_j).$$

For fixed j, the subobjects $\operatorname{im} f_{ij} \subseteq X_j$ increase with $i \le j$, so the middle expression is the limit inferior in (3.6). □

Lemma 3.6. *Let $X : I \to \mathcal{A}$ be a functor from a small filtered category to an abelian category. Suppose it admits a colimit $X_\infty \cong \operatorname{colim}_I X$ with structure maps $u_i : X_i \to X_\infty$. Then $X_\infty = \bigcup_{i \in I} \operatorname{im} u_i$, and every premeasure ν on $\mathcal{A}$ that preserves directed unions (Definition 1.1) satisfies*

$$\nu(X_\infty) = \sup_{i \in I} \nu(\operatorname{im} u_i) \le \sup_{i \in I} \nu(X_i). \tag{3.7}$$

Proof. Filteredness makes the images $\operatorname{im} u_i$ directed. For $Y \in \mathcal{A}$, compatible maps $\operatorname{im} u_i \to Y$ extend uniquely to $X_\infty \to Y$ by composing with $X_i \twoheadrightarrow \operatorname{im} u_i$ and using the colimit property. This proves the union formula. Preservation of this union and $\nu(\operatorname{im} u_i) \le \nu(X_i)$ give (3.7). □

Lemma 3.7. *Let $X : I \to \mathcal{A}$ be a functor from a small filtered category to an abelian category with continuous premeasure ν. Suppose $\sup_i \nu(X_i) < +\infty$. Then X admits a colimit $X_\infty \cong \operatorname{colim}_I X$ with structure maps $u_i : X_i \to X_\infty$. If ν is faithful, there are arrows $i_0 \to i_1 \to \cdots$ such that, for $Y_n = \operatorname{im} u_{i_n}$,*

$$X_\infty = \bigcup_n Y_n, \qquad \nu(X_\infty) = \lim_n \nu(Y_n).$$

Proof. Choose a cofinal functor $P \to I$ from a small directed poset [18, Theorem 9.1.8.1]. Lemma 3.5 gives the colimit X_∞, and (3.7) gives $\nu(\operatorname{im} u_p) \uparrow \nu(X_\infty)$ on P. Choose $p_0 \le p_1 \le \cdots$ with $\nu(\operatorname{im} u_{p_n}) \to \nu(X_\infty)$, and let i_n be their images in I. Continuity gives $\nu(X_\infty / \bigcup_n Y_n) = 0$, so faithfulness gives the asserted union. □

Lemma 3.8. *Let $X : I \to \mathcal{A}$ be a small filtered diagram in an abelian category with continuous premeasure ν. Suppose $\sup_{i\in I} \nu(X_i) < +\infty$, and let $u_i : X_i \to X_\infty$ be the structure maps of its colimit $X_\infty \cong \operatorname{colim}^{\mathcal{A}}_{i\in I} X_i$. There is a filtered diagram (Z_j) of images of transition maps of X such that, for the induced maps $v_j : Z_j \to X_\infty$,*

$$\operatorname{colim}^{\operatorname{Ind}(\mathcal{A})}_j Z_j \cong \operatorname{colim}^{\operatorname{Ind}(\mathcal{A})}_{i\in I} X_i, \qquad \delta_\nu(v_j) \longrightarrow 0, \qquad \nu(Z_j) \longrightarrow \nu(X_\infty). \tag{3.8}$$

Note 3.9. If $I = \mathbb{N}$, one can take $Z_j = \operatorname{im}(X_{n_j} \to X_{n_{j+1}})$ for some $n_0 < n_1 < \cdots$. The same choice applies to the restriction of X along any cofinal sequence $\mathbb{N} \to I$.

Proof of Lemma 3.8 and Note 3.9. Restrict X along a cofinal functor $P \to I$ from a small directed poset as in Lemma 3.7. For $p \le q$, put $Z_{pq} = \operatorname{im}(X_p \to X_q)$ and let $v_{pq} : Z_{pq} \to X_\infty$ be induced by u_q. The factorization $X_p \twoheadrightarrow Z_{pq} \xrightarrow{v_{pq}} X_\infty$ of u_p gives $\operatorname{coker} v_{pq} \cong \operatorname{coker} u_p$ and a short exact sequence $0 \to \ker v_{pq} \to Z_{pq} \to \operatorname{im} u_p \to 0$. For fixed p, additivity and (3.5) give

$$\begin{aligned} \inf_{q\ge p} \nu(\ker v_{pq}) &= \inf_{q\ge p} \nu(Z_{pq}) - \nu(\operatorname{im} u_p) = 0, \\ \inf_{q\ge p} \delta_\nu(v_{pq}) &= \inf_{q\ge p} \nu(\ker v_{pq}) + \nu(\operatorname{coker} u_p) = \nu(\operatorname{coker} u_p). \end{aligned}$$

By (3.7), $\nu(\operatorname{coker} u_p) = \nu(X_\infty) - \nu(\operatorname{im} u_p) \to 0$ as p increases.

Choose $\epsilon_r \downarrow 0$ in $L_{>0}$. The triples $(p \le q, r)$ with $\delta_\nu(v_{pq}) < \epsilon_r$ form a cofinal subposet of $\operatorname{Fun}([1], P) \times \mathbb{N}$. The diagonal $p \mapsto (p \le p)$ is cofinal in $\operatorname{Fun}([1], P)$ and $Z_{pp} \cong X_p$. This proves the isomorphism in (3.8). Along the selected triples $\delta_\nu(v_{pq}) \to 0$, and additivity gives $|\nu(Z_{pq}) - \nu(X_\infty)| \le \delta_\nu(v_{pq})$.

For $P = \mathbb{N}$, start with $n_0 = 0$ and choose $n_{j+1} > n_j$ with $\delta_\nu(v_{n_j,n_{j+1}}) < \nu(\operatorname{coker} u_{n_j}) + \epsilon_j$. The factorizations $X_{n_j} \twoheadrightarrow Z_{n_j,n_{j+1}} \hookrightarrow X_{n_{j+1}}$ give the isomorphism in (3.8) for this sequence. □

Lemma 3.10. *Suppose a premeasure ν on an abelian category $\mathcal{A}$ preserves directed unions in the sense of Definition 1.1. Then ν is continuous if and only if, for every small filtered category I, functors $X : I \to \mathcal{A}$ with $\sup_{i\in I} \nu(X_i) < +\infty$ admit colimits and taking these colimits preserves short exact sequences.*

Proof. Suppose ν is continuous. Lemma 3.7 gives the colimit $X_\infty \cong \operatorname{colim}_I X$ for every X with $\sup_i \nu(X_i) < +\infty$.

For a natural transformation $X' \to X$ with monomorphic components, write $(X'_i, f'_{ij}) \to (X_i, f_{ij})$ for its restriction to a cofinal directed poset and put $K_i = \bigcup_{j\ge i} \ker f_{ij}$. By Lemma 3.4, $\bigcup_{j\ge i} \ker f'_{ij} = X'_i \cap K_i$. Lemma 3.3 therefore gives

$$\operatorname{colim}_i X'_i \cong \bigcup_i (X'_i / (X'_i \cap K_i)) \;\subseteq\; \bigcup_i (X_i / K_i) \cong X_\infty.$$

Colimits preserve cokernels, proving exactness.

Conversely, let $M_\bullet : I \to \mathcal{A}$ be a functor from a small directed poset with monomorphic transitions and $\sup_i \nu(M_i) < +\infty$. Let $u_i : M_i \to M$ be the structure maps of $M \cong \operatorname{colim}_I M_\bullet$. Exactness applied to $0 \to M_i \to M_j \to M_j/M_i \to 0$ over $j \geq i$ shows that each u_i is a monomorphism. Thus $(u_i)_{i\in I}$ is a directed union, so $\nu(M) = \sup_i \nu(M_i)$. For $N \subseteq M$, exactness applied to the kernels of $M_i \to M/N$ gives $\operatorname{colim}_i(N \cap M_i) \cong \ker(M \to M/N) = N$. □

Proposition 3.11. *Let κ be a premeasure on an abelian category $\mathcal{A}$, and let $q : \mathcal{A} \to \mathcal{A}/\mathcal{A}_0$ be the Serre quotient by $\mathcal{A}_0 = \{M \in \mathcal{A} : \kappa(M) = 0\}$. There is a unique measure $\bar\kappa$ on $\mathcal{A}/\mathcal{A}_0$ satisfying*

$$\bar\kappa(qM) = \kappa(M) \qquad (M \in \mathcal{A}). \tag{3.9}$$

It is finite if and only if κ is finite. Its restriction to $\mathcal{A}_{<\infty}/\mathcal{A}_0$ is a finite measure. If κ is continuous, then $\bar\kappa$ and its restriction to $\mathcal{A}_{<\infty}/\mathcal{A}_0$ are continuous. If $\mathcal{A}$ is Grothendieck, then

$$\kappa \text{ is continuous} \quad\Longleftrightarrow\quad \bar\kappa \text{ is continuous and } \mathcal{A}_0 \text{ is localizing in } \mathcal{A}.$$

Proof. Additivity and nonnegativity make $\mathcal{A}_0$ and $\mathcal{A}_{<\infty}$ Serre subcategories. An isomorphism $qM \cong qN$ is represented by a fraction $M \leftarrow M' \to N$ in which both arrows have kernel and cokernel in $\mathcal{A}_0$ [4, Tag 02MS]. Additivity gives $\kappa(M) = \kappa(M') = \kappa(N)$, so (3.9) is well-defined. Every subobject of qM is qU for some $U \subseteq M$, by taking the image of the numerator of a fraction representing its inclusion. Since $qM/qU \cong q(M/U)$, the function $\bar\kappa$ is additive. It is faithful because $qM = 0$ if and only if $M \in \mathcal{A}_0$. Essential surjectivity of q gives uniqueness and the assertion about finiteness. $\mathcal{A}_{<\infty}/\mathcal{A}_0$ identifies with the full subcategory of $\mathcal{A}/\mathcal{A}_0$ on objects of finite value.

Now suppose κ is continuous. Let $(X_i)_{i\in I}$ be a diagram of monomorphisms in $\mathcal{A}/\mathcal{A}_0$ indexed by a small directed poset, with $\lambda = \sup_i \bar\kappa(X_i) < +\infty$. Order the finite subsets $F \subseteq I$ having a greatest element by inclusion. The functor $F \mapsto \max F$ is cofinal since the F with $\max F \geq i$ form a nonempty directed poset for each i. Lift $(X_{\max F})_F$ to a diagram $(M_F)_F$ in $\mathcal{A}$ by induction on $|F|$. Once the lifts on proper subsets $E \subsetneq F$ in this poset are chosen, exactness of q gives

$$q\Big(\operatorname{colim}_{E\subsetneq F} M_E\Big) \cong \operatorname{colim}_{E\subsetneq F} X_{\max E} \longrightarrow X_{\max F}.$$

Writing this map as a fraction with denominator on the target gives $qM_F \cong X_{\max F}$ and a map $\operatorname{colim}_{E\subsetneq F} M_E \to M_F$ inducing all the required transitions.

Let $t_{FH} : M_F \to M_H$ be the transition maps. Since qt_{FH} is monomorphic, $\kappa(\ker t_{FH}) = 0$. The lift satisfies $\sup_F \kappa(M_F) = \lambda$, so Lemma 3.5 gives a colimit $M \cong \operatorname{colim}_F M_F$ with structure maps $u_F : M_F \to M$. Equations (3.5) and (3.7) give

$$\kappa(\operatorname{im} u_F) = \inf_{H\supseteq F} \kappa(\operatorname{im} t_{FH}) = \kappa(M_F),$$
$$\kappa(M) = \sup_F \kappa(\operatorname{im} u_F) = \lambda.$$

Thus $\kappa(\ker u_F) = 0$, so qu_F induces a monomorphism $v_F : X_{\max F} \to qM$.

To verify the colimit property, let $h_F : X_{\max F} \to qY$ be a compatible family. Lift $\operatorname{im}(v_F, h_F) \subseteq q(M \oplus Y)$ to a subobject $G_F \subseteq M \oplus Y$. Replacing G_F by the finite sum $\sum_{E\subseteq F} G_E$ makes the lifts increasing and preserves their images under q. By Lemma 3.3, their directed union is a subobject $G \hookrightarrow M \oplus Y$ of value $\sup_F \kappa(G_F) = \lambda$. The image of the projection $qG \to qM$ contains every $\operatorname{im} v_F$, so it has value $\lambda = \bar\kappa(qG) = \bar\kappa(qM)$. Additivity and faithfulness make this projection an isomorphism. The projection to qY therefore gives a map $qM \to qY$ extending the h_F. It is unique since the kernel of the difference of two extensions contains every $\operatorname{im} v_F$ and hence has value λ.

By cofinality, qM is the directed union of the X_i, with structure maps $v_{\{i\}}$ and $\bar\kappa(qM) = \lambda$. Lemma 1.2 gives Definition 1.1(3). The bounded directed unions just constructed have finite value and hence also give continuity on $\mathcal{A}_{<\infty}/\mathcal{A}_0$.

Suppose $\mathcal{A}$ is Grothendieck. If κ is continuous, then $\mathcal{A}_0$ is closed under small coproducts and hence is localizing [15, Appendix A.2]. Conversely, if $\mathcal{A}_0$ is localizing, the exact functor q has a right adjoint and therefore preserves directed unions. If $\bar\kappa$ is continuous, every directed union $M = \bigcup_i M_i$ in $\mathcal{A}$ with $\sup_i \kappa(M_i) < +\infty$ satisfies

$$\kappa(M) = \bar\kappa(qM) = \bar\kappa\Big(\bigcup_i qM_i\Big) = \sup_i \bar\kappa(qM_i) = \sup_i \kappa(M_i).$$

Filtered colimits in $\mathcal{A}$ exist and are exact, giving conditions (1) and (3) of Definition 1.1. □

4 Cauchy diagrams in abelian categories

Definition 4.1. Let $X : I \to \mathcal{A}$ be a functor from a small filtered category to an abelian category with premeasure κ. The diagram is *Cauchy* if for every $\epsilon \in L_{>0}$ there is an object $i_0 \in I$ such that

$$\delta_\kappa(X(\alpha)) < \epsilon \qquad \text{for every } i_0 \longrightarrow i \xrightarrow{\alpha} j \text{ in } I.$$

A cocone $(u_i : X_i \to X_\infty)_{i\in I}$ is *convergent* if for every $\epsilon \in L_{>0}$ there is an $i_0 \in I$ such that $\delta_\kappa(u_i) < \epsilon$ whenever there is an arrow $i_0 \to i$. We write $\delta_\kappa(u_i) \to 0$ for this condition.

For a directed poset I, write $f_{ij} : X_i \to X_j$ for the transition map when $i \le j$. For $I = \mathbb{N}$, the Cauchy condition reads

$$\forall \epsilon \in L_{>0}\ \exists N\ \forall m \ge n \ge N, \qquad \delta_\kappa(f_{nm}) < \epsilon.$$

The category with its premeasure is *pre-Cauchy complete* if every Cauchy sequence has a convergent colimit cocone. It is *Cauchy complete* if it is pre-Cauchy complete and the premeasure is faithful.

Lemma 4.2. *Let κ be a premeasure on an abelian category $\mathcal{A}$ and $u_n : M_n \to M$ a convergent cocone on a sequence (M_n, f_{nm}). The sequence is Cauchy, and*

$$\kappa(M) = \lim_n \kappa(M_n), \qquad \delta_\kappa(u_n) = \lim_{m\to\infty} \delta_\kappa(f_{nm}) \quad (n \in \mathbb{N}). \tag{4.1}$$

Every natural transformation $(h_n, h)\colon (M_n \xrightarrow{u_n} M) \to (N_n \xrightarrow{v_n} N)$ between convergent cocones satisfies

$$\delta_\kappa(h) = \lim_n \delta_\kappa(h_n). \tag{4.2}$$

Proof. For $m \ge n$, the first two inequalities in (3.3) of Lemma 3.2, applied to $u_n = u_m f_{nm}$, give

$$\delta_\kappa(u_n) \le \delta_\kappa(f_{nm}) + \delta_\kappa(u_m), \qquad \delta_\kappa(f_{nm}) \le \delta_\kappa(u_n) + \delta_\kappa(u_m).$$

By Definition 4.1, $\delta_\kappa(u_m) \to 0$. The second bound implies that (M_n) is Cauchy. For fixed n, the two bounds together give $\delta_\kappa(f_{nm}) \to \delta_\kappa(u_n)$ as $m \to \infty$. Additivity gives

$$\kappa(M_n) \le \kappa(M) + \delta_\kappa(u_n), \qquad \kappa(M) \le \kappa(M_n) + \delta_\kappa(u_n),$$

so $\kappa(M_n) \to \kappa(M)$. Applying (3.4) to the naturality square with vertical maps u_n, v_n gives (4.2), since $\delta_\kappa(u_n) + \delta_\kappa(v_n) \to 0$. □

Lemma 4.3. *Let κ be a premeasure on an abelian category $\mathcal{A}$ and let I be a small filtered category. If $(u_i : X_i \to M)_{i\in I}$ is a convergent cocone on $X : I \to \mathcal{A}$, then X is Cauchy. If κ is faithful and X admits a convergent colimit cocone, then $(u_i)_{i\in I}$ is a colimit cocone.*

In particular, every convergent cocone on a sequence in a Cauchy complete abelian category is a colimit cocone.

Proof. For each arrow $\alpha : i \to j$ in I, the identity $u_j X(\alpha) = u_i$ and (3.3) give $\delta_\kappa(X(\alpha)) \leq \delta_\kappa(u_i) + \delta_\kappa(u_j)$, so X is Cauchy. Let $(v_i : X_i \to C)_{i\in I}$ be a convergent colimit cocone on X. The induced map $h : C \to M$ satisfies $hv_i = u_i$, so (3.3) gives

$$\delta_\kappa(h) \leq \delta_\kappa(u_i) + \delta_\kappa(v_i) \longrightarrow 0.$$

If κ is faithful, then $\ker h = \operatorname{coker} h = 0$, so h is an isomorphism. For a Cauchy sequence, Cauchy completeness supplies $(v_n)_n$ by Definition 4.1. □

Lemma 4.4. *Let κ be a premeasure on an abelian category $\mathcal{A}$, and let $X : I \to \mathcal{A}$ be a Cauchy diagram indexed by a small filtered category. A cocone $(v_i : X_i \to M)_{i\in I}$ is convergent if and only if for every $\epsilon \in L_{>0}$ and every $i \in I$ there is an arrow $i \to j$ with $\delta_\kappa(v_j) < \epsilon$.*

Proof. Convergence implies the condition by filteredness. Conversely, given $\epsilon \in L_{>0}$, choose $a \in I$ such that $\delta_\kappa(X(\alpha)) < \epsilon/2$ for all $a \to i \xrightarrow{\alpha} j$. For each $a \to i$, choose $\alpha : i \to j$ with $\delta_\kappa(v_j) < \epsilon/2$. Then (3.3) gives $\delta_\kappa(v_i) \leq \delta_\kappa(X(\alpha)) + \delta_\kappa(v_j) < \epsilon$. □

Proposition 4.5. *Let $(\mathcal{A}, \kappa)$ be a Cauchy complete abelian category and let $X : I \to \mathcal{A}$ be a Cauchy diagram indexed by a small filtered category. Then X admits a convergent colimit cocone. There is a sequence $i_0 \to i_1 \to \cdots$ in I such that (X_{i_n}) is Cauchy and the induced map*

$$\operatorname{colim}_n X_{i_n} \longrightarrow \operatorname{colim}_{i\in I} X_i$$

is an isomorphism.

If every transition map of X is a monomorphism, then every colimit map $X_i \to \operatorname{colim}_I X$ is a monomorphism and $\kappa(\operatorname{colim}_I X) = \sup_{i\in I} \kappa(X_i)$.

Proof. By cofinality it suffices to take I to be a directed poset [18, Theorem 9.1.8.1]. Choose $\epsilon_n \downarrow 0$ in $L_{>0}$ and $a_0 \leq a_1 \leq \cdots$ in I such that

$$\delta_\kappa(f_{ij}) < \epsilon_n \qquad (a_n \leq i \leq j).$$

Let $\operatorname{Fin}(I)$ be the poset of finite subsets of I ordered by inclusion. Choose an order-preserving map

$$\operatorname{Fin}(I) \times \mathbb{N} \longrightarrow I, \qquad (S, n) \longmapsto b_n(S),$$

with $b_n(S)$ an upper bound for $S \cup \{a_n\}$. This is possible by induction on $|S| + n$, since (S, n) has only finitely many strict predecessors. The map is cofinal since $i \leq b_0(\{i\})$ for every $i \in I$.

For each S, the sequence $(X_{b_n(S)})_n$ is Cauchy. By Cauchy completeness it has a colimit $Y_S \cong \operatorname{colim}_n X_{b_n(S)}$ with convergent cocone. By (4.2),

$$\delta_\kappa(Y_\varnothing \to Y_S) = \lim_n \delta_\kappa\big(f_{b_n(\varnothing), b_n(S)}\big) = 0,$$

since $a_n \leq b_n(\varnothing) \leq b_n(S)$. Faithfulness makes $Y_\varnothing \to Y_S$ an isomorphism for every S. Thus $S \mapsto Y_S$ admits a colimit. Taking colimits first over $\mathbb{N}$ and then over $\operatorname{Fin}(I)$ gives a colimit for $(S, n) \mapsto X_{b_n(S)}$, hence for X by cofinality. Since $S \mapsto Y_S$ is isomorphic to the constant diagram at $Y_\varnothing$, inclusion of the row at $\varnothing$ induces the isomorphism $Y_\varnothing \to \operatorname{colim}_{i\in I} X_i$.

For the colimit maps $v_i : X_i \to \operatorname{colim}_I X$, the row $S = \{i\}$ gives $i \leq b_n(\{i\})$ and $\delta_\kappa(v_{b_n(\{i\})}) \to 0$. Thus (v_i) is convergent by Lemma 4.4.

If the transitions are monomorphisms, f_{ij} restricts to a monomorphism $\ker v_i \hookrightarrow \ker v_j$ for every $j \geq i$. Hence $\kappa(\ker v_i) \leq \delta_\kappa(v_j) \to 0$ as j ranges over $I_{\geq i}$. Faithfulness makes v_i monomorphic, and $\kappa(\operatorname{colim}_I X) = \kappa(X_i) + \delta_\kappa(v_i)$ gives the supremum formula. □

Proposition 4.6.

1. *Let $\mathcal{A}$ be an abelian category with a continuous premeasure κ. Every Cauchy sequence (M_n) in $\mathcal{A}_{<\infty}$ has a colimit M in $\mathcal{A}$ with $\kappa(M) < +\infty$, and its colimit cocone is convergent.*

2. *Let $\mathcal{A}$ be a Grothendieck abelian category with a continuous premeasure κ. Every Cauchy sequence (M_n) in $\mathcal{A}$ has a convergent colimit cocone in $\mathcal{A}$.*

3. *Let $\mathcal{A}$ be an abelian category with a finite premeasure κ. Then $(\mathcal{A}, \kappa)$ is Cauchy complete if and only if κ is faithful and continuous.*

Proof. For (1), let (M_n, f_{nm}) be a Cauchy sequence in $\mathcal{A}_{<\infty}$. Since $|\kappa(M_n) - \kappa(M_m)| \le \delta_\kappa(f_{nm})$ for $m \ge n$, one has $\sup_n \kappa(M_n) < +\infty$. Lemma 3.5 gives a colimit M of finite value, with structure maps $u_n : M_n \to M$. For fixed n, the bound $\delta_\kappa(f_{nm}) \le \kappa(M_n) + \sup_j \kappa(M_j)$ lets us apply the exactness in Lemma 3.10 to the kernels and cokernels of f_{nm}. Thus

$$\ker u_n \oplus \operatorname{coker} u_n \cong \operatorname{colim}_{m \ge n} (\ker f_{nm} \oplus \operatorname{coker} f_{nm}). \tag{4.3}$$

Applying (3.7) to (4.3) gives

$$\delta_\kappa(u_n) \le \sup_{m \ge n} \delta_\kappa(f_{nm}) \longrightarrow 0 \qquad (n \to \infty).$$

For (2), let (M_n, f_{nm}) be a Cauchy sequence in the Grothendieck abelian category $\mathcal{A}$. Exactness of filtered colimits gives (4.3) for the structure maps $u_n : M_n \to \operatorname{colim}_m M_m$. By (3.7) and the Cauchy condition, $\delta_\kappa(u_n) \to 0$.

For (3), a faithful continuous κ gives Cauchy completeness by (1). Conversely, suppose $(\mathcal{A}, \kappa)$ is Cauchy complete. For a diagram of monomorphisms (X_i, f_{ij}) indexed by a small directed poset I with $\lambda = \sup_i \kappa(X_i) < +\infty$,

$$\delta_\kappa(f_{ij}) = \kappa(X_j) - \kappa(X_i) \le \lambda - \kappa(X_i) \quad (j \ge i), \qquad \kappa(X_i) \uparrow \lambda.$$

Thus the diagram is Cauchy. Proposition 4.5 gives a directed union of value λ, proving Definition 1.1(1)–(2). Faithfulness and Lemma 1.2 give (3). □

5 Cauchy completion and approximation by compact objects

Definition 5.1. Let $\mathcal{A}$ be a small abelian category with a finite premeasure ν, and let κ be its continuous extension to $\operatorname{Ind}(\mathcal{A})$ from Theorem 2.1. The *Cauchy completion* of $(\mathcal{A}, \nu)$ is the target of the composite

$$\iota : \mathcal{A} \hookrightarrow \operatorname{Ind}(\mathcal{A})_{\kappa<\infty} \xrightarrow{q} \operatorname{Ind}(\mathcal{A})_{\kappa<\infty} / \operatorname{Ind}(\mathcal{A})_{\kappa=0}, \tag{5.1}$$

equipped with the measure $\bar{\kappa}$ of (3.9). Here q is the Serre quotient functor.

Both functors in (5.1) are exact, and $\bar{\kappa}(\iota F) = \nu(F)$ for $F \in \mathcal{A}$.

Theorem 5.2. *Let $\mathcal{B}$ be an abelian category with a continuous premeasure κ. Then $(\mathcal{B}_{<\infty}/\mathcal{B}_0, \bar{\kappa})$ is Cauchy complete. Thus the Cauchy completion of a small abelian category with a finite premeasure is Cauchy complete.*

Proof. By Proposition 3.11, $\bar{\kappa}$ is a finite continuous measure on $\mathcal{B}_{<\infty}/\mathcal{B}_0$. Proposition 4.6(3) gives Cauchy completeness. □

Lemma 5.3. *Let $\mathcal{B}$ be a compactly generated Grothendieck abelian category with a continuous premeasure κ finite on the full subcategory $\mathcal{B}^c$ of compact objects. Fix $M \in \mathcal{B}_{<\infty}$ and $\epsilon \in L_{>0}$. Every map $Q \to M$ with $Q \in \mathcal{B}^c$ admits a factorization*

$$Q \longrightarrow Q' \xrightarrow{u} M, \qquad Q' \in \mathcal{B}^c, \qquad \delta_\kappa(u) < \epsilon.$$

If $Q \to M$ is an epimorphism, both arrows can be chosen epimorphic. In the slice $\mathcal{B}_{/M}$, the maps u with compact source and $\delta_\kappa(u) < \epsilon$ form a filtered full subcategory cofinal among the maps with compact source.

Proof. Given $Q \to M$ with $Q \in \mathcal{B}^c$, choose $P \in \mathcal{B}^c$ mapping to M such that $\kappa(\operatorname{coker}(Q \oplus P \to M)) < \epsilon/2$. This is possible because the images of maps from compact objects form a directed union equal to M. Take $P = 0$ if $Q \to M$ is epimorphic. The kernel $K = \ker(Q \oplus P \to M)$ has finite value. By the same directed union argument choose $P_0 \in \mathcal{B}^c$ mapping to K with $\kappa(K/\operatorname{im}(P_0 \to K)) < \epsilon/2$. Then

$$Q' = \operatorname{coker}(P_0 \to Q \oplus P) \in \mathcal{B}^c.$$

The induced $u : Q' \to M$ has kernel $K/\operatorname{im}(P_0 \to K)$ and cokernel $\operatorname{coker}(Q \oplus P \to M)$, so $\delta_\kappa(u) < \epsilon$.

Finite colimits in $\mathcal{B}_{/M}$ are computed on sources. Since compact objects are closed under finite colimits, the full subcategory on maps with compact source is filtered. The factorization above gives a morphism from every object of this category to some u with $\delta_\kappa(u) < \epsilon$. The full subcategory on these u is therefore filtered and cofinal [4, Lemma 4.19.3]. □

Theorem 5.4. *Let κ be a continuous premeasure on a compactly generated Grothendieck abelian category $\mathcal{B}$, finite on compact objects. Write $q : \mathcal{B}_{<\infty} \to \mathcal{B}_{<\infty}/\mathcal{B}_0$ for the Serre quotient functor. Every object $X \in \mathcal{B}_{<\infty}/\mathcal{B}_0$ admits a convergent colimit cocone*

$$qF_n \longrightarrow X, \qquad F_\bullet : \mathbb{N} \longrightarrow \mathcal{B}^c.$$

For $X \cong qM$, the transition maps and the cocone maps can be chosen epimorphic if M is finitely generated. They can be chosen monomorphic if M is a directed union of compact subobjects.

Proof. Choose $M \in \mathcal{B}_{<\infty}$ with $qM \cong X$. Starting with $0 \to M$, iterate the factorizations of Lemma 5.3 with $\epsilon_n \downarrow 0$ in $L_{>0}$ to obtain a convergent cocone $u_n : F_n \to M$ with $F_n \in \mathcal{B}^c$. For finitely generated M, start with a compact object surjecting onto M and use the epimorphic factorizations. If M is a directed union of compact subobjects, continuity gives an increasing sequence of them $F_n \subseteq M$ with $\kappa(M/F_n) \to 0$.

Exactness of q and (3.9) give $\delta_{\bar\kappa}(qu_n) = \delta_\kappa(u_n) \to 0$. In the Cauchy complete quotient of Theorem 5.2, Lemma 4.3 shows that $(qu_n)_n$ is a colimit cocone. □

In Definition 5.1, $\operatorname{Ind}(\mathcal{A})^c \simeq \mathcal{A}$, so the sequences in Theorem 5.4 have the form $\iota F_\bullet$ with $F_\bullet : \mathbb{N} \to \mathcal{A}$. The Cauchy completion is small since the Serre quotient is locally small and $\operatorname{Fun}(\mathbb{N}, \mathcal{A})$ is small.

Remark 5.5. Under the hypotheses of Theorem 5.4, the finitely generated subobjects of $M \in \mathcal{B}$ are quotients of compact objects, hence have finite value, and their union is M. If $\kappa(M) = +\infty$, continuity gives an increasing sequence of finitely generated subobjects $M_n \subseteq M$ with $\kappa(M_n) \to +\infty$.

Every $\aleph_1$-compact $M \in \mathcal{B}$ is an increasing union of a sequence of finitely generated subobjects. Indeed, the subobjects admitting such an exhaustion form an $\aleph_1$-filtered family with union M, so one of them equals M.

If κ is faithful, every $M \in \mathcal{B}_{<\infty}$ is $\aleph_1$-compact in $\mathcal{B}$. Indeed, $\mathcal{B}_0 = 0$, so Theorem 5.4 and Proposition 4.6(1) give a convergent colimit cocone $F_n \to M$ in $\mathcal{B}$ with $F_n \in \mathcal{B}^c$. Countable colimits of $\aleph_1$-compact objects are $\aleph_1$-compact [18, Proposition 9.2.5.24]. In particular every finitely generated object is $\aleph_1$-compact, as is every countable increasing union of such objects.

6 Uniformly continuous functors and the Cauchy completion

Definition 6.1. Let $\mathcal{A}$ and $\mathcal{H}$ be abelian categories with premeasures ν and λ. A functor $T : \mathcal{A} \to \mathcal{H}$ is *uniformly continuous* if for every $\epsilon \in L_{>0}$ there is an $\eta \in L_{>0}$ such that for every map f in $\mathcal{A}$,

$$\delta_\nu(f) < \eta \quad \Longrightarrow \quad \delta_\lambda(Tf) < \epsilon.$$

It is *sequentially continuous* if, for every compatible sequential cocone $u_n : M_n \to M$ in $\mathcal{A}$,

$$\delta_\nu(u_n) \longrightarrow 0 \quad \Longrightarrow \quad \delta_\lambda(Tu_n) \longrightarrow 0.$$

Proposition 6.2 (Uniform continuity $\Rightarrow$ sequential continuity)**.** *Let $\mathcal{A}$ and $\mathcal{H}$ be abelian categories with finite premeasures ν and λ, and let $T : \mathcal{A} \to \mathcal{H}$ be uniformly continuous. Then T is sequentially continuous and sends Cauchy diagrams to Cauchy diagrams. For a convergent cocone $u_n : M_n \to M$ in $\mathcal{A}$, one has*

$$\lambda(TM) = \lim_n \lambda(TM_n).$$

If $(\mathcal{H}, \lambda)$ is Cauchy complete, the comparison map in (6.1) *is an isomorphism*

$$\operatorname{colim}_n T(M_n) \xrightarrow{\sim} T(M) \quad \text{in } \mathcal{H}. \tag{6.1}$$

Proof. Apply uniform continuity to the transition maps of a Cauchy diagram and to the maps of a convergent cocone. Formula (4.1) for the convergent cocone $(Tu_n : T(M_n) \to T(M))_n$ gives $\lambda(TM) = \lim_n \lambda(TM_n)$. If $(\mathcal{H}, \lambda)$ is Cauchy complete, Lemma 4.3 shows that $(Tu_n)_n$ is a colimit cocone, proving (6.1). □

Lemma 6.3. *Let κ be the continuous extension of a finite premeasure ν on a small abelian category $\mathcal{A}$ to* $\operatorname{Ind}(\mathcal{A})$*. Let $(\mathcal{H}, \lambda)$ be a Cauchy complete abelian category with finite measure. Every uniformly continuous functor $T : \mathcal{A} \to \mathcal{H}$ admits a uniformly continuous left Kan extension along $j : \mathcal{A} \hookrightarrow \operatorname{Ind}(\mathcal{A})_{\kappa<\infty}$.*

Proof. If the colimits

$$\widetilde{T}(M) := \operatorname{colim}_{(u : jQ \to M),\, Q \in \mathcal{A}} T(Q) \tag{6.2}$$

exist in $\mathcal{H}$ for every $M \in \operatorname{Ind}(\mathcal{A})_{\kappa<\infty}$, they define the left Kan extension of T along j [18, Proposition 7.3.5.1].

Existence of the colimit in (6.2). Fix $M \in \operatorname{Ind}(\mathcal{A})_{\kappa<\infty}$. Proposition 4.6(3) implies that λ is continuous. By Lemma 3.10, it suffices to bound $\lambda(TQ)$ on a filtered cofinal subcategory of the indexing category in (6.2).

Fix $r \in L_{>0}$. By uniform continuity of T, choose $\eta \in L_{>0}$ such that $\delta_\nu(a) < 2\eta$ implies $\delta_\lambda(Ta) < r$ for every map a in $\mathcal{A}$. By Lemma 5.3, the maps $u : Q \to M$ with $Q \in \mathcal{A}$ and $\delta_\kappa(u) < \eta$ form a filtered cofinal full subcategory. For an arrow $a : (Q, u) \to (P, v)$ in this subcategory, $u = va$, so (3.3) gives

$$\delta_\nu(a) = \delta_\kappa(a) \le \delta_\kappa(u) + \delta_\kappa(v) < 2\eta.$$

Hence $\delta_\lambda(Ta) < r$. Fix one object $u_0 : Q_0 \to M$ of this subcategory. For any $u : Q \to M$ in it, filteredness gives arrows $u_0 \to v \leftarrow u$ in the subcategory. Writing $v : P \to M$, additivity

gives $\lambda(TQ) < \lambda(TP) + r < \lambda(TQ_0) + 2r$. This uniform bound proves that the colimit in (6.2) exists.

Error of the structure maps in (6.2). For every $u : Q \to M$ with $Q \in \mathcal{A}$ and $\delta_\kappa(u) < \eta$, we claim

$$\delta_\lambda(TQ \to \widetilde{T}(M)) \le r. \tag{6.3}$$

The undercategory at u in the filtered subcategory above is filtered, and its forgetful functor is cofinal. Thus the structure map $TQ \to \widetilde{T}(M)$ is isomorphic to the colimit of the maps $Ta : TQ \to TP$ over arrows $a : (Q, u) \to (P, v)$ in that subcategory. By the exactness in Lemma 3.10, its kernel and cokernel are the colimits of $\ker(Ta)$ and $\operatorname{coker}(Ta)$, respectively. Applying (3.7) to $\ker(Ta) \oplus \operatorname{coker}(Ta)$ gives (6.3), since each such object has value $\delta_\lambda(Ta) < r$.

Uniform continuity of $\widetilde{T}$. Let $h : M \to N$ in $\operatorname{Ind}(\mathcal{A})_{\kappa<\infty}$ satisfy $\delta_\kappa(h) < \eta/2$. By Lemma 5.3 take $u : Q \to M$ with $Q \in \mathcal{A}$ and $\delta_\kappa(u) < \eta/2$. Then $\delta_\kappa(hu) < \eta$. Hence (3.3) and the bounds (6.3) for u and hu give $\delta_\lambda(\widetilde{T}(h)) \le 2r$. Taking $2r < \epsilon$ for a prescribed $\epsilon \in L_{>0}$ proves uniform continuity. □

Theorem 6.4. *Let $\mathcal{A}$ be a small abelian category with a finite premeasure ν, and let κ be its continuous extension to* $\operatorname{Ind}(\mathcal{A})$. *Let $(\mathcal{H}, \lambda)$ be a Cauchy complete abelian category with finite measure. Write* $\mathrm{Fun}_{\mathrm{uniform}}$ *for the full subcategory of uniformly continuous functors. Restriction along $j : \mathcal{A} \hookrightarrow \operatorname{Ind}(\mathcal{A})_{\kappa<\infty}$ is an equivalence*

$$j^* : \mathrm{Fun}_{\mathrm{uniform}}(\operatorname{Ind}(\mathcal{A})_{\kappa<\infty},\, \mathcal{H}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{uniform}}(\mathcal{A}, \mathcal{H})$$

with inverse Lan_j. *For the Serre quotient q in* (5.1), *restriction along $\iota = qj$ induces an equivalence*

$$\iota^* : \mathrm{Fun}_{\mathrm{uniform}}(\operatorname{Ind}(\mathcal{A})_{\kappa<\infty}/\operatorname{Ind}(\mathcal{A})_{\kappa=0},\, \mathcal{H}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{uniform}}(\mathcal{A}, \mathcal{H}). \tag{6.4}$$

Proof. Lemma 6.3 gives the adjunction $\operatorname{Lan}_j \dashv j^*$ on uniformly continuous functors. Its unit $T \to j^* \operatorname{Lan}_j T$ is an isomorphism because j is fully faithful.

For a uniformly continuous $S : \operatorname{Ind}(\mathcal{A})_{\kappa<\infty} \to \mathcal{H}$, the counit $\operatorname{Lan}_j(j^*S) \to S$ restricts along j to the inverse of the unit at j^*S. For each $M \in \operatorname{Ind}(\mathcal{A})_{\kappa<\infty}$, the proof of Theorem 5.4 supplies a convergent cocone $jF_n \to M$ with $F_n \in \mathcal{A}$. By Proposition 6.2, both S and $\operatorname{Lan}_j(j^*S)$ take this cocone to a colimit cocone in $\mathcal{H}$. Hence the counit is an isomorphism at M, proving that j^* is an equivalence.

The Serre quotient q is the localization at the maps f with $\delta_\kappa(f) = 0$. Every uniformly continuous functor $\operatorname{Ind}(\mathcal{A})_{\kappa<\infty} \to \mathcal{H}$ inverts these maps by faithfulness of λ, and its descent is uniformly continuous by (3.9). Thus q^* is an equivalence on uniformly continuous functors, and $\iota^* = j^*q^*$ gives (6.4). □

Corollary 6.5. *The functor ι of* (5.1) *is an equivalence if and only if $(\mathcal{A}, \nu)$ is Cauchy complete. In particular, the canonical functor from the Cauchy completion of $(\mathcal{A}, \nu)$ to its own Cauchy completion is an equivalence.*

Proof. Since ι preserves the premeasure and its target is Cauchy complete by Theorem 5.2, $(\mathcal{A}, \nu)$ is Cauchy complete whenever ι is an equivalence. Conversely, if $(\mathcal{A}, \nu)$ is Cauchy complete, (6.4) gives a uniformly continuous functor r with $r\iota \simeq \mathrm{id}_{\mathcal{A}}$. Full faithfulness in (6.4), with the completion as target, gives $\iota r \simeq \mathrm{id}$ since both functors restrict to ι. □

Definition 6.6 (Categories with a uniformity)**.** A *category with a uniformity* is a pair $(\mathcal{C}, \mathcal{U})$ consisting of a small category $\mathcal{C}$ and a filter $\mathcal{U}$ on $\pi_0(\mathrm{Fun}([1], \mathcal{C})^{\cong})$ with the following properties. Every $U \in \mathcal{U}$ contains the classes of all isomorphisms. For each $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that, for every composable pair f, g in $\mathcal{C}$, if two of $[f], [g], [gf]$ lie in V, the third lies in U.

A functor $F : (\mathcal{C},\mathcal{U}) \to (\mathcal{D},\mathcal{V})$ is *uniformly continuous* if $\{[f] : [F(f)] \in V\} \in \mathcal{U}$ for every $V \in \mathcal{V}$. These pairs form an $(\infty,2)$-category $\mathrm{Cat}_{\mathrm{unif}}$. Its hom category from $(\mathcal{C},\mathcal{U})$ to $(\mathcal{D},\mathcal{V})$ is the full subcategory $\mathrm{Fun}_{\mathrm{uniform}}(\mathcal{C},\mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C},\mathcal{D})$ on uniformly continuous functors, with all natural transformations between them as 2-morphisms.

Define Cauchy diagrams and convergent cocones as in Definition 4.1, using the tests $[f] \in U$ for $U \in \mathcal{U}$ in place of $\delta_\nu(f) < \epsilon$. The pair $(\mathcal{C},\mathcal{U})$ is *Cauchy complete* if $\bigcap_{U\in\mathcal{U}} U$ consists precisely of the classes of isomorphisms and every small filtered Cauchy diagram admits a convergent colimit cocone.

Remark 6.7. The Cauchy complete objects form a reflective full $(\infty,2)$-subcategory of $\mathrm{Cat}_{\mathrm{unif}}$. For each $(\mathcal{C},\mathcal{U})$, the unit $\eta_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ induces an equivalence of categories

$$\eta_{\mathcal{C}}^* : \mathrm{Fun}_{\mathrm{uniform}}(\widehat{\mathcal{C}},\mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{uniform}}(\mathcal{C},\mathcal{D})$$

for every Cauchy complete $(\mathcal{D},\mathcal{V})$.

For a small abelian category $\mathcal{A}$ with the uniformity generated by $\delta_{\nu_j}(f) < \epsilon$ for a small family $(\nu_j)_{j\in J}$ of finite premeasures and $\epsilon \in L_{>0}$, the reflection is

$$\left(\bigcap_{j\in J} \mathrm{Ind}(\mathcal{A})_{\kappa_j<\infty}\right) \Big/ \left(\bigcap_{j\in J} \mathrm{Ind}(\mathcal{A})_{\kappa_j=0}\right),$$

where κ_j extends ν_j continuously to $\mathrm{Ind}(\mathcal{A})$ and the descended premeasures generate the uniformity. For a single finite premeasure this recovers the completion in (5.1) and the restriction equivalence (6.4). The completeness conditions in Definitions 4.1 and 6.6 agree in this case by Proposition 4.5.

7 Premeasures on the heart of a stable category

Right completeness of a t-structure on a presentable stable category $\mathcal{T}$ means that the truncation maps induce an isomorphism

$$\mathrm{colim}_{n\geq 0}\, \tau_{\geq -n}E \xrightarrow{\sim} E \qquad (E \in \mathcal{T}).$$

The t-structure is right complete and compatible with filtered colimits if and only if $\mathcal{T}_{\geq 0}$ is Grothendieck prestable and the canonical functor $\mathrm{Sp}(\mathcal{T}_{\geq 0}) \to \mathcal{T}$ is a t-exact equivalence [17, Corollary C.3.1.4 and Remark C.3.1.5].

Proposition 7.1. *Let $\mathcal{T}$ be a stable category with a t-structure, let $\mathcal{S} \subseteq \mathcal{T}^\heartsuit$ be a Serre subcategory, and put*

$$\mathcal{T}_{\mathcal{S}} = \{E \in \mathcal{T} : \pi_i E \in \mathcal{S} \text{ for every } i \in \mathbb{Z}\}.$$

1. *The subcategory $\mathcal{T}_{\mathcal{S}}$ is stable and closed under retracts. There is a unique t-structure on the Verdier quotient for which $q : \mathcal{T} \to \mathcal{T}/\mathcal{T}_{\mathcal{S}}$ is t-exact. On hearts q induces an equivalence*
$$\mathcal{T}^\heartsuit/\mathcal{S} \xrightarrow{\sim} (\mathcal{T}/\mathcal{T}_{\mathcal{S}})^\heartsuit.$$

2. *Suppose $\mathcal{T}$ is presentable with a right complete t-structure compatible with filtered colimits and $\mathcal{S}$ is localizing in $\mathcal{T}^\heartsuit$. Then $\mathcal{T}_{\mathcal{S}}$ is a presentable localizing subcategory, $\mathcal{T}/\mathcal{T}_{\mathcal{S}}$ is presentable, and q is an accessible localization with fully faithful right adjoint. The t-structure on $\mathcal{T}/\mathcal{T}_{\mathcal{S}}$ is right complete and compatible with filtered colimits.*

Proof. (1) Stability follows from the long exact sequence of homotopy groups and the Serre property of $\mathcal{S}$. Closure under retracts follows degree by degree.

Take $X \in \mathcal{T}_{\geq 0}$ and $Y \in \mathcal{T}_{\leq -1}$. Represent a map $qX \to qY$ by $X \xleftarrow{s} Z \to Y$ with $\operatorname{cofib}(s) \in \mathcal{T}_{\mathcal{S}}$. For $i < 0$ the surjection $\pi_{i+1} \operatorname{cofib}(s) \twoheadrightarrow \pi_i Z$ gives $\pi_i Z \in \mathcal{S}$. Thus $Z \to Y$ factors through $\tau_{\leq -1} Z \in \mathcal{T}_{\mathcal{S}}$ and the map $qX \to qY$ is zero. Applying this also to $X[j]$ for $j \geq 0$ gives $\Omega^\infty \operatorname{Hom}_{\mathcal{T}/\mathcal{T}_{\mathcal{S}}}(qX, qY) \simeq *$. The essential images of $\mathcal{T}_{\geq 0}$ and $\mathcal{T}_{\leq 0}$ therefore give a t-structure, with truncation sequences

$$q(\tau_{\geq 0} E) \longrightarrow qE \longrightarrow q(\tau_{\leq -1} E).$$

These sequences are forced by t-exactness of q, giving uniqueness.

Write $p : \mathcal{T}^\heartsuit \to \mathcal{T}^\heartsuit/\mathcal{S}$ for the Serre quotient. The exact functor on hearts induced by q has kernel $\mathcal{S}$, so it gives a faithful exact functor from $\mathcal{T}^\heartsuit/\mathcal{S}$. For a map $qM \to qN$ represented by $M \xleftarrow{s} Z \xrightarrow{g} N$ with $M, N \in \mathcal{T}^\heartsuit$, $p(\pi_0 s)$ is an isomorphism since p annihilates $\pi_1 \operatorname{cofib}(s)$ and $\pi_0 \operatorname{cofib}(s)$. By t-exactness, $p(\pi_0 g)\, p(\pi_0 s)^{-1}$ maps to the given morphism. This proves fullness. Every object of the quotient heart is isomorphic to $\pi_0(qE) \cong q(\pi_0 E)$ for some $E \in \mathcal{T}$.

(2) The functor

$$\mathcal{T} \longrightarrow \prod_{i \in \mathbb{Z}} (\mathcal{T}^\heartsuit/\mathcal{S}), \qquad E \longmapsto (p\pi_i E)_{i \in \mathbb{Z}},$$

preserves filtered colimits. Its fiber over zero is $\mathcal{T}_{\mathcal{S}}$, so $\mathcal{T}_{\mathcal{S}}$ is accessible and closed under filtered colimits in $\mathcal{T}$. Stability then gives closure under all small colimits. Thus $\mathcal{T}_{\mathcal{S}}$ is a presentable localizing subcategory. Its Verdier quotient is presentable, and q is an accessible localization with fully faithful right adjoint r.

By t-exactness, applying q to $\operatorname{colim}_{n \geq 0} \tau_{\geq -n} E \cong E$ proves right completeness of the quotient. For a small filtered diagram (Y_j) in $\mathcal{T}/\mathcal{T}_{\mathcal{S}}$, we have

$$\tau_{\leq 0} \operatorname{colim}_j Y_j \cong q(\tau_{\leq 0} \operatorname{colim}_j rY_j) \cong \operatorname{colim}_j q(\tau_{\leq 0} rY_j) \cong \operatorname{colim}_j \tau_{\leq 0} Y_j. \qquad \square$$

Let $\mathcal{T}$ be a stable category with a t-structure and let κ be a premeasure on its heart. For $E \in \mathcal{T}$, write $\pi_i E \in \mathcal{T}^\heartsuit$ for its i-th homotopy group. Put

$$\mu_\kappa(E) = \sum_{i \in \mathbb{Z}} \kappa(\pi_i E), \qquad \mathcal{T}^{\Sigma\kappa < \infty} = \{E \in \mathcal{T} : \mu_\kappa(E) < +\infty\},$$

$$\mathcal{T}_0 = \{E \in \mathcal{T} : \kappa(\pi_i E) = 0 \text{ for every } i\}.$$

The sum is the supremum of its finite partial sums. For $E \in \mathcal{T}^{\Sigma\kappa < \infty}$ the series

$$\chi_\kappa(E) = \sum_{i \in \mathbb{Z}} (-1)^i \kappa(\pi_i E) \in L$$

is absolutely convergent.

For a map $f : E \to F$ in $\mathcal{T}$, the long exact sequence of homotopy groups for $E \xrightarrow{f} F \to \operatorname{cofib}(f)$ gives

$$\mu_\kappa(\operatorname{cofib}(f)) = \sum_i \big(\kappa(\ker(\pi_i f)) + \kappa(\operatorname{coker}(\pi_i f))\big). \tag{7.1}$$

For a cofiber sequence $E \to F \to G$, apply (7.1) to $G[-1] \to E$. The bounds $\kappa(\ker u) \leq \kappa(X)$ and $\kappa(\operatorname{coker} u) \leq \kappa(Y)$ for $u : X \to Y$ in $\mathcal{T}^\heartsuit$, together with $\pi_i(E[1]) \cong \pi_{i-1} E$, give

$$\mu_\kappa(F) \leq \mu_\kappa(E) + \mu_\kappa(G), \qquad \mu_\kappa(E[1]) = \mu_\kappa(E). \tag{7.2}$$

Lemma 7.2. *For a map $f : M \to N$ in $\mathcal{T}^\heartsuit$, the error $\delta_\kappa(f)$ defined in* (3.2) *satisfies*

$$\mu_\kappa(\operatorname{cofib} f) = \kappa(\ker f) + \kappa(\operatorname{coker} f) = \delta_\kappa(f).$$

Proof. The only possibly nonzero homotopy groups of $\operatorname{cofib} f$ are $\pi_1 \operatorname{cofib} f \cong \ker f$ and $\pi_0 \operatorname{cofib} f \cong \operatorname{coker} f$. $\square$

For composable maps $E \xrightarrow{f} F \xrightarrow{g} G$, applying (7.2) to the cofiber sequence $\operatorname{cofib}(f) \to \operatorname{cofib}(gf) \to \operatorname{cofib}(g)$ and its rotations gives (3.3) with $\delta_\kappa(h)$ replaced by $\mu_\kappa(\operatorname{cofib}(h))$. For a commutative square in $\mathcal{T}$, apply these inequalities to the two factorizations $bf \simeq f'a$ in the notation of Lemma 3.2 to obtain (3.4).

Lemma 7.3. *For every cofiber sequence $E \to F \to G$ in $\mathcal{T}^{\Sigma\kappa<\infty}$, one has*

$$\chi_\kappa(F) = \chi_\kappa(E) + \chi_\kappa(G).$$

Proof. We obtain the identity by summing with alternating signs the additivity formulas for κ on the short exact sequences arising from the long exact sequence of homotopy groups of $E \to F \to G$. Each image is a quotient of its source, so absolute convergence follows from

$$\begin{aligned}
\sum_{i\in\mathbb{Z}} \kappa\big(\operatorname{im}(\pi_i E \to \pi_i F)\big) &\le \mu_\kappa(E) < +\infty,\\
\sum_{i\in\mathbb{Z}} \kappa\big(\operatorname{im}(\pi_i F \to \pi_i G)\big) &\le \mu_\kappa(F) < +\infty,\\
\sum_{i\in\mathbb{Z}} \kappa\big(\operatorname{im}(\pi_i G \to \pi_{i-1} E)\big) &\le \mu_\kappa(G) < +\infty.
\end{aligned}$$

□

Use the uniformity on $\mathcal{T}$ generated by the sets $\{[f] : \mu_\kappa(\operatorname{cofib} f) < \epsilon\}$ for $\epsilon \in L_{>0}$, as in Definition 6.6.

Lemma 7.4. *Suppose $\mathcal{T}$ admits small filtered colimits, its t-structure is compatible with them, and κ preserves directed unions in $\mathcal{T}^\heartsuit$. For every small filtered Cauchy diagram $X : I \to \mathcal{T}$, the colimit cocone is convergent. If $\mu_\kappa(X_i) < +\infty$ for every $i \in I$, then $\mu_\kappa(\operatorname{colim}_I X) < +\infty$.*

Proof. For each structure map $u_i : X_i \to \operatorname{colim}_I X$, cofinality of $I_{i/} \to I$ gives

$$\operatorname{cofib} u_i \cong \operatorname{colim}_{(\alpha : i \to j) \in I_{i/}} \operatorname{cofib}(X(\alpha)).$$

The functor $E \mapsto \bigoplus_{d\in\mathbb{Z}} \pi_d E$ preserves filtered colimits. Since κ preserves directed unions, $\kappa(\bigoplus_{d\in\mathbb{Z}} \pi_d E) = \mu_\kappa(E)$. Applying (3.7) to $\alpha \mapsto \bigoplus_{d\in\mathbb{Z}} \pi_d \operatorname{cofib}(X(\alpha))$ gives

$$\mu_\kappa(\operatorname{cofib} u_i) \le \sup_{\alpha : i \to j} \mu_\kappa(\operatorname{cofib}(X(\alpha))).$$

The supremum tends to zero by the Cauchy condition. Hence $\mu_\kappa(\operatorname{cofib} u_i) \to 0$, so the colimit cocone $(u_i)_{i\in I}$ is convergent. If every X_i has finite value, (7.2) gives $\mu_\kappa(\operatorname{colim}_I X) \le \mu_\kappa(X_i) + \mu_\kappa(\operatorname{cofib} u_i) < +\infty$ for all sufficiently large i. □

For the rest of this section, let $\mathcal{T}$ be a presentable stable category with a right complete t-structure compatible with filtered colimits. Suppose that $\mathcal{T}^\heartsuit$ is a compactly generated Grothendieck abelian category with a continuous premeasure κ that is finite on compact objects.

Theorem 7.5. *Write $q_\mathcal{T} : \mathcal{T} \to \mathcal{T}/\mathcal{T}_0$ for the Verdier quotient functor.*

1. *The full subcategories $\mathcal{T}^{\Sigma\kappa<\infty}$ and $\mathcal{T}_0$ of $\mathcal{T}$ are stable and closed under retracts. The subcategory $\mathcal{T}_0$ is presentable and localizing.*

2. *The t-structure on $\mathcal{T}$ descends to a right complete t-structure on $\mathcal{T}/\mathcal{T}_0$ compatible with filtered colimits. Its heart is $\mathcal{T}^\heartsuit/(\mathcal{T}^\heartsuit)_{\kappa=0}$.*

3. *For $X \in \mathcal{T}/\mathcal{T}_0$, put $\bar\mu_\kappa(X) = \sum_{i\in\mathbb{Z}} \bar\kappa(\pi_i X)$, where $\bar\kappa$ is given by (3.9). The quotient $\mathcal{T}^{\Sigma\kappa<\infty}/\mathcal{T}_0$ identifies with the full subcategory on the objects X with $\bar\mu_\kappa(X) < +\infty$. It is idempotent complete, and the t-structure of (2) restricts to it with heart $(\mathcal{T}^\heartsuit)_{\kappa<\infty}/(\mathcal{T}^\heartsuit)_{\kappa=0}$.*

4. *Every sequence (E_n, f_{nm}) in $\mathcal{T}^{\Sigma\kappa<\infty}/\mathcal{T}_0$ that is Cauchy for $\bar{\mu}_\kappa$ admits a colimit E in this quotient. Its structure maps $u_n : E_n \to E$ satisfy*

$$\bar{\mu}_\kappa(E) = \lim_{n\to\infty} \bar{\mu}_\kappa(E_n) < +\infty, \qquad \bar{\mu}_\kappa(\operatorname{cofib} u_n) = \lim_{m\to\infty} \bar{\mu}_\kappa(\operatorname{cofib} f_{nm}) \xrightarrow[n\to\infty]{} 0. \tag{7.3}$$

Proof. (1), (2) The subcategory $(\mathcal{T}^{\heartsuit})_{\kappa=0}$ is localizing by additivity and continuity of κ. Taking $\mathcal{S} = (\mathcal{T}^{\heartsuit})_{\kappa=0}$ in Proposition 7.1 gives the assertions about $\mathcal{T}_0$ and $\mathcal{T}/\mathcal{T}_0$. Equation (7.2) gives stability of $\mathcal{T}^{\Sigma\kappa<\infty}$. For a retract E' of E, each $\pi_i E'$ is a direct summand of $\pi_i E$, so $\mu_\kappa(E') \le \mu_\kappa(E)$.

(3) By t-exactness and (3.9), $\bar{\mu}_\kappa(q_{\mathcal{T}} E) = \mu_\kappa(E)$, so $\bar{\mu}_\kappa$ is faithful. The objects of finite value are closed under retracts and truncations, giving the asserted idempotent completeness and t-structure.

(4) Let $V_\bullet : \mathbb{N} \to \mathcal{T}/\mathcal{T}_0$. Equation (3.1) gives the first inequality in (7.4), and Fatou's lemma gives the second.

$$\begin{aligned} \bar{\mu}_\kappa(\operatorname{colim}_m V_m) &= \sum_{i\in\mathbb{Z}} \bar{\kappa}(\operatorname{colim}_m \pi_i V_m) \\ &\le \sum_{i\in\mathbb{Z}} \liminf_m \bar{\kappa}(\pi_i V_m) \le \liminf_m \bar{\mu}_\kappa(V_m). \end{aligned} \tag{7.4}$$

Let (E_n, f_{nm}) be as in (4), with structure maps $u_n : E_n \to E \cong \operatorname{colim}_n E_n$ in $\mathcal{T}/\mathcal{T}_0$. By (2) and Proposition 3.11, Lemma 7.4 applies in $\mathcal{T}/\mathcal{T}_0$. Thus E has finite value and $\bar{\mu}_\kappa(\operatorname{cofib} u_n) \to 0$. By (3), E is also the colimit in $\mathcal{T}^{\Sigma\kappa<\infty}/\mathcal{T}_0$. The first identity in (7.3) follows from $|\bar{\mu}_\kappa(E) - \bar{\mu}_\kappa(E_n)| \le \bar{\mu}_\kappa(\operatorname{cofib} u_n)$ by (7.2).

For fixed n, (7.2) applied to $\operatorname{cofib}(u_m)[-1] \to \operatorname{cofib}(f_{nm}) \to \operatorname{cofib}(u_n)$ gives the first inequality below. Equation (7.4) applied to $\operatorname{cofib} u_n \cong \operatorname{colim}_{m\ge n} \operatorname{cofib}(f_{nm})$ gives the second.

$$\limsup_{m\to\infty} \bar{\mu}_\kappa(\operatorname{cofib} f_{nm}) \le \bar{\mu}_\kappa(\operatorname{cofib} u_n) \le \liminf_{m\to\infty} \bar{\mu}_\kappa(\operatorname{cofib} f_{nm}). \qquad \square$$

Lemma 7.6. *On $\mathcal{T}^{\Sigma\kappa<\infty}/\mathcal{T}_0$, write $\chi_\kappa(E) = \sum_{i\in\mathbb{Z}} (-1)^i \bar{\kappa}(\pi_i E)$. Let (E_n) be a Cauchy sequence with colimit E and structure maps $u_n : E_n \to E$. Then*

$$|\chi_\kappa(E) - \chi_\kappa(E_n)| \le \bar{\mu}_\kappa(\operatorname{cofib} u_n) \longrightarrow 0.$$

For every natural transformation $(h_n) : (E_n) \to (F_n)$ to another Cauchy sequence in this quotient,

$$\bar{\mu}_\kappa(\operatorname{cofib}(\operatorname{colim}_n h_n)) = \lim_{n\to\infty} \bar{\mu}_\kappa(\operatorname{cofib} h_n).$$

Proof. Lemma 7.3 applied in $\mathcal{T}/\mathcal{T}_0$ gives $\chi_\kappa(E) - \chi_\kappa(E_n) = \chi_\kappa(\operatorname{cofib} u_n)$, whose absolute value is at most $\bar{\mu}_\kappa(\operatorname{cofib} u_n) \to 0$ by (7.3). For the structure maps $v_n : F_n \to \operatorname{colim}_m F_m$, the stable form of (3.4) applied to the naturality square gives

$$|\bar{\mu}_\kappa(\operatorname{cofib}(\operatorname{colim}_m h_m)) - \bar{\mu}_\kappa(\operatorname{cofib} h_n)| \le \bar{\mu}_\kappa(\operatorname{cofib} u_n) + \bar{\mu}_\kappa(\operatorname{cofib} v_n).$$

The right side tends to zero by (7.3). $\square$

Lemma 7.7. *For every $E \in \mathcal{T}^{\Sigma\kappa<\infty}/\mathcal{T}_0$, the cocone $(\tau_{\ge -n} E \to E)_{n\ge 0}$ is a convergent colimit cocone.*

Proof. Right completeness gives $\operatorname{colim}_{n\ge 0} \tau_{\ge -n} E \cong E$. Since $\bar{\mu}_\kappa(E) < +\infty$,

$$\bar{\mu}_\kappa(\operatorname{cofib}(\tau_{\ge -n} E \to E)) = \sum_{i<-n} \bar{\kappa}(\pi_i E) \longrightarrow 0. \qquad \square$$

Lemma 7.8. *Let (X_n, f_{nm}) be a sequence in the full subcategory*

$$\{E \in \mathcal{T} : \kappa(\pi_i E) < +\infty \text{ for every } i \in \mathbb{Z}\}/\mathcal{T}_0 \ \subseteq \ \mathcal{T}/\mathcal{T}_0.$$

Suppose that

$$\forall i \in \mathbb{Z}\ \forall \epsilon \in L_{>0}\ \exists N\ \forall m \geq n \geq N, \qquad \bar{\kappa}(\pi_i \operatorname{cofib} f_{nm}) < \epsilon.$$

Then its colimit X in $\mathcal{T}/\mathcal{T}_0$ belongs to this full subcategory, and its structure maps $u_n : X_n \to X$ satisfy $\bar{\kappa}(\pi_i \operatorname{cofib} u_n) \to 0$ for every $i \in \mathbb{Z}$.

Proof. For each n, we have $\operatorname{cofib} u_n \cong \operatorname{colim}_{m \geq n} \operatorname{cofib} f_{nm}$. Applying (3.1) in the quotient heart gives

$$\bar{\kappa}(\pi_i \operatorname{cofib} u_n) \leq \liminf_{m \to \infty} \bar{\kappa}(\pi_i \operatorname{cofib} f_{nm}) \xrightarrow[n \to \infty]{} 0 \qquad (i \in \mathbb{Z}).$$

By exactness of $\pi_i X_n \to \pi_i X \to \pi_i \operatorname{cofib} u_n$, we obtain $\bar{\kappa}(\pi_i X) \leq \bar{\kappa}(\pi_i X_n) + \bar{\kappa}(\pi_i \operatorname{cofib} u_n) < +\infty$ for each i and all sufficiently large n. Thus X is also the colimit in the stated full subcategory. □

Part II
Applications

8 Normalized length for finite flat local towers

Proposition 8.1 applies Theorem 2.1; see also [8, Remark 2.5]. For the extension argument, see also [7, §14.5], [8, Definition 2.1, Lemma 2.2, and Proposition 2.8], [3, §2.2], [6, Appendix 2, Lemma 7], and [10, Corollary 4.3].

Proposition 8.1. *Let*

$$(R_0, \mathfrak{m}_0) \longrightarrow (R_1, \mathfrak{m}_1) \longrightarrow (R_2, \mathfrak{m}_2) \longrightarrow \cdots$$

be a sequence of finite flat local homomorphisms of discrete Noetherian local rings. Put $R_\infty = \operatorname{colim}_n R_n$, $I = \mathfrak{m}_0 R_\infty$, *and*

$$e_n = \operatorname{length}_{R_n}(R_n/\mathfrak{m}_0 R_n).$$

Let $\mathcal{A}_\infty$ *be the category of finitely presented discrete* R_∞*-modules annihilated by a power of* I*. It is a small abelian category, and scalar extension induces an isomorphism*

$$\operatorname*{colim}_n \operatorname{Mod}_{R_n}^{\heartsuit,\mathrm{fl}} \xrightarrow{\sim} \mathcal{A}_\infty, \tag{8.1}$$

where $\operatorname{Mod}_{R_n}^{\heartsuit,\mathrm{fl}}$ *denotes the category of finite-length discrete* R_n*-modules, and the colimit in* (8.1) *is taken in the category of small categories. The formulas*

$$\nu_\infty(R_\infty \otimes_{R_n} N) = e_n^{-1} \operatorname{length}_{R_n}(N) \qquad (N \in \operatorname{Mod}_{R_n}^{\heartsuit,\mathrm{fl}}) \tag{8.2}$$

define a finite measure on $\mathcal{A}_\infty$*. It extends uniquely to a continuous premeasure* κ_∞ *on all discrete* R_∞*-modules annihilated elementwise by powers of* I*, with*

$$\kappa_\infty(R_\infty/I) = 1.$$

Moreover, (8.2) *induces an ordered isomorphism*

$$K_0(\mathcal{A}_\infty) \xrightarrow{\sim} \bigcup_{n \geq 0} e_n^{-1}\mathbb{Z} \subseteq \mathbb{Q}. \tag{8.3}$$

Proof. Each transition map is faithfully flat, and R_∞ is faithfully flat over every R_n. Since R_m is finite and local over R_n, the ideal $\mathfrak{m}_n R_m$ is $\mathfrak{m}_m$-primary. For $m \geq n$ and a finite-length discrete R_n-module N, a composition series and flatness give

$$\begin{aligned} \operatorname{length}_{R_m}(R_m \otimes_{R_n} N) &= \operatorname{length}_{R_m}(R_m/\mathfrak{m}_n R_m) \operatorname{length}_{R_n}(N), \\ e_m &= \operatorname{length}_{R_m}(R_m/\mathfrak{m}_n R_m) e_n. \end{aligned}$$

Thus e_n divides e_m, and

$$\frac{\operatorname{length}_{R_m}(R_m \otimes_{R_n} N)}{e_m} = \frac{\operatorname{length}_{R_n}(N)}{e_n}. \tag{8.4}$$

Finite presentations and morphisms descend to a finite stage. If $I^a M = 0$ for a finitely presented discrete R_∞-module M, choose a finitely presented discrete R_n-module M_n with $R_\infty \otimes_{R_n} M_n \simeq M$. Since $\mathfrak{m}_0^a$ is finitely generated, increasing n gives $\mathfrak{m}_0^a M_n = 0$. Thus M_n has finite length. Kernels and cokernels of maps between these modules descend by flatness

to finite-length kernels and cokernels over R_n. This proves (8.1) and abelianness of $\mathcal{A}_\infty$. Equation (8.4) proves that ν_∞ is well-defined and additive. Faithful flatness and positivity of finite length prove faithfulness.

Every finitely generated discrete R_∞-module annihilated elementwise by powers of I is a quotient of $(R_\infty/I^a)^{\oplus s}$ for some a, s. Its kernel is the directed union of its finitely generated submodules. Each such submodule is the image of a map from an object of $\mathcal{A}_\infty$, and hence belongs to $\mathcal{A}_\infty$. The given module is the filtered colimit of the corresponding quotients, each of which belongs to $\mathcal{A}_\infty$. Arbitrary elementwise torsion modules are directed unions of their finitely generated submodules. Consequently $\mathrm{Ind}(\mathcal{A}_\infty)$ is the category of discrete R_∞-modules annihilated elementwise by powers of I, and Theorem 2.1 constructs κ_∞.

By dévissage [5, §5, Theorem 4], length identifies each finite-stage Grothendieck group with $\mathbb{Z}$, with nonnegative cone $\mathbb{Z}_{\geq 0}$. Under these identifications, $K_0(\mathrm{Mod}_{R_n}^{\heartsuit,\mathrm{fl}}) \to K_0(\mathrm{Mod}_{R_m}^{\heartsuit,\mathrm{fl}})$ is multiplication by e_m/e_n. Generators and short exact sequence relations descend to a finite stage, so their colimit gives (8.3). □

The values of κ_∞ on arbitrary modules are given by (2.1) and (2.2).

8.1 The ordered Grothendieck group $K_0(\mathrm{Coh}^\heartsuit_{\mathfrak{m}_A}(P))$

Fix a prime p, a perfect field k of characteristic p, and an integer $r \geq 1$. Let

$$A = k[[t_1, \dots, t_r]] \quad \text{or} \quad A = W(k)[[t_2, \dots, t_r]],$$

with $t_1 = p$ in mixed characteristic. Choose compatible p-power roots of $t_1, \dots, t_r$, and put

$$A_n = A[t_1^{1/p^n}, \dots, t_r^{1/p^n}], \qquad R_\infty = \operatorname*{colim}_n A_n, \qquad P = (R_\infty)^\wedge_p.$$

Let $\mathrm{Coh}^\heartsuit_{\mathfrak{m}_A}(P)$ be the category of finitely presented discrete P-modules annihilated by a power of $\mathfrak{m}_A$.

Theorem 8.2. *The category $\mathrm{Coh}^\heartsuit_{\mathfrak{m}_A}(P)$ is abelian. There is an ordered isomorphism*

$$\ell : K_0(\mathrm{Coh}^\heartsuit_{\mathfrak{m}_A}(P)) \xrightarrow{\sim} \mathbb{Z}[1/p], \qquad [P/\mathfrak{m}_A P] \longmapsto 1.$$

In particular, $K_0(\mathrm{Coh}^\heartsuit_{\mathfrak{m}_A}(P)) \otimes_{\mathbb{Z}} \mathbb{R}$ is a real ordered line with distinguished positive element $[P/\mathfrak{m}_A P]$.

Proof. Write $u_{i,n} = t_i^{1/p^n}$. In equal characteristic, $A_n \simeq k[[u_{1,n}, \dots, u_{r,n}]]$. In mixed characteristic,

$$A_n \simeq (W(k)[u_{1,n}]/(u_{1,n}^{p^n} - p))[[u_{2,n}, \dots, u_{r,n}]].$$

In both cases A_n is complete regular local with residue field k and regular parameters $u_{1,n}, \dots, u_{r,n}$. For $m \geq n$, the monomials $\prod_i u_{i,m}^{j_i}$, $0 \leq j_i < p^{m-n}$, form an A_n-basis of A_m. In particular, R_∞ is faithfully flat over every A_n, and

$$e_n = \mathrm{length}_{A_n}(A_n/\mathfrak{m}_A A_n) = p^{rn}.$$

In equal characteristic $P = R_\infty = A^{\mathrm{perf}}$. In mixed characteristic R_∞ is p-torsionfree, so p-adic completion gives

$$R_\infty/p^a R_\infty \xrightarrow{\sim} P/p^a P \qquad (a \geq 1). \tag{8.5}$$

Moreover P is flat over each p-adically complete Noetherian ring A_n by [4, Lemma 15.28.5, Tag 0AGW], applied to the flat A_n-module R_∞. The residue maps $A_n \to k$ extend to $P \to k$, so P is faithfully flat over each A_n. In both characteristics we obtain

$$R_\infty/\mathfrak{m}_A^a R_\infty = \operatorname*{colim}_n A_n/\mathfrak{m}_A^a A_n \xrightarrow{\sim} P/\mathfrak{m}_A^a P \qquad (a \geq 1). \tag{8.6}$$

In mixed characteristic this follows by quotienting (8.5) by $\mathfrak{m}_A^a$, since $p^a \in \mathfrak{m}_A^a$.

For $R = R_\infty, P$ and $M \in \mathrm{Mod}_R^\heartsuit$, put

$$M[\mathfrak{m}_A^a] = \{x \in M : \mathfrak{m}_A^a x = 0\}, \qquad \mathcal{B}_R = \left\{ M \in \mathrm{Mod}_R^\heartsuit : M = \bigcup_{a \geq 1} M[\mathfrak{m}_A^a] \right\}.$$

Extension and restriction of scalars along $R_\infty \to P$ are inverse equivalences

$$\mathcal{B}_{R_\infty} \underset{\mathrm{Res}^P_{R_\infty}}{\overset{P \otimes_{R_\infty} -}{\rightleftarrows}} \mathcal{B}_P \tag{8.7}$$

Indeed, flatness over A gives

$$P \otimes_{R_\infty} (R_\infty/\mathfrak{m}_A^a R_\infty) \simeq P \otimes_A (A/\mathfrak{m}_A^a) \simeq P/\mathfrak{m}_A^a P.$$

On modules killed by $\mathfrak{m}_A^a$, the functors in (8.7) are therefore extension and restriction along the ring isomorphism (8.6). For $M \in \mathcal{B}_{R_\infty}$, the induced P-action on M is given by

$$s \cdot x = tx \qquad (x \in M[\mathfrak{m}_A^a],\ s \in P,\ t \in R_\infty,\ s - t \in \mathfrak{m}_A^a P).$$

These actions agree under the inclusions $M[\mathfrak{m}_A^a] \subseteq M[\mathfrak{m}_A^{a+1}]$. Taking the union over a gives the inverse to restriction in (8.7).

For $R = R_\infty, P$, the ideal $\mathfrak{m}_A^a R$ is finitely generated. A discrete $R/\mathfrak{m}_A^a R$-module is therefore finitely presented over R exactly when it is finitely presented over $R/\mathfrak{m}_A^a R$. Since $\mathrm{Ind}(\mathcal{A}_\infty) \simeq \mathcal{B}_{R_\infty}$ by Proposition 8.1 for $R_n = A_n$, the restriction of (8.7) to compact objects is

$$\mathcal{B}_{R_\infty}^c = \mathcal{A}_\infty \xrightarrow[\sim]{P \otimes_{R_\infty} -} \mathrm{Coh}^\heartsuit_{\mathfrak{m}_A}(P) = \mathcal{B}_P^c. \tag{8.8}$$

By (8.2) and (8.3), the resulting ordered isomorphism is

$$\begin{aligned} K_0(\mathrm{Coh}^\heartsuit_{\mathfrak{m}_A}(P)) &\xrightarrow{\sim} \bigcup_{n \geq 0} p^{-rn}\mathbb{Z} = \mathbb{Z}[1/p], \\ [P \otimes_{A_n} N_n] &\longmapsto p^{-rn} \operatorname{length}_{A_n}(N_n) \qquad (N_n \in \mathrm{Mod}_{A_n}^{\heartsuit,\mathrm{fl}}). \end{aligned} \tag{8.9}$$

Taking $n = 0$ and $N_0 = k$ gives $\ell([P/\mathfrak{m}_A P]) = 1$. □

8.2 Approximation of torsion modules by compact objects

Proposition 8.3. *Let R be a discrete commutative ring, let $I \subseteq R$ be finitely generated, and let $\mathcal{B}_R$ be the Grothendieck abelian category of discrete R-modules annihilated elementwise by powers of I. Suppose a continuous premeasure κ on $\mathcal{B}_R$ satisfies $\kappa(R/I^a) < +\infty$ for all $a \geq 1$. The category $\mathcal{B}_R$ is compactly generated, with*

$$\mathcal{B}_R^c = \{F \in \mathcal{B}_R : F \text{ is finitely presented over } R\}.$$

The approximations of Theorem 5.4 apply to every $M \in \mathcal{B}_R$ of finite value and give colimits in the Serre quotient

$$(\mathcal{B}_R)_{<\infty}/(\mathcal{B}_R)_0. \tag{8.10}$$

If $I^a M = 0$, the approximations $F_n \in \mathcal{B}_R^c$ and compatible maps to M can be chosen with $I^a F_n = 0$ for every n.

Proof. Finite generation of I makes elementwise I-power torsion closed under extensions. The category $\mathcal{B}_R$ is Grothendieck: its colimits are computed in discrete R-modules, filtered colimits are exact, and the objects R/I^a form a family of generators. Every finitely generated object is killed by some I^a and is a quotient of $(R/I^a)^{\oplus s}$. Writing its kernel as a directed union of finitely generated submodules expresses it as a filtered colimit of finitely presented R-modules in $\mathcal{B}_R$. Finite sums combine presentations, and cokernels of differences coequalize parallel morphisms to a fixed target; these operations preserve finite presentation and annihilation by some power of I. Thus every object of $\mathcal{B}_R$ is a filtered colimit of finitely presented R-modules in $\mathcal{B}_R$, which are compact in $\mathcal{B}_R$ because filtered colimits are computed in discrete R-modules. Conversely, a compact object of $\mathcal{B}_R$ is a retract of a term in such a presentation, and hence is finitely presented over R. This proves compact generation and the description of $\mathcal{B}_R^c$.

For $F \in \mathcal{B}_R^c$, choose an epimorphism $(R/I^a)^{\oplus s} \to F$. Then $\kappa(F) \le s\kappa(R/I^a) < +\infty$. Thus Theorem 5.4 gives the asserted compact approximations. When $I^a M = 0$, iterate Lemma 5.3 in the category of discrete R/I^a-modules to construct the cocone $F_n \to M$. Its terms are finitely presented over R because they are finitely presented over R/I^a and I^a is finitely generated. □

9 Continuity of finitely presented endofunctors

We use the following definition of finitely presented additive functors; see [11].

Definition 9.1 (Finitely presented additive functors)**.** Let R be a discrete commutative ring, and let $\mathcal{E}$ be a small R-linear additive category with discrete mapping spaces. For $M \in \mathcal{E}$, let

$$h_M : \mathcal{E} \longrightarrow \mathrm{Mod}_R^{\heartsuit}$$

be the representable functor: the underlying space of $h_M(N)$ is $\mathrm{Hom}_{\mathcal{E}}(M, N)$, with its R-module structure. An additive R-linear functor

$$\Phi : \mathcal{E} \longrightarrow \mathrm{Mod}_R^{\heartsuit}$$

is finitely presented if it admits a pointwise exact presentation

$$h_{M_1} \longrightarrow h_{M_0} \longrightarrow \Phi \longrightarrow 0, \qquad M_0, M_1 \in \mathcal{E}. \tag{9.1}$$

Equivalently, Φ is compact in $\mathrm{Fun}_{R,\mathrm{add}}(\mathcal{E}, \mathrm{Mod}_R^{\heartsuit})$.

Write $\mathrm{Coh}(R)^{\heartsuit}$ for the category of finitely presented discrete R-modules. An additive R-linear endofunctor of $\mathrm{Mod}_R^{\heartsuit}$ is called *continuous* if it preserves filtered colimits. We denote their full subcategory by

$$\mathrm{Fun}_{R,\mathrm{add}}^{\mathrm{continuous}}(\mathrm{Mod}_R^{\heartsuit}, \mathrm{Mod}_R^{\heartsuit}).$$

Restriction induces an isomorphism

$$\begin{aligned} &\mathrm{Fun}_{R,\mathrm{add}}^{\mathrm{continuous}}(\mathrm{Mod}_R^{\heartsuit}, \mathrm{Mod}_R^{\heartsuit}) \\ &\qquad \simeq \mathrm{Fun}_{R,\mathrm{add}}(\mathrm{Coh}(R)^{\heartsuit}, \mathrm{Mod}_R^{\heartsuit}). \end{aligned} \tag{9.2}$$

The inverse of (9.2) takes a functor to its extension preserving filtered colimits. For a finitely presented functor, this extension is given by the same cokernel presentation (9.1) on all discrete R-modules.

Proposition 9.2 (Uniform continuity of finitely presented endofunctors)**.** *Let R be a discrete coherent commutative ring, and let κ be a continuous premeasure on $\mathrm{Mod}_R^{\heartsuit}$ with $\kappa(R) < +\infty$. Let*

$$\Phi \in \mathrm{Fun}_{R,\mathrm{add}}^{\mathrm{continuous}}(\mathrm{Mod}_R^{\heartsuit}, \mathrm{Mod}_R^{\heartsuit})$$

be compact, where continuous means preserving filtered colimits. Then Φ *preserves finitely presented discrete* R*-modules and objects of finite* κ*-value. Its restriction to* $(\mathrm{Mod}_R^{\heartsuit})_{\kappa<\infty}$ *is uniformly continuous with respect to* κ*. Moreover,* Φ *induces a uniformly continuous endofunctor* $\widehat{\Phi}$ *of*

$$\frac{(\mathrm{Mod}_R^{\heartsuit})_{\kappa<\infty}}{(\mathrm{Mod}_R^{\heartsuit})_{\kappa=0}} \tag{9.3}$$

for the induced measure.

Proof. By (9.2), restriction identifies $\mathrm{Fun}_{R,\mathrm{add}}^{\mathrm{continuous}}(\mathrm{Mod}_R^{\heartsuit}, \mathrm{Mod}_R^{\heartsuit})$ with

$$\mathrm{Fun}_{R,\mathrm{add}}(\mathrm{Coh}(R)^{\heartsuit}, \mathrm{Mod}_R^{\heartsuit}).$$

Its representables are compact projective generators. Thus compactness gives a pointwise exact presentation

$$h_{M_1} \longrightarrow h_{M_0} \longrightarrow \Phi \longrightarrow 0, \qquad M_0, M_1 \in \mathrm{Coh}(R)^{\heartsuit},$$

on all discrete R-modules. Choose finite presentations

$$R^{\oplus b_j} \longrightarrow R^{\oplus a_j} \longrightarrow M_j \longrightarrow 0 \qquad (j = 0, 1).$$

They give

$$h_{M_j}(N) = \ker(N^{\oplus a_j} \longrightarrow N^{\oplus b_j}). \tag{9.4}$$

For a map $f : N \to N'$ between objects of finite value, (9.4) gives

$$\kappa(\ker h_{M_j}(f)) \le a_j \kappa(\ker f).$$

Factor f as $N \twoheadrightarrow \mathrm{im}\, f \hookrightarrow N'$. For $N \twoheadrightarrow \mathrm{im}\, f$, the kernel–cokernel exact sequence gives an injection

$$\mathrm{coker}\big(h_{M_j}(N) \to h_{M_j}(\mathrm{im}\, f)\big) \hookrightarrow \mathrm{coker}\big((\ker f)^{\oplus a_j} \to (\ker f)^{\oplus b_j}\big).$$

For $\mathrm{im}\, f \hookrightarrow N'$, one has

$$h_{M_j}(N')/h_{M_j}(\mathrm{im}\, f) \hookrightarrow (\mathrm{coker}\, f)^{\oplus a_j}.$$

Consequently,

$$\kappa(\mathrm{coker}\, h_{M_j}(f)) \le b_j \kappa(\ker f) + a_j \kappa(\mathrm{coker}\, f),$$
$$\delta_\kappa(h_{M_j}(f)) \le (a_j + b_j)\delta_\kappa(f).$$

Comparison of the cokernel presentations of $\Phi(N)$ and $\Phi(N')$ gives

$$\kappa(\ker \Phi(f)) \le \kappa(\ker h_{M_0}(f)) + \kappa(\mathrm{coker}\, h_{M_1}(f)),$$
$$\kappa(\mathrm{coker}\, \Phi(f)) \le \kappa(\mathrm{coker}\, h_{M_0}(f)).$$

It follows that, for some $C_\Phi > 0$,

$$\delta_\kappa(\Phi(f)) \le C_\Phi \delta_\kappa(f). \tag{9.5}$$

Taking $f : 0 \to N$ proves $\kappa(\Phi(N)) \le C_\Phi \kappa(N)$, so Φ preserves objects of finite value. Coherence of R, (9.4), and (9.1) prove that Φ preserves finitely presented discrete R-modules. Estimate (9.5) proves uniform continuity on $(\mathrm{Mod}_R^{\heartsuit})_{\kappa<\infty}$.

Estimate (9.5) also shows that Φ sends maps of error zero to maps of error zero, so it induces $\widehat{\Phi}$ on the quotient in (9.3). Representing quotient morphisms by fractions with denominators of error zero shows that (9.5) holds for $\widehat{\Phi}$ and the induced measure. □

Corollary 9.3 (Sequential continuity)**.** *Under the hypotheses and notation of Proposition 9.2, the following assertions hold. For every compatible cocone $u_n : N_n \to N$ of objects of finite value satisfying $\delta_\kappa(u_n) \to 0$, one has*

$$\begin{aligned} \delta_\kappa(\Phi(u_n)) &\longrightarrow 0, \\ \kappa(\Phi(N)) &= \lim_n \kappa(\Phi(N_n)). \end{aligned} \tag{9.6}$$

The endofunctor $\widehat{\Phi}$ preserves Cauchy diagrams and their convergent colimit cocones. For every Cauchy diagram (X_n) in the Serre quotient (9.3)*, the canonical map in* (9.7) *is an isomorphism*

$$\operatorname{colim}_n \widehat{\Phi}(X_n) \longrightarrow \widehat{\Phi}(\operatorname{colim}_n X_n)\,. \tag{9.7}$$

Proof. Apply Proposition 6.2 to the restriction of Φ to $(\mathrm{Mod}_R^{\heartsuit})_{\kappa<\infty}$ and to $\widehat{\Phi}$. □

10 Normalized length over a regular base

10.1 Perfectoid inputs

Fix a prime p. For a discrete p-adically complete commutative ring R, write $R^\flat = \varprojlim_{x\mapsto x^p} R/p$ for its tilt, $W(-)$ for p-typical Witt vectors, and $[a]$ for the Teichmüller lift of a. For a Witt vector t write $\bar{t}$ for its reduction modulo p. Write $\mathrm{QC}(\operatorname{Spf} R)$ for the category of p-complete R-modules and $M\widehat{\otimes}_R N = \widehat{M \otimes_R N}_p$. For an ideal $J \subset R$, the ring R/J is the discrete quotient.

For a discrete commutative ring S, write S_{perfd}, or $(\widehat{S}_p)_{\mathrm{perfd}}$, for the initial perfectoid S-algebra when it exists. For every perfectoid ring T its structure map induces an isomorphism of discrete spaces

$$\mathrm{Hom}_{\mathrm{CAlg}}(S_{\mathrm{perfd}}, T) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{CAlg}}(S, T).$$

Definition 10.1. A p-adically complete discrete commutative ring R is *perfectoid* if a unit multiple of p is a pth power in R and the Fontaine map

$$\theta_R : W(R^\flat) \longrightarrow R$$

is surjective with principal kernel.

This is the definition in [16, §2.1.2].

Lemma 10.2. *Let R be a discrete p-adically complete ring on which multiplication by p is injective. Suppose that $\varpi \in R$ satisfies $p = u\varpi^p$ for some $u \in R^\times$, and that Frobenius induces an isomorphism*

$$R/(\varpi) \xrightarrow{x\mapsto x^p} R/(\varpi^p).$$

Then R is perfectoid.

This is [14, Lemma 3.10(ii)]. See also [16, §2.1.2, following (2.1.2.4)].

Lemma 10.3 (Generators of the Fontaine kernel)**.** *Let R be a perfectoid ring, and let $\xi \in W(R^\flat)$ generate $\ker \theta_R$. Write $\xi = [a] + pu$, with $a \in R^\flat$ and $u \in W(R^\flat)$. Then u is a unit and $R^\flat$ is a-adically complete.*

See [16, §2.1.2, (2.1.2.1)] and [14, Remark 3.11].

Proof. Reduction modulo p gives $R/p \cong R^\flat/(a)$. Since $R^\flat$ is perfect,

$$R^\flat = \varprojlim_F R/p \cong \varprojlim_F R^\flat/(a) \cong \varprojlim_n R^\flat/(a^{p^n}).$$

Thus a belongs to the Jacobson radical of $R^\flat$, and ξ belongs to the Jacobson radical of $W(R^\flat)$. Choose $\rho \in R$ with $\rho^p = p\varepsilon$ for a unit ε, and lift ρ, ε to $x, w \in W(R^\flat)$ with w a unit. Write $x^p - pw = ([c] + pz)\xi$. Since $x^p \equiv [\bar{x}^p] \pmod{p^2}$, comparison of the coefficients of p gives $-\bar{w} = a\bar{z} + c\bar{u}$. It follows that $\bar{u}$, and hence u, is a unit. □

Lemma 10.4 (Preservation of Fontaine generators)**.** *Let $f : R \to S$ be a map of perfectoid rings, and let $\xi = [a] + pu \in W(R^\flat)$ generate $\ker \theta_R$. Then $W(f^\flat)(\xi)$ generates $\ker \theta_S$, and $S^\flat$ is $f^\flat(a)$-adically complete.*

See [16, §2.1.2, following (2.1.2.1)].

Proof. For a generator $\eta = [b] + pv$ of $\ker \theta_S$, write $W(f^\flat)(\xi) = ([c] + pz)\eta$. Comparison modulo p^2 gives

$$f^\flat(a) = cb, \qquad f^\flat(\bar{u}) = b\bar{z} + c\bar{v}.$$

By Lemma 10.3, the left side of the second equality is a unit and $S^\flat$ is b-adically complete. Thus c and $[c] + pz$ are units. The first equality identifies the two adic topologies on $S^\flat$. □

Lemma 10.5 (Perfectoid rings are reduced)**.** *Every perfectoid ring is reduced.*

Proof. The decomposition in [16, (2.1.3.1)] embeds R in the product of a p-torsionfree perfectoid ring and a perfect $\mathbf{F}_p$-algebra. In the p-torsionfree factor R', choose ϖ with $(\varpi^p) = (p)$ for which $z \mapsto z^p$ induces an isomorphism $R'/(\varpi) \cong R'/p$ [16, (2.1.2.3)]. If $x^p = 0$ in R', then $x = \varpi y$, and p-torsionfreeness gives $y^p = 0$. Iteration gives $x \in \bigcap_n \varpi^n R' = 0$. Both factors are therefore reduced. □

Lemma 10.6. *If S and T are perfectoid algebras over a perfectoid ring R, the pushout of $S \leftarrow R \to T$ in $\mathrm{CAlg}(\mathrm{Sp}_p^\wedge)$ is discrete and perfectoid. Its underlying R-module is $S \widehat{\otimes}_R T$. The colimit in $\mathrm{CAlg}(\mathrm{Sp}_p^\wedge)$ of a filtered diagram of perfectoid rings is also discrete and perfectoid.*

Proof. Choose $\xi = [a] + pu$ generating $\ker \theta_R$. The tensor product $S^\flat \otimes_{R^\flat} T^\flat$ is discrete and perfect [1, Remark 2.1.10]. Its a-adic completion is again discrete and perfect [1, Remark 3.2.7]. Reduction modulo p identifies the map

$$\left(W(S^\flat) \otimes_{W(R^\flat)} W(T^\flat)\right)^\wedge_{(p,[a])} \longrightarrow W\Big(\left(S^\flat \otimes_{R^\flat} T^\flat\right)^\wedge_{(a)}\Big)$$

with the identity on $\left(S^\flat \otimes_{R^\flat} T^\flat\right)^\wedge_{(a)}$. Both sides are p-complete, so this is an isomorphism. Tensor with $R \cong W(R^\flat)/(\xi)$ over $W(R^\flat)$. The ideal $(p, [a])$ has image (p) in R, and Lemma 10.4 identifies $W(S^\flat)/(\xi)$ and $W(T^\flat)/(\xi)$ with S and T. Thus

$$S \widehat{\otimes}_R T \cong W\Big(\left(S^\flat \otimes_{R^\flat} T^\flat\right)^\wedge_{(a)}\Big)/(\xi).$$

This quotient is discrete and perfectoid [16, Proposition 2.1.9].

For a filtered diagram (S_i), restrict to the cofinal category of arrows from one index, so that each S_i is an algebra over a fixed perfectoid ring R. Now $\mathrm{colim}_i S_i^\flat$ is perfect. The same reduction modulo p, followed by tensoring with R over $W(R^\flat)$, gives

$$\widehat{\mathrm{colim}_i} S_{ip} \cong W\Big(\left(\mathrm{colim}_i S_i^\flat\right)^\wedge_{(a)}\Big)/(\xi).$$

This quotient is again discrete and perfectoid. □

The completed tensor product assertion is [1, Corollary 3.2.8]. For colimits see also [1, Remark 3.2.9].

Theorem 10.7 (André's flatness lemma)**.** *For every perfectoid ring R there is a perfectoid R-algebra R' such that $R \to R'$ is p-completely faithfully flat and every monic polynomial of positive degree in $R'[x]$ has a root in R'. In particular every element of R' admits compatible p^nth roots for all $n \geq 0$.*

This is [2, Theorem 7.14].

Proof. For each monic polynomial $f \in R[x]$ of positive degree, form

$$R \longrightarrow \big(R[\widehat{x^{1/p^\infty}]/(f(x))_p}\big)_{\mathrm{perfd}}.$$

This map is p-completely faithfully flat by the monic polynomial calculation in [2, Corollary 7.3, Proposition 7.10 and Remark 7.15]. Indeed the unperfected quotient is p-completely faithfully flat over R and is a quotient of the perfectoid ring $\widehat{R[x^{1/p^\infty}]}_p$ by a regular element. Its prismatic cohomology over $W(R^\flat)$ is $(p, \ker\theta_R)$-completely faithfully flat. Perfection preserves this flatness, since Frobenius is an automorphism of the base and its iterates preserve the topology of $(p, \ker\theta_R)$. Reduction modulo $\ker\theta_R$ gives the displayed algebra. Take the completed tensor products of these algebras over finite sets of polynomials and then their p-completed filtered colimit. The result is perfectoid and p-completely faithfully flat over R, as faithful flatness can be checked modulo p. Every monic polynomial of positive degree over R has a root in it.

Repeat this construction at successor ordinals and take p-completed colimits at limit ordinals. At ω_1, the filtered colimit is already p-complete since ω_1-filtered colimits commute with the countable limit computing completion. The coefficients of each polynomial in the colimit occur at one earlier stage, and the next stage supplies a root. Successively taking roots of $x^p - c$ gives the last assertion. □

Proposition 10.8 (Existence and base change of universal perfectoid quotients)**.** *For a perfectoid ring R and an ideal $J \subset R$, put $\widehat{R/J}_p = \lim_n R/(J + p^n R)$. The universal perfectoid algebra $(\widehat{R/J}_p)_{\mathrm{perfd}}$ exists, and the map*

$$R \longrightarrow (\widehat{R/J}_p)_{\mathrm{perfd}} \tag{10.1}$$

is surjective. Its kernel is denoted

$$J_{\mathrm{perfd}}(R) = \ker\big(R \to (\widehat{R/J}_p)_{\mathrm{perfd}}\big).$$

For every map $R \to R'$ of perfectoid rings,

$$(\widehat{R/J}_p)_{\mathrm{perfd}} \widehat{\otimes}_R R' \cong (\widehat{R'/JR'}_p)_{\mathrm{perfd}}. \tag{10.2}$$

The quotient $(\widehat{R/J}_p)_{\mathrm{perfd}}$ depends only on $\sqrt{J}$.

This is [2, Theorem 7.4 and Proposition 8.13].

Proof. Over the extension R' of Theorem 10.7, choose $g_0 = g$ and $g_{n+1}^p = g_n$ for every member g of a generating set of J. The completed tensor product of the two copies of R' over $R'\langle z_g^{1/p^\infty} \mid g\rangle$ defined by $z_g^{1/p^n} \mapsto g_n$ and $z_g^{1/p^n} \mapsto 0$ is discrete and perfectoid by Lemma 10.6. Here angle brackets denote p-adic completion. Modulo p^m its discrete quotient is $R'/(p^m, (g_n)_{g,n})$, obtained by applying π_0 to the tensor product modulo p^m. Taking the limit

over m identifies it with the ordinary p-completion of $R'/(g_n \mid g,\ n \geq 0)$. Every perfectoid ring receiving a map from R'/JR' kills these roots by Lemma 10.5. This gives the quotient and its universal property over R'. The same description on the completed Čech diagram descends the quotient and detects surjectivity, as in the proof of [2, Theorem 7.4]. The universal property and the completed tensor product give (10.2). Reducedness also gives the assertion about $\sqrt{J}$. □

Proposition 10.9 (Excision for two universal perfectoid quotients)**.** *Let R be perfectoid and let $I, J \subset R$ be ideals. The square*

$$\begin{array}{ccc} (\widehat{R/IJ}_p)_{\mathrm{perfd}} & \longrightarrow & (\widehat{R/I}_p)_{\mathrm{perfd}} \\ \downarrow & & \downarrow \\ (\widehat{R/J}_p)_{\mathrm{perfd}} & \longrightarrow & (\widehat{R/(I+J)}_p)_{\mathrm{perfd}} \end{array} \tag{10.3}$$

is cartesian in commutative ring spectra. Its lower-right ring is the completed tensor product

$$(\widehat{R/I}_p)_{\mathrm{perfd}} \widehat{\otimes}_{(\widehat{R/IJ}_p)_{\mathrm{perfd}}} (\widehat{R/J}_p)_{\mathrm{perfd}} \cong (\widehat{R/(I+J)}_p)_{\mathrm{perfd}}. \tag{10.4}$$

Applying $T \mapsto \mathrm{QC}(\operatorname{Spf} T)$ gives a cartesian square of symmetric monoidal categories. In $\mathrm{QC}(\operatorname{Spf} R)$ there are natural isomorphisms compatible with the maps to R,

$$\begin{aligned} I_{\mathrm{perfd}}(R) \widehat{\otimes}_R J_{\mathrm{perfd}}(R) &\cong (IJ)_{\mathrm{perfd}}(R), \\ I_{\mathrm{perfd}}(R) \amalg_{(IJ)_{\mathrm{perfd}}(R)} J_{\mathrm{perfd}}(R) &\cong (I+J)_{\mathrm{perfd}}(R). \end{aligned} \tag{10.5}$$

The ring square and its tensor product are the excision and base-change formulas of [2, Corollary 8.12 and Proposition 8.13].

Proof. In a valuation ring receiving a map from R/IJ on which the image of J is nonzero, $IJ = 0$ forces I to vanish. Thus the map factors uniquely through R/I. This verifies the valuation hypothesis for excision, giving (10.3). Proposition 10.8 gives (10.4) and the same tensor-product formula over R. Each of the two quotient algebras of the upper-left ring is idempotent for the completed tensor product. A compatible pair of modules on the other three vertices glues by taking its pullback. Tensoring that pullback with either quotient recovers the specified module by idempotence and (10.4). Conversely, tensoring the ring square with any module over its upper-left vertex recovers that module as the pullback of its restrictions.

Replace the upper-left ring by R. The total fiber of the new square is $I_{\mathrm{perfd}}(R) \widehat{\otimes}_R J_{\mathrm{perfd}}(R)$ by the tensor-product formula over R. By cartesianness of (10.3) it is also $(IJ)_{\mathrm{perfd}}(R)$. Taking the fibers of the maps from R to all four rings in (10.3) gives a cartesian square of ideals. This square is also cocartesian in $\mathrm{QC}(\operatorname{Spf} R)$, giving the second formula in (10.5). □

Proposition 10.10 (Existence of universal perfectoid algebras for integral algebras)**.** *Let R be perfectoid and S a discrete integral R-algebra. There is an initial perfectoid S-algebra $(\widehat{S}_p)_{\mathrm{perfd}}$. The functor $S \mapsto (\widehat{S}_p)_{\mathrm{perfd}}$ takes filtered colimits of integral R-algebras to p-completed filtered colimits and commutes with base change to perfectoid R-algebras.*

Existence is [2, Theorem 10.11 and Corollary 8.14].

Proof. Choose $R \to R'$ as in Theorem 10.7. For a split monic polynomial $(x-b)h(x) \in R'[x]$, with h monic, successive linear factors give the formula

$$(R'[x]/((x-b)h))_{\mathrm{perfd}} \cong (R'[x]/h)_{\mathrm{perfd}} \times_{(R'/h(b)R')_{\mathrm{perfd}}} R'.$$

Indeed, let T be a perfectoid $R'[x]/((x-b)h)$-algebra. Apply (10.3) in T to the ideals $(x-b)$ and $(h(x))$. Their product is zero. Taking the limit of this square over all such T gives the displayed pullback: passage to each perfectoid quotient is left adjoint to the inclusion of the corresponding quotient algebras, hence is initial for limits. The map $R' \to (R'/h(b)R')_{\mathrm{perfd}}$ is surjective. The pullback is therefore perfectoid by the Witt presentations of its three rings: in tilts it is a pullback along a surjection, and tensoring its Witt vectors with R' over $W((R')^\flat)$ recovers the displayed pullback. It carries the map from $R'[x]/((x-b)h)$ and is thus the initial perfectoid algebra.

Every monic polynomial over R' splits, so induction on the degree constructs the initial algebra of each monic quotient. Every finitely generated integral R'-algebra is a quotient of a tensor product of finitely many such monic quotients. Take the completed tensor product of their initial algebras and then its perfectoid quotient by the relations. An arbitrary integral R'-algebra is the filtered colimit of its finitely generated subalgebras, so Lemma 10.6 gives its initial perfectoid algebra.

Whenever S_{perfd} exists over R, the universal properties of the two tensor products give, for a perfectoid R-algebra T,

$$(\widehat{S}_p)_{\mathrm{perfd}} \widehat{\otimes}_R T \cong \big(\pi_0(S \otimes_R T)\big)_{\mathrm{perfd}}. \tag{10.6}$$

Apply (10.6) over R' to the initial perfectoid algebra of $\pi_0(S \otimes_R R')$. It gives the descent datum on the completed Čech nerve of $R \to R'$. Effective descent for perfectoid algebras descends this algebra to R, and faithfully flat descent of maps to each perfectoid test algebra gives its initial property. Finally, maps from a completed filtered colimit of perfectoid rings to a perfectoid ring form the limit of the mapping spaces, giving the assertion about filtered colimits. □

Definition (Almost coefficients)**.** For a perfectoid ring R and an ideal $J \subset R$, put

$$\mathrm{QC}((\mathrm{Spf}\, R)^{a_J}) = \ker\Big(\mathrm{QC}(\mathrm{Spf}\, R) \longrightarrow \mathrm{QC}(\mathrm{Spf}((\widehat{R/J}_p)_{\mathrm{perfd}}))\Big), \qquad M^{a_J} = J_{\mathrm{perfd}}(R)\widehat{\otimes}_R M.$$

The functor in the kernel is extension of scalars. The tensor product is inherited from $\mathrm{QC}(\mathrm{Spf}\, R)$, with unit $J_{\mathrm{perfd}}(R)$. We use the t-structure induced by the quotient

$$\mathrm{QC}((\mathrm{Spf}\, R)^{a_J}) \cong \frac{\{M \in \mathrm{Mod}_R : M[1/p] = 0\}}{\{M \in \mathrm{Mod}_R : M[1/p] = 0,\ J_{\mathrm{perfd}}(R)\pi_i M = 0 \text{ for every } i\}}. \tag{10.7}$$

Indeed, completion identifies p-torsion modules with p-complete modules, with inverse $N \mapsto \mathrm{fib}(N \to N[1/p])$. Under this identification the idempotent algebra $(\widehat{R/J}_p)_{\mathrm{perfd}}$ has as its modules the subcategory in the denominator of (10.7). On discrete p-torsion modules this is restriction of scalars along the surjection (10.1). For bounded modules it follows by successive truncations and full faithfulness of restriction. For arbitrary modules, restriction of scalars commutes with the truncation limits $\lim_m \tau_{\le m}$ and colimits $\mathrm{colim}_n \tau_{\ge -n}$ in module categories. In particular, a discrete p-torsion module has zero image in (10.7) precisely when it is annihilated by $J_{\mathrm{perfd}}(R)$. See [2, §10.1] for the corresponding almost categories modulo p^n.

Lemma 10.11 (Comparison of integral models)**.** *Let R be perfectoid, $J \subset R$ an ideal, and $F \to G$ a map of discrete integral R-algebras inducing an isomorphism $F[1/g] \to G[1/g]$ for every $g \in J$. Then*

$$\big((\widehat{F}_p)_{\mathrm{perfd}}\big)^{a_J} \xrightarrow{\sim} \big((\widehat{G}_p)_{\mathrm{perfd}}\big)^{a_J} \quad \textit{in } \mathrm{CAlg}\big(\mathrm{QC}((\mathrm{Spf}\, R)^{a_J})\big).$$

Every $g \in J$ acts by a null map on $\mathrm{cofib}(F_{\mathrm{perfd}} \to G_{\mathrm{perfd}})$ *in* Mod_R.

This follows from [2, Corollary 8.12]. The valuation test gives both assertions.

Proof. Let V be a p-complete valuation ring of rank at most one receiving a map from F with $JV \neq 0$. Choose $g \in J$ nonzero in V. Any extension to G must be

$$G \longrightarrow G[1/g] \cong F[1/g] \longrightarrow \mathrm{Frac}(V).$$

Every element in its image is integral over the image of F, hence integral over V and therefore in V. Excision and the universal property of the perfectoid quotients now give

$$\mathrm{fib}(F_{\mathrm{perfd}} \to G_{\mathrm{perfd}}) \cong \mathrm{fib}((F/JF)_{\mathrm{perfd}} \to (G/JG)_{\mathrm{perfd}}). \tag{10.8}$$

The right side is a module over $(F/JF)_{\mathrm{perfd}}$, which receives a map from $(R/J)_{\mathrm{perfd}}$. Thus every $g \in J$ acts on it by a null map, and tensoring it with $J_{\mathrm{perfd}}(R)$ in $\mathrm{QC}(\mathrm{Spf}\, R)$ gives zero by (10.2). □

Lemma 10.12 (Splitting finite étale covers after André's extension)**.** *Let R be perfectoid, let $J \subset R$ be finitely generated, put $U = \mathrm{Spec}\, R \setminus V(J)$, and let $Y \to U$ be finite étale. Let $R \to R'$ be as in Theorem 10.7. There are $f_1, \dots, f_m \in JR'$ and integers $d_j \geq 0$ such that*

$$\mathrm{Spec}\, R' \times_{\mathrm{Spec}\, R} U = \bigcup_{j=1}^{m} D(f_j), \qquad Y \times_U D(f_j) \cong \coprod_{i=1}^{d_j} D(f_j). \tag{10.9}$$

Proof. Every monic polynomial over a local ring of R' splits, by clearing its denominators and rescaling its variable. Its residue field k is algebraically closed. For a finite étale algebra E of positive rank d over such a local ring H, choose $x \in E$ whose image in $E \otimes_H k \cong k^d$ has pairwise distinct coordinates. The characteristic polynomial of x splits over H with pairwise unit root differences. Evaluation at x gives $H^d \cong H[t]/(\chi_x) \xrightarrow{\sim} E$, since the last map is an isomorphism modulo the maximal ideal between free modules of rank d. Rank zero gives the empty cover. Finite presentation spreads each splitting to a principal open contained in the inverse image of U. The inverse image of U is quasi-compact because J is finitely generated. A finite subcover gives (10.9). □

Theorem 10.13 (Almost purity)**.** *Let R be perfectoid, let $J \subset R$ be finitely generated, and let Y be finite étale over $U = \mathrm{Spec}\, R \setminus V(J)$. Define the discrete ring*

$$D = \{b \in \Gamma(Y, \mathcal{O}_Y) : b \text{ is integral over the image of } R\}.$$

The algebra

$$((\widehat{D}_p)_{\mathrm{perfd}})^{a_J} \in \mathrm{CAlg}(\mathrm{QC}((\mathrm{Spf}\, R)^{a_J}))$$

is dualizable as an object of $\mathrm{QC}((\mathrm{Spf}\, R)^{a_J})$, with perfect trace pairing. Tensoring with this algebra is t-exact, and its multiplication admits a section linear over its tensor square. If $Y \to U$ is surjective, tensoring with this algebra is conservative. The t-structure is the one in (10.7)*.*

For the almost purity theorem see [2, Theorems 1.17 and 10.9].

Proof. For $g \in J$, the algebra of $Y \times_U D(g)$ is $\Gamma(Y, \mathcal{O}_Y)[1/g]$, since Y is quasi-compact and quasi-separated. Clearing denominators in a monic equation shows that $D[1/g]$ is this algebra. Any discrete integral R-algebra S with $\mathrm{Spec}\, S \times_{\mathrm{Spec}\, R} U \cong Y$ maps to D by restriction, and Lemma 10.11 gives $(S_{\mathrm{perfd}})^{a_J} \cong (D_{\mathrm{perfd}})^{a_J}$.

Take R', f_j, d_j as in Lemma 10.12. In $R'[1/f_j]^{d_j}$ form the subalgebra generated by the image of $\pi_0(D \otimes_R R')$ and the coordinate idempotents. It is integral over each of $\pi_0(D \otimes_R R')$ and $(R')^{d_j}$, and both maps become isomorphisms on inverting f_j. Lemma 10.11 and (10.6) give

$$(D_{\mathrm{perfd}} \widehat{\otimes}_R R')^{a_{(f_j)}} \cong \left((R')^{a_{(f_j)}}\right)^{\times d_j}. \tag{10.10}$$

Complete modules descend along $R \to R'$ by faithfully flat descent on the reductions $R \otimes_{\mathbf{Z}} \mathbf{Z}/p^n \to R' \otimes_{\mathbf{Z}} \mathbf{Z}/p^n$ and passage to the limit over n. By (10.2), this descent restricts to the kernels defining the almost categories. Over R', the second formula in (10.5), tensored with a module, glues its restrictions to two principal opens from their common restriction. Iteration gives descent for the finite cover (10.9). These restrictions are jointly conservative and t-exact. Indeed restriction from J to (f_j) is the quotient functor in (10.7) induced by the inclusion of its denominators. Scalar extension along $R \to R'$ is t-exact and conservative on p-torsion modules and commutes with almost localization by (10.2).

On the jth open, the trace of (10.10) is the sum of the coordinates, and multiplication is projection onto the diagonal factors of its tensor square. The trace identifies the algebra with its dual. Inclusion of the diagonal factors is the unique section linear over the tensor square, so it descends, as does the duality. Tensoring is the direct sum of d_j copies of the identity and is therefore t-exact. If $Y \to U$ is surjective, every nonempty $D(f_j)$ has $d_j > 0$, so these functors also detect zero objects. □

Corollary 10.14 (Almost flatness and detection of zero for modules with p-power torsion)**.** *Let R be perfectoid, let S be a discrete integral R-algebra, and let $J \subset R$ be an ideal such that $S[1/g]$ is finite étale over $R[1/g]$ for every $g \in J$. If M is a discrete R-module killed elementwise by powers of p, then*

$$J_{\mathrm{perfd}}(R)\pi_i(M \otimes_R S_{\mathrm{perfd}}) = 0 \qquad (i > 0). \tag{10.11}$$

If $\operatorname{Spec} S \to \operatorname{Spec} R$ *is surjective away from $V(J)$, then also*

$$\begin{aligned} J_{\mathrm{perfd}}(R) \ker\bigl(M \to \pi_0(M \otimes_R S_{\mathrm{perfd}})\bigr) &= 0, \\ J_{\mathrm{perfd}}(R)\pi_0(M \otimes_R S_{\mathrm{perfd}}) = 0 \quad &\Longleftrightarrow \quad J_{\mathrm{perfd}}(R)M = 0. \end{aligned} \tag{10.12}$$

For finitely presented finite S see also [3, Proposition 2.4.9].

Proof. For $J = (g)$ apply Theorem 10.13 to $Y = \operatorname{Spec} S[1/g]$ over $\operatorname{Spec} R[1/g]$, using the integral model S. Under (10.7), t-exactness gives (10.11). In the surjective case the functor on the quotient hearts is exact and conservative. Tensoring the unit map $M \to \pi_0(M \otimes_R S_{\mathrm{perfd}})$ with the almost algebra gives a split monomorphism in the quotient heart, with retraction induced by multiplication on S_{perfd}. Exactness and conservativity imply that the unit is monic in the quotient heart and detect whether the extended module is zero. By (10.7) these are the two assertions of (10.12).

For arbitrary J and any generating set $\mathcal{G}$,

$$J_{\mathrm{perfd}}(R) = \overline{\sum_{g \in \mathcal{G}} (g)_{\mathrm{perfd}}(R)}^{\,p}.$$

Indeed the perfectoid quotient killing the generators is the completed tensor product of their quotients, taken over finite subsets and then p-completed. Its kernel is the displayed closed sum. On a module killed elementwise by powers of p, annihilation by this closure is equivalent to annihilation by each $(g)_{\mathrm{perfd}}(R)$. Apply the principal case to every $g \in \mathcal{G}$. □

Proposition 10.15 (Scalar bounds on $\operatorname{cofib}(S \to S_{\mathrm{perfd}})$)**.** *Let R be perfectoid, let S be a discrete R-algebra finitely generated as an R-module, and let $J \subset R$ be an ideal such that $S[1/g]$ is finite étale over $R[1/g]$ for every $g \in J$. For every $g \in J$, some power g^b acts by a null map on $C = \operatorname{cofib}(S \to S_{\mathrm{perfd}}) \in \operatorname{Mod}_R$. If J is finitely generated, there exists $N \geq 1$ such that every element of J^N acts by a null map on C.*

For S finitely presented as an R-module with bounded p-power torsion, the assertion for each $g \in J$ is [3, Proposition 2.4.7].

Proof. Fix $g \in J$. We first reduce to the case where S is finitely presented as an R-module. Choose generators $s_1, \dots, s_l$ of S as an R-algebra and monic $h_i \in R[t_i]$ with $h_i(s_i) = 0$. The map

$$T = R[t_1, \dots, t_l]/(h_1(t_1), \dots, h_l(t_l)) \twoheadrightarrow S, \qquad t_i \longmapsto s_i,$$

has finite free source. Since $S[1/g]$ is projective over $R[1/g]$, the kernel of $T[1/g] \to S[1/g]$ is a direct summand of a finite free module. Choose $r_1, \dots, r_m \in \ker(T \to S)$ whose images generate that kernel. Then

$$F = T/(r_1, \dots, r_m) \twoheadrightarrow S, \qquad F[1/g] \cong S[1/g],$$

and F is finitely presented as an R-module. Suppose g^b acts by a null map on $\operatorname{cofib}(F \to F_{\mathrm{perfd}})$. For $x \in \ker(F \to S)$ the image of x in F_{perfd} is killed by a power of g, hence by g because F_{perfd} is reduced. Thus $gx \in \ker(F \to F_{\mathrm{perfd}})$ and $g^{b+1}x = 0$. As $F \to S$ is surjective, g^{b+1} acts by a null map on its cofiber. By Lemma 10.11, g acts by a null map on $\operatorname{cofib}(F_{\mathrm{perfd}} \to S_{\mathrm{perfd}})$. In the square

$$\begin{array}{ccc} F & \longrightarrow & F_{\mathrm{perfd}} \\ \downarrow & & \downarrow \\ S & \longrightarrow & S_{\mathrm{perfd}} \end{array}$$

the cofibers of the top map and both vertical maps are thus killed by powers of g. The same holds for the cofiber of the bottom map, since the objects on which some power of g acts by a null map are closed under cofiber sequences.

Now assume that S is finitely presented as an R-module. The identity of the finite projective $R[1/g]$-module $S[1/g]$ factors through a finite free module. Since S is finitely presented over R, localization at g commutes with $\pi_0 \operatorname{Hom}_R(S, -)$ on discrete modules. Clearing denominators in the factorization gives $a \geq 0$ and maps

$$S \longrightarrow R^{\oplus l} \longrightarrow S \quad \text{with composite } g^a \operatorname{id}_S .$$

The discrete module S is p-complete by finite presentation [2, proof of Theorem 10.11, footnote 19]. In particular C is p-complete.

Take $R \to R'$ from Theorem 10.7 and put $S' = \pi_0(S \otimes_R R')$. Applying π_0 to the factorization over R' makes $g^a : S' \to S'$ factor through $(R')^{\oplus l}$. Applying $\tau_{\geq 1}$ makes g^a null on $\tau_{\geq 1}(S \otimes_R R')$. Lemma 10.12 for $Y = \operatorname{Spec} S[1/g]$ gives $f_j \in gR'$ with

$$D(g) = \bigcup_{j=1}^{m} D(f_j) \subseteq \operatorname{Spec} R', \qquad S'[1/f_j] \cong R'[1/f_j]^{d_j}. \tag{10.13}$$

In this product let T_j be the subalgebra generated by the image of S' and the coordinate idempotents. The integral maps

$$S' \longrightarrow T_j \longleftarrow (R')^{d_j} \tag{10.14}$$

become isomorphisms after inverting f_j.

The kernel of either map in (10.14) consists of the elements of its source killed by a power of f_j. Since R' is reduced, $f_j R'[f_j^{\infty}] = 0$. The finite free factorization of g^a on S' therefore gives

$$f_j g^a \ker(S' \to T_j) = 0, \qquad f_j \ker\big((R')^{d_j} \to T_j\big) = 0.$$

Here $f_j \in gR'$, so f_j^{a+1} kills the first kernel. Both cokernels are finitely generated, since T_j is finite over R'. They vanish on inverting f_j, so a power of f_j kills each of them. The two

cofibers in $\mathrm{Mod}_{R'}$ have homotopy groups only in degrees $0, 1$. Their two truncation terms therefore give a power of f_j acting by a null map on each cofiber.

Compare the rows of

$$\begin{array}{ccccc} S' & \longrightarrow & T_j & \longleftarrow & (R')^{d_j} \\ \downarrow & & \downarrow & & \| \\ N_{\mathrm{perfd}} & \longrightarrow & (T_j)_{\mathrm{perfd}} & \longleftarrow & (R')^{d_j}. \end{array} \tag{10.15}$$

By Lemma 10.11 the cofibers of both bottom maps are killed by f_j. Apply closure under cofiber sequences first to the right square of (10.15), whose right vertical arrow is the identity, and then to its left square. Some power of f_j acts by a null map on $\mathrm{cofib}(S' \to N_{\mathrm{perfd}})$. The scalars acting by a null map form an ideal. By (10.13), g is in the radical of that ideal, so some power of g acts by a null map on $\mathrm{cofib}(S' \to N_{\mathrm{perfd}})$.

The finitely presented discrete R'-module S' is p-complete, so the fiber of $S\widehat{\otimes}_R R' \to S'$ is the p-completion of $\tau_{\geq 1}(S \otimes_R R')$. Combining the bound g^a on this fiber with (10.6) gives $k \geq 1$ such that g^k acts by a null map on $C\widehat{\otimes}_R R'$.

To descend the bound, put $T = \mathrm{fib}(C \to C[1/p])$. Its homotopy groups are p-torsion and vanish outside $\{-1, 0, 1\}$. The cofiber of $C \otimes_R R' \to C\widehat{\otimes}_R R'$ has p invertible. Taking the fibers of localization at p therefore gives

$$T \otimes_R R' \cong \mathrm{fib}\big(C\widehat{\otimes}_R R' \longrightarrow (C\widehat{\otimes}_R R')[1/p]\big).$$

Thus g^k acts by a null map on $T \otimes_R R'$. On p-torsion modules scalar extension along $R \to R'$ is t-exact and conservative, so it is faithful on the heart. Consequently $g^k \pi_i T = 0$ for all i. The three truncation terms of T imply that g^{3k} is null on T and hence on $\widehat{T}_p \cong C$.

If $J = (g_1, \ldots, g_s)$, choose bounds b_j for its generators. Every monomial of degree $1 + \sum_j (b_j - 1)$ is divisible by some $g_j^{b_j}$, so

$$J^{\,1+\sum_j (b_j-1)} \subseteq \ker\big(R \to \pi_0 \,\mathrm{Hom}_R(C, C)\big).$$

□

10.2 Setup and statements

Let A, A_n, R_∞ and $P = (R_\infty)_p^\wedge$ be as in Theorem 8.2. Let $A \subset B$ be a finite generically separable inclusion of complete Noetherian local domains inducing the identity on residue field k, and write $d = [\mathrm{Frac}(B) : \mathrm{Frac}(A)]$. Set

$$D = (B \otimes_A P)_{\mathrm{perfd}}.$$

By Proposition 10.10, D is initial among perfectoid $B \otimes_A P$-algebras.

We call the natural map $(\mathrm{Coh}^\heartsuit_{\mathfrak{m}_A}(P))^{\cong} \to (K_0(\mathrm{Coh}^\heartsuit_{\mathfrak{m}_A}(P)) \otimes_{\mathbb{Z}} \mathbb{R})_{\geq 0}$, $N \mapsto [N] \otimes 1$, the positive Grothendieck class function. Its image is dense in this cone. With $+\infty$ adjoined to the target, it has a unique additive extension κ_A to $\mathrm{Ind}(\mathrm{Coh}^\heartsuit_{\mathfrak{m}_A}(P))$ preserving directed unions by Theorem 2.1. This extension is surjective.

For $R = P$ or D, let $\mathrm{Mod}_{\mathfrak{m}_A}(R) \subseteq \mathrm{Mod}_R$ consist of modules whose homotopy groups are annihilated elementwise by powers of $\mathfrak{m}_A$. For D the function κ_A is defined by restriction of scalars to P. Put

$$\mathrm{Mod}_{\mathfrak{m}_A}^{\sum \kappa_A < \infty}(R) = \left\{ E \in \mathrm{Mod}_{\mathfrak{m}_A}(R) : \sum_{i \in \mathbb{Z}} \kappa_A(\pi_i E) < +\infty \right\}.$$

We extend κ_A by

$$\kappa_A : (\mathrm{Mod}_{\mathfrak{m}_A}^{\sum \kappa_A < \infty}(R))^{\cong} \longrightarrow K_0(\mathrm{Coh}_{\mathfrak{m}_A}^{\heartsuit}(P)) \otimes_{\mathbb{Z}} \mathbb{R}, \qquad \kappa_A(E) = \sum_{i \in \mathbb{Z}} (-1)^i \kappa_A(\pi_i E) \tag{10.16}$$

The sum converges absolutely, and κ_A is additive on cofiber sequences by Lemma 7.3.

The proof of Theorem 11.1 uses foundational perfectoid almost purity and the ordinary identity $[B] = d[A]$ in $K_0(\mathrm{Perf}(A))$. Proposition 10.22 proves that $B \otimes_A P \to D$ is injective and that $D/(B \otimes_A P)$ is annihilated by g^b for some $0 \neq g \in \mathfrak{m}_A$ and $b \geq 1$. For the parameter ideal $\mathfrak{q} = (x_2, \dots, x_r, g^b)$ in (11.5), we show

$$\kappa_A\big((D/(B \otimes_A P)) \otimes_A A/\mathfrak{q}\big) = 0.$$

Together with $[B] = d[A]$, this gives

$$\kappa_A(D/\mathfrak{q}D) = d \operatorname{length}_A(A/\mathfrak{q})[P/\mathfrak{m}_A P].$$

10.3 The ring P and the category $\mathrm{Coh}_{\mathfrak{m}_A}(P)$

Lemma 10.16 ([8, §2])**.** *The ring $P = (R_\infty)_p^\wedge$ is perfectoid.*

Proof. In equal characteristic $P = A^{\mathrm{perf}}$ is perfect. In mixed characteristic, P is p-torsionfree by flatness over A and $p = t_1$. We verify the Frobenius criterion of Lemma 10.2 using

$$P/(p^{1/p}) \xrightarrow{x \mapsto x^p} P/p. \tag{10.17}$$

Surjectivity follows by taking one further root stage and using that k is perfect. To check injectivity of (10.17), choose a representative $x \in A_n$ with $n \geq 1$. If $x^p \in pP$, faithful flatness gives $x^p \in pA_n$. Since $u_{1,n} = p^{1/p^n}$ is a prime element of A_n, one has $x \in u_{1,n}^{p^{n-1}} A_n = p^{1/p} A_n$. This proves injectivity. Together with p-completeness and p-torsionfreeness, Lemma 10.2 shows that P is perfectoid. □

Lemma 10.17. *For $R = A_n$ or P, let $\mathrm{Coh}_{\mathfrak{m}_A}(R) \subset \mathrm{Mod}_R$ consist of those modules N for which $\pi_i N = 0$ for all but finitely many i, and every $\pi_i N$ is finitely presented and annihilated by a power of $\mathfrak{m}_A$. Scalar extension induces an isomorphism of stable categories*

$$\operatorname*{colim}_n \mathrm{Coh}_{\mathfrak{m}_A}(A_n) \xrightarrow{\sim} \mathrm{Coh}_{\mathfrak{m}_A}(P). \tag{10.18}$$

The colimit in (10.18) *is taken in the category of small categories, and* (10.18) *respects the t-structures inherited from the module categories. Consequently, every object of $\mathrm{Coh}_{\mathfrak{m}_A}^{\heartsuit}(P)$ admits a resolution by finite free P-modules of length at most r.*

Proof. Equations (8.1) and (8.8) identify the functor on hearts in (10.18) with the equivalence $\mathrm{colim}_n \mathrm{Mod}_{A_n}^{\heartsuit,\mathrm{fl}} \xrightarrow{\sim} \mathrm{Coh}_{\mathfrak{m}_A}^{\heartsuit}(P)$ induced by $P \otimes_{A_n} -$.

For $N \in \mathrm{Coh}_{\mathfrak{m}_A}^{\heartsuit}(P)$, choose a finite-length discrete A_n-module N_n with $N \simeq P \otimes_{A_n} N_n$. The regular local ring A_n has dimension r, so N_n admits a finite free resolution of length at most r [4, Proposition 10.110.1, Tag 00O7]. Flat scalar extension to P preserves this resolution. The subcategories $\mathrm{Coh}_{\mathfrak{m}_A}(R)$ are therefore stable and inherit bounded t-structures; finite Postnikov filtrations show that their objects are perfect.

For $M_n, N_n \in \mathrm{Coh}_{\mathfrak{m}_A}(A_n)$, the module M_n is perfect and each $\pi_i \mathrm{Hom}_{A_n}(M_n, N_n)$ is a finite-length discrete A_n-module. Flatness and (8.6) therefore give an isomorphism of mapping spectra

$$\operatorname*{colim}_{m \geq n} \mathrm{Hom}_{A_m}(A_m \otimes_{A_n} M_n, A_m \otimes_{A_n} N_n) \simeq \mathrm{Hom}_P(P \otimes_{A_n} M_n, P \otimes_{A_n} N_n).$$

This proves full faithfulness of (10.18). Essential surjectivity of (10.18) follows by finite Postnikov filtrations from descent of the objects of the heart and of their attaching maps. Faithful flatness identifies the t-structures. □

Remark 10.18. Inclusion induces an isomorphism

$$K_0(\mathrm{Coh}^{\heartsuit}_{\mathfrak{m}_A}(P)) \xrightarrow{\sim} K_0(\mathrm{Perf}_{\mathfrak{m}_A P}(P)), \qquad [P/\mathfrak{m}_A P] \longmapsto [P \otimes_A k].$$

10.4 Normalized length on $\mathrm{Ind}(\mathrm{Coh}^{\heartsuit}_{\mathfrak{m}_A}(P))$

Write

$$L_A = K_0(\mathrm{Coh}^{\heartsuit}_{\mathfrak{m}_A}(P)) \otimes_{\mathbb{Z}} \mathbb{R}, \qquad e_A = [P/\mathfrak{m}_A P].$$

For $N \in \mathrm{Coh}^{\heartsuit}_{\mathfrak{m}_A}(P)$, put

$$\nu_A(N) = [N] \otimes 1 \in (L_A)_{\geq 0}. \tag{10.19}$$

If $N \simeq P \otimes_{A_n} N_n$ for a finite-length discrete A_n-module N_n, Theorem 8.2 gives

$$\begin{aligned} \nu_A(N) &= \mathrm{length}_{A_n}(N_n)([P \otimes_{A_n} k] \otimes 1) \\ &= p^{-rn}\,\mathrm{length}_{A_n}(N_n) e_A. \end{aligned} \tag{10.20}$$

Corollary 10.19 ([6, Appendix 2], [8, Definition 2.1, Lemma 2.2, and Proposition 2.8])**.** *The finite measure ν_A extends uniquely to a continuous premeasure κ_A on the Grothendieck abelian category of discrete P-modules annihilated elementwise by powers of $\mathfrak{m}_A$. It takes values in $(L_A)_{\geq 0} \cup \{+\infty\}$ and is surjective.*

Proof. Taking ind-categories in (8.8) and applying Proposition 8.1 and (8.7) gives

$$\mathrm{Ind}(\mathrm{Coh}^{\heartsuit}_{\mathfrak{m}_A}(P)) \simeq \mathrm{Ind}(\mathcal{A}_\infty) \simeq \mathcal{B}_{R_\infty} \simeq \mathcal{B}_P.$$

Extend the coordinate $\ell_{\mathbb{R}} : L_A \xrightarrow{\sim} \mathbb{R}$ from (8.9) by $\ell_{\mathbb{R}}(+\infty) = +\infty$. For $N \in \mathcal{A}_\infty$, equations (8.2) and (8.9) give $\ell_{\mathbb{R}}(\nu_A(P \otimes_{R_\infty} N)) = \nu_\infty(N)$. The unique continuous extension in Proposition 8.1 thus gives

$$\ell_{\mathbb{R}}(\kappa_A(M)) = \kappa_\infty(\mathrm{Res}^P_{R_\infty} M) \qquad (M \in \mathcal{B}_P).$$

For $\lambda \in \mathbb{R}_{\geq 0}$, write $\lambda = \sum_{n \geq 0} a_n p^{-rn}$ with $a_n \in \mathbb{Z}_{\geq 0}$. Equation (10.20) gives

$$\kappa_A\left(\bigoplus_{n \geq 0} (P \otimes_{A_n} k)^{\oplus a_n}\right) = \lambda e_A.$$

A countable direct sum of copies of $P/\mathfrak{m}_A P$ has value $+\infty$. □

The normalization $e_A \mapsto 1$ identifies κ_A with Faltings' normalized length [6, Appendix 2]. See also [8, Definition 2.1 and Lemma 2.3] and [3, §2.2]. On discrete D-modules annihilated elementwise by powers of $\mathfrak{m}_A$, restriction of scalars defines

$$\kappa_A(M) = \kappa_A(\mathrm{Res}^D_P M).$$

Restriction is exact and preserves directed unions, so this is again a continuous premeasure.

10.5 Vanishing of κ_A

Lemma 10.20. *Suppose that A has mixed characteristic. Let Q be a perfectoid P-algebra and let $q = p^n$, where $n \geq 0$. The map of discrete quotient rings in* (10.21)

$$Q/(p^{1/q}) \longrightarrow Q/p, \qquad z \longmapsto z^q \tag{10.21}$$

is an isomorphism.

This is also [16, (2.1.2.3)–(2.1.2.4)].

Proof. Let $x = (p^{1/p^m} \bmod p)_{m\geq 0} \in P^\flat$. Its Teichmüller lift satisfies $\theta_P([x]) = p$. For a generator $\xi = [a] + pu$ of $\ker\theta_P$, write $[x] - p = ([c] + pz)\xi$. Comparison modulo p^2 gives

$$x = ca, \qquad -1 = a\bar{z} + c\bar{u} \quad \text{in } P^\flat.$$

Since $P^\flat$ is a-adically complete by Lemma 10.3, c and $[c] + pz$ are units. Thus $[x] - p$ generates $\ker\theta_P$. If $\bar{x}$ denotes the image of x in $Q^\flat$, Lemma 10.4 gives $Q \cong W(Q^\flat)/([\bar{x}] - p)$. The element $[\bar{x}^{1/q}]$ maps to $p^{1/q}$, so

$$Q/(p^{1/q}) \cong Q^\flat/(\bar{x}^{1/q}), \qquad Q/p \cong Q^\flat/(\bar{x}). \tag{10.22}$$

Under (10.22), (10.21) is induced by the nth iterate of Frobenius on the perfect ring $Q^\flat$. □

Theorem 10.21. *Let $0 \neq J \subseteq A$ be an ideal, and put $Q = (P/JP)_{\mathrm{perfd}}$. For every discrete Q-module M annihilated elementwise by powers of $\mathfrak{m}_A$, one has*

$$\kappa_A(\mathrm{Res}^Q_P M) = 0.$$

Restriction of scalars identifies these modules with the objects $M \in \mathrm{Ind}(\mathrm{Coh}^\heartsuit_{\mathfrak{m}_A}(P))$ satisfying $(JP)_{\mathrm{perfd}}(P)M = 0$.

Proof. First reduce to the case $J = (g)$: choose $0 \neq g \in J$ and restrict scalars along

$$(P/gP)_{\mathrm{perfd}} \twoheadrightarrow (P/JP)_{\mathrm{perfd}}.$$

The map exists by the universal property and is surjective by Proposition 10.8. We now write $Q = (P/gP)_{\mathrm{perfd}}$.

It suffices to prove

$$\kappa_A(Q/\mathfrak{m}_A Q) = 0. \tag{10.23}$$

Indeed, the filtration by powers of $\mathfrak{m}_A$ then gives $\kappa_A(Q/\mathfrak{m}_A^a Q) = 0$ for every $a \geq 1$. Every finitely generated P-submodule of M is a quotient of $(Q/\mathfrak{m}_A^a Q)^{\oplus s}$ for some a, s, so preservation of directed unions gives $\kappa_A(M) = 0$.

To prove (10.23), we use the following colength estimate. Let $H = k[[z_1, \ldots, z_s]]$, where $s \geq 1$, and let $0 \neq h \in H$ have $(z_1, \ldots, z_s)$-adic order b. For $N \geq b$, multiplication by h gives an injection

$$H/(z_1, \ldots, z_s)^{N-b} \longrightarrow H/(z_1, \ldots, z_s)^N.$$

It follows that

$$\mathrm{length}_H\, H/((z_1, \ldots, z_s)^N, h) \leq \binom{N+s-1}{s} - \binom{N-b+s-1}{s}.$$

Since $(z_1, \ldots, z_s)^{sq} \subseteq (z_1^q, \ldots, z_s^q)$, this gives

$$\mathrm{length}_H\, H/(z_1^q, \ldots, z_s^q, h) = O(q^{s-1}) \qquad (q \longrightarrow \infty). \tag{10.24}$$

Put $q = p^n$. In equal characteristic, $g^{1/q} = 0$ in Q, since Q is reduced. The finite-stage formula (10.20) gives

$$0 \le \kappa_A(Q/\mathfrak{m}_A Q) \le q^{-r} \operatorname{length}_A(A/(t_1^q, \dots, t_r^q, g))e_A \longrightarrow 0 \tag{10.25}$$

by (10.24).

Suppose that A has mixed characteristic. For $g = p$, reducedness gives $p^{1/q} = 0$ in Q, and

$$0 \le \kappa_A(Q/\mathfrak{m}_A Q) \le \kappa_A\big(P/(p^{1/q}, t_2, \dots, t_r)P\big) = q^{-1}e_A \longrightarrow 0.$$

Next suppose $p \nmid g$. If $r = 1$, then g is a unit and $Q = 0$. For $r \ge 2$, choose $g_n \in A_n$ whose image modulo $p^{1/q}$ is the qth root of $\bar{g} \in A/p$. Then $g_n^q = g$ modulo pP. Lemma 10.20 gives

$$g_n Q \subseteq p^{1/q} Q.$$

The filtration of $Q/\mathfrak{m}_A Q$ by powers of $p^{1/q}$ has q successive quotients, each a quotient of $Q/(p^{1/q}, t_2, \dots, t_r)Q$. Consequently

$$\begin{aligned} \kappa_A(Q/\mathfrak{m}_A Q) &\le q\,\kappa_A(Q/(p^{1/q}, t_2, \dots, t_r)Q) \\ &\le q^{1-r} \operatorname{length}_{A_n}(A_n/(p^{1/q}, t_2, \dots, t_r, g_n))e_A \\ &= q^{1-r} \operatorname{length}_{A/p}((A/p)/(t_2^q, \dots, t_r^q, \bar{g}))e_A \longrightarrow 0 \end{aligned} \tag{10.26}$$

by (10.24) with $s = r - 1$.

For $g = p^b h$ with $b \ge 1$ and $p \nmid h$, Proposition 10.8 and (10.5) give

$$(p)_{\mathrm{perfd}}(P)\,(h)_{\mathrm{perfd}}(P) \subseteq (ph)_{\mathrm{perfd}}(P) = (g)_{\mathrm{perfd}}(P).$$

Thus in the short exact sequence

$$0 \longrightarrow (h)_{\mathrm{perfd}}(P)M \longrightarrow M \longrightarrow M/(h)_{\mathrm{perfd}}(P)M \longrightarrow 0,$$

the first nonzero term is a discrete $(P/pP)_{\mathrm{perfd}}$-module and the last nonzero term is a discrete $(P/hP)_{\mathrm{perfd}}$-module. The cases $g = p$ and $g = h$, together with the reduction to (10.23), give value zero for both, so additivity proves the theorem. □

Proposition 10.22. *Choose $0 \ne g \in A$ such that $B[1/g]$ is finite étale over $A[1/g]$. The map $P \otimes_A B \to D$ is injective, and for some $b \ge 1$,*

$$g^b(D/(P \otimes_A B)) = 0.$$

Proof. The algebra $P \otimes_A B$ is discrete by flatness of P over A and finitely generated as a P-module because B is finite over A. Its localization at g is finite étale over $P[1/g]$. Proposition 10.15, with $J = (g)$, applies and shows that a power of g annihilates both the kernel and the cokernel of $P \otimes_A B \to D$. The module B is g-torsionfree, so flatness makes $P \otimes_A B$ g-torsionfree; hence the kernel is zero. □

11 Normalized length under finite base change

In mixed characteristic, the case $i = 0$ and $V = A/(y_1, \dots, y_r)$ of Theorem 11.1, where $y_1, \dots, y_r$ is a system of parameters of A, follows from [3, Propositions 4.0.14 and 4.0.16]. The proof of Theorem 11.1 retains the cancellation argument of [3, Propositions 4.0.14 and 4.0.16], expressed here in K_0, and then uses composition series to prove

$$\kappa_A\big(\pi_i(D \otimes_A V)\big) = \begin{cases} d \operatorname{length}_A(V)[P/\mathfrak{m}_A P], & i = 0, \\ 0, & i \ne 0, \end{cases}$$

for every finite-length discrete A-module V.

Theorem 11.1 (Finite base change; mixed characteristic: [3, Theorem 4.0.3(a) and Propositions 4.0.10, 4.0.16])**.** *Let $A \subseteq B$ be a finite generically separable inclusion of complete noetherian local domains, and put $d = [\mathrm{Frac}(B) : \mathrm{Frac}(A)]$. Set $D = (P \otimes_A B)_{\mathrm{perfd}}$. For every finite-length discrete A-module V one has*

$$\kappa_A\big(\pi_i(D \otimes_A V)\big) = \begin{cases} d\,\mathrm{length}_A(V)[P/\mathfrak{m}_A P], & i = 0, \\ 0, & i \neq 0. \end{cases} \tag{11.1}$$

In particular, for positive integers $a_1, \dots, a_r$,

$$\kappa_A\big(D/(t_1^{a_1}, \dots, t_r^{a_r})D\big) = d\,a_1 \cdots a_r [P/\mathfrak{m}_A P]. \tag{11.2}$$

For every $\mathfrak{m}_B$-primary ideal $I \subset B$, the value $\kappa_A(D/ID)$ is finite.

Proof. Choose $0 \neq g \in \mathfrak{m}_A$ such that $B[1/g]$ is finite étale over $A[1/g]$. Apply Corollary 10.14 with $R = P$, $S = P \otimes_A B$, $J = (g)$, and $M = P \otimes_A k$. Flatness of P over A makes M discrete, and $pM = 0$ in both characteristics. Thus

$$(g)_{\mathrm{perfd}}(P)\pi_i(D \otimes_A k) = 0 \qquad (i > 0).$$

Theorem 10.21 gives $\kappa_A(\pi_i(D \otimes_A k)) = 0$ for $i > 0$. A composition series of V and the long exact sequences of homotopy groups then give

$$\kappa_A\big(\pi_i(D \otimes_A V)\big) = 0 \qquad (i > 0) \tag{11.3}$$

for every finite-length V. For every short exact sequence $0 \to V' \to V \to V'' \to 0$ of finite-length discrete A-modules, the long exact sequence of homotopy groups of $D \otimes_A -$ gives

$$\kappa_A\big(\pi_0(D \otimes_A V)\big) = \kappa_A\big(\pi_0(D \otimes_A V')\big) + \kappa_A\big(\pi_0(D \otimes_A V'')\big). \tag{11.4}$$

Let $S = P \otimes_A B$ and $C = D/S$. By Proposition 10.22, $g^b C = 0$ for some $b \geq 1$. Since $0 \neq g^b \in \mathfrak{m}_A$, the ring $A/(g^b)$ has dimension $r - 1$. Lift a system of parameters of $A/(g^b)$ to $x_2, \dots, x_r \in \mathfrak{m}_A$. Then $x_2, \dots, x_r, g^b$ is a system of parameters of A, hence a regular sequence since A is Cohen–Macaulay. Put

$$\mathfrak{q} = (x_2, \dots, x_r, g^b), \qquad E = C \otimes_A A/(x_2, \dots, x_r). \tag{11.5}$$

Multiplication by g^b is injective on $A/(x_2, \dots, x_r)$ and is zero on C. Thus in Mod_P one has

$$C \otimes_A A/\mathfrak{q} \simeq \mathrm{cofib}(g^b : E \to E) \simeq E \oplus E[1]. \tag{11.6}$$

For every i, both $\pi_i E$ and $\pi_i(C \otimes_A A/\mathfrak{q})$ are annihilated by $\mathfrak{q}$ and therefore by a power of $\mathfrak{m}_A$.

All homotopy groups of $S \otimes_A A/\mathfrak{q}$ have finite κ_A-value: they are the flat scalar extensions to P of finite-length A-modules. By (11.3), one has $\kappa_A(\pi_i(D \otimes_A A/\mathfrak{q})) = 0$ for $i > 0$. The cofiber sequence obtained by tensoring $S \to D \to C$ with $A/\mathfrak{q}$ therefore makes $\kappa_A(\pi_i(C \otimes_A A/\mathfrak{q}))$ finite for every $i \geq 1$. The splitting (11.6) makes every $\kappa_A(\pi_i E)$ finite, including $i = 0$, which occurs as a summand of $\pi_1(C \otimes_A A/\mathfrak{q})$. Thus the degree-zero values for $C \otimes_A A/\mathfrak{q}$ and $D \otimes_A A/\mathfrak{q}$ are finite as well. All these modules have only finitely many nonzero homotopy groups.

Thus $S \otimes_A A/\mathfrak{q}$, $D \otimes_A A/\mathfrak{q}$, $C \otimes_A A/\mathfrak{q}$, and E belong to $\mathrm{Mod}_{\mathfrak{m}_A}^{\sum \kappa_A < \infty}(P)$. For $F \in \mathrm{Mod}_{\mathfrak{m}_A}^{\sum \kappa_A < \infty}(P)$, recall (10.16):

$$\kappa_A(F) = \sum_{i \in \mathbb{Z}} (-1)^i \kappa_A(\pi_i F) \in L_A = K_0(\mathrm{Coh}^{\heartsuit}_{\mathfrak{m}_A}(P)) \otimes_{\mathbb{Z}} \mathbb{R}.$$

The splitting (11.6) gives

$$\kappa_A(C \otimes_A A/\mathfrak{q}) = \kappa_A(E) + \kappa_A(E[1]) = 0.$$

In $K_0(\mathrm{Perf}(A))$ one has $[B] = d[A]$, since A is regular local and this Grothendieck group is given by rank. The exact functor $U \mapsto P \otimes_A U \otimes_A A/\mathfrak{q}$ gives a homomorphism

$$K_0(\mathrm{Perf}(A)) \longrightarrow L_A, \qquad [U] \longmapsto \kappa_A(P \otimes_A U \otimes_A A/\mathfrak{q}).$$

Using the cofiber sequence $S \to D \to C$ and (11.3) gives

$$\begin{aligned} \kappa_A(D/\mathfrak{q}D) &= \kappa_A(D \otimes_A A/\mathfrak{q}) \\ &= \kappa_A(S \otimes_A A/\mathfrak{q}) \\ &= d\,\kappa_A(P/\mathfrak{q}P) \\ &= d\,\mathrm{length}_A(A/\mathfrak{q})[P/\mathfrak{m}_A P]. \end{aligned}$$

The third equality applies the homomorphism to $[B] = d[A]$. By (11.4), a composition series of $A/\mathfrak{q}$ gives

$$\kappa_A(D/\mathfrak{q}D) = \mathrm{length}_A(A/\mathfrak{q})\,\kappa_A(D/\mathfrak{m}_A D).$$

Cancelling the positive integer $\mathrm{length}_A(A/\mathfrak{q})$ gives $\kappa_A(D/\mathfrak{m}_A D) = d[P/\mathfrak{m}_A P]$. A composition series of an arbitrary V proves the degree-zero assertion in (11.1); negative homotopy groups vanish by connectivity.

Equation (11.2) follows from (11.1) by taking $V = A/(t_1^{a_1}, \dots, t_r^{a_r})$. Finally, if I is $\mathfrak{m}_B$-primary, then $\mathfrak{m}_A^a B \subseteq I$ for some a. The surjection $D/\mathfrak{m}_A^a D \to D/ID$ and positivity yield

$$\kappa_A(D/ID) \le d\,\mathrm{length}_A(A/\mathfrak{m}_A^a)[P/\mathfrak{m}_A P] < +\infty.$$

□

12 Normalized length on P-modules and D-modules

12.1 Abelian and stable module categories

Apply Proposition 8.3 with $R = P$ or D, $I = \mathfrak{m}_A R$, and $\kappa = \kappa_A$. For P, the required finiteness follows from (10.20). For D, Theorem 11.1 gives

$$\kappa_A(D/\mathfrak{m}_A^a D) = d\,\mathrm{length}_A(A/\mathfrak{m}_A^a)e_A < +\infty.$$

For P, one has $\mathcal{B}_P^c = \mathrm{Coh}^{\heartsuit}_{\mathfrak{m}_A}(P)$.

For $R = P$ or D, write $\mathrm{Mod}_{\mathfrak{m}_A}(R) \subseteq \mathrm{Mod}_R$ for the modules whose homotopy groups are annihilated elementwise by powers of $\mathfrak{m}_A$. Its objects are precisely the E with $E[1/t_j] = 0$ for $1 \le j \le r$. The compact R-module $K = \bigotimes_{j=1}^r \mathrm{cofib}(R \xrightarrow{t_j} R)$ generates this subcategory. Indeed, $\mathrm{Hom}_R(K, E) = 0$ gives $K \otimes_R E = 0$ by self-duality up to shift. Removing the factors in reverse order gives $E = 0$, since t_j can act invertibly on a module V with $V[1/t_j] = 0$ only if $V = 0$. Thus $\mathrm{Mod}_{\mathfrak{m}_A}(R) \simeq \mathrm{Ind}(\mathrm{Mod}_{\mathfrak{m}_A}(R)^c)$. Truncations and filtered colimits are computed in Mod_R, and the heart is $\mathcal{B}_R$. The category from Section 10 is

$$\mathrm{Mod}_{\mathfrak{m}_A}^{\sum \kappa_A < \infty}(R) = \left\{ E \in \mathrm{Mod}_{\mathfrak{m}_A}(R) : \sum_{i \in \mathbb{Z}} \kappa_A(\pi_i E) < +\infty \right\}. \tag{12.1}$$

Write $\mu_A(E) = \sum_i \kappa_A(\pi_i E)$ and $\mathrm{Mod}_{\mathfrak{m}_A}^{\kappa_A = 0}(R) = \{E : \kappa_A(\pi_i E) = 0 \text{ for every } i\}$. By Theorem 7.5(3), the stable quotient

$$\mathrm{Mod}_{\mathfrak{m}_A}^{\sum \kappa_A < \infty}(R) / \mathrm{Mod}_{\mathfrak{m}_A}^{\kappa_A = 0}(R) \tag{12.2}$$

has heart $(\mathcal{B}_R)_{<\infty}/(\mathcal{B}_R)_0$. Restriction of scalars from D to P preserves μ_A, κ_A, and the corresponding errors, and induces a t-exact functor on these stable quotients.

Remark 12.1. For $D = P$, the finite partial sums of $\bigoplus_{j\geq 1}(P \otimes_{A_j} k)[j]$ form a Cauchy diagram whose colimit has nonzero value in every degree $j \geq 1$. For $A = k[[t_1, t_2]]$ and $D = P$, put $Q_j = P/(t_1^j, t_2^{1/p^j})$. Then $\kappa_A(Q_j) = jp^{-j}e_A$, so the finite partial sums form a convergent cocone, but

$$\kappa_A(t_1^a Q_j) = (j-a)p^{-j}e_A > 0 \qquad (j > a).$$

Their colimit is therefore not isomorphic in the abelian quotient to a module killed by any fixed power of $\mathfrak{m}_A$.

12.2 Scalar extension

Proposition 12.2. *Scalar extension defines an exact functor*

$$D \otimes_P - : \mathrm{Mod}_{\mathfrak{m}_A}^{\Sigma\kappa_A<\infty}(P) \longrightarrow \mathrm{Mod}_{\mathfrak{m}_A}^{\Sigma\kappa_A<\infty}(D).$$

For every object E of its domain and every integer i,

$$\kappa_A(\pi_i(D \otimes_P E)) = d\,\kappa_A(\pi_i E). \tag{12.3}$$

Consequently $\mu_A(D \otimes_P E) = d\,\mu_A(E)$, and scalar extension induces a t-exact functor between the stable quotients in (12.2).

Proof. Choose $0 \neq g \in A$ such that $B[1/g]$ is finite étale over $A[1/g]$. Let N be a discrete P-module annihilated elementwise by powers of $\mathfrak{m}_A$. Since $p \in \mathfrak{m}_A$, Corollary 10.14, with $R = P$, $S = P \otimes_A B$, and $J = (g)$, gives $(g)_{\mathrm{perfd}}(P)\pi_j(D \otimes_P N) = 0$ for $j > 0$. Write N as a filtered colimit of objects of $\mathrm{Coh}^{\heartsuit}_{\mathfrak{m}_A}(P)$. Each is annihilated by a power of $\mathfrak{m}_A$ and has a finite free resolution of length at most r by Lemma 10.17. Thus the homotopy groups of $D \otimes_P N$ are elementwise $\mathfrak{m}_A$-power torsion and vanish outside $[0, r]$. Theorem 10.21 gives

$$\kappa_A(\pi_j(D \otimes_P N)) = 0 \qquad (j > 0). \tag{12.4}$$

For $N \in \mathrm{Coh}^{\heartsuit}_{\mathfrak{m}_A}(P)$, the value $\kappa_A(\pi_0(D \otimes_P N))$ is finite, since this module is a quotient of a finite direct sum of $D/\mathfrak{m}_A^a D$. Equation (12.4) makes this assignment additive. Theorem 8.2 and $\kappa_A(D/\mathfrak{m}_A D) = de_A$ identify it with $d\,\nu_A$.

For arbitrary discrete N of finite value, write $\lambda(N) = \kappa_A(\pi_0(D \otimes_P N))$. For finitely generated N, taking the infimum over finitely presented surjections gives $\lambda(N) \leq d\kappa_A(N)$. For arbitrary N, take the directed union of finitely generated submodules; their images after scalar extension exhaust $\pi_0(D \otimes_P N)$ and give $\lambda(N) \leq d\kappa_A(N)$. For finitely generated N, choose $Q \twoheadrightarrow N$ with $Q \in \mathrm{Coh}^{\heartsuit}_{\mathfrak{m}_A}(P)$ and kernel K. Equation (12.4) and additivity give

$$d\kappa_A(Q) = \lambda(K) + \lambda(N) \leq d\kappa_A(K) + \lambda(N),$$

which proves $\lambda(N) \geq d\kappa_A(N)$. For a finitely generated $N' \subseteq N$, the kernel of $\pi_0(D \otimes_P N') \to \pi_0(D \otimes_P N)$ has value zero, by applying (12.4) to N/N'. Taking the supremum over N' proves $\lambda(N) = d\kappa_A(N)$.

For bounded E, its finite Postnikov filtration gives (12.3) for every i: all contributions other than $\pi_0(D \otimes_P \pi_i E)$ have value zero, and differentials to or from the other contributions do not change the value. The finite Postnikov filtration also shows that scalar extension takes a torsion object with homotopy groups in $[b, c]$ to one with homotopy groups in $[b, c + r]$. For E with $\pi_j E = 0$ for $j > c$, passing to the filtered colimit of $\tau_{\geq -n}E$ gives $\pi_j(D \otimes_P E) = 0$ for $j > c + r$. If $\pi_j E = 0$ for $j < b$, tensor connectivity gives $\pi_j(D \otimes_P E) = 0$ for $j < b$. Thus $\pi_i(D \otimes_P E)$ is computed from the finite truncation interval $[i - r, i]$ of E. For arbitrary E, applying the argument for bounded objects to $\tau_{\geq i-r}\tau_{\leq i}E$ proves (12.3). Summing (12.3) over i gives $\mu_A(D \otimes_P E) = d\mu_A(E)$; exactness and (12.3) prove the remaining claims. □

12.3 Continuity of endofunctors on module quotients

Lemma 12.3. *Let S be a discrete commutative P-algebra which is finitely presented as a P-module and annihilated by a power of $\mathfrak{m}_A$. Use κ_A on discrete S-modules by restriction to P. Then $\mathrm{Coh}(S)^{\heartsuit}$ is abelian and κ_A is finite on it. Let*

$$q_S : (\mathrm{Mod}_S^{\heartsuit})_{\kappa_A<\infty} \longrightarrow (\mathrm{Mod}_S^{\heartsuit})_{\kappa_A<\infty}/(\mathrm{Mod}_S^{\heartsuit})_{\kappa_A=0} \tag{12.5}$$

be the Serre quotient functor. For every U in this source, there is a sequential diagram (Q_n) in $\mathrm{Coh}(S)^{\heartsuit}$ and a compatible family $u_n : Q_n \to U$ satisfying

$$\delta_{\kappa_A}(u_n) \longrightarrow 0, \qquad q_S U \cong \mathrm{colim}_n\, q_S Q_n. \tag{12.6}$$

The colimit in (12.6) *is taken in the quotient category in* (12.5).

Proof. Restriction to P takes finitely presented discrete S-modules to $\mathrm{Coh}^{\heartsuit}_{\mathfrak{m}_A}(P)$, so κ_A is finite on them. For a map f between two such modules, Lemma 10.17 gives $\ker f \in \mathrm{Coh}^{\heartsuit}_{\mathfrak{m}_A}(P)$. Choose a surjection $S^{\oplus a} \twoheadrightarrow \ker f$ with a finite. The kernel of $S^{\oplus a} \twoheadrightarrow \ker f$ also belongs to $\mathrm{Coh}^{\heartsuit}_{\mathfrak{m}_A}(P)$, hence is finitely generated over S. Thus $\ker f$ is finitely presented over S, proving that $\mathrm{Coh}(S)^{\heartsuit}$ is abelian.

Restriction to P gives a continuous premeasure on $\mathrm{Mod}_S^{\heartsuit}$. Now $\mathrm{Mod}_S^{\heartsuit} \simeq \mathrm{Ind}(\mathrm{Coh}(S)^{\heartsuit})$. For each $U \in (\mathrm{Mod}_S^{\heartsuit})_{\kappa_A<\infty}$, the construction in the proof of Theorem 5.4 with $\mathcal{B} = \mathrm{Mod}_S^{\heartsuit}$ and $M = U$ gives a cocone $u_n : Q_n \to U$ satisfying (12.6). □

For every $\mathfrak{m}_B$-primary ideal I, put $R = P \otimes_A (B/I)$. Lemma 12.3 verifies the hypotheses of Corollary 9.3 for R and κ_A. Thus every compact continuous additive R-linear endofunctor Φ of $\mathrm{Mod}_R^{\heartsuit}$ restricts to a sequentially continuous endofunctor of $(\mathrm{Mod}_R^{\heartsuit})_{\kappa_A<\infty}$, in the sense of Definition 6.1.

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