Primer on Perfectoid Mathematics

These AI-generated notes (by Astra) develop perfectoid algebra and almost perfectoid geometry, including arc descent, excision, almost purity, and 0-affineness of perfectoidizations. They were produced while studying the background needed for the Serre multiplicity proof. The note is purely expository. No originality is claimed. The exposition draws on Bhatt's lecture notes [3, 4, 17] and work of Bhatt–Lurie [2], Bhatt–Mathew [6], Bhatt–Morrow–Scholze [8, 9], Bhatt–Scholze [1], Česnavičius–Scholze [7], and Mathew [14]. For further background on almost algebra and perfectoid geometry, we refer to Faltings [15], Gabber–Ramero [19], Kedlaya–Liu [20], and Scholze [21].

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Conventions

Fix a prime p. Categories are $(\infty,1)$-categories. Write Spaces for the category of spaces. For $X, Y \in \mathcal{C}$, the space of morphisms from X to Y is denoted by $\mathrm{Map}_{\mathcal{C}}(X,Y) \in \mathrm{Spaces}$. A module over a ring A is an object of the stable category Mod_A of modules in spectra. For $M, N \in \mathrm{Mod}_A$, the mapping spectrum is denoted by $\mathrm{Hom}_A(M,N)$.

Derived commutative rings are algebras for the derived Sym monad on $\mathrm{Mod}_{\mathbf{Z}}$; connective derived commutative rings are also called animated rings. We write DAlg_H for derived commutative H-algebras.

A ring A is *discrete* if $\pi_i A = 0$ for every $i \neq 0$. For a discrete ring A and an ideal $J \subset A$, the notation A/J denotes the discrete quotient ring.

Complete modules. Write $\mathrm{Sp}_p^\wedge$ for the category of derived p-complete spectra. For $A \in \mathrm{CAlg}(\mathrm{Sp})$ and a finitely generated ideal $K \subset \pi_0 A$, let $\mathrm{Mod}_A^{\wedge K}$ denote the symmetric monoidal localization of Mod_A whose kernel is generated, as a localizing subcategory, by the $A[1/f]$ for $f \in K$. Write

$$\Gamma_A : \mathrm{Mod}_A \rightleftarrows \mathrm{Mod}_A^{\wedge K} : \Lambda_A, \qquad \widehat{M}_K = \Lambda_A \Gamma_A M.$$

Here Γ_A is the localization and Λ_A its fully faithful right adjoint. The ideal of definition is (p) unless it is specified otherwise. The same localization on spectra at (p) is written $\Gamma : \mathrm{Sp} \to \mathrm{Sp}_p^\wedge$. For $A \in \mathrm{CAlg}(\mathrm{Sp}_p^\wedge)$ one has $\mathrm{Mod}_A^{\wedge p} \cong \mathrm{Mod}_A(\mathrm{Sp}_p^\wedge)$. The notation $M \widehat{\otimes}_A N$ always means the tensor product of complete modules for the indicated ideal of definition. Its realization in Mod_A is

$$\Lambda_A \widehat{M \otimes_A} \Lambda_A N_K.$$

K-complete flatness. Let A be a discrete ring with a finitely generated ideal of definition $K = (f_1, \ldots, f_r)$. Put

$$E_K = \bigotimes_{i=1}^{r} \mathrm{fib}(A \to A[1/f_i]) \in \mathrm{Mod}_A\,.$$

An A-module is K-torsion if each element of each homotopy group is annihilated by a power of K. The functor $M \mapsto E_K \otimes_A \Lambda_A M$ identifies $\mathrm{Mod}_A^{\wedge K}$ with the full subcategory of K-torsion A-modules; its inverse is the restriction of the localization functor. Indeed, $E_K \otimes_A A[1/f_i] = 0$, the map $E_K \to A$ is an isomorphism in $\mathrm{Mod}_A^{\wedge K}$, and $E_K \otimes_A N \cong N$ for every K-torsion A-module N. The *torsion t-structure* on $\mathrm{Mod}_A^{\wedge K}$ is transported from the standard t-structure on these K-torsion modules. An object $M \in \mathrm{Mod}_A^{\wedge K}$ is *K-completely flat* if

$$M \widehat{\otimes}_A - : \mathrm{Mod}_A^{\wedge K} \longrightarrow \mathrm{Mod}_A^{\wedge K}$$

is t-exact for the torsion t-structure. It is *K-completely faithfully flat* if this functor is also conservative.

1 Witt vectors and perfectoid rings

1.1 Witt vectors are left adjoint to tilting

For a discrete ring A, define its tilt by

$$A^\flat = \lim_{F} (A/p).$$

The inverse limit is in discrete $\mathbf{F}_p$-algebras, so $A^\flat$ is perfect. Here W denotes the p-typical Witt functor.

Proposition 1.1 (Witt vectors are left adjoint to tilting)**.** *For a perfect discrete $\mathbf{F}_p$-algebra B and an ordinary p-adically complete ring A, there is a natural isomorphism*

$$\mathrm{Map}_{\mathrm{CAlg}}(W(B), A) = \mathrm{Map}_{\mathrm{CAlg}_{\mathbf{F}_p}}(B, A^\flat).$$

There is also an isomorphism of multiplicative monoids

$$A^\flat = \lim_{x \mapsto x^p} A.$$

Projection to the zeroth coordinate defines $b \mapsto b^\sharp$. The counit of the adjunction is the Fontaine map

$$\theta_A : W(A^\flat) \longrightarrow A, \qquad \theta_A([b]) = b^\sharp.$$

Proof. For a compatible sequence $(\bar{b}_n)$ in A/p, choose lifts $\widetilde{b}_n \in A$. The limits $b_n = \lim_{j\to\infty} \widetilde{b}_{n+j}^{p^j}$ exist and satisfy $b_{n+1}^p = b_n$. The binomial formula shows that changing a lift modulo p changes its p^jth power by a multiple of p^{j+1}. The construction of (b_n) is independent of the chosen lifts $(\widetilde{b}_n)$. The sequence (b_n) is the unique lift of $(\bar{b}_n)$ to $\lim_{x\mapsto x^p} A$.

The Witt addition and multiplication identities, applied modulo each p^m, show that the formula

$$\theta_A : W(A^\flat) \longrightarrow A, \qquad \theta_A\left(\sum_{n\geq 0} p^n [x_n]\right) = \sum_{n\geq 0} p^n x_n^\sharp \qquad (x_n \in A^\flat)$$

defines a ring map. Every $w \in W(B)$ has a unique p-adically convergent Teichmüller expansion $w = \sum_{n\geq 0} p^n[c_n]$, with $c_n \in B$. Its standard Witt coordinates are $(c_0, c_1^p, c_2^{p^2}, \ldots)$. Given a ring map $f : B \to A^\flat$, the associated ring map is

$$\theta_A \circ W(f) : W(B) \longrightarrow A, \qquad \theta_A(W(f)(w)) = \sum_{n\geq 0} p^n \big(f(c_n)\big)^\sharp.$$

Conversely, a ring map $W(B) \to A$ carries the compatible roots of each $[b]$ to compatible p^nth roots in A. The two constructions are inverse. □

1.2 Perfectoid rings and their Witt presentations

Definition 1.2. A p-adically complete discrete commutative ring R is *perfectoid* if a unit multiple of p is a pth power in R and the Fontaine map

$$\theta_R : W(R^\flat) \longrightarrow R$$

is surjective with principal kernel [7, §2.1.2].

Lemma 1.3 (Quotients by distinguished Witt vectors are perfectoid)**.** *Let B be a perfect discrete $\mathbf{F}_p$-algebra, let $a \in B$ such that B is a-adically complete, and let $u \in W(B)^\times$. Put $\xi = [a] + pu$ and $R = W(B)/(\xi)$, and let $\varpi \in R$ be the image of $[a^{1/p}] \in W(B)$.*

(i) *Multiplication by ξ on $W(B)$ is injective, and $R[p^\infty] = R[p]$.*

(ii) *The induced map $B \to R^\flat$ is an isomorphism identifying the quotient map $W(B) \to R$ with θ_R. The ring R is perfectoid.*

(iii) *Frobenius induces an isomorphism*

$$R/(\varpi) \xrightarrow{x\mapsto x^p} R/p.$$

Proof. (i). The ring $W(B)$ is p-torsionfree and every element has a unique expansion $\sum_{n\geq 0} p^n[b_n]$. If $\xi x \in p^2W(B)$, write $x = [x_0] + py$. Reduction modulo p gives $ax_0 = 0$, hence $[a][x_0] = 0$. Dividing by p and reducing once more gives $ay_0 + u_0x_0 = 0$, where $y_0, u_0 \in B$ are the reductions of y, u modulo p. Multiplication by x_0 gives $u_0x_0^2 = 0$. Since u_0 is a unit and Frobenius on B is injective, $x_0 = 0$. Thus

$$\xi x \in p^2W(B) \quad \Longrightarrow \quad x \in pW(B). \tag{1.1}$$

Iteration proves regularity of ξ. If $p^2b = \xi x$ in $W(B)$, (1.1) gives $x = py$ for some $y \in W(B)$. Cancelling p gives $pb = \xi y$. Hence $R[p^2] = R[p]$, and induction gives $R[p^\infty] = R[p]$.

(ii). By (i), the discrete $W(B)$-module R is the cofiber of multiplication by ξ on $W(B)$. It is therefore p-complete as a module; the bounded torsion in (i) makes it classically p-adically complete.

Since B is perfect and a-adically complete, $B \cong \lim_F B/(a)$. The identification $R/p \cong B/(a)$ therefore gives $B \cong R^\flat$. Under this isomorphism, the adjunction of proposition 1.1 identifies the quotient map $W(B) \to R$ with θ_R. The identity $\varpi^p = -p\theta_R(u)$ exhibits ϖ^p as a unit multiple of p. Thus R satisfies definition 1.2.

(iii). The Frobenius map identifies with the isomorphism

$$R/(\varpi) \cong B/(a^{1/p}) \xrightarrow{x\mapsto x^p} B/(a) \cong R/p.$$

□

Lemma 1.4 (Generators of the Fontaine kernel)**.** *Let R be a perfectoid ring, and let $\xi \in W(R^\flat)$ generate $\ker\theta_R$. Write $\xi = [a] + pu$, with $a \in R^\flat$ and $u \in W(R^\flat)$. Then u is a unit and $R^\flat$ is a-adically complete.*

Proof. Reduction modulo p identifies $R/p \cong R^\flat/(a)$. Since $R^\flat$ is perfect,

$$R^\flat = \lim_F R/p \cong \lim_F R^\flat/(a) \cong \lim_n R^\flat/(a^{p^n}).$$

Thus $R^\flat$ is a-adically complete. In particular a lies in its Jacobson radical, so ξ lies in the Jacobson radical of $W(R^\flat)$.

Choose $\rho \in R$ and $\varepsilon \in R^\times$ with $\rho^p = p\varepsilon$, and lift them to $x, w \in W(R^\flat)$, respectively, with w a unit. Write

$$x^p - pw = ([c] + pz)\xi, \qquad c \in R^\flat, \quad z \in W(R^\flat).$$

For a Witt vector t, let $\bar{t}$ denote its reduction modulo p. The binomial formula gives $x^p \equiv [\bar{x}^p] \pmod{p^2}$. Comparing the coefficients of p in the Teichmüller expansions gives

$$-\bar{w} = a\bar{z} + c\bar{u}.$$

Since a lies in the Jacobson radical and $\bar{w}$ is a unit, $c\bar{u}$ is a unit. Hence $u \in W(R^\flat)^\times$. □

Lemma 1.5 (Preservation of Fontaine generators)**.** *Let $f : R \to S$ be a map of perfectoid rings, and let $\xi = [a] + pu \in W(R^\flat)$ generate $\ker\theta_R$. Then $W(f^\flat)(\xi)$ generates $\ker\theta_S$, and $S^\flat$ is $f^\flat(a)$-adically complete.*

Proof. Choose a generator $\eta = [b] + pv$ of $\ker\theta_S$. By lemma 1.4, u and v are units and $S^\flat$ is b-adically complete. Write $W(f^\flat)(\xi) = ([c] + pz)\eta$, with $c \in S^\flat$ and $z \in W(S^\flat)$. Reduction modulo p and comparison of the coefficients of p give

$$f^\flat(a) = cb, \qquad f^\flat(\bar{u}) = b\bar{z} + c\bar{v}.$$

Here bars denote reduction modulo p. The left side of the second equality is a unit, and b lies in the Jacobson radical of $S^\flat$. Hence c and $[c] + pz$ are units. This proves the assertion about $\ker \theta_S$; the equality $f^\flat(a) = cb$ gives completeness. □

Lemma 1.6 (Tilting of perfectoid algebras)**.** *Let R be a perfectoid ring, and choose a generator $\xi = [a] + pu \in W(R^\flat)$ of $\ker \theta_R$. Tilting identifies perfectoid R-algebras with a-adically complete perfect $R^\flat$-algebras; the inverse sends C to $W(C)/(\xi)$.*

Proof. For a perfectoid R-algebra S, lemma 1.5 shows that $S^\flat$ is a-adically complete and that $W(S^\flat)/(\xi) \cong S$. Conversely, lemmas 1.3 and 1.4 show that $W(C)/(\xi)$ is a perfectoid R-algebra with tilt C for every a-adically complete perfect $R^\flat$-algebra C. The adjunction of proposition 1.1 identifies these constructions as inverse functors. □

Lemma 1.7 (Untilting a pullback along a surjection)**.** *Let $B_1 \twoheadrightarrow B_{12} \leftarrow B_2$ be a diagram of perfect $\mathbf{F}_p$-algebras. Choose $\xi = [a] + pu$ in $W(B_1 \times_{B_{12}} B_2)$ with u a unit. If B_1, B_2, B_{12} are complete for the images of (a), then $B_1 \times_{B_{12}} B_2$ is a-adically complete and the canonical map*

$$W(B_1 \times_{B_{12}} B_2)/(\xi) \longrightarrow W(B_1)/(\xi) \times_{W(B_{12})/(\xi)} W(B_2)/(\xi)$$

is an isomorphism. All four Witt quotients are perfectoid.

Proof. Surjectivity makes the pullback square of the given rings a derived pullback. All four rings are perfect, so their a-power torsion is killed by a. Thus ordinary and derived a-adic completeness agree for them, and a-adic completeness passes to the pullback. Witt vectors preserve the pullback and the surjection. Multiplication by ξ is injective on each Witt ring by lemma 1.3. Taking its cofiber gives the displayed pullback, whose terms are perfectoid by the same lemma. □

Corollary 1.8 (Pullbacks of perfectoid algebras over a perfectoid ring)**.** *Let $R_1 \twoheadrightarrow R_{12} \leftarrow R_2$ be a diagram of perfectoid algebras over a perfectoid ring P. The pullback $R_1 \times_{R_{12}} R_2$ is perfectoid with tilt $R_1^\flat \times_{R_{12}^\flat} R_2^\flat$.*

Proof. Choose $\xi = [a] + pu$ generating $\ker \theta_P$. The surjection $R_1/p \to R_{12}/p$ identifies with $R_1^\flat/(a) \to R_{12}^\flat/(a)$. Successive lifting in powers of a and completeness give a surjection on tilts. Apply lemmas 1.6 and 1.7 with the image of ξ. □

Proposition 1.9 (Decomposition of a perfectoid ring by its p-torsion and its reduced special fiber)**.** *For a perfectoid ring R, the following square is cartesian:*

$$\begin{array}{ccc} R & \longrightarrow & R/R[p] \\ \downarrow & & \downarrow \\ (R/p)_{\mathrm{red}} & \longrightarrow & (R/(R[p] + pR))_{\mathrm{red}}. \end{array} \tag{1.2}$$

The ring $R/R[p]$ is p-torsionfree and perfectoid. The two rings in the bottom row are perfect $\mathbf{F}_p$-algebras.

Proof. Choose a generator $\xi = [a] + pu$ of $\ker \theta_R$ and set $I = \ker(a : R^\flat \to R^\flat)$ and $J = \sqrt{(a)} = (a^{1/p^n} \mid n \geq 0)$. For $b \in I$ the element ba^{1/p^n} has zero p^nth power. Thus $IJ = 0$, and reducedness of $R^\flat$ gives $I \cap J = 0$. Consequently

$$R^\flat \cong (R^\flat/I) \times_{R^\flat/(I+J)} (R^\flat/J).$$

The ideals I and J are preserved by Frobenius and its inverse, so the three quotients are perfect. Multiplication by a identifies $R^\flat/I$ with the closed ideal $aR^\flat$ with its a-adic topology. The other quotients are killed by a.

Apply lemma 1.7. The two untilts in the bottom row are $R^\flat/J$ and $R^\flat/(I+J)$, since ξ maps to p times a unit. Set $S = W(R^\flat/I)/(\xi)$. Multiplication by a on $R^\flat/I$ is injective because $\ker(a^2 : R^\flat \to R^\flat) = \ker(a : R^\flat \to R^\flat)$. Thus $px = \xi y$ in $W(R^\flat/I)$ implies $y = pz$ by reduction modulo p, and then $x = \xi z$ by cancellation. This proves that S is p-torsionfree. The map $R \to S$ is surjective. Its kernel embeds in $R^\flat/J$ by the pullback, so the kernel is killed by p and equals $R[p]$. Finally $R/p \cong R^\flat/(a)$ and $S/p \cong R^\flat/(I+(a))$ identify their reduced quotients with $R^\flat/J$ and $R^\flat/(I+J)$, respectively. □

Corollary 1.10 (Perfectoid rings are reduced)**.** *Every perfectoid ring is reduced.*

Proof. By (1.2) it suffices to treat a p-torsionfree perfectoid ring S. Choose ϖ as in lemma 1.3. If $x^p = 0$, injectivity of $S/\varpi \to S/p$, $z \mapsto z^p$, gives $x = \varpi y$. Since ϖ^p is a unit multiple of p, we have $y^p = 0$. Iteration gives $x \in \bigcap_n \varpi^n S = 0$. □

See also [3, Proposition 3.2 and Corollary 3.3].

1.3 Completed tensor products preserve perfectoid rings

Proposition 1.11 (Completed tensor products over a perfectoid ring are perfectoid)**.** *If S and T are perfectoid algebras over a perfectoid ring R, then the pushout of $S \leftarrow R \to T$ in $\mathrm{CAlg}(\mathrm{Sp}^\wedge_p)$ is discrete and perfectoid. Its underlying R-module is $S \widehat{\otimes}_R T \in \mathrm{Mod}_R^{\wedge p}$.*

Proof. Choose a generator $\xi = [a] + pu \in W(R^\flat)$ of $\ker \theta_R$ and form $U = S^\flat \otimes_{R^\flat} T^\flat$ in derived commutative rings. The ring U is discrete and perfect. Indeed, Frobenius is invertible on U and acts by zero on π_i for $i > 0$ in every connective derived commutative $\mathbf{F}_p$-algebra. To check the latter assertion, represent an element by a map from $\mathrm{Sym}_{\mathbf{F}_p}(\mathbf{F}_p[i])$. In the polynomial-degree decomposition

$$\mathrm{Sym}_{\mathbf{F}_p}(\mathbf{F}_p[i]) = \bigoplus_{n \geq 0} \mathrm{Sym}^n_{\mathbf{F}_p}(\mathbf{F}_p[i]),$$

Frobenius takes the generator to the weight-p summand $\mathrm{Sym}^p_{\mathbf{F}_p}(\mathbf{F}_p[i])$ of $\mathrm{Sym}_{\mathbf{F}_p}(\mathbf{F}_p[i])$. This summand satisfies

$$\mathrm{Sym}^p_{\mathbf{F}_p}(\mathbf{F}_p[i]) \cong \left(\textstyle\bigwedge^p_{\mathbf{F}_p}(\mathbf{F}_p[i-1])\right)[p] \in (\mathrm{Mod}_{\mathbf{F}_p})_{\geq i+1}.$$

Reduction modulo p and uniqueness of the p-complete Witt lift give

$$W(S^\flat) \widehat{\otimes}_{W(R^\flat)} W(T^\flat) \cong W(U). \tag{1.3}$$

Set $C = \widehat{U}_{(a)}$. Put $M_n = \mathrm{cofib}(a^n : U \to U) \in \mathrm{Mod}_U$. Its only possibly nonzero homotopy groups are

$$\pi_0 M_n = U/a^n U, \qquad \pi_1 M_n = U[a].$$

The transition from $n+1$ to n induces the quotient map on π_0 and multiplication by a, which is zero, on π_1. Taking limits of the Postnikov fiber sequences gives

$$\lim_n (\pi_1 M_n)[1] \longrightarrow C = \lim_n M_n \longrightarrow \lim_n \pi_0 M_n.$$

The first term is zero because the transition maps are zero. The last term is discrete because the transition maps are surjective. Thus $C \cong \lim_n U/a^n U$ is discrete. Frobenius induces

an isomorphism on C because it is invertible on U and sends (a^n) to (a^{pn}). Since C is a-adically complete, $W(C)$ is $(p,[a])$-complete. The canonical map

$$\widehat{W(U)}_{(p,[a])} \longrightarrow W(C) \tag{1.4}$$

reduces modulo p to the isomorphism $\widehat{U}_{(a)} \cong C$. Both sides are p-complete, so (1.4) is an isomorphism. Completing (1.3) at $[a]$ therefore gives

$$W(S^\flat) \mathbin{\widehat{\otimes}_{W(R^\flat)}} W(T^\flat)_{(p,[a])} \cong W(C). \tag{1.5}$$

Tensoring (1.5) with $R \cong W(R^\flat)/(\xi)$ over $W(R^\flat)$ gives

$$S \mathbin{\widehat{\otimes}_R} T \cong W(C)/(\xi).$$

By lemma 1.3, $W(C)/(\xi)$ is discrete and perfectoid. □

Lemma 1.12 (p-completed localizations of perfectoid rings are perfectoid)**.** *For a perfectoid ring R and $f \in R$, the ring $\widehat{R[1/f]}_p$ is discrete and perfectoid.*

Proof. Choose a generator $\xi = [a] + pu$ of $\ker \theta_R$ and an element $b \in R^\flat$ whose image in R/p is $f \bmod p$. The p-completion of $W(R^\flat)[1/[b]]$ is $W(R^\flat[1/b])$. Apply (1.4) to $U = R^\flat[1/b]$ and tensor with $R \cong W(R^\flat)/(\xi)$ to obtain

$$\widehat{R[1/f]}_p \cong \widehat{R[1/b^\sharp]}_p \cong W\Big(\widehat{R^\flat[1/b]}_{(a)}\Big)/(\xi).$$

The first isomorphism uses $f \equiv b^\sharp \pmod p$. Lemma 1.3 makes the last ring perfectoid. □

Lemma 1.13 (Products of perfectoid rings are perfectoid)**.** *For every family $(R_i)_{i\in I}$ of perfectoid rings, the ring $\prod_{i\in I} R_i$ is perfectoid.*

Proof. Choose generators $\xi_i \in W(R_i^\flat)$ of $\ker \theta_{R_i}$. Since Witt vectors preserve products,

$$\prod_{i\in I} R_i \cong W\Big(\prod_{i\in I} R_i^\flat\Big)/((\xi_i)_{i\in I}).$$

Lemma 1.3 shows that this product is perfectoid. □

Lemma 1.14 (Completed filtered colimits of perfectoid rings are perfectoid)**.** *The colimit in $\mathrm{CAlg}(\mathrm{Sp}^\wedge_p)$ of a filtered diagram of perfectoid rings is discrete and perfectoid.*

Proof. Restricting to the cofinal undercategory of one index, we may regard the diagram (R_i) as a diagram of algebras over a fixed perfectoid ring R. Choose a generator $\xi = [a] + pu \in W(R^\flat)$ of $\ker \theta_R$ and put

$$C = \widehat{\operatorname*{colim}_i R_i^\flat}_{(a)}.$$

The filtered colimit of the tilts is perfect, and the completion argument in proposition 1.11 shows that C is discrete and perfect. The canonical map

$$\widehat{\operatorname*{colim}_i W(R_i^\flat)}_{(p,[a])} \longrightarrow W(C) \tag{1.6}$$

is an isomorphism: both sides are p-complete, and reduction modulo p identifies this map with $\widehat{\operatorname{colim}_i R_i^\flat}_{(a)} \cong C$.

Tensor (1.6) with R over $W(R^\flat)$. The image of $(p,[a])$ in R is (p), and lemma 1.5 identifies $W(R_i^\flat)\otimes_{W(R^\flat)} R \cong R_i$. Thus

$$\widehat{\operatorname*{colim}_i R_i}_p \cong W(C)/(\xi).$$

The last ring is discrete and perfectoid by lemma 1.3. □

For a perfectoid ring R, write $R\langle x_1,\dots,x_d\rangle$ for the p-completion of its ordinary polynomial algebra, and $R\langle x_i^{1/p^\infty} \mid i\in I\rangle$ for the p-completion of the ordinary algebra with generators $x_{i,n}$ and relations $x_{i,n+1}^p = x_{i,n}$. These completions are discrete.

Corollary 1.15 (Adjoining compatible p-power roots of polynomial variables preserves perfectoidness)**.** *For a perfectoid ring R and a set I, the ring $R\langle x_i^{1/p^\infty} \mid i\in I\rangle$ is perfectoid.*

Proof. Choose a generator $\xi = [a]+pu \in W(R^\flat)$ of $\ker\theta_R$. Applying lemma 1.3 to the a-completion of $R^\flat[x_i^{1/p^\infty} \mid i\in I]$ shows that $R\langle x_i^{1/p^\infty} \mid i\in I\rangle$ is perfectoid; its tilt is the a-completion of that perfect algebra. □

1.4 Perfect modules and finite étale algebras

For a commutative ring A, write $\mathrm{Perf}(A)\subset \mathrm{Mod}_A$ for the full subcategory of dualizable modules.

Lemma 1.16 (Lifting a perfect module from reduction modulo p)**.** *Let P be a discrete p-adically complete ring and let $M\in \mathrm{Mod}_P$ be derived p-complete. If $M\otimes_{\mathbf{Z}_p}\mathbf{F}_p$ is perfect over $P\otimes_{\mathbf{Z}_p}\mathbf{F}_p$, then M is perfect over P.*

Proof. Put $A = P\otimes_{\mathbf{Z}_p}\mathbf{F}_p$ and $N = M\otimes_{\mathbf{Z}_p}\mathbf{F}_p$. The perfect A-module N has Tor-amplitude in some interval $[r,s]$, meaning $\pi_i(N\otimes_A T)=0$ for every discrete A-module T and every $i\notin[r,s]$. First M is r-connective. Indeed, $M\otimes_{\mathbf{Z}_p}\mathbf{Z}/p^n$ is r-connective for every n by the filtration with successive quotients $M\otimes_{\mathbf{Z}_p}\mathbf{F}_p$. The transition maps are surjective on π_r since their fibers are r-connective. Taking the limit proves the assertion and gives a surjection $\pi_r M\to\pi_r(M\otimes_{\mathbf{Z}_p}\mathbf{F}_p)$.

For $r<s$, lift finitely many generators of the latter module and take the cofiber of the resulting map $P^{\oplus m}[r]\to M$. Its reduction is perfect of Tor-amplitude in $[r+1,s]$, since tensoring with any discrete P/p-module preserves surjectivity in degree r. Iteration reduces to $r=s$. A perfect module of Tor-amplitude $[r,r]$ over a connective ring is a shift of a finite projective module. Lift its idempotent matrix from P/p to P by successive lifting modulo p^n. For the resulting finite projective P-module E, lift the identification $E/pE\cong\pi_r(M\otimes_{\mathbf{Z}_p}\mathbf{F}_p)$ to a map $E\to\pi_r M$ by projectivity. This gives $E[r]\to M$ whose reduction is an isomorphism between shifts of finite projective modules. Its p-complete cofiber has zero reduction modulo p, hence vanishes. □

Consequently $\mathrm{Perf}(P)$ identifies with the dualizable objects of $\mathrm{Mod}_P^{\wedge p}$. A perfect P-module and its dual are p-complete. Conversely, reduction modulo p is symmetric monoidal, so the complete realization of a dualizable object satisfies lemma 1.16.

For a discrete ring A, write $\mathrm{F\acute{E}t}(A)$ for the category of finite étale A-algebras.

Lemma 1.17 (Reduction modulo an ideal of completeness preserves finite étale algebras)**.** *Let A be a discrete I-adically complete ring. Reduction induces an isomorphism of categories*

$$\mathrm{F\acute{E}t}(A)\longrightarrow \mathrm{F\acute{E}t}(A/I).$$

Proof. We first prove full faithfulness. For finite étale A-algebras B, C, formal étaleness lifts every A/I-algebra map $B/IB \to C/IC$ uniquely through the quotients C/I^nC. Since $C \cong \lim_n C/I^nC$, these lifts give a unique A-algebra map $B \to C$.

For essential surjectivity, start with a finite étale A/I-algebra B_1. Lifting an étale polynomial presentation across $A/I^{n+1} \to A/I^n$ and inverting its Jacobian determinant gives compatible étale algebras B_n over A/I^n. Each B_n is finite: a finite type algebra whose reduction modulo a nilpotent ideal is finite is finite. Set $B = \lim_n B_n$.

To prove that B is finite projective over A, choose an idempotent matrix $e_1 \in M_r(A/I)$ presenting the underlying module of B_1, and lift it compatibly to an idempotent $e \in M_r(A)$. Projectivity lifts the identification $e_1(A/I)^r \cong B_1$ to compatible module isomorphisms $(e \bmod I^n)(A/I^n)^r \cong B_n$. Hence $B \cong eA^r$ as A-modules. Since $B/IB \cong B_1$ is étale and I lies in the Jacobson radical of A, the finite projective A-algebra B is étale. □

Proposition 1.18 (Tilting identifies finite étale algebras over R and $R^\flat$)**.** *Let R be perfectoid. Tilting induces an isomorphism between the categories of finite étale R-algebras and finite étale $R^\flat$-algebras.*

Proof. Choose a generator $\xi \in W(R^\flat)$ of $\ker \theta_R$, and put $a = \xi \bmod p$. Lemma 1.17 gives

$$\text{FÉt}(R^\flat) \cong \text{FÉt}(R^\flat/(a)) \cong \text{FÉt}(R/p) \cong \text{FÉt}(R).$$

Write Φ for the composite. A finite étale $R^\flat$-algebra C is perfect by relative Frobenius and is a-adically complete by finite projectivity. Hence

$$\Phi(C)^\flat = \lim_F \Phi(C)/p \cong \lim_F C/(a) \cong C.$$

The last isomorphism follows from a-adic completeness and inverse Frobenius, as in lemma 1.3. To see that $D = \Phi(C)$ is perfectoid, put $E = W(C)/(\xi)$. The Fontaine map of D induces an R-algebra map $f : E \to D$ whose reduction modulo p is the isomorphism $C/(a) \cong D/p$. Since D is formally étale over R and E is p-adically complete, the inverse modulo p lifts uniquely to $g : D \to E$. Formal étaleness gives $fg = \mathrm{id}_D$. The endomorphism gf induces the identity on E/p, hence on $E^\flat = \lim_F E/p$. The ring E is perfectoid by lemma 1.3, so lemma 1.6 gives $gf = \mathrm{id}_E$. Thus $D \cong E$. The identifications $\Phi(C)^\flat \cong C$ are natural in C, so tilting is inverse to Φ. □

Proposition 1.19 (Comparison of quotients of a perfectoid ring and its tilt)**.** *Let R be a perfectoid ring, and let $\varpi \in R$ and $\varpi^\flat \in R^\flat$ satisfy $((\varpi^\flat)^\sharp) = (\varpi)$. If R is ϖ-adically complete and ϖ^p divides p, then $b \mapsto b^\sharp \bmod \varpi$ induces an isomorphism*

$$R^\flat/(\varpi^\flat) \cong R/(\varpi).$$

Proof. Choose a generator $\xi = [a] + pu$ of $\ker \theta_R$. Since $p \in (\varpi)$, the isomorphism $R^\flat/(a) \cong R/p$ identifies $R/(\varpi)$ with $R^\flat/(a, \varpi^\flat)$. We prove $a \in ((\varpi^\flat)^p)$. The element a belongs to the Jacobson radical of $R^\flat$ by a-adic completeness. The image of $\varpi^\flat$ in R/p generates the same ideal as $\varpi \bmod p$, which lies in the Jacobson radical by ϖ-adic completeness of R. Thus $\varpi^\flat$ also belongs to the Jacobson radical of $R^\flat$.

Since $((\varpi^\flat)^\sharp)^p$ divides p, surjectivity of θ_R gives $g, h \in W(R^\flat)$ with $p - [\varpi^\flat]^p g = \xi h$. For each maximal ideal $\mathfrak{m} \subset R^\flat$, the field $R^\flat/\mathfrak{m}$ is perfect. In $W(R^\flat/\mathfrak{m})$ the equality becomes $p = puh$, so cancellation of p shows that the image of h is a unit. Consequently $h \bmod p$ is a unit in $R^\flat$. Reducing $p - [\varpi^\flat]^p g = \xi h$ modulo p gives $-(\varpi^\flat)^p (g \bmod p) = a(h \bmod p)$, as claimed. Hence

$$R^\flat/(a, \varpi^\flat) = R^\flat/(\varpi^\flat).$$

□

2 Flatness and descent of perfectoid algebras

2.1 Detection and preservation of K-complete flatness

Lemma 2.1 (Tensoring with discrete modules supported on the closed fiber)**.** *Let A be a discrete ring, let $K \subset A$ be finitely generated, and let $M \in \mathrm{Mod}_A$. If $M \otimes_A A/K$ is discrete and flat over A/K, then $M \otimes_A -$ is an exact functor on discrete K-torsion A-modules.*

Proof. A discrete K-torsion module is the filtered union of its submodules annihilated by K^n. The filtration by powers of K of each such submodule has A/K-modules as its successive quotients. Tensoring these filtrations with M proves discreteness; tensoring a short exact sequence then proves exactness. □

Lemma 2.2 (Detection of K-complete flatness on the closed fiber)**.** *Let A be a discrete ring with a finitely generated ideal of definition K, and let $M \in \mathrm{Mod}_A^{\wedge K}$. Then M is K-completely flat if and only if $\Lambda_A M \otimes_A A/K$ is discrete and flat over A/K.*

Proof. Suppose the reduction is discrete and flat. Lemma 2.1 gives exactness on discrete K-torsion modules. Write $K = (f_1, \dots, f_r)$ and put

$$Q_n = \bigotimes_{j=1}^{r} \mathrm{cofib}(f_j^n : A \to A).$$

Each Q_n is bounded and connective, and its homotopy groups are K-torsion. Its finite Postnikov filtration shows that $\Lambda_A M \otimes_A Q_n$ is connective. Since $\Lambda_A M \cong \lim_n (\Lambda_A M \otimes_A Q_n)$ by derived K-adic completeness, $\Lambda_A M$ belongs to $(\mathrm{Mod}_A)_{\geq -1}$.

For a connective K-torsion module X, tensoring the fiber of $X \to \tau_{\leq n} X$ with $\Lambda_A M$ gives a module in degrees $\geq n$. The tensor product of $\Lambda_A M$ with $\tau_{\leq n} X$ is connective by its finite Postnikov filtration. It follows that $\Lambda_A M \otimes_A X$ is connective. A coconnective K-torsion module X is the filtered colimit of its bounded truncations $\tau_{\geq -n} X$. The same finite filtrations show that $\Lambda_A M \otimes_A X$ is coconnective. Conversely, restriction of this t-exactness to discrete A/K-modules proves flatness of the reduction. □

Lemma 2.3 (Detection of K-complete faithful flatness on the closed fiber)**.** *Let A be a discrete ring with a finitely generated ideal of definition K, and let $M \in \mathrm{Mod}_A^{\wedge K}$. Then M is K-completely faithfully flat if and only if $\Lambda_A M \otimes_A A/K$ is discrete and faithfully flat over A/K.*

Proof. By lemma 2.2, it remains to detect conservativity. Every nonzero discrete K-torsion module has a nonzero submodule annihilated by K. Exactness and faithful flatness on A/K-modules therefore detect nonzero discrete K-torsion modules, and t-exactness detects nonzero homotopy groups. Conversely, conservativity restricted to A/K-modules gives faithful flatness. □

Lemma 2.4 (Preservation of K-complete flatness under completed base change)**.** *Let $A \to A'$ be a map of discrete rings, with ideals of definition $K \subset A$ and $KA' \subset A'$, where K is finitely generated. If $M \in \mathrm{Mod}_A^{\wedge K}$ is K-completely flat, then $M \widehat{\otimes}_A A'$ is KA'-completely flat.*

Proof. Apply lemma 2.2 to

$$(M \widehat{\otimes}_A A') \otimes_{A'} A'/KA' \cong (\Lambda_A M \otimes_A A/K) \otimes_{A/K} A'/KA'.$$

□

Corollary 2.5 (Preservation of K-complete faithful flatness under completed base change)**.** *Under the hypotheses of lemma 2.4, if M is K-completely faithfully flat, then $M\widehat{\otimes}_A A'$ is KA'-completely faithfully flat.*

Proof. Use the same reduction formula and lemma 2.3. □

Lemma 2.6 (Preservation of K-complete flatness under filtered colimits)**.** *Let A be a discrete ring with a finitely generated ideal of definition K. A filtered colimit of K-completely flat objects in $\mathrm{Mod}_A^{\wedge K}$ is K-completely flat.*

Proof. Reduction modulo K commutes with these colimits. Apply lemma 2.2 and preservation of ordinary flatness under filtered colimits. □

Lemma 2.7 (Preservation of K-complete faithful flatness under filtered colimits of algebras)**.** *Let A be a discrete ring with a finitely generated ideal of definition K. A filtered colimit of K-completely faithfully flat commutative algebras in $\mathrm{Mod}_A^{\wedge K}$ is K-completely faithfully flat.*

Proof. Flatness follows from lemma 2.6. Over each residue field of A/K, the reductions form a filtered diagram of nonzero unital algebras. Their colimit is nonzero. Apply lemma 2.3. □

Lemma 2.8 (Discreteness of p-completely flat modules over rings with bounded p-power torsion)**.** *Let A be a discrete ring with bounded p-power torsion. If $M \in \mathrm{Mod}_A^{\wedge p}$ is p-completely flat, then $\Lambda_A M$ is discrete and*

$$\Lambda_A M \cong \lim_n (\Lambda_A M \otimes_A A/p^n).$$

Every term on the right is discrete and flat over A/p^n.

Proof. The finite filtration of an A/p^n-module by powers of p and lemma 2.2 prove the assertion about each term. The fiber of $\mathrm{cofib}(p^n : A \to A) \to A/p^n$ is $A[p^n][1]$. In this inverse system, the transition on $A[p^n]$ is multiplication by p. Bounded p-power torsion therefore makes the fiber system pro-zero. Tensoring with $\Lambda_A M$ and taking limits gives the displayed isomorphism. The transition maps between its discrete terms are surjective, so the limit is discrete. This is the completion argument of [9, Lemma 4.7]. □

Lemma 2.9 (Preservation of a regular element under (p,d)-completely flat base change)**.** *Let A be a discrete (p,d)-complete ring, where $d \in A$ is a nonzerodivisor and A/d has bounded p-power torsion. If $M \in \mathrm{Mod}_A^{\wedge(p,d)}$ is (p,d)-completely flat, then $\Lambda_A M$ is discrete and multiplication by d on it is injective.*

Proof. Since A/d is the cofiber of multiplication by d on A, $\Lambda_A M \otimes_A A/d$ is p-complete. It is p-completely flat by lemma 2.4, hence discrete by lemma 2.8. Tensor the short exact sequences

$$A/d \xrightarrow{d^{n-1}} A/d^n \longrightarrow A/d^{n-1} \qquad (n \geq 2)$$

with $\Lambda_A M$. Induction shows that every $\Lambda_A M \otimes_A A/d^n$ is discrete and that their transition maps are surjective. Their limit is $\Lambda_A M$, which is therefore discrete. The cofiber sequence for multiplication by d now proves injectivity. □

Lemma 2.10 (Detection of a lower homotopy bound modulo an element of completeness)**.** *Let A be a discrete ring, let $t \in A$, and let $M \in \mathrm{Mod}_A$ be t-complete. For every $r \in \mathbf{Z}$,*

$$M \in (\mathrm{Mod}_A)_{\geq r} \quad \Longleftrightarrow \quad \mathrm{cofib}(t : M \to M) \in (\mathrm{Mod}_A)_{\geq r}.$$

Proof. For the reverse implication, the successive fibers of $\mathrm{cofib}(t^{n+1} : M \to M) \to \mathrm{cofib}(t^n : M \to M)$ are $\mathrm{cofib}(t : M \to M)$. Induction gives the bound for every term and surjectivity of the transition maps on π_r. Their inverse limit is M and has the same bound. □

2.2 Effective faithfully flat descent of perfectoid algebras

Lemma 2.11 (Faithfully flat detection of discrete complete modules with bounded torsion)**.** *Let $A \to B$ be a p-completely faithfully flat map of discrete p-complete rings, let $M \in \mathrm{Mod}_A^{\wedge p}$, and let $c \geq 0$. Then $\Lambda_A M$ is discrete with $p^c \Lambda_A M[p^\infty] = 0$ if and only if $\Lambda_B(M \widehat{\otimes}_A B)$ has these properties.*

Proof. Put $N = \mathrm{fib}(\Lambda_A M \to (\Lambda_A M)[1/p]) \in \mathrm{Mod}_A$. The asserted properties of $\Lambda_A M$ are equivalent to

$$\pi_i N = 0 \ (i \neq -1, 0), \qquad p^c \pi_0 N = 0, \qquad p : \pi_{-1} N \twoheadrightarrow \pi_{-1} N.$$

For the converse implication, complete the Postnikov fiber sequence $\pi_0 N \to N \to (\pi_{-1} N)[-1]$ at p. The first term is already complete and discrete. If L is a p-divisible p-torsion module, then

$$\widehat{L[-1]}_p \cong \lim_n L[p^n]$$

is discrete, since the transition maps are multiplication by p and are surjective. This proves discreteness of $\Lambda_A M$; then $\pi_0 N = \Lambda_A M[p^\infty]$ gives its torsion bound. Finally, $N \otimes_A B$ is the torsion realization of $M \widehat{\otimes}_A B$. Tensoring with B is exact and conservative on p-torsion modules, so it preserves and detects the three displayed conditions. □

Lemma 2.12 (Faithfully flat descent for modules and complete modules)**.** *Let $A \to B$ be a faithfully flat map of connective commutative rings. With $B^q = B^{\otimes_A (q+1)}$, extension of scalars gives*

$$\mathrm{Mod}_A \xrightarrow{\sim} \lim_{[q] \in \Delta} \mathrm{Mod}_{B^q}.$$

Let $A \to B$ instead be a map of connective t-complete commutative algebras over a discrete ring H, where $t \in H$ is a nonzerodivisor. If $A \otimes_H H/t \to B \otimes_H H/t$ is faithfully flat, then

$$\mathrm{Mod}_A^{\wedge(t)} \xrightarrow{\sim} \lim_{[q] \in \Delta} \mathrm{Mod}^{\wedge(t)}_{B \widehat{\otimes}_A (q+1)}$$

for the t-adic topology. Both isomorphisms are symmetric monoidal.

Proof. For faithfully flat $A \to B$, the right adjoint takes the limit of the underlying A-modules. First fix d and a descent datum $N^\bullet$ with each N^q in degrees $\leq d$. Flatness gives

$$B \otimes_A \mathrm{Tot}\, N^\bullet \ \cong \ \mathrm{Tot}(B \otimes_A N^\bullet).$$

Indeed, tensoring commutes with finite partial totalizations. The fiber from the mth partial totalization to the preceding one is the m-fold loop of the normalized degree-m term, hence is in degrees $\leq d - m$. Flatness preserves these bounds. Thus both sides in each fixed degree are computed by a finite partial totalization. After base change to B, multiplication $B \otimes_A B \to B$ splits the cover. The augmented Čech diagram for the datum over B therefore has an extra codegeneracy and its limit is N^0. This proves the counit in degree zero, hence in every degree by scalar extension. The same split diagram gives the adjunction unit for an A-module in degrees $\leq d$ after tensoring with B. Faithful flatness detects this isomorphism.

For an arbitrary descent datum the truncations $\tau_{\leq d} N^q$ again form descent data. In fact, every N^q is extended from a vertex along a flat map $B \to B^q$, so truncation commutes with these extensions and is compatible with every map of simplices. The descended modules form a Postnikov tower by the bounded-above case. Tensoring with B carries its limit to the limit of the extended tower, since it is t-exact and module categories over connective rings are left complete. This proves the first formula.

Put $A_n = A \otimes_H H/t^n$. Reduction gives a symmetric monoidal isomorphism

$$\mathrm{Mod}_A^{\wedge(t)} \xrightarrow{\sim} \lim_n \mathrm{Mod}_{A_n}. \tag{2.1}$$

To see effectivity, let (M_n) be a compatible family and set $M = \lim_n M_n$ in Mod_A. The module M is t-complete. For $n \geq k$, the fiber of $H/t^n \otimes_H H/t^k \to H/t^k$ is $(t^{n-k}H/t^n)[1] \cong (H/t^k)[1]$ as an H/t^n-module. The transition from $n+1$ to n is multiplication by t under this identification. Tensoring over H/t^n with M_n gives the fiber of $M_n \otimes_H H/t^k \to M_k$. Its tower has zero limit because every composite of k transitions is null. The H-module H/t^k is dualizable, so $M \otimes_H H/t^k \cong M_k$. Conversely, a t-complete module is the limit of its reductions.

Write $B_n = B \otimes_H H/t^n$. Each $A_n \to B_n$ is faithfully flat. A discrete $\pi_0 A_n$-module has a finite filtration by powers of t, and scalar extension on each quotient is computed along $A_1 \to B_1$. Exactness and faithfulness on discrete modules follow by this filtration. Apply the first formula at every n and commute the two limits in (2.1). The Čech terms modulo t^n are $B_n^{\otimes_{A_n}(q+1)}$, as required. □

Taking commutative algebra objects in either formula gives effective descent for commutative algebras. Taking endomorphisms of the units gives the corresponding formula for the rings. Dualizable modules satisfy the same descent since the duals, evaluations and coevaluations are determined in the limit of symmetric monoidal categories.

Lemma 2.13 (Functorial trivialization of the Fontaine conormal module modulo p)**.** *For a perfectoid ring R, put $I_R = \ker \theta_R \subset W(R^\flat)$. There is a functorial isomorphism of discrete R/p-modules*

$$I_R/(I_R^2 + pI_R) \longrightarrow R/p, \qquad x = [\bar{x}] + px_1 \longmapsto \theta_R(x_1) \bmod p.$$

Here $\bar{x} \in R^\flat$ is the reduction of x modulo p.

Proof. Choose a generator $\xi = [a] + pu$ of I_R and write $t = [b] + pv \in W(R^\flat)$. Then

$$t\xi = [ab] + p\big([b]u + [a]v + puv\big).$$

Under $R/p \cong R^\flat/(a)$, the proposed map sends $t\xi$ to $b\bar{u}$. Since $\bar{u}$ is a unit, it induces the stated isomorphism. Functoriality follows from the functoriality of the Witt expansion and the Fontaine map. □

Lemma 2.14 (Descent of Fontaine thickenings along a p-completely faithfully flat map)**.** *Let $A \to B$ be a p-completely faithfully flat map of discrete p-complete rings, with B and $B\widehat{\otimes}_A B$ perfectoid. Put*

$$B^q = B^{\widehat{\otimes}_A(q+1)}, \qquad I^q = \ker\big(W((B^q)^\flat) \xrightarrow{\theta} B^q\big).$$

The modules $I^q/(I^q)^2$ descend to an invertible discrete A-module L. For $m \geq 1$, the rings

$$H_m = \lim_{[q] \in \Delta} W((B^q)^\flat)/(I^q)^m$$

are discrete and fit into short exact sequences of discrete $\mathbf{Z}_p$-modules

$$0 \longrightarrow L^{\otimes_A m} \longrightarrow H_{m+1} \longrightarrow H_m \longrightarrow 0.$$

There are canonical isomorphisms

$$H_1 \cong A, \qquad W(A^\flat) \cong \lim_m H_m,$$

under which the projection to H_1 is θ_A.

Proof. Every B^q is perfectoid by proposition 1.11. Lemma 1.5 makes the conormal modules compatible with base change. Lemma 2.12 identifies $H_1 \cong A$ and descends the conormal modules to $L \in \mathrm{Mod}_A^{\wedge p}$. Its realization $\Lambda_A L$ is discrete by lemma 2.11, since the conormal module over B is free of rank one and $pB[p^\infty] = 0$. The functorial trivializations of lemma 2.13 descend along the faithfully flat map $A/p \to B/p$ to give $\Lambda_A L \otimes_A A/p \cong A/p$. Choose a lift of 1 to $\Lambda_A L$. The resulting map $A \to \Lambda_A L$ is an isomorphism: its cofiber is p-complete and vanishes on tensoring with A/p, hence also with $\mathrm{cofib}(p : A \to A)$, whose homotopy groups are A/p and $A[p]$. Thus L is represented by a free rank-one A-module, also denoted by L. Descent of $(I^q)^m/(I^q)^{m+1} \cong (I^q/(I^q)^2)^{\otimes_{B^q} m}$ gives fiber sequences

$$L^{\otimes_A m} \longrightarrow H_{m+1} \longrightarrow H_m.$$

Induction makes every H_m discrete and every transition map surjective. Hence $\lim_m H_m$ is discrete. Completeness gives

$$\lim_m H_m \cong \lim_{[q]\in\Delta} W((B^q)^\flat).$$

Since this limit is discrete, preservation of ordinary limits by Witt vectors and tilting identifies it with $W(A^\flat)$. The projection to $H_1 \cong A$ is induced by the Fontaine maps. □

Theorem 2.15. *Let $A \to B$ be a p-completely faithfully flat map of discrete p-complete rings. If B and $B\widehat{\otimes}_A B$ are perfectoid, then A is perfectoid.*

Proof. Use the rings B^q, H_m, the ideals I^q, and the invertible A-module L of lemma 2.14. Choose a generator of the free rank-one A-module L and lift its image in H_2 through the surjection $W(A^\flat) \to H_2$ to an element $d \in \ker \theta_A$. Its image d_q generates $I^q/(I^q)^2$, hence I^q, because I^q lies in the Jacobson radical of $W((B^q)^\flat)$.

Multiplication by d on $W(A^\flat) \cong \lim_{[q]\in\Delta} W((B^q)^\flat)$ is injective, since multiplication by each d_q is injective. Taking cofibers gives

$$W(A^\flat)/(d) \cong \lim_{[q]\in\Delta} W((B^q)^\flat)/(d_q) \cong \lim_{[q]\in\Delta} B^q \cong A.$$

Write $d = [a] + pu$. The images of u in all the Witt rings $W((B^q)^\flat)$ are units, so u is a unit. Completeness of those Witt rings for (d_q) makes $W(A^\flat)$ d-complete and $A^\flat$ a-adically complete. Apply lemma 1.3. □

Corollary 2.16 (Effective p-completely faithfully flat descent of perfectoid algebras)**.** *Let $S \to S'$ be a p-completely faithfully flat map of perfectoid rings, and let A be a p-complete S-algebra. Then*

$$A \text{ is perfectoid} \quad \Longleftrightarrow \quad A\widehat{\otimes}_S S' \text{ is perfectoid.}$$

Writing $\mathrm{PerfAlg}_T \subseteq \mathrm{CAlg}_T$ *for the full subcategory of perfectoid T-algebras, completed extension of scalars induces an isomorphism*

$$\mathrm{PerfAlg}_S \longrightarrow \lim_{[q]\in\Delta} \mathrm{PerfAlg}_{(S')^{\widehat{\otimes}_S (q+1)}}.$$

Proof. Put $B = A\widehat{\otimes}_S S'$. If A is perfectoid, B is perfectoid by proposition 1.11. Conversely, suppose B is perfectoid. By lemmas 1.3 and 2.11, A is discrete with $pA[p^\infty] = 0$. The map $A \to B$ is p-completely faithfully flat by corollary 2.5. Its tensor square is

$$B\widehat{\otimes}_A B \cong B\widehat{\otimes}_{S'}(S'\widehat{\otimes}_S S'),$$

which is perfectoid by two applications of proposition 1.11. Theorem 2.15 now gives perfectoidness of A. The isomorphism of categories follows by restricting descent for commutative algebras in lemma 2.12 to the perfectoid algebras. □

Corollary 2.17 (Perfectoidness is local for finite completed principal covers)**.** *Let A be a discrete p-complete ring, and let $f_1, \dots, f_n \in A$ have images generating the unit ideal in A/p. Then A is perfectoid if and only if every $\widehat{A[1/f_i]}_p$ is perfectoid.*

Proof. The forward implication is lemma 1.12. For the converse, put $B = \prod_i \widehat{A[1/f_i]}_p$, which is perfectoid by lemma 1.13. The map $A \to B$ is p-completely faithfully flat by lemma 2.3, since

$$B \otimes_A A/p \cong \prod_i (A/p)[1/f_i].$$

Each $\widehat{A[1/(f_i f_j)]}_p$ is a p-completed localization of $\widehat{A[1/f_i]}_p$. Thus

$$B \widehat{\otimes}_A B \cong \prod_{i,j} \widehat{A[1/(f_i f_j)]}_p$$

is perfectoid by lemmas 1.12 and 1.13. Theorem 2.15 proves that A is perfectoid. □

Lemma 2.18 (Faithfully flat detection of p-complete flatness)**.** *Let $S \to S'$ be a p-completely faithfully flat map of discrete p-complete rings. An object $M \in \mathrm{Mod}_S^{\wedge p}$ is p-completely flat if and only if $M \widehat{\otimes}_S S'$ is p-completely flat over S'.*

Proof. Use lemma 2.2 and faithfully flat descent over $S/p \to S'/p$ in

$$(M \widehat{\otimes}_S S') \otimes_{S'} S'/p \cong (\Lambda_S M \otimes_S S/p) \otimes_{S/p} S'/p.$$

□

Corollary 2.19 (Faithfully flat detection of p-complete faithful flatness)**.** *Under the hypotheses of lemma 2.18, M is p-completely faithfully flat if and only if $M \widehat{\otimes}_S S'$ is p-completely faithfully flat over S'.*

Proof. Apply lemma 2.3 to the same reduction formula. □

Lemma 2.20 (Faithfully flat detection of surjectivity of complete modules)**.** *Let $S \to S'$ be a p-completely faithfully flat map of discrete p-complete rings. A map $u : M \to N$ of discrete p-complete S-modules is surjective if and only if $\pi_0(u \widehat{\otimes}_S S')$ is surjective.*

Proof. Let Q and Q' be the ordinary cokernels of u and $\pi_0(u \widehat{\otimes}_S S')$, respectively. For a p-complete module M, let s be the shift on $\prod_{n \geq 0} M$ whose components are multiplication by p. The fiber of $1 - s$ is $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[1/p], M) = 0$. Thus $1 - s$ is also an isomorphism on $\prod_{n \geq 0} \pi_j M$ for each j, proving that the discrete module $\pi_j M$ is p-complete. Apply this to the two cofibers to see that Q and Q' are p-complete. Reduction gives

$$Q'/pQ' \cong (Q/pQ) \otimes_{S/p} S'/p.$$

A discrete p-complete module L with $L/pL = 0$ vanishes: successive p-divisions of any element define a map $\mathbf{Z}[1/p] \to L$, which vanishes by p-completeness. Faithful flatness of $S/p \to S'/p$ now gives $Q = 0$ if and only if $Q' = 0$. □

3 Prismatic cohomology

3.1 Relative prismatic cohomology

Definition 3.1. A discrete δ-ring is a discrete ring H with an operation δ satisfying $\delta(0) = \delta(1) = 0$ and

$$\delta(x+y) = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i},$$
$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y).$$

Its Frobenius lift is $\varphi(x) = x^p + p\delta(x)$. An element d is distinguished if $\delta(d)$ is a unit. For $d = [a] + pu \in W(B)$, one has $\delta(d) \bmod p = (u \bmod p)^p$, so d is distinguished exactly when u is a unit. The free δ-H-algebra on y has underlying ring $H[y_0, y_1, \ldots]$, with $\delta(y_i) = y_{i+1}$. A discrete δ-ring is *perfect* if φ is an isomorphism.

Definition 3.2 (The derived free-δ monad)**.** Equip $\mathbf{Z}_p$ with $\delta(a) = (a - a^p)/p$. On finite free discrete $\mathbf{Z}_p$-modules the free-δ functor has underlying module

$$F_\delta(\mathbf{Z}_p^{\oplus S}) = \mathbf{Z}_p[x_{s,j} \mid s \in S,\ j \geq 0], \qquad \delta(x_{s,j}) = x_{s,j+1}.$$

Give $x_{s,j}$ weight p^j, and write $F_{\delta,\leq d}$ for the subfunctor spanned by monomials of weight at most d. Let $\mathrm{Sym}^{\delta,\leq d}$ be its sifted-colimit-preserving polynomial extension of degree at most d to $\mathrm{Mod}_{\mathbf{Z}_p}$, and put

$$\mathrm{Sym}^\delta = \operatorname*{colim}_d \mathrm{Sym}^{\delta,\leq d}.$$

Here degree at most d means that every $(d+1)$-cube whose two-dimensional faces are pushouts is carried to a cartesian cube. The categorical extension theorem supplies these extensions and extends substitution to a monad structure on Sym^δ. Define $\mathrm{DAlg}^\delta_{\mathbf{Z}_p}$ to be its category of algebras; their underlying modules may be nonconnective.

We use this categorical extension theorem and the following formal properties without proof [10, Theorem 2.3.9 and Proposition 2.3.12]. The inclusion of the variables $x_{s,0}$ induces a map of monads $\mathrm{Sym} \to \mathrm{Sym}^\delta$. The resulting forgetful functor

$$U : \mathrm{DAlg}^\delta_{\mathbf{Z}_p} \longrightarrow \mathrm{DAlg}_{\mathbf{Z}_p}$$

preserves small limits and colimits and admits both a left adjoint $F^\delta_{\mathbf{Z}_p}$ and a right adjoint W. For $H \in \mathrm{DAlg}^\delta_{\mathbf{Z}_p}$, put $\mathrm{DAlg}^\delta_H = (\mathrm{DAlg}^\delta_{\mathbf{Z}_p})_{H/}$; the relative adjoints are denoted by F^δ_H and W_H. The forgetful functors

$$\mathrm{DAlg}^\delta_H \longrightarrow \mathrm{DAlg}_H \longrightarrow \mathrm{CAlg}_H$$

preserve pushouts, whose underlying ring spectra are therefore relative tensor products [11, Proposition 4.2.27].

Fix a perfectoid ring R and a generator d of $\ker(\theta_R : W(R^\flat) \to R)$, so $R = W(R^\flat)/(d)$. Write $\mathrm{DAlg}^{\delta,(p,d)\text{-comp}}_{W(R^\flat)}$ for derived δ-$W(R^\flat)$-algebras whose underlying modules are (p,d)-complete, and U for the forgetful functor.

Proposition 3.3 (The prismatic adjunction)**.** *Reduction modulo d has a left adjoint*

$$\Delta_{-/W(R^\flat)} : \mathrm{DAlg}^{p\text{-comp}}_R \rightleftarrows \mathrm{DAlg}^{\delta,(p,d)\text{-comp}}_{W(R^\flat)} : (D \longmapsto UD \otimes_{W(R^\flat)} R).$$

Thus, for every $T \in \mathrm{DAlg}_R^{p\text{-comp}}$ *and complete derived* δ*-*$W(R^\flat)$*-algebra* D*,*

$$\mathrm{Map}_{\mathrm{DAlg}^\delta_{W(R^\flat)}}(\Delta_{T/W(R^\flat)}, D) \cong \mathrm{Map}_{\mathrm{DAlg}_R}(T, UD \otimes_{W(R^\flat)} R).$$

The algebra $\Delta_{T/W(R^\flat)}$ *is the relative prismatic cohomology of* T*.*

Proof. The $W(R^\flat)$-module $R = \mathrm{cofib}(d : W(R^\flat) \to W(R^\flat))$ is dualizable. The reduction functor therefore preserves limits; it is also accessible. The adjoint functor theorem supplies its left adjoint. □

3.2 Properties of prismatic cohomology

Proposition 3.4 (Prismatic cohomology preserves completed pushouts and filtered colimits)**.** *Let* P *be perfectoid, let* d *generate* $\ker \theta_P$*, and let* $T_1 \leftarrow T_0 \to T_2$ *be a diagram in* $\mathrm{DAlg}_P^{p\text{-comp}}$*. Then*

$$\Delta_{T_1 \widehat{\otimes}_{T_0} T_2/W(P^\flat)} \cong \Delta_{T_1/W(P^\flat)} \widehat{\otimes_{\Delta_{T_0/W(P^\flat)}}} \Delta_{T_2/W(P^\flat)(p,d)}.$$

Relative prismatic cohomology also preserves completed filtered colimits.

Proof. The adjunction in proposition 3.3 makes Δ preserve colimits in complete derived δ-algebras. Their underlying derived algebras have the same colimits because the forgetful functor has the cofree Witt functor as a right adjoint. □

Proposition 3.5 (Change of perfectoid coefficients in prismatic cohomology)**.** *Let* $P \to P'$ *be a map of perfectoid rings, let* d *generate* $\ker \theta_P$*, and let* $T \in \mathrm{DAlg}_P^{p\text{-comp}}$*. Then*

$$\Delta_{(T \widehat{\otimes}_P P')/W(P'^\flat)} \cong \Delta_{T/W(P^\flat)} \widehat{\otimes_{W(P^\flat)}} W(P'^\flat)_{(p,d)}.$$

Proof. Maps from either side to a complete derived δ-$W(P'^\flat)$-algebra D identify with

$$\mathrm{Map}_{\mathrm{DAlg}_P}(T, D \otimes_{W(P'^\flat)} P').$$

Here $W(P'^\flat) \otimes_{W(P^\flat)} P \cong P'$ by the Witt presentations in lemma 1.5. □

Lemma 3.6 (Prismatic cohomology recovers the Witt presentation of a perfectoid algebra)**.** *Let* C *be a perfect* $R^\flat$*-algebra complete at* $(d \bmod p)$*. For* $D \in \mathrm{DAlg}^{\delta,(p,d)\text{-comp}}_{W(R^\flat)}$*, reduction gives*

$$\mathrm{Map}_{\mathrm{DAlg}^\delta_{W(R^\flat)}}(W(C), D) \cong \mathrm{Map}_{\mathrm{DAlg}_R}(W(C)/d, D \otimes_{W(R^\flat)} R).$$

In particular $\Delta_{(W(C)/d)/W(R^\flat)} = W(C)$*.*

Proof. For every perfect discrete $\mathbf{F}_p$-algebra E, the p-completed cotangent module of $W(E)$ over $\mathbf{Z}_p$ vanishes. Its reduction modulo p is the cotangent module of E over $\mathbf{F}_p$, on which Frobenius induces both an isomorphism and the zero map. Consequently, for any p-complete derived commutative $\mathbf{Z}_p$-algebra B, reduction induces an isomorphism

$$\mathrm{Map}_{\mathrm{DAlg}_{\mathbf{Z}_p}}(W(E), B) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{DAlg}_{\mathbf{Z}_p}}(W(E), B \otimes_{\mathbf{Z}_p} \mathbf{F}_p).$$

Indeed, for each square-zero extension $B \otimes_{\mathbf{Z}_p} \mathbf{Z}/p^{n+1} \to B \otimes_{\mathbf{Z}_p} \mathbf{Z}/p^n$, the obstruction and the space of lifts are computed by maps from the cotangent module into p-complete modules. Taking the limit over n gives the formula.

Suppose B also has a derived δ-structure. By [10, Theorem 2.4.4], a derived δ-map over a given algebra map $f : W(E) \to B$ is a homotopy $f \circ \varphi_{W(E)} \simeq \varphi_B \circ f$ lifting the specified

naturality homotopy for Frobenius modulo p. The displayed isomorphism identifies the two path spaces with these endpoints. Thus the space of lifts of the specified homotopy is contractible. Applying this to $E = C$ and $E = R^\flat$, then taking the fibers of restriction along $W(R^\flat) \to W(C)$ at the structure map to D, identifies the spaces of δ-maps and algebra maps over $W(R^\flat)$.

The transitivity sequence gives vanishing of the p-completed cotangent module of $W(C)$ over $W(R^\flat)$. Square-zero lifting through $D \otimes_{W(R^\flat)} W(R^\flat)/(d^n)$ and d-completeness of D now identify

$$\mathrm{Map}_{\mathrm{DAlg}_{W(R^\flat)}}(W(C), D) \cong \mathrm{Map}_{\mathrm{DAlg}_R}(W(C)/d, D \otimes_{W(R^\flat)} R).$$

Here $W(C) \otimes_{W(R^\flat)} R$ is discrete and equal to $W(C)/d$ by lemma 1.3. The assertion about prismatic cohomology follows from proposition 3.3. □

Lemma 3.7 (Filtered colimits commute with totalizations of uniformly bounded-above cosimplicial modules)**.** *Let $\{M_\alpha^\bullet\}_\alpha$ be a filtered diagram of cosimplicial objects of* $\mathrm{Mod}_{\mathbf{Z}}$ *with $M_\alpha^q \in (\mathrm{Mod}_{\mathbf{Z}})_{\leq N}$ for one integer N and all α, q. Then*

$$\operatorname*{colim}_\alpha \mathrm{Tot}\, M_\alpha^\bullet \cong \mathrm{Tot}(\operatorname*{colim}_\alpha M_\alpha^\bullet).$$

Lemma 3.8 (Polynomial transmutation is computed by the Witt translation action)**.** *Let P be perfectoid and let d generate* $\ker \theta_P$*. The algebra $\Delta_{P\langle x\rangle/W(P^\flat)}$ is the (p,d)-completion of the functions on the Witt translation action $x \mapsto x + dz$. Equivalently,*

$$\Delta_{P\langle x\rangle/W(P^\flat)} \cong \mathrm{Tot}_{[q]\in\Delta} \widehat{W(P^\flat)\{x, z_1, \ldots, z_q\}}_{\delta_{(p,d)}}.$$

This identification respects polynomial substitution.

Proof. Let H be a discrete δ-$\mathbf{Z}_p$-algebra and let $e \in H$ be a nonzerodivisor. Denote by $K_e^\bullet$ the cosimplicial finite free H-module with $K_e^q = Hx \oplus \bigoplus_{j=1}^q Hz_j$ associated with the additive action $x \mapsto x + ez$. Its coface maps insert an initial translation, add two consecutive translation parameters, or omit the last parameter; its codegeneracies set a translation parameter to zero. Its normalized terms are $H \xrightarrow{e} H$ in degrees 0 and 1. Hence

$$\mathrm{Tot}\, K_e^\bullet = \mathrm{fib}(e : H \to H) = (H/e)[-1].$$

The polynomial Kan extension defining the free-δ functor gives

$$\mathrm{Sym}_H^\delta((H/e)[-1]) \cong \mathrm{Tot}_{[q]\in\Delta} H\{x, z_1, \ldots, z_q\}_\delta. \tag{3.1}$$

For a bounded-weight piece of weight at most r, the normalized cosimplicial terms vanish above degree r: a normalized monomial contains a variable from each translation parameter. Its totalization is therefore finite and computes the right Kan extension on this finite projective presentation. Taking the filtered colimit over r gives (3.1). The terms are discrete, so lemma 3.7 justifies this interchange of colimit and totalization. The monad operations give the algebra structures in this formula.

Duality for H/e identifies the left adjoint to reduction on a polynomial algebra with the left side of (3.1). In particular,

$$\Delta_{P\langle x\rangle/W(P^\flat)} \cong \widehat{\mathrm{Sym}^\delta_{W(P^\flat)}(P[-1])}_{(p,d)}.$$

The cosimplicial algebra on the right of (3.1) is the algebra of functions on the action groupoid for multiplication by e on the additive Witt group. Indeed, a map from a free

δ-algebra to the cofree Witt algebra is specified by the images of its free generators; the coface $x \mapsto x + ez$ becomes Witt addition and multiplication by the image of e. Multiplication of arrows with parameters z, z' and initial points x, x' has parameter $xz' + x'z + ezz'$, since

$$(x + ez)(x' + ez') - xx' = e(xz' + x'z + ezz').$$

Thus the isomorphism (3.1) respects the ring operations on the quotient and, in particular, the substitution $x \mapsto x^p$.

□

Smith ideal quotients. Equip $\mathrm{Fun}(\Delta^1, \mathrm{Sp})$ with the pushout-product tensor product

$$(I \to A) \,\square\, (J \to B) = \big((I \otimes B) \amalg_{I \otimes J} (A \otimes J) \longrightarrow A \otimes B\big).$$

An E_∞ *Smith ideal* is a commutative algebra object for this tensor product, whose unit is $0 \to \mathbb{S}$. The symmetric monoidal isomorphism

$$(I \to A) \longmapsto (A \to \mathrm{cofib}(I \to A))$$

to the arrow category with pointwise tensor product identifies Smith ideals with morphisms of E_∞-rings. Write $A \to [A/I]$ for the quotient morphism. Equivalently, its Čech nerve is the internal groupoid

$$G_n = \underbrace{A \times_{[A/I]} \cdots \times_{[A/I]} A}_{n+1}, \qquad |G_\bullet| \cong [A/I].$$

These correspondences also hold for sheaves of spectra. If I and A are connective, the groupoid is connective, and passage to underlying spaces preserves this realization. For the pushout-product tensor product, see [18, §§1,4].

Lemma 3.9 (The Witt quotient by p is the divided-power translation quotient)**.** *Let k be an animated $\mathbf{F}_p$-algebra. Work on the site of affine formal schemes $\mathrm{Spf}\, T$ for animated k-algebras T, with the descendable topology. Write W for the sheaf of animated $W(k)$-algebras $T \mapsto W(T)$, and $\mathbf{G}_a$ for the sheaf of animated k-algebras $T \mapsto T$. Their underlying sheaves of spaces are represented by $\mathrm{Spf}\, k[w_0, w_1, \ldots]$ and $\mathrm{Spf}\, k[x]$, respectively. Let $\Gamma(z)$ be the extension to k of the ordinary divided-power polynomial algebra over $\mathbf{F}_p$, so*

$$\Gamma(z) \cong \bigoplus_{n \geq 0} k z^{[n]}, \qquad z^{[i]} z^{[j]} = \binom{i+j}{i} z^{[i+j]}, \qquad \Delta(z^{[n]}) = \sum_{i+j=n} z^{[i]} \otimes z^{[j]}.$$

Let $i : \mathbf{G}_a^\sharp \to \mathbf{G}_a$ be the Smith ideal represented by

$$k[x] \longrightarrow \Gamma(z), \qquad x \longmapsto z^{[1]}.$$

Its addition is defined by Δ, and its $\mathbf{G}_a$-module action is represented by

$$\Gamma(z) \longrightarrow k[x] \otimes_k \Gamma(z), \qquad z^{[n]} \longmapsto x^n \otimes z^{[n]}.$$

There is an isomorphism of quotient sheaves of animated k-algebras

$$[W/pW] \cong [\mathbf{G}_a / \mathbf{G}_a^\sharp].$$

On the left, the coefficient action is induced by Witt Frobenius: $W(k) \xrightarrow{F} W(k) \to [W/pW]$ factors through k.

Proof. By base change, assume $k = \mathbf{F}_p$.

The Smith ideal identity $i(u)v = i(v)u$ follows from

$$(z^{[1]})^n \otimes z^{[n]} = n!\, z^{[n]} \otimes z^{[n]} = z^{[n]} \otimes (z^{[1]})^n.$$

This common map defines the multiplication uv on $\mathbf{G}_a^\sharp$. The translation groupoid has objects $\mathbf{G}_a$ and arrows $\mathbf{G}_a \oplus \mathbf{G}_a^\sharp$, with source $(x,u) \mapsto x$ and target $(x,u) \mapsto x + i(u)$. The arrow algebra has coordinatewise addition, unit $(1,0)$, and multiplication

$$(x,u)(y,v) = (xy,\ xv + yu + uv).$$

Its identity arrows are $(x,0)$ and its composition is $(x,u),(x+i(u),v) \mapsto (x,u+v)$. These coordinate formulas define an internal groupoid in sheaves of animated rings; the equality $i(u)v = i(v)u$ ensures that composition preserves multiplication. Its realization in that category defines $[\mathbf{G}_a/\mathbf{G}_a^\sharp]$ and its animated ring structure. The same construction for the Smith ideal $p : W \to W$ defines $[W/pW]$.

Witt Frobenius $F : W \to W$ raises every coordinate to its pth power. Its coordinate algebra is free over the source, with basis the monomials whose exponents are less than p. The zeroth-coordinate map $W \to \mathbf{G}_a$ has a polynomial coordinate algebra. Both bases contain 1, so both maps are descendable and hence epimorphisms of sheaves. Their kernels are $W[F]$ and $V(W)$, respectively.

We identify $W[F]$ with $\mathbf{G}_a^\sharp$. For the length-n Witt functor W_n, the coordinate Hopf algebra of $W_n[F]$ is

$$A_n = k[w_0, \dots, w_{n-1}]/(w_0^p, \dots, w_{n-1}^p).$$

Its linear dual $A_n^\vee$ is a discrete commutative Hopf algebra. Let $e_i \in A_n^\vee$ extract the coefficient of w_i. These functionals form a basis of its primitive elements. Give w_i degree p^i; Witt addition preserves this grading. The primitive element e_i^p has degree p^{i+1}, so it is a scalar multiple of e_{i+1} for $i < n-1$, and $e_{n-1}^p = 0$. For $n \geq 2$, the coefficient of $t_1 \cdots t_p$ in Witt coordinate 1 of $[t_1]+\cdots+[t_p]$ is $-(p-1)! = 1$. Thus $e_0^p = e_1$. Verschiebung $W_{n-1}[F] \to W_n[F]$ sends e_i to e_{i+1} on the dual Hopf algebras, so

$$e_i^p = e_{i+1} \quad (0 \leq i < n-1), \qquad e_{n-1}^p = 0.$$

The products $\prod_i e_i^{a_i}$ with $0 \leq a_i < p$ form a basis: their pairing with the coordinate monomials is triangular for total degree, with diagonal entries $\prod_i a_i!$. Consequently

$$A_n^\vee \cong k[t]/(t^{p^n}), \qquad t \longmapsto e_0, \qquad \Delta(t) = t \otimes 1 + 1 \otimes t.$$

Dualizing and taking the inverse limit of the $W_n[F]$ gives $\mathbf{G}_a^\sharp \cong W[F]$, with coordinate map $w_i \mapsto z^{[p^i]}$. In particular, the maps to $\mathbf{G}_a$ agree.

Consider the Smith ideal

$$W[F] \oplus W \longrightarrow W, \qquad (a,b) \longmapsto a + V(b),$$

where $c \in W$ acts by $c \cdot (a,b) = (ca, F(c)b)$. Its induced multiplication is $(a,b)(a',b') = (aa', pbb')$, since $aV(b') = V(F(a)b') = 0$ for $a \in W[F]$. Projection to the two summands gives the following morphisms of Smith ideals, using $FV = p$:

$$\begin{array}{ccccc} \mathbf{G}_a^\sharp & \xleftarrow{\mathrm{pr}_1} & W[F] \oplus W & \xrightarrow{\mathrm{pr}_2} & W \\ {\scriptstyle i}\downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ \mathbf{G}_a & \xleftarrow{w_0} & W & \xrightarrow{F} & W. \end{array}$$

Both squares are pullbacks by the kernel identifications above, and the bottom maps are epimorphisms of sheaves. Hence both morphisms induce isomorphisms of quotient sheaves of animated rings. The map to the quotient by p is W-linear for the action induced by F, giving the stated coefficient map. □

Lemma 3.10 (Invariants of a divided-power coaction defined by a locally nilpotent endomorphism)**.** *Let k be an animated $\mathbf{F}_p$-algebra, let $M \in \mathrm{Mod}_k$ be discrete, and let $\partial : M \to M$ be a locally nilpotent k-linear endomorphism. The invariants for the coaction*

$$M \longrightarrow M \otimes_k \Gamma(z), \qquad m \longmapsto \sum_{n\geq 0} \partial^n(m) z^{[n]}$$

are $\mathrm{fib}(\partial : M \to M) \in \mathrm{Mod}_k$.

Proof. The corresponding $\Gamma(z)$-comodule has the cofree resolution

$$0 \longrightarrow M \xrightarrow{m \mapsto \sum_n \partial^n(m) z^{[n]}} M \otimes_k \Gamma(z) \xrightarrow{\partial_z - \partial} M \otimes_k \Gamma(z) \longrightarrow 0, \qquad \partial_z(z^{[n]}) = z^{[n-1]}.$$

The convention is $\partial_z(1) = 0$. Conjugation by the finite Taylor sum reduces exactness to the fact that ∂_z is surjective with kernel M. A cofree comodule has no higher invariants: its bar diagram has the extra codegeneracy given by its counit. Taking invariants of the displayed resolution therefore gives $\mathrm{fib}(\partial : M \to M)$ in Mod_k. □

Lemma 3.11 (Functions on the divided-power translation quotient are the de Rham algebra)**.** *For an animated $\mathbf{F}_p$-algebra k, the functions on $[\mathbf{G}_a/\mathbf{G}_a^\sharp]$ identify naturally with the de Rham E_∞-k-algebra of $k[x]$. Its underlying object of* Mod_k *is*

$$\mathrm{fib}\big(k[x] \xrightarrow{\partial_x} k[x]\, dx\big).$$

Proof. First take $k = \mathbf{F}_p$. The translation action is given by

$$f(x + z^{[1]}) = \sum_{n\geq 0} \partial_x^n(f) z^{[n]}.$$

Apply lemma 3.10 to $M = k[x]$ and $\partial = \partial_x$. Its cofree resolution is multiplicative. Use the derivation $\partial_z - \partial_x$ and the exterior product on one symbol dz, with $dz \wedge dz = 0$. The Taylor map is an algebra map. Taking invariants gives the de Rham E_∞-k-algebra of $k[x]$.

Extension of scalars carries this cofree resolution of $\mathbf{F}_p[x]$ to a cofree resolution of $k[x]$ for every animated k, proving the assertion. □

For lemma 3.9, see also [17, §2.6].

Theorem 3.12 (Reduction of polynomial prismatic cohomology to the de Rham algebra)**.** *Let P be perfectoid, put $k = P \otimes_{\mathbf{Z}_p} \mathbf{F}_p$, and give k the coefficient map $W(P^\flat) \xrightarrow{\varphi} W(P^\flat) \xrightarrow{\theta_P} P \to k$. The only possibly nonzero homotopy groups of k are $\pi_0 k = P/p$ and $\pi_1 k = P[p]$. There is an isomorphism of E_∞-k-algebras*

$$\Delta_{P\langle x\rangle/W(P^\flat)} \otimes_{W(P^\flat)} k \;\cong\; \mathrm{dR}_{k[x]/k}$$

natural for polynomial substitutions. Extension of scalars along $k \to \pi_0 k = P/p$ gives the same formula over the ordinary quotient.

Proof. Let d generate $\ker \theta_P$. With the indicated coefficient map, the underlying $W(P^\flat)$-module of k is

$$\mathrm{cofib}\Big(\varphi^{-1}(d) : W(P^\flat)/p \longrightarrow W(P^\flat)/p\Big). \tag{3.2}$$

This is the cofiber presentation of $P \cong W(P^\flat)/(d)$, twisted by φ and tensored over $\mathbf{Z}_p$ with $\mathbf{F}_p$. Since p is regular on $W(P^\flat)$, formula (3.2) makes k a perfect $W(P^\flat)$-module. Thus

tensoring with k preserves the totalization in lemma 3.8. It also removes the (p, d)-completion, since $(p, \varphi^{-1}(d))$ defines the same topology.

Let $h_0 : W(P^\flat) \to W(P)$ be the δ-map adjoint to θ_P. Its zeroth coordinate on d is zero in P, so $h_0(d) = V(v)$ for a unique $v \in W(P)$. The zeroth coordinate of v is $\theta_P(\delta(d))$, a unit. The δ-map adjoint to $\theta_P\varphi$ is Fh_0. It sends d to $FV(v) = pv$. Under $W(P) \to W(k)$ the image of v is invertible: its zeroth coordinate is a unit in $\pi_0 k$. Rescaling the translation parameter by this image identifies the action in lemma 3.8 over k with $x \mapsto x + pz$.

The two maps of Smith ideals in lemma 3.9 also identify the P-actions. On objects they carry $h_0(c)$, for $c \in W(P^\flat)$, to $Fh_0(c)$ on the Witt side and to the image of $\theta_P(c)$ in k on the divided-power side. Write $\bar{v}$ for the image of v in $W(k)$. The arrow with parameter $(0, \bar{v})$ has boundary $V(\bar{v})$ and maps to the parameters $\bar{v}$ and 0, respectively. Thus the nullhomotopy for d in the Witt quotient corresponds to the one supplied by $\theta_P(d) = 0$ on the divided-power side. The quotient over k is therefore $[\mathbf{G}_a/\mathbf{G}_a^\sharp]$ with its P-action through $P \to k$. Its functions are $\mathrm{dR}_{k[x]/k}$ by lemma 3.11. These isomorphisms respect polynomial substitutions since the maps of Smith ideals respect the ring operations and the P-actions. □

Compare [1, Theorem 6.4 and Corollary 15.4].

Corollary 3.13 (Reduction of the prismatic map adjoining all p-power roots)**.** *Let P be perfectoid. Put $k = P \otimes_{\mathbf{Z}_p} \mathbf{F}_p$ with the $W(P^\flat)$-algebra structure of theorem 3.12. Tensoring the map*

$$\Delta_{P\langle x\rangle/W(P^\flat)} \longrightarrow W(P\langle x^{1/p^\infty}\rangle^\flat)$$

over $W(P^\flat)$ with k gives the map of E_∞-k-algebras

$$k[t] \oplus k[t][-1] \longrightarrow k[t^{1/p^\infty}]$$

given by the augmentation of the split square-zero algebra and the inclusion of polynomial rings.

Proof. By theorem 3.12, the source is $\mathrm{dR}_{k[x]/k}$. Over $\mathbf{F}_p$, the derivative maps x^m to $mx^{m-1}dx$. The summands with $p \nmid m$ cancel by isomorphisms, leaving $\mathbf{F}_p[x^p]$ in degree 0 and $\mathbf{F}_p[x^p]x^{p-1}dx$ in degree -1. Extending this calculation to k, the map of de Rham algebras sending

$$t \longmapsto x^p, \qquad \eta \longmapsto x^{p-1}dx$$

therefore identifies the de Rham algebra with $k[t] \oplus k[t]\eta$, where $|\eta| = -1$ and $\eta^2 = 0$. The substitution $x \mapsto x^p$ sends $t \mapsto t^p$ and $\eta \mapsto 0$.

The ring $P\langle x^{1/p^\infty}\rangle$ is the completed colimit of these polynomial substitutions. The prismatic left adjoint preserves this colimit. Taking the colimit of $t \mapsto t^p$, $\eta \mapsto 0$ gives $k[t^{1/p^\infty}]$. The map from $k[t] \oplus k[t]\eta$ is the augmentation followed by the inclusion. By lemma 3.6, $\Delta_{P\langle x^{1/p^\infty}\rangle/W(P^\flat)} = W(P\langle x^{1/p^\infty}\rangle^\flat)$.

□

Lemma 3.14 (Tensor-square nilpotence for a split square-zero algebra)**.** *Let T be a connective commutative ring, and let S be a T-algebra whose underlying T-module admits a basis containing the unit of S. For the split square-zero T-algebra $\Omega = T \oplus T[-1]$ and the composite $\Omega \to T \to S$, one has*

$$\mathrm{Hom}_\Omega\Big(\mathrm{fib}(\Omega \to S)^{\otimes_\Omega 2}, \Omega\Big) \in (\mathrm{Mod}_T)_{\geq 1}.$$

Consequently the tensor square of $\mathrm{fib}(\Omega \to S) \to \Omega$ *is null.*

Proof. Choose a basis $\mathcal{B}$ containing the unit and put $F = \mathrm{fib}(\Omega \to S)$. The splitting $S \cong T \oplus \bigoplus_{\mathcal{B}\setminus\{1\}} T$ identifies F with $\bigoplus_{\mathcal{B}} T[-1]$ as an Ω-module, where Ω acts through its augmentation.

The two-sided bar construction for $T \otimes_\Omega T$ splits by tensor length. Its term of length n is $(T[-1])^{\otimes_T n}[n] = T$; multiplication between two augmentation-ideal factors is zero. Thus

$$T \otimes_\Omega T \cong \bigoplus_{n\geq 0} T \qquad \text{in } \mathrm{Mod}_T\,.$$

Projection onto $T[-1]$ gives a perfect pairing $\Omega \otimes_T \Omega \to T[-1]$, and hence an isomorphism $\mathrm{Hom}_T(\Omega, T) \cong \Omega[1]$ of Ω-modules. The restriction–coinduction adjunction therefore gives

$$\begin{aligned}\mathrm{Hom}_\Omega(T[-1] \otimes_\Omega T[-1], \Omega) &\cong \mathrm{Hom}_T\big((T \otimes_\Omega T)[-2], T\big)[-1] \\ &\cong \prod_{n\geq 0} T[1].\end{aligned}$$

Taking products over $\mathcal{B} \times \mathcal{B}$ proves the asserted bound and the vanishing of π_0. □

Theorem 3.15 (Prismatic descent along polynomial root extensions)**.** *Let P be perfectoid, let d generate $\ker \theta_P$, and set*

$$Q = P\langle x_1, \dots, x_r\rangle, \qquad Q_\infty = P\langle x_1^{1/p^\infty}, \dots, x_r^{1/p^\infty}\rangle \quad (r < \infty).$$

The maps

$$\Delta_{Q/W(P^\flat)} \longrightarrow \Delta_{Q_\infty/W(P^\flat)}$$

and

$$\operatorname*{colim}_\varphi \widehat{\Delta_{Q/W(P^\flat)}}_{(p,d)} \longrightarrow \operatorname*{colim}_\varphi \widehat{\Delta_{Q_\infty/W(P^\flat)}}_{(p,d)}$$

are descendable. For $r = 1$, their fiber maps have null second and fourth tensor powers, respectively.

Let R be a connective p-complete derived commutative Q-algebra and put $R' = R\widehat{\otimes}_Q Q_\infty$. Then

$$\Delta_{R/W(P^\flat)} \cong \mathrm{Tot}_{[q]\in\Delta}\, \Delta_{(R')^{\widehat{\otimes}_R(q+1)}/W(P^\flat)}.$$

The same formula holds with completed Frobenius perfections on both sides.

Proof. By proposition 3.4, the several-variable map is a tensor product of the one-variable maps. It suffices to treat $r = 1$.

Let

$$F = \mathrm{fib}(\Delta_{P\langle x\rangle/W(P^\flat)} \to W(P\langle x^{1/p^\infty}\rangle^\flat)).$$

We prove

$$\mathrm{Hom}_{\Delta_{P\langle x\rangle/W(P^\flat)}}\Big(F \otimes_{\Delta_{P\langle x\rangle/W(P^\flat)}} F, \Delta_{P\langle x\rangle/W(P^\flat)}\Big) \in (\mathrm{Mod}_{W(P^\flat)})_{\geq 1}.$$

This mapping object is (p,d)-complete. Put $k = P \otimes_{\mathbf{Z}_p} \mathbf{F}_p$ with the coefficient map of theorem 3.12. Formula (3.2) makes k dualizable over $W(P^\flat)$, so tensoring the displayed mapping object with k gives the mapping object for

$$k[t] \oplus k[t][-1] \longrightarrow k[t^{1/p^\infty}]$$

by corollary 3.13. Its target is free over the connective algebra $k[t]$ on the basis t^q for $q \in \mathbf{Z}[1/p] \cap [0,1)$. The reduced mapping object is therefore 1-connective by lemma 3.14.

Apply lemma 2.10 first to the reduction of the displayed mapping object modulo p, with the element $\varphi^{-1}(d)$, then to the mapping object itself, with p. The ideals $(p, \varphi^{-1}(d))$ and (p, d) define the same topology. The mapping object therefore belongs to degrees at least 1, so the square of the fiber map is null.

For the Frobenius sequence, write $A_n \to B_n$ for its maps of rings, with coefficients twisted by the nth Frobenius. Set $A_\infty = \operatorname{colim}_n A_n$ and $I_n = A_\infty \otimes_{A_n} \operatorname{fib}(A_n \to B_n)$. The fiber at the colimit is $\operatorname{colim}_n I_n$. For $e = 2, 4$ put

$$E_n(e) = \operatorname{Hom}_{A_\infty}(I_n^{\otimes_{A_\infty} e}, A_\infty).$$

The eth tensor power of the fiber map defines a class in $\pi_0 \lim_n E_n(e)$, since the diagonal in the product of e filtered indexing categories is cofinal. For $e = 2$ its restriction to every $\pi_0 E_n(2)$ vanishes by the preceding calculation. As $\pi_0 \lim_n \tau_{\leq 0} E_n(2) = \lim_n \pi_0 E_n(2)$, this class lifts to $\pi_0 \lim_n \tau_{\geq 1} E_n(2)$. The tensor pairing of two such lifts factors through $\lim_n \tau_{\geq 2} E_n(4)$. This limit is 1-connective: it is the fiber of $1 -$ shift on the product of the 2-connective terms. Thus the fourth tensor power is null. Symmetric monoidal completion preserves its nullhomotopy. This proves descendability also for the completed Frobenius perfections. Compare [13, Lemma 11.22].

Finally the pushout property identifies the prismatic algebras of $(R')^{\widehat{\otimes}_R(q+1)}$ with the completed Čech terms of $\Delta_{R/W(P^\flat)} \to \Delta_{R'/W(P^\flat)}$. This map is a base change of the polynomial map, so (5.1) gives the asserted totalization. Inverting Frobenius and completing also preserve pushouts, so the same argument proves the perfected formula. □

Lemma 3.16 ((p, d)-complete faithful flatness of prismatic cohomology for a monic equation)**.** *Let R be perfectoid, let d generate $\ker \theta_R$, and let $f \in R[x]$ be monic of positive degree. The prismatic algebra*

$$\Delta_{\widehat{R[x^{1/p^\infty}]/(f(x))}_{(p)} \,/\, W(R^\flat)}$$

is discrete and (p, d)-completely faithfully flat over $W(R^\flat)$. Multiplication by d on this algebra is injective. Its reduction modulo d contains the root of f given by the image of x under the structure map from $\widehat{R[x^{1/p^\infty}]/(f(x))}_{(p)}$.

Proof. Write $T = \widehat{R[x^{1/p^\infty}]/(f(x))}_{(p)}$. Choose a monic lift $\widetilde{f} \in W(R^\flat)[x]$ and set $P = \widehat{W(R^\flat)[x^{1/p^\infty}]}_{(p,d)}$, with $\delta(x^{1/p^n}) = 0$. By lemma 3.6, a map from T to $D \otimes_{W(R^\flat)} R$ consists of a δ-map $P \to D$ and a nullhomotopy of $\widetilde{f}$ modulo d. The latter datum is an element $y \in D$ with a specified homotopy $dy \sim \widetilde{f}$. Thus the adjunction computes

$$\Delta_{T/W(R^\flat)} = P\{y\} \underset{W(R^\flat)\{z\}}{\widehat{\otimes}} W(R^\flat)_{(p,d)}, \qquad z \longmapsto dy - \widetilde{f}, \quad z \longmapsto 0.$$

The underlying derived algebra of this pushout is obtained by imposing the equations

$$r_n = \delta^n(dy_0 - \widetilde{f}) = 0 \qquad (n \geq 0)$$

in $P[y_0, y_1, \ldots]$.

The identities for δ give $r_n = \varphi^n(d) y_n + H_n(x, y_0, \ldots, y_{n-1})$. Modulo (p, d), the first relation is $-\overline{f}(x)$, and each subsequent relation is a degree-p polynomial in the next variable y_n:

$$\overline{r}_0 = -\overline{f}, \qquad \overline{r}_{n+1} = \overline{\varphi^n(\delta d)} y_n^p + v_n y_n + w_n,$$

where

$$v_n, w_n \in (W(R^\flat)/(p, d))[x^{1/p^\infty}, y_0, \ldots, y_{n-1}].$$

The coefficient of y_n^p is a unit: δd is a unit modulo (p,d), and φ preserves (p,d) modulo p. The term $\delta(H_n)$ is linear in y_n modulo p, as follows by substituting $\varphi(y_{n-1}) = y_{n-1}^p + py_n$.

For every discrete ring k,

$$k[x^{1/p^\infty}] = \bigoplus_{q \in \mathbf{Z}[1/p] \cap [0,1)} k[x]x^q.$$

Taking $k = W(R^\flat)/(p,d)$, the quotient by $\overline{f}$ is therefore free over k with basis x^{q+i} for $q \in \mathbf{Z}[1/p] \cap [0,1)$ and $0 \leq i < \deg f$. Dividing each subsequent equation by its invertible leading coefficient makes it monic of degree p in its new variable. Every finite initial sequence is regular, and its quotient is faithfully flat over $W(R^\flat)/(p,d)$. Filtered colimits give

$$\Delta_{T/W(R^\flat)} \otimes_{W(R^\flat)} W(R^\flat)/(p,d) \quad \text{discrete and faithfully flat over } W(R^\flat)/(p,d).$$

Apply lemmas 2.3 and 2.9. □

3.3 The prismatic proof for monic polynomial root quotients

For a discrete ring D, a universal perfectoid D-algebra is an initial object among perfectoid rings equipped with a map from D. When it exists, write it as $(\widehat{D}_p)_{\mathrm{perfd}}$. For every perfectoid ring P, its defining map induces an isomorphism of discrete spaces

$$\mathrm{Map}_{\mathrm{CAlg}}((\widehat{D}_p)_{\mathrm{perfd}}, P) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{CAlg}}(D, P).$$

Maps into P extend uniquely across ordinary p-adic completion. The same notation applies to a connective input because maps to a discrete ring factor uniquely through its zeroth homotopy group.

Lemma 3.17 (Connective reduction of perfected prismatic cohomology gives the universal perfectoid algebra)**.** *Let P be perfectoid, let d generate $\ker \theta_P$, and let $A \in \mathrm{DAlg}_P^{p\text{-comp}}$ be connective. If*

$$\operatorname*{colim}_{\varphi} \widehat{\Delta_{A/W(P^\flat)}}_{(p,d)} \otimes_{W(P^\flat)} P$$

is connective, it is discrete and perfectoid and is the universal perfectoid A-algebra.

Proof. Frobenius becomes invertible on the colimit and remains invertible on its completion, since $(p, \varphi(d)) = (p, d^p)$. The quotients of the completed Frobenius colimit by d^n have finite filtrations with successive terms its reduction modulo d. They are connective, and their transition maps are surjective on π_0. Completeness therefore makes $\widehat{\operatorname{colim}_\varphi \Delta_{A/W(P^\flat)}}_{(p,d)}$ connective. Its reduction modulo p is a connective derived commutative $\mathbf{F}_p$-algebra with invertible Frobenius. Frobenius acts by zero on π_i for $i > 0$, by proposition 1.11, so this reduction has only π_0, which is perfect. Completeness at p makes the completed Frobenius colimit discrete and p-torsionfree; proposition 1.1 identifies it with the Witt ring of this perfect ring. The image of d is distinguished, so lemma 1.3 proves that its reduction modulo d is discrete and perfectoid.

For every perfectoid P-algebra T, tilting, Frobenius perfection, and proposition 3.3 give

$$\begin{aligned}
&\mathrm{Map}_{\mathrm{PerfAlg}_P}\left(\widehat{\operatorname*{colim}_{\varphi} \Delta_{A/W(P^\flat)}}_{(p,d)} \otimes_{W(P^\flat)} P, T\right) \\
&\quad \cong \mathrm{Map}_{\mathrm{DAlg}^{\delta}_{W(P^\flat)}}(\Delta_{A/W(P^\flat)}, W(T^\flat)) \\
&\quad \cong \mathrm{Map}_{\mathrm{DAlg}_P}(A, T).
\end{aligned}$$

□

Proposition 3.18 (p-complete faithful flatness of the universal perfectoid algebra for a monic quotient of $R[x^{1/p^\infty}]$)**.** *Let R be perfectoid and let $f \in R[x]$ be monic of positive degree. The p-completion of $R[x^{1/p^\infty}]/(f(x))$ admits a universal discrete perfectoid algebra, and the map*

$$R \longrightarrow \left(\widehat{R[x^{1/p^\infty}]/(f(x))}_{(p)}\right)_{\mathrm{perfd}}$$

is p-completely faithfully flat.

Proof. For the prismatic calculation put $T_f = \widehat{R[x^{1/p^\infty}]/(f(x))}_{(p)}$. Choose a generator d of $\ker \theta_R$ and define

$$\Delta^{\mathrm{perf}}_{T_f/W(R^\flat)} = \mathrm{colim}(\Delta_{T_f/W(R^\flat)} \xrightarrow{\widehat{\varphi}} \Delta_{T_f/W(R^\flat)} \xrightarrow{\varphi} \cdots)_{(p,d)}.$$

By lemma 3.16, $\Delta_{T_f/W(R^\flat)}$ is (p,d)-completely faithfully flat over $W(R^\flat)$. The $W(R^\flat)$-structure on the nth term is twisted by $\varphi^n_{W(R^\flat)}$. This is an automorphism, and $(p, \varphi^n_{W(R^\flat)}(d)) = (p, d^{p^n})$ has the same adic topology as (p,d). Each term is therefore (p,d)-completely faithfully flat over $W(R^\flat)$. By lemma 2.7, the completed filtered colimit is (p,d)-completely faithfully flat. Its base change

$$\Delta^{\mathrm{perf}}_{T_f/W(R^\flat)} \otimes_{W(R^\flat)} R$$

is p-completely faithfully flat by corollary 2.5 and discrete by lemma 2.8. Lemma 3.17 identifies this base change with the universal perfectoid algebra of T_f. □

4 André's extension and universal perfectoid algebras

4.1 Splitting monic polynomials over a faithfully flat perfectoid extension

Corollary 4.1 (André's flatness lemma)**.** *For every perfectoid ring R there is a perfectoid R-algebra R' such that $R \to R'$ is p-completely faithfully flat and every monic polynomial of positive degree in $R'[x]$ has a root in R'. In particular every element of R' admits compatible p^nth roots for all $n \geq 0$.*

Proof. For each monic polynomial $f \in R[x]$ of positive degree, put $T_f = \widehat{R[x^{1/p^\infty}]/(f(x))}_{(p)}$ and take the universal perfectoid algebra $(T_f)_{\mathrm{perfd}}$ of proposition 3.18. Form their completed tensor products over finite sets of polynomials and then their filtered colimit in $\mathrm{Mod}_R^{\wedge p}$. By proposition 1.11, lemmas 1.14 and 2.7, and corollary 2.5, the result is a p-completely faithfully flat perfectoid R-algebra in which every monic polynomial over R has a root.

Repeat at successor stages; at a limit stage take the p-completed filtered colimit. Lemma 1.14 and reduction modulo the ordinary R/p preserve perfectoidness and p-complete faithful flatness. At the first uncountable ordinal ω_1, the colimit is already p-complete: ω_1-filtered colimits commute with the countable limit computing completion. Every finite set of coefficients occurs at an earlier stage, and its monic polynomial acquires a root at the next stage. Successive roots of $x^p - c$ give the last assertion. □

Remark 4.2. A discrete ring is *absolutely integrally closed* if every monic polynomial of positive degree over it has a root. Successive division by linear factors shows that every monic polynomial over such a ring splits.

4.2 Universal perfectoid quotients and flatness of their principal kernels

Proposition 4.3 (Existence and base change of universal perfectoid quotients)**.** *For a perfectoid ring R and an ordinary ideal $J \subset R$, put $\widehat{R/J}_p = \lim_n R/(J+p^nR)$. The universal perfectoid algebra $(\widehat{R/J}_p)_{\mathrm{perfd}}$ exists, and $R \to (\widehat{R/J}_p)_{\mathrm{perfd}}$ is surjective. Its kernel is denoted*

$$J_{\mathrm{perfd}}(R) = \ker\bigl(R \to (\widehat{R/J}_p)_{\mathrm{perfd}}\bigr).$$

For every map $R \to R'$ of perfectoid rings,

$$(\widehat{R/J}_p)_{\mathrm{perfd}} \widehat{\otimes}_R R' \cong (\widehat{R'/JR'}_p)_{\mathrm{perfd}}.$$

The quotient $(\widehat{R/J}_p)_{\mathrm{perfd}}$ depends only on $\sqrt{J}$.

Proof. After extending scalars along the map in corollary 4.1, choose $g_0 = g$ and $g_{n+1}^p = g_n$ for every $g \in J$. The two maps

$$R\langle z_g^{1/p^\infty} \mid g \in J\rangle \rightrightarrows R, \qquad z_g^{1/p^n} \longmapsto g_n \quad \text{or} \quad 0,$$

have a completed tensor product that is discrete and perfectoid by proposition 1.11. Its ordinary degree-zero description is the p-completion of the quotient by all g_n. Every map from R/J to a perfectoid ring kills these roots by corollary 1.10. This proves the universal property and surjectivity over that extension. The universal property supplies the identifications on its completed Čech diagram. Corollary 2.16 descends the algebra, and lemma 2.20 detects surjectivity. Base change is immediate. Since perfectoid rings are reduced, $(\widehat{R/J}_p)_{\mathrm{perfd}}$ depends only on $\sqrt{J}$. □

Lemma 4.4 (Generation of a perfectoid ideal from generators of the original ideal)**.** *Let R be perfectoid, let $J \subset R$ be an ideal, and let $\mathcal{G}$ be a set of generators of J. Then*

$$J_{\mathrm{perfd}}(R) = \overline{\sum_{g \in \mathcal{G}} (g)_{\mathrm{perfd}}(R)}^{\,p}.$$

For a discrete R-module M killed elementwise by powers of p, $J_{\mathrm{perfd}}(R)M = 0$ if and only if $(g)_{\mathrm{perfd}}(R)M = 0$ for every $g \in \mathcal{G}$.

Proof. The perfectoid quotient killing all $g \in \mathcal{G}$ is the completed tensor product of their quotients, taken as a filtered colimit over finite subsets. Its ordinary kernel is the indicated closed sum. For an element of a module killed by p^n, the action of a p-adic closure is tested modulo p^n. □

Lemma 4.5 (Flatness of $\ker(R \to (\widehat{R/c}_p)_{\mathrm{perfd}})$)**.** *Let R be perfectoid and $c \in R$. The ideal $(c)_{\mathrm{perfd}}(R) = \ker(R \to (\widehat{R/c}_p)_{\mathrm{perfd}})$ is a discrete, p-completely flat object of $\mathrm{Mod}_R^{\wedge p}$. If $c_0 = c$ and $c_{n+1}^p = c_n$ in R, then*

$$(c)_{\mathrm{perfd}}(R) = (\widehat{c_n \mid n \geq 0})_{(p)},$$
$$(c_n \mid n \geq 0) = \operatorname{colim}\left(R \xrightarrow{c_1^{p-1}} R \xrightarrow{c_2^{p-1}} R \longrightarrow \cdots\right).$$

The nth copy maps to R by multiplication by c_n.

Proof. Suppose first that $c_0 = c$ and $c_{n+1}^p = c_n$ in R. Put $A = R[x^{1/p^\infty}]$ and $x_n = x^{1/p^n}$. The kernel of the augmentation $A \to R$, $x_n \mapsto 0$, is the union of the ideals $x_n A$. Each x_n is a nonzerodivisor in A. Consequently this kernel is the colimit of copies of A with transitions given by multiplication by x_{n+1}^{p-1}.

Tensor the fiber sequence $(x_n \mid n \geq 0) \to A \to R$ over A with R via $x_n \mapsto c_n$ and take p-completions. Its last term is the completed tensor product of the two copies of R over $R\langle x^{1/p^\infty}\rangle$. The proof of proposition 4.3 identifies this tensor product with the discrete perfectoid quotient $(\widehat{R/c_p})_{\mathrm{perfd}}$. The first term is therefore $(c)_{\mathrm{perfd}}(R)$, and the fiber sequence identifies it with the p-completion of the displayed colimit of copies of R.

The map from that colimit to R has image $(c_n \mid n \geq 0)$. If $ac_n = 0$, then $(ac_{n+1})^p = 0$, so $ac_{n+1} = 0$ by corollary 1.10. Thus $ac_{n+1}^{p-1} = 0$, proving injectivity. The completion of the colimit is p-completely flat by lemma 2.6. For general c, choose a p-completely faithfully flat map $R \to R'$ as in corollary 4.1. By proposition 4.3, $(c)_{\mathrm{perfd}}(R)\widehat{\otimes}_R R' \cong (c)_{\mathrm{perfd}}(R')$. Descend p-complete flatness by lemma 2.18. □

4.3 Universal perfectoid algebras of integral algebras

Lemma 4.6 (The universal perfectoid algebra of a monic algebra with a linear factor)**.** *Let R be perfectoid, let $b \in R$, and let $q \in R[x]$ be monic. Suppose $R[x]/q$ admits a universal perfectoid algebra. Then $R[x]/((x-b)q)$ admits a universal perfectoid algebra, and*

$$(R[x]/((x-b)q))_{\mathrm{perfd}} \cong (R[x]/q)_{\mathrm{perfd}} \times_{(\widehat{R/q(b)}_p)_{\mathrm{perfd}}} R. \tag{4.1}$$

Proof. Put $A = R[x]/((x-b)q)$, $s = x - b$ and $t = q(x)$. For a perfectoid A-algebra T, write $T_s = (\widehat{T/sT}_p)_{\mathrm{perfd}}$ and $I_s = \mathrm{fib}(T \to T_s)$, and use the same notation for t. We first show that $I_s \widehat{\otimes}_T I_t = 0$. This can be checked after the p-completely faithfully flat map $T \to T'$ of corollary 4.1, by proposition 4.3 and lemma 2.12. Choose compatible p^nth roots s_n, t_n of s, t in T'. By lemma 4.5, $(I_s \widehat{\otimes}_T I_t)\widehat{\otimes}_T T'$ is the p-completion of the colimit of copies of T' with transition maps given by multiplication by $(s_{n+1}t_{n+1})^{p-1}$, using the cofinal diagonal in $\mathbf{N}^2$. These maps vanish because $(s_n t_n)^{p^n} = st = 0$ and T' is reduced.

The total fiber of the square from T to T_s, T_t and $T_s \widehat{\otimes}_T T_t$ is $I_s \widehat{\otimes}_T I_t$, so the square is cartesian. Take its limit over the perfectoid A-algebras T. The functor $T \mapsto T_s$ is left adjoint to the inclusion of perfectoid A/s-algebras and hence is initial for limits. The same holds for t and for the simultaneous quotient $T_s \widehat{\otimes}_T T_t$. The resulting limit of the perfectoid A-algebras is the pullback in (4.1): the three quotient algebras of A are $A/(s) \cong R$, $A/(t) \cong R[x]/q$, and $A/(s,t) \cong R/q(b)$, each of which has an initial perfectoid algebra. The map $R \to (\widehat{R/q(b)}_p)_{\mathrm{perfd}}$ is surjective, so this pullback is perfectoid by corollary 1.8. It inherits the map from A and is therefore initial among perfectoid A-algebras. Indeed, its limit projections give maps to every such algebra. The projection to the limit object itself is the identity by the universal property of the limit. Naturality then forces every map from that object to a perfectoid A-algebra to be the corresponding projection; all these mapping spaces are discrete. □

Proposition 4.7 (Existence of universal perfectoid algebras for integral algebras)**.** *Let R be perfectoid and S a discrete integral R-algebra. There is an initial perfectoid S-algebra $(\widehat{S}_p)_{\mathrm{perfd}}$. The functor $S \mapsto (\widehat{S}_p)_{\mathrm{perfd}}$ takes filtered colimits of integral R-algebras to p-completed filtered colimits and commutes with base change to perfectoid R-algebras.*

Bhatt–Scholze [1, Theorem 10.11].

Proof. Choose $R \to R'$ by corollary 4.1. Over R', every monic polynomial splits. Induction on its degree using (4.1) constructs the universal perfectoid algebra of every monic algebra $R'[x]/f$. A finitely generated integral R'-algebra is a quotient of a tensor product of finitely many such algebras. These tensor products are discrete since the factors are finite free. Take the completed tensor product of the universal perfectoid algebras and then its perfectoid quotient by the relations, using propositions 1.11 and 4.3. Completed filtered colimits of these perfectoid rings give the universal algebra for every integral R'-algebra by lemma 1.14.

If B is the initial perfectoid algebra of an integral R'-algebra D, then $B\widehat{\otimes}_{R'}P$ is the initial perfectoid algebra of $D \otimes^{\mathrm{ord}}_{R'} P$ for every perfectoid R'-algebra P, by proposition 1.11 and the universal mapping properties of the two tensor products. Apply this to $D = S \otimes^{\mathrm{ord}}_{R} R'$ on the Čech nerve of $R \to R'$. Effective descent of perfectoid algebras (corollary 2.16) descends B to R. The representing property follows by faithfully flat descent on each perfectoid test. It gives the asserted compatibility with base change and filtered colimits over R. □

For a diagram $D \leftarrow E \to F$ of ordinary integral R-algebras, their initial perfectoid algebras satisfy

$$\widehat{(D \otimes^{\mathrm{ord}}_{E} F}_p)_{\mathrm{perfd}} \;\cong\; (\widehat{D}_p)_{\mathrm{perfd}} \widehat{\otimes}_{(\widehat{E}_p)_{\mathrm{perfd}}} (\widehat{F}_p)_{\mathrm{perfd}}. \tag{4.2}$$

Indeed, maps from either side to a perfectoid R-algebra are pairs of maps from D and F whose restrictions to E agree.

5 Descendability and abstract excision

5.1 Descendability

Let $\mathcal{D} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$. Write

$$\mathrm{Mod}(\mathcal{D}) = \mathrm{Mod}_{\mathcal{D}}(\mathrm{Pr}^L_{\mathrm{st}}), \qquad 2\,\mathrm{QC}(\mathcal{D}) = \mathrm{Mod}(\mathcal{D}).$$

Thus $2\,\mathrm{QC}(\mathcal{D})$ is the category of presentable stable $\mathcal{D}$-linear categories and colimit-preserving $\mathcal{D}$-linear functors. For $B \in \mathrm{CAlg}(\mathcal{D})$ and $\mathcal{C} \in \mathrm{Mod}(\mathcal{D})$, write $\mathrm{Mod}_B(\mathcal{C})$ for the category of B-module objects in $\mathcal{C}$. All the general results in this section are used without proof.

Definition 5.1 (Descendability and absolute limits)**.** An algebra $B \in \mathrm{CAlg}(\mathcal{D})$ is *descendable* if $\mathbf{1}$ belongs to the thick tensor ideal generated by B. This ideal is the smallest stable full subcategory of $\mathcal{D}$ containing B and closed under retracts and tensor products with arbitrary objects of $\mathcal{D}$. An augmented diagram in $\mathcal{D}$ is an *absolute limit diagram for exact functors* if every exact functor from $\mathcal{D}$ to an idempotent-complete stable category carries it to a limit diagram.

Theorem 5.2 (Criteria for descendability)**.** *For $B \in \mathrm{CAlg}(\mathcal{D})$, put*

$$v : I = \mathrm{fib}(\mathbf{1} \longrightarrow B) \longrightarrow \mathbf{1}, \qquad B^{[q]} = B^{\otimes(q+1)} \quad (q \geq 0).$$

The following conditions are equivalent.

(i) *B is descendable.*

(ii) *For some integer $n \geq 1$, the morphism $v^{\otimes n} : I^{\otimes n} \to \mathbf{1}$ is null.*

(iii) *The canonical coaugmentation*

$$\mathbf{1} \longrightarrow \{\mathrm{Tot}_{\leq n}(B^{[\bullet]})\}_{n \geq 0}$$

is an isomorphism of pro-objects, where the source is the constant pro-object.

(iv) *The augmented Čech diagram $\mathbf{1} \to B^{[\bullet]}$ is an absolute limit diagram for exact functors.*

These criteria are preserved by colimit-preserving symmetric monoidal functors. Descendable algebra maps compose; if a composite $A \to B \to C$ in $\mathrm{CAlg}(\mathcal{D})$ is descendable, then $A \to B$ is descendable. Here a map out of A is tested in $\mathrm{Mod}_A(\mathcal{D})$. See [14, §3, especially Proposition 3.20].

Proposition 5.3 (Tensor exponents and completion)**.** *With v as above, suppose that $M' \to M \to M''$ is a fiber sequence in a stable $\mathcal{D}$-linear category, and let $a, b \geq 0$ be integers. If $v^{\otimes a} \otimes M'$ and $v^{\otimes b} \otimes M''$ are null, then $v^{\otimes(a+b)} \otimes M$ is null.*

Let $A \to B$ be a map of K-complete commutative rings, where $K \subset \pi_0 A$ is a finitely generated ideal and B is complete for its extension. For $I = \mathrm{fib}(A \to B) \in \mathrm{Mod}_A$, completion induces

$$\mathrm{Hom}_{\mathrm{Mod}_A^{\wedge K}}(\widehat{I^{\otimes_A n}}_K, A) \cong \mathrm{Hom}_A(I^{\otimes_A n}, A).$$

Consequently ordinary and completed tensor powers of the fiber map have the same nullity criterion.

Theorem 5.4 (Descent for modules and categories of modules)**.** *Suppose that $B \in \mathrm{CAlg}(\mathcal{D})$ is descendable, and put $B^{[q]} = B^{\otimes(q+1)}$. For every $\mathcal{C} \in \mathrm{Mod}(\mathcal{D})$, extension of scalars gives an isomorphism*

$$\mathcal{C} \xrightarrow{\sim} \lim_{[q]\in\Delta} \mathrm{Mod}_{B^{[q]}}(\mathcal{C}). \tag{5.1}$$

Extension of scalars also gives descent for categories of linear categories:

$$2\,\mathrm{QC}(\mathcal{D}) \xrightarrow{\sim} \lim_{[q]\in\Delta} 2\,\mathrm{QC}(\mathrm{Mod}_{B^{[q]}}(\mathcal{D})). \tag{5.2}$$

The transition functors in both limits are extension of scalars.

The corresponding colimit with restriction of scalars is

$$\operatorname*{colim}_{[q]\in\Delta^{\mathrm{op}}} \mathrm{Mod}_{B^{[q]}}(\mathcal{D}) \xrightarrow{\sim} \mathcal{D} \qquad \text{in } \mathrm{Mod}(\mathcal{D}). \tag{5.3}$$

Here every transition functor and every map to $\mathcal{D}$ forgets the corresponding algebra action. For an arbitrary B, the colimit identity (5.3) is equivalent to (5.1) for every $\mathcal{C} \in \mathrm{Mod}(\mathcal{D})$.

For $\mathcal{E} \in \mathrm{CAlg}(\mathrm{Mod}(\mathcal{D}))$, let $X_{\mathcal{E}}$ denote its object in

$$\mathrm{CAlg}\big(\mathrm{Mod}(\mathcal{D})\big)^{\mathrm{op}}.$$

There is also the affine Čech colimit

$$\operatorname*{colim}_{[q]\in\Delta^{\mathrm{op}}} X_{\mathrm{Mod}_{B^{[q]}}(\mathcal{D})} \xrightarrow{\sim} X_{\mathcal{D}}. \tag{5.4}$$

This last colimit is preserved by every base change in the displayed category of coefficient affines. These are the relative forms of the categorical descent statements in [14, §3.7].

Remark 5.5. When $\mathcal{D} = \mathrm{Mod}_R$ for $R \in \mathrm{CAlg}(\mathrm{Sp})$, each of the following conversely implies descendability: descent (5.1) for every presentable stable R-linear category, descent (5.2) for categories of modules, the restriction colimit (5.3), or the universally preserved affine colimit (5.4). Thus in this case these conditions and the absolute limit and tensor-ideal criteria are equivalent [14, Corollary 3.42 and Propositions 3.43, 3.45].

5.2 Finite excision

Theorem 5.6 (Abstract Zariski excision)**.** *Let $\mathcal{D} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ and let $A, B \in \mathrm{CAlg}(\mathcal{D})$ be idempotent. The following conditions are equivalent.*

(i) *The functor $(A \times B) \otimes - : \mathcal{D} \to \mathcal{D}$ is conservative.*

(ii) *The tensor unit $\mathbf{1}$ belongs to the thick tensor ideal of $\mathcal{D}$ generated by $A \times B$.*

(iii) *The canonical square*

$$\begin{array}{ccc} \mathbf{1} & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & A \otimes B \end{array} \tag{5.5}$$

is cartesian in $\mathrm{CAlg}(\mathcal{D})$.

(iv) *Extension of scalars gives $2\,\mathrm{QC}$ descent, that is, an isomorphism of categories*

$$\mathrm{Mod}(\mathcal{D}) \xrightarrow{\sim} \mathrm{Mod}(\mathrm{Mod}_A(\mathcal{D})) \underset{\mathrm{Mod}(\mathrm{Mod}_{A\otimes B}(\mathcal{D}))}{\times} \mathrm{Mod}(\mathrm{Mod}_B(\mathcal{D})). \tag{5.6}$$

(v) *For every $\mathcal{C} \in \mathrm{Mod}(\mathcal{D})$, extension of scalars induces an isomorphism in $\mathrm{Pr}^L_{\mathrm{st}}$*

$$\mathcal{C} \xrightarrow{\sim} \mathrm{Mod}_A(\mathcal{C}) \times_{\mathrm{Mod}_{A\otimes B}(\mathcal{C})} \mathrm{Mod}_B(\mathcal{C}). \tag{5.7}$$

(vi) *In $\mathrm{CAlg}(\mathrm{Mod}(\mathcal{D}))^{\mathrm{op}}$, the canonical map*

$$X_{\mathrm{Mod}_A(\mathcal{D})} \amalg_{X_{\mathrm{Mod}_{A\otimes B}(\mathcal{D})}} X_{\mathrm{Mod}_B(\mathcal{D})} \longrightarrow X_{\mathcal{D}}$$

is an isomorphism. This colimit identity remains true after every base change in that category.

In condition (iii)*, the underlying square is an absolute limit diagram for exact functors. All six conditions remain true after any colimit-preserving symmetric monoidal change of coefficient category.*

Theorem 5.7 (Abstract Nisnevich excision)**.** *Let $\mathcal{D} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ and let $A, B \in \mathrm{CAlg}(\mathcal{D})$. Assume that A is idempotent. Write $\mu : B \otimes B \to B$ for multiplication, and suppose the map*

$$\mathrm{fib}(\mathbf{1} \to A) \otimes (B \otimes B) \xrightarrow{\mathrm{id}\otimes\mu} \mathrm{fib}(\mathbf{1} \to A) \otimes B$$

is an isomorphism. Then conditions (i)–(vi) *of theorem 5.6 are equivalent. For idempotent A, the cartesian square* (5.5) *already implies* (5.7) *for every $B \in \mathrm{CAlg}(\mathcal{D})$ and every $\mathcal{C} \in \mathrm{Mod}(\mathcal{D})$.*

Corollary 5.8 (Finite families of idempotent algebras)**.** *Let $n \geq 1$ and let $A_1, \ldots, A_n$ be idempotent commutative algebras in $\mathcal{D}$. Then conservativity of $(\prod_i A_i) \otimes -$ is equivalent to membership of $\mathbf{1}$ in the thick tensor ideal generated by $\prod_i A_i$. Under these conditions the finite-family form of theorem 5.6 gives, for every $\mathcal{C} \in \mathrm{Mod}(\mathcal{D})$,*

$$\mathbf{1} \cong \lim_{\varnothing \neq L \subseteq \{1,\ldots,n\}} \bigotimes_{i\in L} A_i, \qquad \mathcal{C} \cong \lim_{\varnothing \neq L \subseteq \{1,\ldots,n\}} \mathrm{Mod}_{\bigotimes_{i\in L} A_i}(\mathcal{C}).$$

The first limit is absolute for exact functors. The analogous limit of $2\,\mathrm{QC}$ categories and the corresponding affine colimit hold, universally after change of coefficient category.

The quotient by modules over an idempotent algebra. Let $\mathcal{C} \in \mathrm{Mod}(\mathcal{D})$. For an idempotent $A \in \mathrm{CAlg}(\mathcal{D})$ put $I = \mathrm{fib}(\mathbf{1} \to A)$. For $M \in \mathcal{C}$, the fiber sequence $I \otimes M \to M \to A \otimes M$ identifies $\ker(A \otimes -)$ with the full subcategory on which $I \otimes M \to M$ is an isomorphism. Tensoring with I realizes the quotient $\mathcal{C}/\operatorname{Mod}_A(\mathcal{C})$ as this kernel. For $\mathcal{C} = \mathcal{D}$, its tensor product is inherited from $\mathcal{D}$ and its tensor unit is I. For any $B \in \mathrm{CAlg}(\mathcal{D})$, extension of scalars induces

$$\ker(\mathcal{D} \to \mathrm{Mod}_A(\mathcal{D})) \longrightarrow \ker(\mathrm{Mod}_B(\mathcal{D}) \to \mathrm{Mod}_{A\otimes B}(\mathcal{D})). \tag{5.8}$$

This functor is an isomorphism precisely when $I \to B \otimes I$ is an isomorphism, or equivalently when (5.5) is cartesian.

6 Arc topology

6.1 Arc covers

Definition 6.1 (Arc covers)**.** A map $A \to B$ of ordinary commutative rings is an *arc-cover* if every map $A \to V$ to a valuation ring of rank at most one fits in a commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ V & \longrightarrow & W, \end{array}$$

where $V \to W$ is a faithfully flat extension of valuation rings of rank at most one. Fields are included. The arc topology on ordinary affine schemes is generated by these maps; a finite family is covering when its disjoint union is covering. A *v-cover* satisfies the same condition without the rank bound. For connective commutative rings the condition is imposed on $\pi_0 A \to \pi_0 B$.

For ordinary affine schemes, arc covers are precisely universal spectral submersions [6, Proposition 2.19]. The following idempotent formulation is a consequence.

Proposition 6.2 (Universal descent of idempotents)**.** *A map $A \to B$ of ordinary rings is an arc-cover if and only if, for every ordinary A-algebra C, the map*

$$\mathrm{Idem}(C) \longrightarrow \mathrm{Eq}_{\mathrm{Spaces}}\Big(\mathrm{Idem}(C \otimes_A^{\mathrm{ord}} B) \rightrightarrows \mathrm{Idem}(C \otimes_A^{\mathrm{ord}} B \otimes_A^{\mathrm{ord}} B)\Big) \tag{6.1}$$

is an isomorphism. Here $\mathrm{Idem}(C) = \{e \in C : e^2 = e\}$ is a discrete space. Equivalently, for every ordinary A-algebra C, the map on prime spectra induced by $C \to C \otimes_A^{\mathrm{ord}} B$ is a spectral submersion: it is surjective, and a subset of its target is open whenever its inverse image is quasi-compact open.

Proof. Idempotents correspond to open-and-closed subsets. A compatible idempotent on a cover specifies such a subset saturated over the target. A spectral submersion descends it and its complement, proving the equalizer condition. Conversely the universal equalizer condition forces surjectivity, by base change to residue fields. Over a rank-one valuation ring, failure of the valuation lifting condition means that no specialization in the source lifts the generic-to-closed specialization. The generic fiber is then a quasi-compact open stable under specialization, hence open and closed. Its nonconstant idempotent is saturated but cannot descend to the connected valuation spectrum. This contradiction proves the lifting condition; the field case is surjectivity. □

Corollary 6.3 (Integral surjections are arc covers)**.** *For an integral map $A \to B$, the map is an arc-cover if and only if every prime ideal of A is the contraction of a prime ideal of B. In particular, if $I_1 \cdots I_n \subset \sqrt{0}$, then $A \to \prod_i A/I_i$ is an arc-cover.*

Proof. Every base change of an integral surjection on spectra is a closed surjection, hence a spectral submersion. Conversely, residue-field tests detect surjectivity. For the last assertion, every prime contains at least one I_i. □

For a discrete $\mathbf{F}_p$-algebra D, write $D^{\mathrm{perf}} = \mathrm{colim}(D \xrightarrow{x \mapsto x^p} D \to \cdots)$. By proposition 1.11, tensor products of perfect discrete $\mathbf{F}_p$-algebras over a perfect ring are discrete and perfect. For maps of perfect rings, the valuation rings in the definition of a cover may also be taken perfect: adjoining their unique p-power roots preserves their ranks and faithfully flat extensions.

We will use one elementary identity for valuation rings. If V is a valuation ring and $\mathfrak{r} \subset V$ is prime, then $\mathfrak{r}V_{\mathfrak{r}} = \mathfrak{r}$ inside its fraction field. Indeed, for $x \in \mathfrak{r}$ and $s \notin \mathfrak{r}$, the valuation property gives $x/s \in V$, and primality gives $x/s \in \mathfrak{r}$. Consequently

$$V \cong V_{\mathfrak{r}} \times_{\kappa(\mathfrak{r})} V/\mathfrak{r}. \tag{6.2}$$

6.2 Perfect rings and elementary descent

Theorem 6.4 (The perfected plane is the affinization of the perfected blowup)**.** *Let $k = \mathbf{F}_p$. The morphism*

$$\pi : (\mathrm{Bl}_0 \mathbb{A}^2_k)^{\mathrm{perf}} \longrightarrow (\mathbb{A}^2_k)^{\mathrm{perf}}$$

is the affinization morphism. Its fiber over the origin is $(\mathbf{P}^1_k)^{\mathrm{perf}}$. In the chart coordinates $y = xt$ and $x = ys$, the isomorphism $\mathcal{O}_{(\mathbb{A}^2_k)^{\mathrm{perf}}} \xrightarrow{\sim} \pi_* \mathcal{O}_{(\mathrm{Bl}_0 \mathbb{A}^2_k)^{\mathrm{perf}}}$ is expressed on global sections by the pullback square of E_∞-rings

$$\begin{array}{ccc} k[x,y]^{\mathrm{perf}} & \xrightarrow{y \mapsto xt} & k[x,t]^{\mathrm{perf}} \\ {\scriptstyle x \mapsto ys}\downarrow & & \downarrow{\scriptstyle \text{invert } t} \\ k[y,s]^{\mathrm{perf}} & \xrightarrow[y \mapsto xt,\ s \mapsto t^{-1}]{} & k[x,t^{\pm 1}]^{\mathrm{perf}}. \end{array} \tag{6.3}$$

Proof. The Rees algebra $k[x,y][xz,yz] \subset k[x,y,z]$ has presentation $k[x,y,u,v]/(xv - yu)$ by $u \mapsto xz$ and $v \mapsto yz$. Thus the blowup is the incidence scheme $xv = yu$ in $\mathbb{A}^2_k \times \mathbf{P}^1_k$. Setting $x = y = 0$ gives $\mathbf{P}^1_k$, and perfection commutes with this fiber product. The opens $u \neq 0$ and $v \neq 0$ of the perfected blowup give the square

$$\begin{array}{ccc} \mathrm{Spf}(k[x,t^{\pm 1}]^{\mathrm{perf}}) & \hookrightarrow & \mathrm{Spf}(k[x,t]^{\mathrm{perf}}) \\ \downarrow & & \downarrow{\scriptstyle u \neq 0} \\ \mathrm{Spf}(k[y,s]^{\mathrm{perf}}) & \xrightarrow[v \neq 0]{} & (\mathrm{Bl}_0 \mathbb{A}^2_k)^{\mathrm{perf}} \xrightarrow{\pi} (\mathbb{A}^2_k)^{\mathrm{perf}}. \end{array} \tag{6.4}$$

Here $t = v/u$ and $s = u/v$. By Zariski descent, the global sections of the perfected blowup form the pullback of the two chart rings over $k[x,t^{\pm 1}]^{\mathrm{perf}}$. Since the target of π is affine, it suffices to identify the map from $k[x,y]^{\mathrm{perf}}$ to this pullback.

The overlap has basis $x^r t^q$ for $r \in \mathbf{Z}[1/p]_{\geq 0}$ and $q \in \mathbf{Z}[1/p]$. The first chart contains the basis monomials with $q \geq 0$, and the second those with $q \leq r$. The difference of the two restriction maps is therefore surjective. Its kernel has basis the common monomials $x^r t^q = x^{r-q} y^q$ for $0 \leq q \leq r$. These are precisely the images of the monomial basis of $k[x,y]^{\mathrm{perf}}$ under pullback along π. □

Corollary 6.5 (Pullback of the chart rings over a perfect ring)**.** *Let D be a perfect $\mathbf{F}_p$-algebra and $a, b \in D$. The square*

$$\begin{array}{ccc} D & \longrightarrow & (D[t]/(at-b))^{\mathrm{perf}} \\ \downarrow & & \downarrow \\ (D[s]/(bs-a))^{\mathrm{perf}} & \xrightarrow[s\mapsto t^{-1}]{} & (D[t^{\pm 1}]/(at-b))^{\mathrm{perf}} \end{array} \tag{6.5}$$

is cartesian in E_∞-rings. The product of the two chart algebras is descendable over D.

Proof. Tensor (6.3) along $\mathbf{F}_p[x,y]^{\mathrm{perf}} \to D$, $x \mapsto a$, $y \mapsto b$. Tensoring preserves the pullback of modules, and proposition 1.11 identifies the resulting rings with those in (6.5). The two chart rings and the ring of their intersection are modules over the product of the chart rings. Their pullback D therefore belongs to the thick tensor ideal generated by that product. □

Lemma 6.6 (Reduction to perfect valuation rings of rank at most one)**.** *Let A be a perfect $\mathbf{F}_p$-algebra and let $\mathcal{P}$ be a property of perfect A-algebras preserved under extension of scalars. Assume that $\mathcal{P}$ holds for the zero ring and satisfies the following conditions.*

(i) If $\mathcal{P}$ holds for a filtered colimit, it holds at some stage.

(ii) For $a, b \in D$, it suffices to check $\mathcal{P}(D)$ on the two rings $(D[t]/(at-b))^{\mathrm{perf}}$ and $(D[s]/(bs-a))^{\mathrm{perf}}$.

(iii) For a perfect valuation A-algebra V and a prime $\mathfrak{r} \subset V$, it suffices to check $\mathcal{P}(V)$ on $V_{\mathfrak{r}}$ and $V/\mathfrak{r}$.

If $\mathcal{P}$ holds for all perfect valuation A-algebras of rank at most one, it holds for every perfect A-algebra.

Proof. Suppose $\mathcal{P}(D)$ fails. By (ii), for each $a, b \in D$ we can make a divide b or b divide a while preserving this failure. Well-order the pairs, perform the extensions successively, and use (i) at limit ordinals. Repeat for the new rings and take the colimit after countably many rounds. In the resulting nonzero perfect ring V, principal ideals are comparable. If $ab = 0$ and $a = bc$, then $a^2 = 0$, hence $a = 0$. Interchanging a, b proves that V is a domain, hence a valuation ring.

Well-order the primes of V. Start with $[0, \mathfrak{m}_V]$. Retain an interval $[\mathfrak{p}, \mathfrak{q}]$ on which $\mathcal{P}((V/\mathfrak{p})_{\mathfrak{q}/\mathfrak{p}})$ fails. When an interior prime $\mathfrak{r}$ is considered, (iii) permits us to retain one of $[\mathfrak{p}, \mathfrak{r}]$ and $[\mathfrak{r}, \mathfrak{q}]$. At a limit ordinal the corresponding ring is the filtered colimit, with endpoints $\bigcup_\alpha \mathfrak{p}_\alpha$ and $\bigcap_\alpha \mathfrak{q}_\alpha$. By (i) the failure persists. The final interval has no interior prime, which gives a counterexample of rank at most one. □

For family (ii) of lemma 6.6, the product of the chart rings is descendable by corollary 6.5. For family (iii), (6.2) gives the ordinary pullback

$$\begin{array}{ccc} V & \longrightarrow & V_{\mathfrak{r}} \\ \downarrow & & \downarrow \\ V/\mathfrak{r} & \longrightarrow & \kappa(\mathfrak{r}). \end{array} \tag{6.6}$$

The map $V_{\mathfrak{r}} \to \kappa(\mathfrak{r})$ is surjective, so this is also a pullback of E_∞-rings. Its other three terms are modules over $V_{\mathfrak{r}} \times V/\mathfrak{r}$; hence V belongs to the thick tensor ideal generated by this product, which proves its descendability.

Lemma 6.7 (Testing a map of functors on valuation rings)**.** *Let F, G be functors from perfect A-algebras to* Spaces*, where A is a perfect $\mathbf{F}_p$-algebra. Suppose F and G preserve filtered colimits, send the zero ring to a point, and satisfy Čech descent for the two families in lemma 6.6(ii),(iii). Let $F \to G$ have n-truncated fibers for a fixed n. If $F(V) \to G(V)$ is an isomorphism for every perfect valuation A-algebra V of rank at most one, then $F \to G$ is an isomorphism.*

Proof. First suppose the fibers are empty or contractible. For a perfect A-algebra D and $y \in G(D)$, apply lemma 6.6 over D to the existence of a lift of y. The fibers preserve filtered colimits. A lift on each member of a covering family gives a lift on every member of its Čech nerve. The corresponding fibers are contractible, so their limit is contractible by descent. The valuation hypothesis now gives a lift over D. For $n \geq 0$, apply induction to the diagonal $F \to F \times_G F$, whose fibers are $(n-1)$-truncated. Consequently the fibers of $F \to G$ are empty or contractible, and the first case applies. □

Proposition 6.8 (Arc descent from faithfully flat descent and elementary covers)**.** *Let U be a perfect $\mathbf{F}_p$-algebra and let F be a sheaf of spaces on perfect affine U-schemes for the faithfully flat topology. Suppose that F preserves filtered colimits of perfect U-algebras, takes values in* $\mathrm{Spaces}_{\leq n}$ *for some integer $n \geq 0$, and satisfies Čech descent for the families in lemma 6.6(ii),(iii) and their base changes. Then F is an arc-sheaf.*

Proof. For an arc-cover $A \to B$ of perfect U-algebras, apply lemma 6.7 to

$$F(D) \longrightarrow \lim_{[q] \in \Delta} F(D \otimes_A B^{\otimes_A (q+1)}), \qquad D \text{ a perfect } A\text{-algebra.}$$

Both sides satisfy descent for the elementary families. The right side preserves filtered colimits because totalizations of n-truncated spaces are determined through cosimplicial degree $n+1$. For a perfect valuation A-algebra D of rank at most one, the arc-cover provides $B \to W$ over a faithfully flat extension $D \to W$ of perfect valuation rings. The map $W \to W \otimes_A B$ has a retraction. The comparison is therefore an isomorphism over W and over every Čech term of W/D. Faithfully flat descent proves the comparison over D, and lemma 6.7 proves it for every D. □

The same criterion applies to presheaves of categories whose spaces of objects and arrows satisfy its hypotheses.

For a connective commutative ring L, let $\mathrm{Perf}(L) \subset \mathrm{Mod}_L$ be the full subcategory of dualizable modules. For integers $r \leq s$, write $\mathrm{Perf}^{[r,s]}(L)$ for its full subcategory consisting of modules M such that

$$\pi_i(M \otimes_L N) = 0 \quad \text{for every discrete } L\text{-module } N \text{ and every } i \notin [r, s].$$

Proposition 6.9 (Arc descent for perfect modules after base change from a perfect ring)**.** *Let $U \to V$ be an arc-cover or a v-cover of perfect $\mathbf{F}_p$-algebras. For every connective commutative U-algebra L whose underlying U-module is perfect, extension of scalars induces a symmetric monoidal isomorphism*

$$\mathrm{Perf}(L) \longrightarrow \lim_{[q] \in \Delta} \mathrm{Perf}(L \otimes_U V^{\otimes_U (q+1)}).$$

Proof. Fix integers $r \leq s$. Apply the categorical form of proposition 6.8 to

$$D \longmapsto \mathrm{Perf}^{[r,s]}(L \otimes_U D), \qquad D \text{ a perfect } U\text{-algebra.}$$

The stacks on connective L-algebras classifying perfect modules with Tor-amplitude in $[r,s]$, and arrows between these modules, are geometric and locally of finite presentation by Antieau–Gepner [5, Theorems 5.2 and 5.6]. Their functors of points therefore preserve filtered colimits.

Choose d such that the U-module L has Tor-amplitude in $[0,d]$. For $M,N \in \mathrm{Perf}^{[r,s]}(L\otimes_U D)$,

$$\pi_i \operatorname{Hom}_{L\otimes_U D}(M,N) = 0 \qquad (i > d+s-r).$$

Indeed, $M^\vee \otimes_{L\otimes_U D} N$ has Tor-amplitude in $[r-s, s-r]$, and $\pi_i(L\otimes_U D) = 0$ for $i > d$. Thus the spaces of objects and arrows are $(d+s-r+1)$-truncated.

Let $D \to E$ be a base change, among perfect U-algebras, of the map to the product of either family in lemma 6.6(ii),(iii). This map is descendable, so (5.1) gives

$$\mathrm{Mod}_{L\otimes_U D} \cong \lim_{[q]\in\Delta} \mathrm{Mod}_{L\otimes_U E^{\otimes_D (q+1)}}.$$

Taking dualizable objects gives descent for perfect modules. Testing Tor-amplitude over residue fields restricts this isomorphism to $\mathrm{Perf}^{[r,s]}$, since $\operatorname{Spf} E \to \operatorname{Spf} D$ is surjective. Faithfully flat descent follows in the same way from lemma 2.12. The criterion gives

$$\mathrm{Perf}^{[r,s]}(L\otimes_U D) \;\cong\; \lim_{[q]\in\Delta} \mathrm{Perf}^{[r,s]}\Big(L\otimes_U D\otimes_U V^{\otimes_U (q+1)}\Big). \tag{6.7}$$

Every perfect module in a descent datum has the Tor-amplitude bound of its degree-zero term. Taking the union over $[r,s]$ proves the assertion. □

A map $A \to B$ of connective p-complete rings is a *p-complete arc-cover* if

$$\pi_0 A \longrightarrow \pi_0 B \times (\pi_0 A)[1/p]$$

is an arc-cover in the sense of definition 6.1. Equivalently, every map $A \to V$ to a p-complete valuation ring of rank at most one extends to a commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ V & \longrightarrow & W \end{array}$$

in which $V \to W$ is a faithfully flat extension of p-complete valuation rings of rank at most one. Fields are included among the valuation rings. To prove the equivalence, use proposition 6.2. A valuation test in which p is a unit lifts through $(\pi_0 A)[1/p]$. In every other test, $p = 0$ or completion at p is a faithfully flat extension of valuation rings of rank one [6, Proposition 6.2]. Thus these tests may be completed before applying the displayed lifting condition. Conversely, for a p-complete valuation test, a lift to the product must factor through $\pi_0 B$, since p remains a nonunit in a faithfully flat valuation extension; completing that extension gives the required p-complete lift. The lifting condition is preserved by composition and completed base change, by the universal property of the completed tensor product.

6.3 Perfectoid valuation rings and tilting

Lemma 6.10 (A Frobenius criterion for perfectoidness of p-torsionfree rings)**.** *Let A be a discrete p-adically complete, p-torsionfree ring. Suppose $\varpi \in A$ and $u \in A^\times$ satisfy*

$$\varpi^p = pu, \qquad A/(\varpi) \xrightarrow{x\mapsto x^p} A/p \text{ is an isomorphism.}$$

Then A is perfectoid.

Proof. We prove that θ_A is surjective with principal kernel. Frobenius on A/p is surjective, so choose $b \in A^\flat$ with $\tau = b^\sharp \equiv \varpi \pmod{p}$. The element τ is a unit multiple of ϖ. Put $\tau_n = (b^{1/p^n})^\sharp$ for $n \geq 0$. Quotienting the assumed Frobenius isomorphism by the image of τ_{n+1} gives

$$A/(\tau_{n+1}) \xrightarrow[x \mapsto x^p]{\sim} A/(\tau_n) \qquad (n \geq 0).$$

The map $A^\flat \to A/(\tau)$ is surjective. For $(x_n) \in \lim_{x \mapsto x^p} A$ with $x_0 \in \tau A$, these isomorphisms give $x_n \in \tau_n A$ for every n. Since each τ_n is a nonzerodivisor, the quotients x_n/τ_n form a compatible sequence. Thus $x \mapsto x^\sharp$ induces the first isomorphism below; Frobenius on both sides gives the second:

$$A^\flat/(b) \xrightarrow{\sim} A/(\tau), \qquad A^\flat/(b^p) \xrightarrow{\sim} A/p.$$

Surjectivity of $A^\flat \to A/p$ and successive p-adic lifting make θ_A surjective. Choose $w \in W(A^\flat)$ with $\theta_A(w) = \tau^p/p$. This lift is a unit: its reduction in $A^\flat = \lim_F A/p$ has a unit in every coordinate. Set $\xi = [b^p] - pw \in \ker \theta_A$.

For $x = [c] + px_1 \in \ker \theta_A$, one has $c^\sharp \in pA$. The isomorphism $A^\flat/(b^p) \cong A/p$ gives $c = b^p d$ for some $d \in A^\flat$. Then $x - \xi[d]$ is divisible by p in $W(A^\flat)$, and its quotient belongs to $\ker \theta_A$ because A is p-torsionfree. Iteration gives

$$x = \xi \sum_{n \geq 0} p^n [d_n], \qquad \ker \theta_A = (\xi).$$

This proves that A is perfectoid. □

Compare [8, Lemma 3.10(ii)].

Corollary 6.11 (A Frobenius criterion for complete rank-one valuation rings)**.** *Let V be a p-adically complete rank-one valuation ring of mixed characteristic. Suppose Frobenius on V/p is surjective and there is $\varpi \in V$ with $(\varpi^p) = (p)$. Then V is perfectoid.*

Proof. Valuations give $x^p \in pV$ if and only if $x \in \varpi V$. Thus $V/(\varpi) \xrightarrow{x \mapsto x^p} V/p$ is an isomorphism. Apply lemma 6.10. □

Lemma 6.12 (Perfectoid extensions of complete rank-one valuation rings)**.** *Every p-complete valuation ring of rank at most one admits a faithfully flat extension to a p-complete perfectoid valuation ring of rank at most one.*

Proof. In characteristic p, take the perfection. In mixed characteristic, let L be an algebraic closure of $\mathrm{Frac}(V)$. A maximal local subring $D \subset L$ dominating V exists by Zorn's lemma and is a valuation ring. Indeed, for $x \in L^\times$ at least one of $\mathfrak{m}_D D[x]$, $\mathfrak{m}_D D[x^{-1}]$ is proper. If both contain 1, the two polynomial identities give monic equations for x^{-1} and x with all nonleading coefficients in $\mathfrak{m}_D$ (divide by the unit constant $1 - c_0$ in each identity). Then $D[x, x^{-1}]$ is finite over D, yet both x and x^{-1} are nilpotent modulo $\mathfrak{m}_D$, contradicting Nakayama's lemma. Localizing the indicated proper extension at a maximal ideal over $\mathfrak{m}_D$ and using maximality of D gives $x \in D$ or $x^{-1} \in D$. The value group of D embeds into the divisible hull of that of V: two terms of least value in a polynomial relation over $\mathrm{Frac}(V)$ have equal value. Hence D has rank one. Completing its fraction field for this valuation yields a complete rank-one valuation ring $\widehat{D}_p$ with the same value group; a Cauchy sequence with nonzero limit has eventually constant value. The local injection $V \to \widehat{D}_p$ is faithfully flat (torsionfree modules over a valuation ring are flat, since its finitely generated ideals are principal). Frobenius on $\widehat{D}_p/p$ is surjective, and $\widehat{D}_p$ contains a pth root of p. Corollary 6.11 makes it perfectoid. □

Lemma 6.13 (Passing an arc-cover to a universal perfectoid algebra)**.** *Let R be perfectoid and let S be a discrete R-algebra whose p-completion admits a universal perfectoid algebra $(\widehat{S}_p)_{\mathrm{perfd}}$. If $R \to S$ is an arc-cover, then*

$$R \longrightarrow (\widehat{S}_p)_{\mathrm{perfd}} \quad \textit{is a } p\textit{-complete arc-cover.}$$

In particular, this applies when S is integral over R and every prime ideal of R is the contraction of a prime ideal of S.

Proof. Given a map $R \to V$ to a p-complete valuation ring of rank at most one, the arc-cover gives $S \to W$ over a faithfully flat valuation extension $V \to W$. Completion does nothing if $p = 0$ in W. Otherwise p is a nonzero nonunit in a rank-one valuation ring, and completion gives a local injection of the same rank, hence a faithfully flat valuation extension. Apply lemma 6.12 to make this extension perfectoid. The map from S then factors through $(\widehat{S}_p)_{\mathrm{perfd}}$ by its universal property. The last assertion uses corollary 6.3 and proposition 4.7. □

Lemma 6.14 (Tilting preserves complete rank-one valuation tests)**.** *Let P be perfectoid, write $\ker \theta_P = (\xi)$ with $\xi = [a] + pu$ and $u \in W(P^\flat)^\times$, and let V be an a-adically complete perfect valuation $P^\flat$-algebra of rank at most one. Then $W(V)/(\xi)$ is a p-complete perfectoid valuation ring of rank at most one, with tilt V. Local injective maps between these valuation rings correspond under tilting.*

Proof. Perfectoidness and the identification of the tilt follow from lemma 1.3. We check the valuation assertion. If $a = 0$ in V, the quotient is V. Otherwise V is a domain and reduction modulo p shows that $W(V)/(\xi)$ is p-torsionfree: from $px = \xi y$ one obtains $a(y \bmod p) = 0$, hence $y = pz$ and $x = \xi z$.

Every nonzero element of $W(V)/(\xi)$ is p^n times an element nonzero modulo p, for some $n \geq 0$. Write $\theta : W(V) \to W(V)/(\xi)$ for the quotient map and $b^\sharp = \theta([b])$ for $b \in V$. An element of the latter kind has the form $b^\sharp + pt$ with $b \in V$ and $v(b) < v(a)$. Since

$$p = -a^\sharp \theta(u)^{-1} = b^\sharp(-(a/b)^\sharp \theta(u)^{-1}),$$

and the expression in parentheses is topologically nilpotent, $b^\sharp + pt$ is $b^\sharp$ times a unit. Consequently every nonzero element of $W(V)/(\xi)$ is $c^\sharp$ times a unit for some $0 \neq c \in V$. Here $c^\sharp \neq 0$: choose N with $a^N/c \in V$ and use $(a^N)^\sharp = (-p\theta(u))^N \neq 0$. Products of nonzero elements are therefore nonzero, and divisibility is determined by divisibility in V. This proves that the quotient is a rank-one valuation ring. The same factorization shows that untilting carries local injections to local injections.

Conversely, the multiplicative inverse-limit formula shows that the tilt of a perfectoid valuation ring is a valuation ring and that tilting preserves local injections: both divisibility and being a unit can be checked on the zeroth coordinate. These local injections are precisely the faithfully flat maps between the valuation rings in the statement. □

Lemma 6.15 (Covers by products of valuation rings)**.** *Every connective p-complete ring H admits a p-complete arc-cover by a product of p-complete perfectoid valuation rings of rank at most one.*

Proof. To construct a set of factors, fix an algebraic closure of $\kappa(\mathfrak{q})$ for each prime $\mathfrak{q} \subset \pi_0 H$. Use the p-adic completions of all valuation subrings of these fields that contain the image of $\pi_0 H$, have rank at most one, and contain p in their maximal ideal. These completions are perfectoid by the proof of lemma 6.12. Given $H \to V$ with V a p-complete valuation ring of rank at most one, extend its valuation to an algebraic closure of $\mathrm{Frac}(V)$. For $\mathfrak{q} = \ker(\pi_0 H \to V)$, restriction to an embedded algebraic closure of $\kappa(\mathfrak{q})$ gives one of the

chosen subrings. After p-adic completion, projection onto this factor gives the required lift over a faithfully flat extension of V. □

Lemma 6.16 (Tilting a p-complete arc-cover and adjoining the open where a is invertible)**.** *Let $P \to Q$ be a p-complete arc-cover of perfectoid rings, let ξ generate $\ker \theta_P$, and put $a = \xi \bmod p$. The following map is an arc-cover of perfect rings.*

$$P^\flat \longrightarrow Q^\flat \times P^\flat[1/a]. \tag{6.8}$$

Proof. Consider a map $P^\flat \to V$ to a perfect valuation ring of rank at most one. If a is a unit, it factors through $P^\flat[1/a]$. If $a = 0$, or if a is a nonzero nonunit and we replace V by its faithfully flat a-adic completion, which remains perfect, lemma 6.14 gives a p-complete valuation ring $W(V)/(\xi)$ over P. The covering hypothesis gives a lift from Q after a faithfully flat valuation extension. Lemma 6.12 makes the extension perfectoid. Tilting it, and using lemma 6.14, gives the required lift from $Q^\flat$. □

6.4 p-complete arc descent

Theorem 6.17 (p-complete arc descent for perfect modules on perfectoid rings)**.** *Let $P \to Q$ be a p-complete arc-cover of perfectoid rings. Extension of scalars induces a symmetric monoidal isomorphism*

$$\mathrm{Perf}(P) \;\longrightarrow\; \lim_{[q]\in\Delta} \mathrm{Perf}(Q^{\widehat{\otimes}_P(q+1)}).$$

Proof. Choose $\xi = [a] + pu$ generating $\ker \theta_P$ and put $K = P \otimes_{\mathbf{Z}_p} \mathbf{F}_p$. For every perfectoid P-algebra T, the Witt presentation gives a natural isomorphism

$$T \otimes_{\mathbf{Z}_p} \mathbf{F}_p \;\cong\; T^\flat \otimes_{\mathbf{F}_p[t]} \mathbf{F}_p \qquad (t \mapsto a \text{ in } T^\flat, \quad t \mapsto 0 \text{ in } \mathbf{F}_p). \tag{6.9}$$

Indeed, $T = W(T^\flat) \otimes_{\mathbf{Z}_p[t]} \mathbf{Z}_p$ for $t \mapsto \xi$ and $t \mapsto 0$, since ξ is regular. Reducing this pushout modulo p gives (6.9). In particular, K is perfect over $P^\flat$, with underlying module $\mathrm{cofib}(a : P^\flat \to P^\flat)$.

Apply proposition 6.9 to the arc-cover (6.8) with coefficient algebra K. Tensoring its Čech rings with K kills every factor containing $P^\flat[1/a]$. On the remaining factor, the derived quotient by a is unchanged by a-adic completion. By proposition 1.11 this completion in degree q is $(Q^{\widehat{\otimes}_P(q+1)})^\flat$. Thus (6.9) identifies the resulting descent isomorphism with

$$\mathrm{Perf}(K) \;\cong\; \lim_{[q]\in\Delta} \mathrm{Perf}(Q^{\widehat{\otimes}_P(q+1)} \otimes_{\mathbf{Z}_p} \mathbf{F}_p). \tag{6.10}$$

Let $M^\bullet$ be a descent datum of perfect modules and set $M = \lim_\Delta M^q$ in Mod_P by restriction of scalars. Each M^q is p-complete, so M is p-complete. Tensoring over $\mathbf{Z}_p$ with the perfect module $\mathbf{F}_p$ commutes with this limit. Let $N \in \mathrm{Perf}(K)$ correspond to the reduced datum under (6.10). Taking mapping spectra from the unit to N in that isomorphism gives $N \cong \lim_\Delta (M^q \otimes_{\mathbf{Z}_p} \mathbf{F}_p) \cong M \otimes_{\mathbf{Z}_p} \mathbf{F}_p$ as K-modules. Lemma 1.16 makes M perfect over P. The maps $M \otimes_P Q^{\widehat{\otimes}_P(q+1)} \to M^q$ are isomorphisms modulo p by (6.10), hence are isomorphisms between p-complete modules. Conversely, for a perfect P-module N, its map to the limit of its scalar extensions is an isomorphism by the same reduction modulo p. Thus taking the limit is inverse to extension of scalars. □

Corollary 6.18 (Descent of the structure sheaf on perfectoid rings)**.** *For a p-complete arc-cover $P \to Q$ of perfectoid rings,*

$$P \;\cong\; \lim_{[q]\in\Delta} Q^{\widehat{\otimes}_P(q+1)} \qquad \textit{in } \mathrm{CAlg}(\mathrm{Sp}^\wedge_p).$$

Proof. Take endomorphism spectra of the tensor units in theorem 6.17. □

Proposition 6.19 (Arc-covers detected by universal structure-sheaf descent)**.** *Let $R \to S$ be a map of perfectoid rings. For a perfectoid R-algebra T, set $S_T = T\widehat{\otimes}_R S$. The following are equivalent:*

1. *$R \to S$ is a p-complete arc-cover.*
2. *For every perfectoid R-algebra T, the canonical map*

$$HT \longrightarrow \lim_{[n]\in\Delta} H\left(S_T^{\widehat{\otimes}_T(n+1)}\right)$$

   is an isomorphism in $\mathrm{Sp}_p^\wedge$.
3. *For every perfectoid R-algebra T, the canonical map*

$$T \longrightarrow \mathrm{Eq}_{\mathrm{CRing}}\left(S_T \rightrightarrows S_T\widehat{\otimes}_T S_T\right)$$

   is an isomorphism of ordinary rings.

Here $HA \in \mathrm{Sp}_p^\wedge$ is the Eilenberg–Mac Lane spectrum of a perfectoid ring A; its only possibly nonzero homotopy group is $\pi_0(HA) = A$. The category CRing *consists of ordinary commutative rings. Each completed tensor power is the coproduct of the indicated copies in perfectoid T-algebras.*

Proof. The structure presheaf with values in spectra satisfies descent for p-complete arc-covers by corollary 6.18. Apply this to every base change to obtain $1 \Rightarrow 2$. The functor $\Omega^\infty : \mathrm{Sp}_p^\wedge \to$ Spaces preserves limits. Since every Čech term has a discrete underlying space, it gives $2 \Rightarrow 3$.

The universal ordinary equalizer condition in 3 is preserved by composition. To verify this for $A \to B \to C$, an element of C whose images agree in $C\widehat{\otimes}_A C$ first descends to B by using $C\widehat{\otimes}_B C$. Its two images in $B\widehat{\otimes}_A B$ agree because $B\widehat{\otimes}_A B \to C\widehat{\otimes}_A C$ is a composite of two injective base changes of $B \to C$. Now descend to A, and repeat after any perfectoid base change.

Assume 3. To check the valuation lifting property, start with a map $R \to V$ to a p-complete valuation ring of rank at most one. Take a faithfully flat extension making V perfectoid, by lemma 6.12. Form $C = V\widehat{\otimes}_R S$, and choose the p-complete arc-cover

$$C \longrightarrow P = \prod_i W_i$$

by p-complete perfectoid valuation rings of rank at most one constructed in lemma 6.15. The already proved implication $1 \Rightarrow 3$, and the composition argument, give

$$V \cong \mathrm{Eq}_{\mathrm{CRing}}(P \rightrightarrows P\widehat{\otimes}_V P). \tag{$*$}$$

If V is a field, $(*)$ implies $P \neq 0$, and any factor provides an injective local map $V \to W_i$. Suppose V has rank one and $p \neq 0$ in V. An injective nonlocal map $V \to W_i$ would factor through $\mathrm{Frac}(V)$, making p invertible in the nonzero p-complete ring W_i. Hence an injective map is local. If no factor were injective, every factor would kill the maximal ideal of V, and therefore p. This contradicts the injectivity $V \to P$ in $(*)$.

It remains to treat a rank-one V of characteristic p. Put $K = \mathrm{Frac}(V)$ and $k = V/\mathfrak{m}_V$. If no $V \to W_i$ is injective and local, each factors through K or k. Thus $V \to P$ factors through the perfectoid algebra $E = K \times k$. The multiplication

$$E\widehat{\otimes}_V E = E \otimes_V^{\mathrm{ord}} E \longrightarrow E$$

is an isomorphism, since $K \otimes_V^{\mathrm{ord}} k = 0$. Consequently $E \to P$ has equal composites into $P \widehat{\otimes}_V P$. Equation $(*)$ yields a V-algebra map $E \to V$. Since V is a domain, this would factor through K or k; neither admits a V-algebra map to the nonfield valuation ring V. This contradiction proves that some $V \to W_i$ is injective and local in every case, hence faithfully flat. The composite $S \to C \to W_i$ is the required lift. This proves $3 \Rightarrow 1$. □

Thus the full Čech limit in $\mathrm{Sp}_p^\wedge$ is the descent assertion, and its universal ordinary equalizer consequence already detects the covers.

6.5 Descent for perfectoid ideals

Corollary 6.20 (Arc descent for the ideal of a perfectoid quotient)**.** *Let $P \to Q$ be a p-complete arc-cover of perfectoid rings, $J \subset P$ an ideal, and $Q^q = Q^{\widehat{\otimes}_P(q+1)}$. For a perfectoid P-algebra D, write $I_D = (JD)_{\mathrm{perfd}}(D)$. Then*

$$I_P \;\cong\; \lim_{[q]\in\Delta} I_{Q^q}, \qquad \mathrm{Hom}_P(I_P, I_P) \;\cong\; \lim_{[q]\in\Delta} \mathrm{Hom}_{Q^q}(I_{Q^q}, I_{Q^q}).$$

Proof. The perfectoid quotient commutes with base change, so that $I_D \cong I_P \widehat{\otimes}_P D$. Apply corollary 6.18 to $P \to Q$ and to its base change along $P \to (\widehat{P/J}_p)_{\mathrm{perfd}}$. Taking fibers gives the first formula. Applying $\mathrm{Hom}_P(I_P, -)$ and using extension of scalars gives the second. □

7 Excision and almost purity

7.1 Almost coefficients

For a p-complete commutative algebra A, use

$$\mathrm{QC}(\mathrm{Spf}\, A) = \mathrm{Mod}_A^{\wedge p} = \mathrm{Mod}_A(\mathrm{Sp}_p^\wedge).$$

Tensor products and colimits in this category are p-complete. For perfectoid R-algebras B and C, the relative module-category formula gives

$$\mathrm{QC}(\mathrm{Spf}\, B) \otimes_{\mathrm{QC}(\mathrm{Spf}\, R)} \mathrm{QC}(\mathrm{Spf}\, C) \cong \mathrm{QC}(\mathrm{Spf}(B \widehat{\otimes}_R C)). \tag{7.1}$$

The tensor product on the left is formed in presentable stable categories. The ring $B \widehat{\otimes}_R C$ is perfectoid by proposition 1.11.

Definition 7.1 (Almost coefficients)**.** For a perfectoid ring R and an ordinary ideal $J \subset R$, put

$$J_{\mathrm{perfd}}(R) = \mathrm{fib}(R \longrightarrow (\widehat{R/J}_p)_{\mathrm{perfd}}).$$

Define

$$(\mathrm{Mod}_R^{\wedge p})^{a_J} = \ker(\mathrm{Mod}_R^{\wedge p} \longrightarrow \mathrm{Mod}^{\wedge p}_{(\widehat{R/J}_p)_{\mathrm{perfd}}}),$$

where the arrow is extension of scalars. The tensor product is inherited from $\mathrm{Mod}_R^{\wedge p}$, and its unit is $J_{\mathrm{perfd}}(R)$. Almost localization is

$$M^{a_J} = J_{\mathrm{perfd}}(R) \widehat{\otimes}_R M.$$

By proposition 4.3,

$$(\widehat{R/J}_p)_{\mathrm{perfd}} \widehat{\otimes}_R (\widehat{R/J}_p)_{\mathrm{perfd}} \cong (\widehat{R/J}_p)_{\mathrm{perfd}}.$$

The complementary tensor ideal is therefore the quotient

$$(\mathrm{Mod}_R^{\wedge p})^{a_J} \cong \mathrm{Mod}_R^{\wedge p} \,/\, \mathrm{Mod}_{(\widehat{R/J}_p)_{\mathrm{perfd}}}^{\wedge p}$$

of presentable stable categories. In particular, it depends only on $\sqrt{J}$.

Remark 7.2 (Almost coefficients at $\sqrt{pR}$). For $J = \sqrt{pR}$, the perfectoid quotient is the perfect $\mathbf{F}_p$-algebra $R/\sqrt{pR}$, and $J_{\mathrm{perfd}}(R) = \sqrt{pR}$. For $M \in \mathrm{Mod}_R^{\wedge p}$ the fibre sequence

$$\sqrt{pR} \otimes_R M \longrightarrow M \longrightarrow (R/\sqrt{pR}) \otimes_R M$$

has p-complete second and third terms. Thus $\sqrt{pR} \otimes_R M$ is already p-complete and agrees with M^{a_J}. For $R = \mathcal{O}_{\mathbf{C}_p}$ and $\mathfrak{m} = \sqrt{pR}$, the unit $\mathfrak{m}$ of this almost category is nonzero. The almost image of R/p is represented by $\mathfrak{m}/p\mathfrak{m} \neq 0$, concentrated in degree zero.

7.2 The affine site

All affine objects below belong to

$$\mathscr{A} = \mathrm{CAlg}(\mathrm{Mod}_{\mathrm{Sp}_p^\wedge}(\mathrm{Pr}^L_{\mathrm{st}}))^{\mathrm{op}}.$$

Write $X_{\mathcal{D}}$ for the object specified by the $\mathrm{Sp}_p^\wedge$-linear symmetric monoidal category $\mathcal{D}$, and put $\mathrm{QC}(X_{\mathcal{D}}) = \mathcal{D}$. Thus

$$\mathrm{Map}(X_{\mathcal{E}}, X_{\mathcal{D}}) = \mathrm{Fun}^{L,\otimes}_{\mathrm{Sp}_p^\wedge}(\mathcal{D}, \mathcal{E})^{\simeq} \quad \text{in Spaces.}$$

For $A \in \mathrm{CAlg}(\mathrm{Sp}_p^\wedge)$ and a perfectoid ring R with an ordinary ideal $J \subset R$, define

$$\mathrm{Spf}\, A = X_{\mathrm{Mod}_A^{\wedge p}}, \qquad (\mathrm{Spf}\, R)^{a_J} = X_{(\mathrm{Mod}_R^{\wedge p})^{a_J}}.$$

The latter is called a *full almost affine chart*; its case $J = (1)$ is the full perfectoid formal affine $\mathrm{Spf}\, R$.

For $X = X_{\mathcal{D}}$ and $s \in \pi_0 \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, \mathbf{1})$, define its *principal open* by

$$X[1/s] = X_{\mathrm{Mod}_{\mathbf{1}[1/s]}(\mathcal{D})}.$$

Here $\mathbf{1}[1/s]$ is formed in $\mathrm{CAlg}(\mathcal{D})$. A map $X_{\mathcal{E}} \to X$ factors through this object precisely when it makes s invertible on $\mathbf{1}_{\mathcal{E}}$; the space of factorizations is then contractible. Consequently principal opens are monomorphisms, commute with base change, and satisfy $X[1/s] \times_X X[1/t] \cong X[1/(st)]$.

Definition 7.3 (Affine tests and their Zariski topology)**.** Fix universes containing the rings and diagrams in use. Let $\mathscr{A}_{\mathrm{perf}} \subset \mathscr{A}$ be the full subcategory generated by the full almost affine charts under finite fibre products and principal localizations. Give it the topology generated by finite families

$$\{X[1/s_i] \longrightarrow X\}_{i=1}^m$$

for which extension of scalars $\mathrm{QC}(X) \to \prod_i \mathrm{QC}(X[1/s_i])$ is conservative. The empty family covers the zero coefficient category.

Proposition 7.4 (Descent on the affine site)**.** *The displayed covering families are stable under base change. This topology is subcanonical, and* QC *satisfies descent for it with values in* $\mathrm{CAlg}(\mathrm{Mod}_{\mathrm{Sp}_p^\wedge}(\mathrm{Pr}^L_{\mathrm{st}}))$.

These are the finite scalar-localization consequences of the abstract descent and excision statements. We identify an affine test with its representable sheaf of Spaces. In particular, for perfectoid rings R and B,

$$\mathrm{Map}(\mathrm{Spf}\, B, \mathrm{Spf}\, R) \cong \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp}_p^\wedge)}(R, B) \cong \mathrm{Hom}_{\mathrm{CRing}}(R, B).$$

The last space is discrete because R and B have only their degree-zero homotopy groups.

Definition 7.5 (Perfectoid formal schemes)**.** Let $\mathscr{T} = \mathrm{Shv}_{\mathrm{Zar}}(\mathscr{A}_{\mathrm{perf}}, \mathrm{Spaces})$. A subobject $W \hookrightarrow X$ in $\mathscr{T}$ is *Zariski open* if its pullback along every affine test $U \to X$ is a union of subobjects obtained by finite successive principal localizations of U. A *perfectoid formal scheme*, respectively an *almost perfectoid formal scheme*, is a sheaf in $\mathscr{T}$ admitting an open cover by finite successive principal localizations of $\mathrm{Spf}\, R$, respectively of $(\mathrm{Spf}\, R)^{a_J}$. A cover here means that the coproduct of its members maps effectively epimorphically to the sheaf; there is no finiteness condition on an atlas. Morphisms are maps of sheaves of Spaces.

For $X \in \mathscr{T}$, define

$$\mathrm{QC}(X) = \lim_{(U \to X) \in (\mathscr{A}_{\mathrm{perf}}/X)^{\mathrm{op}}} \mathrm{QC}(U).$$

Coefficient descent identifies this definition on an affine test with its specified category, and on a formal scheme it computes $\mathrm{QC}(X)$ from any open atlas. In particular,

$$\mathrm{QC}((\mathrm{Spf}\, R)^{a_J}) = (\mathrm{Mod}_R^{\wedge p})^{a_J}.$$

For $f \in R$ one has

$$(\mathrm{Spf}\, R)[1/f] \cong \mathrm{Spf}\big(\widehat{R[1/f]}_p\big).$$

The ring on the right is perfectoid by lemma 1.12.

Proposition 7.6 (Base change for almost affine charts)**.** *Let $R \to B$ be a map of perfectoid rings and let $J \subset R$ be an ordinary ideal. Then*

$$(\mathrm{Spf}\, B)^{a_{JB}} \cong (\mathrm{Spf}\, R)^{a_J} \times_{\mathrm{Spf}\, R} \mathrm{Spf}\, B.$$

The associated commutative algebra is

$$B^{a_J} = J_{\mathrm{perfd}}(R) \widehat{\otimes}_R B \in \mathrm{CAlg}\big(\mathrm{QC}((\mathrm{Spf}\, R)^{a_J})\big),$$

and extension of scalars induces a symmetric monoidal isomorphism

$$\mathrm{QC}((\mathrm{Spf}\, B)^{a_{JB}}) \cong \mathrm{Mod}_{B^{a_J}}\big(\mathrm{QC}((\mathrm{Spf}\, R)^{a_J})\big).$$

Proof. By proposition 4.3,

$$B \widehat{\otimes}_R (\widehat{R/J}_p)_{\mathrm{perfd}} \cong (\widehat{B/JB}_p)_{\mathrm{perfd}}.$$

Thus extension of scalars to B commutes with the almost quotient. The universal properties of relative module categories and of this quotient give the assertions. □

7.3 The t-structure on almost coefficients

Lemma 7.7 (The torsion t-structure on almost coefficients)**.** *For a perfectoid ring R and an ideal $J \subset R$, put*

$$E_{(p)} = \mathrm{fib}(R \to R[1/p]), \qquad \mathrm{Mod}_R^{p\text{-tors}} = \{M \in \mathrm{Mod}_R : M[1/p] = 0\}.$$

The torsion incarnation of complete modules identifies

$$(\mathrm{Mod}_R^{\wedge p})^{a_J} \cong \mathrm{Mod}_R^{p\text{-tors}} / \{M : J_{\mathrm{perfd}}(R)\pi_i M = 0 \text{ for all } i \in \mathbf{Z}\}.$$

The standard t-structure on p-torsion modules descends to this quotient. Equip $\mathrm{QC}((\mathrm{Spf}\, R)^{a_J})$ *with it. For $N \in \mathrm{Mod}_R^{\wedge p}$, put $N_{\mathrm{tors}} = E_{(p)} \otimes_R \Lambda_R N$. Then*

$$\begin{aligned} N^{a_J} = 0 &\iff J_{\mathrm{perfd}}(R)\pi_i N_{\mathrm{tors}} = 0 \quad (i \in \mathbf{Z}), \\ N^{a_J} \in \mathrm{QC}((\mathrm{Spf}\, R)^{a_J})_{\geq 0} &\iff J_{\mathrm{perfd}}(R)\pi_i N_{\mathrm{tors}} = 0 \quad (i < 0), \\ N^{a_J} \in \mathrm{QC}((\mathrm{Spf}\, R)^{a_J})_{\leq 0} &\iff J_{\mathrm{perfd}}(R)\pi_i N_{\mathrm{tors}} = 0 \quad (i > 0). \end{aligned}$$

Its connective part is generated under colimits and extensions by $\mathrm{cofib}(p^n : \mathbf{1} \to \mathbf{1})$ *for $n \geq 1$. In particular this t-structure is determined by the symmetric monoidal category* $\mathrm{QC}((\mathrm{Spf}\, R)^{a_J})$.

Proof. The torsion incarnation carries restriction from $(\widehat{R/J}_p)_{\mathrm{perfd}}$ to restriction on p-torsion modules. This functor is fully faithful, commutes with truncations, and has image on discrete modules exactly those annihilated by $J_{\mathrm{perfd}}(R)$. Its image on all modules is therefore the subcategory displayed in the quotient. It is stable under truncations; its intersection with the heart is a localizing Serre subcategory. The standard quotient construction gives the stated t-structure and criteria. Finally the connective p-torsion modules are generated under colimits and extensions by $\mathrm{cofib}(p^n : R \to R)$, $n \geq 1$: these lift elements annihilated by p^n, and successive attachments resolve every connective p-torsion module. Pass this description through the quotient. □

7.4 Integral excision

Definition 7.8 (p-adic integrality)**.** A map $R \to B$ of perfectoid rings is *p-adically integral* if the map of ordinary rings $R/pR \to B/pB$ is integral.

Proposition 7.9 (Characterizations and locality of p-adic integrality)**.** *For a map $R \to B$ of perfectoid rings, the following conditions are equivalent:*

(i) *$R \to B$ is p-adically integral.*
(ii) *$R/p^n R \to B/p^n B$ is integral for every $n \geq 1$.*
(iii) *$R/\sqrt{pR} \to B/\sqrt{pB}$ is integral.*
(iv) *$\mathrm{Spf}(B/pB) \to \mathrm{Spf}(R/pR)$ is universally closed.*

These conditions are preserved by composition and by completed base change to perfectoid rings. They descend along every p-completely faithfully flat map of perfectoid bases.

Proof. If a monic polynomial $f \in R[t]$ satisfies $f(b) \in pB$, then $f(t)^n$ gives a monic equation for b modulo $p^n B$. This proves (i)⇒(ii); the converse includes $n = 1$. Integrality passes to quotients. Conversely, a monic equation modulo a nil ideal has nilpotent value, so a power of that polynomial is a monic equation in the original ring. This proves the equivalence with (iii). The equivalence with (iv) is the ordinary affine criterion for integrality [12, Tag 01WM].

For a perfectoid R-algebra R', proposition 1.11 gives

$$(B\widehat{\otimes}_R R')/p(B\widehat{\otimes}_R R') \cong (B/pB) \otimes^{\mathrm{ord}}_{R/pR} (R'/pR').$$

These are ordinary rings: the formula is obtained by taking the degree-zero homotopy group of the reduction modulo p of the completed tensor product. Ordinary integrality is preserved by composition and base change and descends along faithfully flat maps [12, Tag 02L9]. If $R \to R'$ is p-completely faithfully flat, then $R/pR \to R'/pR'$ is faithfully flat, proving the assertions. □

Lemma 7.10 (The universal perfectoid algebra of an integral algebra is p-adically integral)**.** *Let R be perfectoid and let F be an ordinary integral R-algebra. Then $R \to (\widehat{F}_p)_{\mathrm{perfd}}$ is p-adically integral.*

Proof. The reduced special fibre of a perfectoid ring is a perfect $\mathbf{F}_p$-algebra. Testing the universal property of $(\widehat{F}_p)_{\mathrm{perfd}}$ on perfect $\mathbf{F}_p$-algebras identifies

$$(\widehat{F}_p)_{\mathrm{perfd}}/\sqrt{p(\widehat{F}_p)_{\mathrm{perfd}}} \cong (F/pF)^{\mathrm{perf}}.$$

Every element of the right side has a p-power in the image of F/pF, which is integral over R/pR. Thus $R/\sqrt{pR} \to (\widehat{F}_p)_{\mathrm{perfd}}/\sqrt{p(\widehat{F}_p)_{\mathrm{perfd}}}$ is integral. Apply proposition 7.9(iii). □

Perfectoid rings in this subsection are regarded as commutative ring spectra concentrated in degree zero. For a perfectoid ring R and an ordinary ideal $J \subset R$, write

$$J_{\mathrm{perfd}}(R) = \mathrm{fib}(R \longrightarrow (\widehat{R/J}_p)_{\mathrm{perfd}}).$$

This fibre is concentrated in degree zero and is the ordinary kernel, because $R \to (\widehat{R/J}_p)_{\mathrm{perfd}}$ is surjective by proposition 4.3.

Proposition 7.11 (A valuation criterion for excision)**.** *Let $R \to B$ be a map of perfectoid rings and $J \subset R$ an ordinary ideal. Assume that, for every p-complete perfectoid valuation R-algebra V of rank at most one with $JV \neq 0$, the space $\mathrm{Map}_{\mathrm{CAlg}_R}(B, V)$ is contractible. Then the square*

$$\begin{array}{ccc} R & \longrightarrow & B \\ \downarrow & & \downarrow \\ (\widehat{R/J}_p)_{\mathrm{perfd}} & \longrightarrow & (\widehat{B/JB}_p)_{\mathrm{perfd}} \end{array} \tag{7.2}$$

is cartesian in commutative ring spectra, and

$$(\widehat{B/JB}_p)_{\mathrm{perfd}} \cong B\widehat{\otimes}_R(\widehat{R/J}_p)_{\mathrm{perfd}}.$$

Every element of J acts by a null map on either horizontal fiber. Extension of scalars induces a symmetric monoidal isomorphism

$$(\mathrm{Mod}_R^{\wedge p})^{a_J} \cong (\mathrm{Mod}_B^{\wedge p})^{a_{JB}}$$

and a symmetric monoidal isomorphism

$$\mathrm{Mod}_R^{\wedge p} \cong \mathrm{Mod}_B^{\wedge p} \underset{\mathrm{Mod}^{\wedge p}_{(\widehat{B/JB}_p)_{\mathrm{perfd}}}}{\times} \mathrm{Mod}^{\wedge p}_{(\widehat{R/J}_p)_{\mathrm{perfd}}}.$$

In particular the associated almost perfectoid formal affines are isomorphic.

Proof. Work in sheaves of spaces on perfectoid rings with the p-complete arc topology. For a perfectoid ring A, let $h_A(P) = \mathrm{Map}_{\mathrm{CAlg}}(A, P)$. This is a sheaf by corollary 6.18. We first prove

$$h_B \amalg_{h_{(\widehat{B/JB}_p)_{\mathrm{perfd}}}} h_{(\widehat{R/J}_p)_{\mathrm{perfd}}} \cong h_R. \tag{7.3}$$

Each representable takes values in discrete spaces. Since $B \to (\widehat{B/JB}_p)_{\mathrm{perfd}}$ is surjective, $h_{(\widehat{B/JB}_p)_{\mathrm{perfd}}} \to h_B$ is a monomorphism; the pushout is therefore also a sheaf of discrete spaces.

Given $R \to P$, choose a p-complete arc-cover $P \to \prod_\alpha V_\alpha$ by rank-at-most-one perfectoid valuation rings, as constructed in lemma 6.15. Partition the product according to whether JV_α is zero. On the factors where it is nonzero, the hypothesis supplies a unique map from B. On the other factors, the map from R factors through $(\widehat{R/J}_p)_{\mathrm{perfd}}$. These two subproducts are complementary open and closed factors. Thus the left side of (7.3) maps onto h_R locally.

For injectivity, two maps $B \rightrightarrows P$ agreeing on R become equal on every factor with $JV_\alpha \neq 0$. On the remaining factors they factor through $(\widehat{B/JB}_p)_{\mathrm{perfd}}$, and the pushout identifies their images with the same map from $(\widehat{R/J}_p)_{\mathrm{perfd}}$. A map from B and a map from $(\widehat{R/J}_p)_{\mathrm{perfd}}$ agreeing on R are identified already over P: the first kills JB and hence factors through $(\widehat{B/JB}_p)_{\mathrm{perfd}}$. Two maps from $(\widehat{R/J}_p)_{\mathrm{perfd}}$ agreeing on R are equal because $R \to (\widehat{R/J}_p)_{\mathrm{perfd}}$ is surjective. Representatives of sections of the pushout are locally of these forms, proving (7.3).

For any $M \in \mathrm{Sp}_p^\wedge$, the functor $P \mapsto \mathrm{Map}_{\mathrm{Sp}_p^\wedge}(M, P)$ is an arc sheaf by corollary 6.18. Mapping (7.3) into this sheaf and using Yoneda proves that (7.2) is cartesian on underlying spectra, hence in commutative ring spectra. The tensor-product formula follows from proposition 4.3. The lower horizontal fiber is a $(\widehat{R/J}_p)_{\mathrm{perfd}}$-module, so J acts on it by null maps; the cartesian square identifies the two horizontal fibers.

The same square identifies its vertical fibers:

$$J_{\mathrm{perfd}}(R) \cong (JB)_{\mathrm{perfd}}(B) \cong J_{\mathrm{perfd}}(R) \widehat{\otimes}_R B.$$

The almost-unit criterion (5.8) gives the assertion for almost modules. Finally, theorem 5.7, applied to the cartesian square (7.2) with idempotent quotient algebra, gives the isomorphism of module categories. □

Lemma 7.12 (A bound from finite generation on a principal open)**.** *Let $R \to B$ be a map of perfectoid rings, let $g \in R$, and let $b_1, \ldots, b_d \in B$. If these elements generate $B[1/g]$ as an $R[1/g]$-module, there are integers $N, n \geq 0$ such that*

$$p^N g^n B \subseteq \sum_{i=1}^{d} R b_i.$$

In particular, if $R[1/g] \to B[1/g]$ is surjective, take $d = 1$ and $b_1 = 1$.

Proof. Reducedness (corollary 1.10) implies $R[p^\infty] = R[p]$ and $B[p^\infty] = B[p]$: if $p^m x = 0$, then $(px)^m = 0$. The quotients $\overline{R} = R/R[p]$ and $\overline{B} = B/B[p]$ are p-complete and p-torsionfree. Indeed multiplication by p identifies each quotient with the corresponding closed subgroup pR or pB, and identifies their p-adic filtrations up to a shift. Consequently $R[1/p]$ and $B[1/p]$ are Banach $\mathbf{Q}_p$-spaces with unit balls $\overline{R}$ and $\overline{B}$. Let $\alpha : R[1/p]^d \to B[1/p]$ be the continuous linear map $(a_i) \mapsto \sum_i a_i b_i$. For each $n \geq 0$, the subspace

$$E_n = \{(a, b) : \alpha(a) = g^n b\} \subset R[1/p]^d \times B[1/p]$$

is closed, hence Banach. Localized surjectivity implies that the images of its projections to $B[1/p]$, as n varies, cover $B[1/p]$.

We recall the required complete-metric argument, including the bound. For a continuous linear map $u : E \to H$ of nonarchimedean Banach $\mathbf{Q}_p$-spaces, if the closure of the image of its unit ball E° contains a ball H_r about zero, then $H_r \subset u(E^\circ)$. For $y \in H_r$, successively choose $x_m \in E^\circ$ so that

$$\left\| y - u\left(\sum_{i=0}^{m} p^i x_i\right)\right\| \leq |p|^{m+1} r.$$

The convergent sum $\sum_{m\geq 0} p^m x_m$ supplies the lift. The closure of the image of a ball is an additive subgroup, so nonempty interior implies that it contains a ball about zero. Thus, if u is not surjective, each closed set $\overline{u(p^{-m}E^\circ)}$ has empty interior. A complete metric space cannot be covered by countably many such closed sets: choose nested nonempty closed balls with radii tending to zero, the mth avoiding the mth closed set, and take the limit of their centers. Applying this to the projections $E_n \to B[1/p]$ shows that one is surjective. Applying it again to the images of the scaled unit balls of that E_n shows that this projection has bounded lifts.

Its unit ball is $E_n \cap (\overline{R}^d \times \overline{B})$. Scaling the bound by a power of p gives $p^N g^n \overline{B} \subseteq \sum_i \overline{R}\,\overline{b}_i$ for some $N \geq 0$, where $\overline{b}_i$ is the image of b_i in $\overline{B}$. Lifting the coefficients to R introduces an error in $B[p]$; one further factor of p kills it. □

Theorem 7.13 (Excision for p-adically integral maps)**.** *Let $R \to B$ be a p-adically integral map of perfectoid rings and let $J \subset R$ be an ordinary ideal. If*

$$R[1/g] \longrightarrow B[1/g] \quad \textit{is an isomorphism for every } g \in J,$$

all conclusions of proposition 7.11 hold for $R \to B$.

Proof. Fix a valuation test $R \to V$ with $JV \neq 0$ and choose $g \in J$ nonzero in V. The isomorphism $R[1/g] \cong B[1/g]$ forces any extension to be the map

$$\phi : B \longrightarrow B[1/g] \cong R[1/g] \longrightarrow \mathrm{Frac}(V),$$

provided its image belongs to V. If $p = 0$ in V, this map factors through B/pB. Integrality modulo p makes its values integral over V, so they belong to V.

If $p \neq 0$ in V, then V has rank one: a nonzero p-complete field cannot have p invertible. By lemma 7.12, some $N, n \geq 0$ satisfy $p^N g^n B \subseteq \mathrm{im}(R \to B)$. For $b \in B$ with $\phi(b) \neq 0$ and every integer $m \geq 1$, apply this bound to b^m. For a real-valued additive valuation v defining V it gives

$$N v(p) + n v(g) + m\, v(\phi(b)) \geq 0.$$

Hence $v(\phi(b)) \geq 0$, and again $\phi(b) \in V$. In both cases the extension is unique because g is nonzero in the domain V. The valuation criterion applies. □

Remark 7.14. The integrality hypothesis is used only for valuation rings of characteristic p. More generally the theorem holds if, for every perfect valuation R-algebra V of characteristic p and rank at most one with $JV \neq 0$, the unique map $B \to \mathrm{Frac}(V)$ prescribed by these localized isomorphisms has image in V.

Corollary 7.15 (Excision for ideals contained in $\sqrt{pR}$)**.** *Let $R \to B$ be any map of perfectoid rings and let $J \subseteq \sqrt{pR}$ be an ordinary ideal. If $R[1/g] \to B[1/g]$ is an isomorphism for every $g \in J$, all conclusions of proposition 7.11 hold.*

Proof. Every map from R to a characteristic-p valuation ring kills J, because valuation rings are reduced. Only the mixed-characteristic case in the preceding proof can therefore occur, and that case uses no integrality hypothesis. □

Corollary 7.16 (Excision for a universal perfectoid algebra)**.** *Let R be perfectoid, let $J \subset R$ be an ordinary ideal, and let D be an ordinary integral R-algebra such that*

$$R[1/g] \longrightarrow D[1/g] \quad \textit{is an isomorphism for every } g \in J.$$

The universal perfectoid algebra $(\widehat{D}_p)_{\mathrm{perfd}}$ exists by proposition 4.7. All conclusions of proposition 7.11 hold for $R \to (\widehat{D}_p)_{\mathrm{perfd}}$. In particular,

$$(\mathrm{Mod}_R^{\wedge p})^{a_J} \cong \left(\mathrm{Mod}^{\wedge p}_{(\widehat{D}_p)_{\mathrm{perfd}}}\right)^{a_{J(\widehat{D}_p)_{\mathrm{perfd}}}}.$$

Proof. Let $R \to V$ be a valuation test in that proposition. Choose $g \in J$ nonzero in V. Any extension $D \to V$ must be the map

$$D \longrightarrow D[1/g] \cong R[1/g] \longrightarrow \mathrm{Frac}(V).$$

Its values are integral over V and hence belong to V, because a valuation ring is integrally closed in its fraction field. Thus $\mathrm{Map}_{\mathrm{CAlg}_R}(D, V)$ is contractible. The universal property of $(\widehat{D}_p)_{\mathrm{perfd}}$ identifies this space with $\mathrm{Map}_{\mathrm{CAlg}_R}((\widehat{D}_p)_{\mathrm{perfd}}, V)$, so the proposition applies. □

Corollary 7.17 (Excision for two universal perfectoid quotients)**.** *Let R be perfectoid and let $I, J \subset R$ be ordinary ideals. The square*

$$\begin{array}{ccc} (\widehat{R/IJ}_p)_{\mathrm{perfd}} & \longrightarrow & (\widehat{R/I}_p)_{\mathrm{perfd}} \\ \downarrow & & \downarrow \\ (\widehat{R/J}_p)_{\mathrm{perfd}} & \longrightarrow & (\widehat{R/(I+J)}_p)_{\mathrm{perfd}} \end{array}$$

is cartesian. Its lower-right ring is the completed tensor product over the upper-left ring. Applying $A \mapsto \mathrm{QC}(\mathrm{Spf}\, A) = \mathrm{Mod}_A^{\wedge p}$ gives a cartesian square of symmetric monoidal categories. Moreover, in $\mathrm{Mod}_R^{\wedge p}$ there are natural isomorphisms compatible with the maps to R:

$$\begin{aligned} I_{\mathrm{perfd}}(R) \widehat{\otimes}_R J_{\mathrm{perfd}}(R) &\cong (IJ)_{\mathrm{perfd}}(R), \\ I_{\mathrm{perfd}}(R) \amalg_{(IJ)_{\mathrm{perfd}}(R)} J_{\mathrm{perfd}}(R) &\cong (I+J)_{\mathrm{perfd}}(R). \end{aligned} \tag{7.4}$$

Proof. Apply proposition 7.11 to $(\widehat{R/IJ}_p)_{\mathrm{perfd}} \to (\widehat{R/I}_p)_{\mathrm{perfd}}$ and the image of J in the source. For a valuation test $(\widehat{R/IJ}_p)_{\mathrm{perfd}} \to V$ on which J is nonzero, the equality $IJ = 0$ forces the image of I to be zero. The universal property of the perfectoid quotient gives the unique map $(\widehat{R/I}_p)_{\mathrm{perfd}} \to V$. The other two quotients in that proposition are the displayed lower rings, by proposition 4.3.

Replace the upper-left ring of the displayed square by R. Its total fiber is $I_{\mathrm{perfd}}(R) \widehat{\otimes}_R J_{\mathrm{perfd}}(R)$, because the lower-right ring is also the completed tensor product of the upper-right and lower-left rings over R. The cartesian square identifies this total fiber with $(IJ)_{\mathrm{perfd}}(R)$. Taking fibers from R throughout the cartesian square gives the second formula, since finite cartesian and cocartesian squares agree in a stable category. □

Proposition 7.18 (Zariski descent for almost coefficients)**.** *Let R be a perfectoid ring, let $n \geq 0$, and let $K, K_1, \ldots, K_n \subset R$ be finitely generated ordinary ideals such that*

$$\sqrt{K} = \sqrt{K_1 + \cdots + K_n}.$$

Almost localization induces a symmetric monoidal isomorphism

$$(\mathrm{Mod}_R^{\wedge p})^{a_K} \xrightarrow{\sim} \lim_{\varnothing \neq L \subseteq \{1,\dots,n\}} (\mathrm{Mod}_R^{\wedge p})^{a \prod_{i \in L} K_i}.$$

The indexing category is ordered by inclusion. The restriction functors are t-exact for the torsion t-structures and jointly detect both bounds of these t-structures.

Proof. For $n = 0$ both sides are the zero category. For $n = 1$ the assertion follows from dependence only on the radical of the ideal. Suppose $n \geq 2$. By corollary 7.17,

$$(K_1)_{\mathrm{perfd}}(R) \widehat{\otimes}_R (K_2)_{\mathrm{perfd}}(R) \cong (K_1 K_2)_{\mathrm{perfd}}(R),$$
$$(K_1)_{\mathrm{perfd}}(R) \amalg_{(K_1 K_2)_{\mathrm{perfd}}(R)} (K_2)_{\mathrm{perfd}}(R) \cong (K_1 + K_2)_{\mathrm{perfd}}(R).$$

The gluing statement for the corresponding tensor ideals is therefore the two-term case of abstract excision. Iteration, together with dependence of almost localization only on the radical of the ideal, gives the displayed limit for n ideals. Each restriction is a further almost localization, hence is t-exact by lemma 7.7. Joint conservativity and truncation give the assertions about bounds. □

7.5 Almost purity

For a perfectoid ring R and an ordinary integral R-algebra F, the perfectoid ring $(\widehat{F}_p)_{\mathrm{perfd}}$ exists by proposition 4.7. We first record the consequence of integral excision used below.

Lemma 7.19 (Comparison of integral models)**.** *Let R be perfectoid, let $J \subset R$ be an ideal, and let $F \to G$ be a map of ordinary integral R-algebras. If $F[1/f] \to G[1/f]$ is an isomorphism for every $f \in J$, then*

$$\left((\widehat{F}_p)_{\mathrm{perfd}}\right)^{a_J} \longrightarrow \left((\widehat{G}_p)_{\mathrm{perfd}}\right)^{a_J}$$

is an isomorphism in $\mathrm{CAlg}((\mathrm{Mod}_R^{\wedge p})^{a_J})$*. It also induces an isomorphism of almost perfectoid formal affines.*

Proof. The ordinary base change of $F \to G$ gives an integral map

$$(\widehat{F}_p)_{\mathrm{perfd}} \longrightarrow G \otimes_F^{\mathrm{ord}} (\widehat{F}_p)_{\mathrm{perfd}}$$

which becomes an isomorphism after inverting any $f \in J$. The universal perfectoid algebra of the target is $(\widehat{G}_p)_{\mathrm{perfd}}$, by the universal property of perfectoidization. Apply corollary 7.16 over $(\widehat{F}_p)_{\mathrm{perfd}}$ and use base change of perfectoid ideals from proposition 4.3. □

Construction 7.20 (The almost algebra of a finite étale cover)**.** Let R be perfectoid, let $J \subset R$ be finitely generated, and let $Y \to U = \operatorname{Spec} R \setminus V(J)$ be an ordinary finite étale morphism. Define the ordinary ring

$$D = \{b \in \Gamma(Y, \mathcal{O}_Y) : b \text{ is integral over the image of } R\}.$$

The canonical morphism

$$\left(\operatorname{Spf}\left((\widehat{D}_p)_{\mathrm{perfd}}\right)\right)^{a_{J(\widehat{D}_p)_{\mathrm{perfd}}}} \longrightarrow (\operatorname{Spf} R)^{a_J}$$

has associated commutative algebra

$$\left((\widehat{D}_p)_{\mathrm{perfd}}\right)^{a_J} = J_{\mathrm{perfd}}(R) \widehat{\otimes}_R (\widehat{D}_p)_{\mathrm{perfd}} \quad \text{in } \mathrm{CAlg}(\mathrm{QC}((\operatorname{Spf} R)^{a_J})).$$

For every $f \in J$, the scheme $Y \times_U D(f)$ is affine, finite over $R[1/f]$, and has ring of functions $\Gamma(Y, \mathcal{O}_Y)[1/f]$. Here sections commute with this localization because Y is quasi-compact and quasi-separated. Clearing denominators in monic equations gives

$$D[1/f] = \Gamma(Y \times_U D(f), \mathcal{O}_Y).$$

Consequently $\operatorname{Spec} D \times_{\operatorname{Spec} R} U \cong Y$.

More generally, if F is an ordinary integral R-algebra equipped with $\operatorname{Spec} F \times_{\operatorname{Spec} R} U \cong Y$, restriction of functions gives an integral map $F \to D$ which is an isomorphism over U. Lemma 7.19 therefore gives the same almost algebra and the same almost affine using $(\widehat{F}_p)_{\mathrm{perfd}}$. For a perfectoid R-algebra R', the ordinary integral R'-algebra $D \otimes_R^{\mathrm{ord}} R'$ is an integral model of $Y \times_{\operatorname{Spec} R} \operatorname{Spec} R'$ over $\operatorname{Spec} R' \setminus V(JR')$. Proposition 4.7 gives the natural isomorphism

$$(\widehat{D}_p)_{\mathrm{perfd}} \widehat{\otimes}_R R' \cong \left(D \widehat{\otimes_R^{\mathrm{ord}}} R'_p \right)_{\mathrm{perfd}}.$$

Independence of the integral model therefore identifies the completed base change of the displayed almost affine with the almost affine constructed from $Y \times_{\operatorname{Spec} R} \operatorname{Spec} R'$.

Corollary 7.21 (The almost extension of the identity cover)**.** *Let R be perfectoid, let $J \subset R$ be finitely generated, put $U = \operatorname{Spec} R \setminus V(J)$, and define the ordinary ring*

$$D = \{b \in \Gamma(U, \mathcal{O}_U) : b \text{ is integral over the image of } R\}.$$

The canonical morphism

$$\left(\operatorname{Spf}((\widehat{D}_p)_{\mathrm{perfd}})\right)^{a_{J(\widehat{D}_p)_{\mathrm{perfd}}}} \longrightarrow (\operatorname{Spf} R)^{a_J}$$

is an isomorphism.

Proof. The ordinary integral R-algebra R is an integral model of $U \to U$. The restriction map $R \to D$ is integral and gives isomorphisms $R[1/f] \cong D[1/f]$ for every $f \in J$. Since R is already perfectoid, lemma 7.19 gives

$$R^{a_J} \xrightarrow{\sim} ((\widehat{D}_p)_{\mathrm{perfd}})^{a_J},$$

which is the asserted isomorphism of almost affines. □

Lemma 7.22 (A finite integral model with prescribed localizations)**.** *Let $R \to B$ be a map of ordinary rings, and let $J \subset R$ be finitely generated. If $B[1/g]$ is finite as an $R[1/g]$-module for every $g \in J$, there is an ordinary R-subalgebra $F \subset B$, finite as an R-module, such that $F[1/g] \to B[1/g]$ is an isomorphism for every $g \in J$.*

Proof. Choose generators $g_1, \ldots, g_m$ of J. For each i, choose finitely many elements $b_{ij} \in B$ whose images generate $B[1/g_i]$ as an $R[1/g_i]$-module. Fix one such element $b = b_{ij}$ and write $g = g_i$. A monic equation for b over $R[1/g]$, with denominators cleared after replacing b by $c = g^a b$ for a sufficiently large integer $a \geq 0$, gives

$$h(t) = t^d + \sum_{k<d} r_k t^k \in R[t], \qquad h(c) = 0 \quad \text{in } B[1/g].$$

Choose $e \geq 0$ such that $g^e h(c) = 0$ in B. The element $g^e c$ then satisfies the monic polynomial

$$t^d + \sum_{k<d} g^{e(d-k)} r_k t^k,$$

whose value is $g^{ed}h(c) = 0$. Thus some $g_i^{N_{ij}} b_{ij}$ is integral over R. Let $F \subset B$ be the R-subalgebra generated by these finitely many elements. It is finite as an R-module, and $F[1/g_i] = B[1/g_i]$ for every i. For $g \in J$, the images of the g_i generate the unit ideal in $R[1/g]$. Localization at these elements jointly detects isomorphisms of $R[1/g]$-modules, so $F[1/g] = B[1/g]$. □

Theorem 7.23 (Comparison with a finite integral model)**.** *Let $R \to B$ be a map of perfectoid rings, and let $J \subset R$ be finitely generated. Assume that $B[1/g]$ is finite as an $R[1/g]$-module for every $g \in J$ and that either $R \to B$ is p-adically integral or $J \subseteq \sqrt{pR}$. For any finite ordinary R-subalgebra $F \subset B$ as in lemma 7.22, the canonical map*

$$(\widehat{F}_p)_{\mathrm{perfd}} \longrightarrow B$$

satisfies all the conclusions of proposition 7.11, with base ring $(\widehat{F}_p)_{\mathrm{perfd}}$ and ideal $J(\widehat{F}_p)_{\mathrm{perfd}}$. In particular, it induces an isomorphism

$$(\operatorname{Spf} B)^{a_{JB}} \xrightarrow{\sim} \Big(\operatorname{Spf}\big((\widehat{F}_p)_{\mathrm{perfd}}\big)\Big)^{a_{J(\widehat{F}_p)_{\mathrm{perfd}}}}.$$

Put $U = \operatorname{Spec} R \setminus V(J)$. If the ordinary scheme $Y = \operatorname{Spec} B \times_{\operatorname{Spec} R} U$ is finite étale over U, define the ordinary ring

$$D = \{b \in \Gamma(Y, \mathcal{O}_Y) : b \text{ is integral over the image of } R\}.$$

Then restriction of functions induces a canonical isomorphism

$$\Big(\operatorname{Spf}\big((\widehat{D}_p)_{\mathrm{perfd}}\big)\Big)^{a_{J(\widehat{D}_p)_{\mathrm{perfd}}}} \xrightarrow{\sim} \Big(\operatorname{Spf}\big((\widehat{F}_p)_{\mathrm{perfd}}\big)\Big)^{a_{J(\widehat{F}_p)_{\mathrm{perfd}}}}.$$

Proof. The universal property of perfectoidization extends $F \to B$ to $(\widehat{F}_p)_{\mathrm{perfd}} \to B$. Let $(\widehat{F}_p)_{\mathrm{perfd}} \to V$ be a map to a p-complete perfectoid valuation ring of rank at most one such that $JV \neq 0$, and choose $g \in J$ with nonzero image in V. Any extension to B must be the ordinary ring homomorphism

$$\phi : B \longrightarrow B[1/g] \cong F[1/g] \longrightarrow \operatorname{Frac}(V),$$

provided that its image belongs to V.

If $p = 0$ in V, the alternative $J \subseteq \sqrt{pR}$ is impossible. Thus B/pB is integral over R/pR. The values of ϕ are integral over V and hence belong to V.

If $p \neq 0$ in V, the ring V has rank one. Choose finite R-module generators $b_1, \dots, b_d$ of F. The map $R[1/g]^d \to B[1/g]$ defined by these elements is surjective. By lemma 7.12, there are $N, n \geq 0$ with

$$p^N g^n B \subseteq \sum_{i=1}^{d} R b_i = F.$$

The prescribed map $F \to V$ sends the right side into V. For $b \in B$ with $\phi(b) \neq 0$, apply this inclusion to every b^m, $m \geq 1$. A real-valued additive valuation v defining V gives

$$N v(p) + n v(g) + m\, v(\phi(b)) \geq 0 \qquad (m \geq 1).$$

It follows that $v(\phi(b)) \geq 0$. Thus $\phi(B) \subseteq V$.

In either case $\phi : B \to V$ extends the given map $F \to V$. Its composite with $(\widehat{F}_p)_{\mathrm{perfd}} \to B$ is the prescribed map $(\widehat{F}_p)_{\mathrm{perfd}} \to V$ by the universal property of perfectoidization. Uniqueness follows from $F[1/g] \cong B[1/g]$ and injectivity of $V \to \operatorname{Frac}(V)$. Therefore $\operatorname{Map}_{\mathrm{CAlg}_{(\widehat{F}_p)_{\mathrm{perfd}}}}(B, V)$ is contractible, and proposition 7.11 applies.

If $Y \to U$ is finite étale, F is an ordinary integral model of Y. Construction 7.20 gives the final displayed isomorphism from the integral map $F \to D$. The identification is independent of F: any two choices lie in the finite R-subalgebra they generate, and all the comparison maps are induced by the inclusions into B and restriction to Y. □

Lemma 7.24 (Splitting finite étale covers after André's extension)**.** *Let R be perfectoid, let $J \subset R$ be finitely generated, put $U = \operatorname{Spec} R \setminus V(J)$, and let $Y \to U$ be finite étale. Let $R \to R'$ be the p-completely faithfully flat perfectoid extension of corollary 4.1, in which every monic polynomial splits. There are $f_1, \dots, f_m \in JR'$ and integers $d_j \geq 0$ such that*

$$\operatorname{Spec} R' \times_{\operatorname{Spec} R} U = \bigcup_{j=1}^{m} D(f_j), \qquad Y \times_U D(f_j) \cong \coprod_{i=1}^{d_j} D(f_j).$$

Proof. Clearing denominators shows that every monic polynomial splits over each local ring A of R'. Its residue field k is therefore algebraically closed. A finite étale A-algebra E of rank d satisfies $E \otimes_A^{\mathrm{ord}} k \cong k^d$. For $d > 0$, choose $x \in E$ with pairwise distinct residue coordinates. Its characteristic polynomial χ_x splits over A, and the differences of distinct roots are units. Evaluation at x gives

$$A^d \cong A[t]/(\chi_x) \xrightarrow{\sim} E :$$

the last map reduces to an isomorphism over k and is a map between free A-modules of rank d. If $d = 0$, then $E = 0$. Finite presentation spreads each splitting to a principal neighborhood inside $\operatorname{Spec} R' \times_{\operatorname{Spec} R} U$. Such a neighborhood can be chosen as $D(f_j)$ with $f_j \in JR'$. Quasi-compactness gives a finite cover. □

Theorem 7.25 (Almost purity)**.** *Let R be perfectoid, let $J \subset R$ be finitely generated, and let Y be finite étale over $U = \operatorname{Spec} R \setminus V(J)$. Define the ordinary ring*

$$D = \{b \in \Gamma(Y, \mathcal{O}_Y) : b \text{ is integral over the image of } R\}.$$

The algebra

$$((\widehat{D}_p)_{\mathrm{perfd}})^{a_J} \in \mathrm{CAlg}(\mathrm{QC}((\operatorname{Spf} R)^{a_J}))$$

is dualizable with perfect trace pairing. Tensoring with this algebra is t-exact, and its multiplication admits a section linear over its tensor square. If $Y \to U$ is surjective, tensoring with this algebra is conservative. The t-structure on $\mathrm{QC}((\operatorname{Spf} R)^{a_J})$ is that of lemma 7.7.

Proof. Choose R' and the finite cover $D(f_j)$ from lemma 7.24. Then

$$(\widehat{D}_p)_{\mathrm{perfd}} \widehat{\otimes}_R R' \cong \left(\widehat{D \otimes_R^{\mathrm{ord}} R'_p} \right)_{\mathrm{perfd}},$$
$$(D \otimes_R^{\mathrm{ord}} R')[1/f_j] \cong R'[1/f_j]^{d_j}.$$

Inside $R'[1/f_j]^{d_j}$, let T_j be the subalgebra generated by the image of $D \otimes_R^{\mathrm{ord}} R'$ and the coordinate idempotents. There are integral maps

$$D \otimes_R^{\mathrm{ord}} R' \longrightarrow T_j \longleftarrow (R')^{d_j}$$

which become isomorphisms after inverting f_j. Indeed, adjoining these idempotents is integral, and T_j is integral over R'. Lemma 7.19 gives

$$((\widehat{D}_p)_{\mathrm{perfd}} \widehat{\otimes}_R R')^{a_{(f_j)}} \cong \left((R')^{a_{(f_j)}}\right)^{\times d_j}.$$

Complete almost modules satisfy faithfully flat descent along $R \to R'$: use lemma 2.12, the realization as the kernel of extension to the perfectoid quotient, and base change of that quotient in proposition 4.3. They satisfy descent for the finite principal cover by proposition 7.18. Both sets of restriction functors are t-exact and jointly conservative for the torsion t-structure of lemma 7.7.

On each of these charts the algebra is a finite product of the tensor unit. Its trace is the sum of the coordinate projections, and its trace pairing gives a self-duality. Multiplication is the projection from the product indexed by ordered pairs to the diagonal factors; inclusion of those factors is a section linear over the tensor square. Duality and the perfect trace pairing descend. The space of tensor-square-linear sections of multiplication, if nonempty, is contractible: such a section identifies the target with its idempotent direct factor in the source. The sections therefore descend as well.

Tensoring on each chart is a finite direct sum of the identity functor, so it is t-exact. The jointly conservative t-exact restriction functors detect both bounds of the t-structure. If $Y \to U$ is surjective, every nonempty $D(f_j)$ has $d_j > 0$, so the same restrictions detect conservativity. This proves all assertions. □

Corollary 7.26 (Almost purity for p-adically integral maps)**.** *Let $R \to B$ be a map of perfectoid rings, let $J \subset R$ be finitely generated, and assume that $B[1/g]$ is a finite étale $R[1/g]$-algebra for every $g \in J$. If $R \to B$ is p-adically integral, or $J \subseteq \sqrt{pR}$, the algebra of*

$$(\operatorname{Spf} B)^{a_{JB}} \longrightarrow (\operatorname{Spf} R)^{a_J}$$

has all the properties in theorem 7.25: it is dualizable with perfect trace pairing, tensoring with it is t-exact, and its multiplication admits a section linear over its tensor square. Tensoring is conservative if $R[1/g] \to B[1/g]$ is faithfully flat for every $g \in J$. These assertions are in $\mathrm{QC}((\operatorname{Spf} R)^{a_J})$.

Proof. Apply theorem 7.23 to identify the displayed map with the almost extension of its finite étale restriction, then apply theorem 7.25. □

8 Realizations of perfectoid stacks

8.1 Restriction to perfectoids

Let $\mathscr{A}_p$ be the opposite of the category of connective p-complete E_∞-$\mathbb{Z}_p$-algebras. Give it the topology whose covers are induced by p-completely faithfully flat maps. Let $\mathscr{P}_p \subset \mathscr{A}_p$ be the full subcategory on perfectoid rings, with the same covering condition. These are auxiliary sites of ring tests. A *perfectoid stack* is a sheaf of Spaces on $\mathscr{P}_p$ with this topology.

Write $\operatorname{Spf} A$ for the representable sheaf on $\mathscr{A}_p$ associated with a connective p-complete algebra A. Let

$$i : \mathscr{P}_p \hookrightarrow \mathscr{A}_p.$$

Restriction along i gives

$$i^* : \mathrm{Shv}(\mathscr{A}_p; \mathrm{Spaces}) \longrightarrow \mathrm{Shv}(\mathscr{P}_p; \mathrm{Spaces}).$$

For a connective p-complete algebra A, put $h_A = i^*(\operatorname{Spf} A)$. Thus, for a perfectoid ring T,

$$h_A(T) = \mathrm{Map}_{\mathrm{CAlg}}(A, T) \in \mathrm{Spaces}.$$

If A is perfectoid, h_A is representable on $\mathscr{P}_p$. For an ordinary ring D, use

$$h_D(T) = \mathrm{Map}_{\mathrm{CAlg}}(D, T).$$

It equals $h_{\widehat{D}_p}$, where $\widehat{D}_p = \lim_n^{\mathrm{ord}} D/p^n D$.

The inclusion of sites has the adjunction

$$i_! : \mathrm{Shv}(\mathscr{P}_p; \mathrm{Spaces}) \rightleftarrows \mathrm{Shv}(\mathscr{A}_p; \mathrm{Spaces}) : i^*.$$

Here $i_!$ is left Kan extension followed by sheafification on the auxiliary site $\mathscr{A}_p$. This functor is used only for complete-coefficient calculations on ring tests; it is not the realization defined below.

For $F \in \mathrm{Shv}(\mathscr{A}_p; \mathrm{Spaces})$, set

$$\mathrm{QC}(F) = \lim_{(\mathrm{Spf}\, B \to F) \in (\mathscr{A}_p/F)^{\mathrm{op}}} \mathrm{Mod}_B^{\wedge p},$$
$$\Gamma(F, \mathcal{O}) = \lim_{(\mathrm{Spf}\, B \to F) \in (\mathscr{A}_p/F)^{\mathrm{op}}} B.$$

For $F \in \mathrm{Shv}(\mathscr{P}_p; \mathrm{Spaces})$, define

$$\mathrm{QC}(F) = \lim_{(h_B \to F) \in (\mathscr{P}_p/F)^{\mathrm{op}}} \mathrm{Mod}_B^{\wedge p},$$
$$\Gamma(F, \mathcal{O}) = \lim_{(h_B \to F) \in (\mathscr{P}_p/F)^{\mathrm{op}}} B.$$

The limits defining QC are formed in presentable stable symmetric monoidal categories; those defining Γ are formed in p-complete commutative algebras. Each definition uses the indicated auxiliary site.

In particular,

$$\mathrm{QC}(h_A) \cong \lim_{T \in \mathrm{PerfAlg}_A} \mathrm{Mod}_T^{\wedge p},$$

and

$$\Gamma(h_A, \mathcal{O}) \cong \lim_{T \in \mathrm{PerfAlg}_A} T. \tag{8.1}$$

For $F \in \mathrm{Shv}(\mathscr{P}_p; \mathrm{Spaces})$ one has

$$i_! F \cong \operatorname*{colim}_{(h_T \to F) \in \mathscr{P}_p/F} \mathrm{Spf}\, T \quad \text{in } \mathrm{Shv}(\mathscr{A}_p; \mathrm{Spaces}).$$

Descent for complete module coefficients on the auxiliary sites gives

$$\mathrm{QC}(i_! F) \cong \mathrm{QC}(F), \qquad \Gamma(i_! F, \mathcal{O}) \cong \Gamma(F, \mathcal{O}). \tag{8.2}$$

These equalities concern $i_!$, not realization.

A map $h_P \to F$ on $\mathscr{P}_p$ is an *affine atlas* if for every $h_T \to F$ its pullback is h_C for a perfectoid T-algebra C, and $T \to C$ is p-completely faithfully flat. On $\mathscr{A}_p$ the same definition uses $\mathrm{Spf}\, P$, $\mathrm{Spf}\, T$, $\mathrm{Spf}\, C$ and connective p-complete algebras. This is a definition about auxiliary test sheaves.

A universal perfectoid A-algebra is an initial object A_{pfd} of $\mathrm{PerfAlg}_A$. When it exists,

$$h_A \cong h_{A_{\mathrm{pfd}}}, \qquad \Gamma(h_A, \mathcal{O}) \cong A_{\mathrm{pfd}}.$$

Conversely, if the ring in (8.1) is perfectoid, it is an initial perfectoid A-algebra. Indeed, it is then the limit of the identity diagram inside $\mathrm{PerfAlg}_A$; the universal cone makes it initial.

8.2 Realization of a perfectoid stack

Use the affine site $\mathscr{A}_{\mathrm{perf}}$ of definition 7.3 and its category $\mathscr{T}$ of sheaves of Spaces. There is a functor

$$\nu : \mathscr{P}_p \longrightarrow \mathscr{A}_{\mathrm{perf}}, \qquad P \longmapsto \operatorname{Spf} P.$$

It is fully faithful: for perfectoid rings P and T, the universal property of complete module categories gives

$$\operatorname{Map}(\operatorname{Spf} T, \operatorname{Spf} P) \cong \operatorname{Fun}^{L,\otimes}_{\mathrm{Sp}^{\wedge}_p}(\mathrm{Mod}_P^{\wedge p}, \mathrm{Mod}_T^{\wedge p})^{\simeq} \cong \operatorname{Map}_{\mathrm{CAlg}(\mathrm{Sp}^{\wedge}_p)}(P, T).$$

The last mapping space is discrete because P and T have only their degree-zero homotopy groups.

For a perfectoid stack F, let $\underline{F}$ denote its underlying presheaf. Define its realization by

$$\mathrm{Real}_{\mathrm{perf}}(F) := a_{\mathrm{Zar}} \operatorname{Lan}_{\nu^{\mathrm{op}}}(\underline{F}) \quad \text{in } \mathscr{T}. \tag{8.3}$$

Equivalently,

$$\mathrm{Real}_{\mathrm{perf}}(F) \cong \operatorname*{colim}_{(h_P \to F) \in \mathscr{P}_p/F} \operatorname{Spf} P,$$

where the colimit is in $\mathscr{T}$. Define the perfectoidization of an input stack X by

$$X^{\mathrm{pfd}} := \mathrm{Real}_{\mathrm{perf}}(i^* X).$$

For an affine input, the notation is $(\operatorname{Spf} A)^{\mathrm{pfd}} = \mathrm{Real}_{\mathrm{perf}}(h_A)$. It is a sheaf of Spaces on the affine site; a representability theorem is required before calling it a perfectoid formal scheme.

Lemma 8.1 (Realization of a representable sheaf)**.** *For a perfectoid ring P there is a canonical isomorphism*

$$\mathrm{Real}_{\mathrm{perf}}(h_P) \cong \operatorname{Spf} P.$$

Consequently, if a sheaf F on $\mathscr{P}_p$ is represented by P, its realization is the perfectoid formal scheme $\operatorname{Spf} P$*. In particular, if a universal perfectoid A-algebra A_{pfd} exists, then*

$$(\operatorname{Spf} A)^{\mathrm{pfd}} \cong \operatorname{Spf}(A_{\mathrm{pfd}}).$$

Proof. Left Kan extension of the representable presheaf h_P along ν^{op} is the representable presheaf of $\operatorname{Spf} P$. This follows directly from Yoneda: mapping that extension to a presheaf G is evaluation $G(\operatorname{Spf} P)$. The Zariski topology is subcanonical by proposition 7.4, so its sheafification is the same representable. Apply this to an isomorphism $F \cong h_P$ and then to $h_A \cong h_{A_{\mathrm{pfd}}}$. □

Lemma 8.2 (Restriction depends on the ordinary completed ring)**.** *Let A be a connective p-complete ring. For every perfectoid ring T,*

$$h_A(T) \cong \operatorname{Map}_{\mathrm{CAlg}}(\overset{\mathrm{ord}}{\lim_n}(\pi_0 A)/p^n, T).$$

Proof. The target T has only its degree-zero homotopy group, so maps from the connective ring A to T are exactly maps from $\pi_0 A$. Ordinary p-adic completeness of T gives the displayed completion formula. □

Lemma 8.3 (Restriction preserves limits)**.** *The restriction functor i^* on the auxiliary test sheaves preserves limits. In particular,*

$$i^*(X \times_Z Y) \cong i^* X \times_{i^* Z} i^* Y.$$

Proof. It is restriction along i. □

Lemma 8.4 (Extension of an atlas to all complete algebra tests)**.** *Let $F \in \mathrm{Shv}(\mathscr{P}_p; \mathrm{Spaces})$ and let $h_T \to F$ be an affine atlas. Then $\mathrm{Spf}\, T \to i_! F$ is an affine atlas on $\mathscr{A}_p$ with the same coordinate rings in its Čech nerve.*

Proof. The nerve of $h_T \to F$ is a groupoid of perfectoid-ring representables with p-completely faithfully flat source and target maps. Its pullbacks over h_T are also the pullbacks on $\mathscr{A}_p$ by proposition 1.11. The functor $i_!$ preserves colimits and takes h_P to $\mathrm{Spf}\, P$ for perfectoid P. Therefore $i_! F$ is the quotient of this same groupoid on $\mathscr{A}_p$. Faithfully flat descent of affine complete algebras gives the stated atlas and nerve. This assertion does not concern $\mathrm{Real}_{\mathrm{perf}}$. □

Definition 8.5 (An atlas by perfectoid rings)**.** A stack X on $\mathscr{A}_p$ has a *perfectoid atlas* if it has an affine atlas $\mathrm{Spf}\, B \to X$ with B perfectoid such that, for every perfectoid T and every $\mathrm{Spf}\, T \to X$, the pullback is $\mathrm{Spf}\, C$ for a perfectoid ring C.

Every term of the atlas nerve is then represented by a perfectoid ring. The same nerve presents X on $\mathscr{A}_p$ and $i^* X$ on $\mathscr{P}_p$. Consequently $i_! i^* X \to X$ is an isomorphism, and (8.2) compares their complete coefficients. No assertion about an atlas on the affine site is used here.

Lemma 8.6. *Let $(T_i)_{i \in \mathcal{I}}$ be a small diagram of perfectoid algebras over a perfectoid ring P. If its limit in $\mathrm{CAlg}(\mathrm{Sp}_p^{\wedge})$ is connective, the limit is a discrete perfectoid ring.*

Proof. Write $A = \lim_i T_i$. Since each T_i is discrete, $\pi_j A = 0$ for $j > 0$. Hence A is discrete and is the ordinary limit of the rings T_i. Choose a generator $\xi = [a] + pu$ of $\ker \theta_P$. By lemma 1.5, multiplication by ξ on each $W(T_i^{\flat})$ is injective with cokernel T_i.

Put $H = \lim_i W(T_i^{\flat})$, with the limit taken in ring spectra. It is ξ-complete because each $W(T_i^{\flat})$ is ξ-complete. Taking cofibers of multiplication by ξ gives $\mathrm{cofib}(\xi : H \to H) \cong A$ in modules. The cofibers of ξ^n are successive extensions of copies of A; they are discrete and their transition maps are surjective. Their limit H is therefore discrete. Witt vectors preserve ordinary limits, and the multiplicative description of the tilt in proposition 1.1 gives

$$H \cong W\left(\lim_i{}^{\mathrm{ord}}\, T_i^{\flat}\right) \cong W(A^{\flat}).$$

The map $H \to A$ is θ_A. The ring $A^{\flat}$ is perfect and a-adically complete: it is perfect as an ordinary limit of perfect rings, and $A^{\flat} = \lim_F A/p$ with $A/p \cong A^{\flat}/(a)$. Inverse Frobenius identifies the latter limit with $\lim_n A^{\flat}/(a^{p^n})$. Thus $W(A^{\flat})/(\xi) \cong A$ is perfectoid by lemma 1.3. □

In particular, let F be a sheaf on $\mathscr{P}_p$ with a map to h_P, where P is perfectoid. If $\Gamma(F, \mathcal{O})$ is connective, the preceding lemma applied to its diagram of perfectoid P-algebra tests makes it a discrete perfectoid ring.

8.3 Prismatic computation of functions on the perfectoid site

Lemma 8.7 (Perfectoidization of semiperfect algebras over a perfectoid ring)**.** *Let B be a connective p-complete derived commutative algebra over a perfectoid ring R. Suppose Frobenius on $(\pi_0 B)/p$ is surjective. Then B receives a map from a perfectoid ring surjective on π_0 and has a universal perfectoid algebra B_{perfd}.*

Proof. Put $C = \lim_F((\pi_0 B)/p)$, a perfect discrete $\mathbf{F}_p$-algebra. The projection $C \to (\pi_0 B)/p$ is surjective. The vanishing of the completed cotangent module of $W(C)$, proved in lemma 3.6, lifts this map uniquely through the Postnikov truncations of B/p and then through B/p^n for every n. It gives a map $W(C) \to B$. The resulting map

$$R\widehat{\otimes}_{\mathbf{Z}_p} W(C) \longrightarrow B$$

is surjective on π_0 modulo p. Its source is discrete and perfectoid: its Witt lift is the (p,d)-completion of $W(R^\flat) \otimes_{\mathbf{Z}_p} W(C)$, a perfect δ-ring, and lemma 1.3 applies. The cofiber of the displayed map is connective and p-complete; its reduction modulo p belongs to $(\mathrm{Mod}_{\mathbf{Z}_p})_{\geq 1}$. The same bound holds modulo every p^n and in their inverse limit. The displayed map is therefore surjective on π_0. By lemma 8.2 and proposition 4.3, the perfectoid quotient associated with the kernel on π_0 supplies B_{perfd}. □

Corollary 8.8 (Universal perfectoid algebras for tensor products)**.** *Let A be a connective p-complete ring, and let T_1, T_2 be perfectoid A-algebras. The completed tensor product $T_1 \widehat{\otimes}_A T_2$ admits a universal map*

$$T_1 \widehat{\otimes}_A T_2 \longrightarrow (T_1 \widehat{\otimes}_A T_2)_{\mathrm{perfd}}$$

to a perfectoid ring.

Proof. Maps to perfectoid rings depend only on the discrete base $\pi_0 A$. Form

$$B = T_1 \widehat{\otimes_{\pi_0 A}} T_{2p}$$

in derived commutative T_1-algebras. This ring is connective, and

$$(\pi_0 B)/p \cong (T_1/p) \otimes^{\mathrm{ord}}_{(\pi_0 A)/p} (T_2/p)$$

has surjective Frobenius. Apply lemma 8.7. The universal mapping problem is unchanged by the replacement of the base and by completion. □

Proposition 8.9 (An affine atlas on the perfectoid site)**.** *For every connective p-complete ring A, the sheaf h_A on $\mathscr{P}_p$ admits an affine atlas h_T with T perfectoid. Its diagonal is affine.*

Proof. Replace A by $\widehat{\pi_0 A_p}$ using lemma 8.2. Put $P = \mathbf{Z}_p[p^{1/p^\infty}]^\wedge_p$ and choose $a_i \in A$ whose images generate A/p as an $\mathbf{F}_p$-algebra. Form the p-complete derived commutative algebra

$$A' = (A\widehat{\otimes}_{\mathbf{Z}_p} P)\widehat{\otimes}_{P\langle x_i | i\in I\rangle} P\langle x_i^{1/p^\infty} \mid i \in I\rangle, \qquad x_i \longmapsto a_i.$$

Frobenius on $(\pi_0 A')/p$ is surjective, so lemma 8.7 gives $T = A'_{\mathrm{perfd}}$. On perfectoid tests, the map $h_T \to h_A$ adjoins compatible p-power roots of p and the a_i. Lemma 3.18, followed by completed filtered colimits, makes each of its pullbacks p-completely faithfully flat. Corollary 8.8 represents these pullbacks by perfectoid rings.

Thus $h_T \to h_A$ is an affine atlas. Since $h_T \times_{h_A} h_T$ is affine, faithfully flat descent makes the diagonal of h_A affine. □

Lemma 8.10 (Prismatic construction of the universal perfectoid algebra for semiperfect reduction)**.** *Let P be perfectoid, let d generate $\ker \theta_P$, and let R be a connective p-complete derived commutative P-algebra with surjective Frobenius on $(\pi_0 R)/p$. There are natural isomorphisms*

$$\operatorname*{colim}_{\varphi} \widehat{\Delta_{R/W(P^\flat)}}_{(p,d)} \otimes_{W(P^\flat)} P \cong R_{\mathrm{perfd}} \cong \Gamma(h_R, \mathcal{O}).$$

The ring R_{perfd} is discrete and perfectoid.

Proof. By lemma 8.7, choose a perfectoid P-algebra C mapping to R surjectively on π_0. We prove that $\Delta_{R/W(P^\flat)}$ is connective. Starting from C, a connective derived commutative P-algebra with this surjectivity property can be obtained by adjoining generators in degrees $n \geq 1$, adjoining nullhomotopies of elements in degrees $n \geq 0$, and taking colimits and p-completion. To construct such a presentation, kill the kernel on π_0, then successively make the maps on π_n surjective and kill their kernels. The resulting map is an isomorphism on every homotopy group.

Write $\mathrm{Sym}^{\delta}_{W(P^\flat)}(N)$ for the free derived δ-$W(P^\flat)$-algebra on a module $N \in \mathrm{Mod}_{W(P^\flat)}$. Duality for $P = W(P^\flat)/(d)$ and proposition 3.3 give

$$\Delta_{\mathrm{Sym}_P(P[n])/W(P^\flat)} \cong \widehat{\mathrm{Sym}^{\delta}_{W(P^\flat)}(P[n-1])}_{(p,d)}.$$

For $n \geq 1$ the module $P[n-1]$ is connective. The free δ-algebra functor preserves connective objects, as follows from its polynomial Kan extension. Pushouts of connective derived δ-algebras and adic completions are connective. Consequently adjoining a positive-degree generator or killing a positive-degree element preserves connectivity of prismatic cohomology.

For a degree-zero relation, let D be the connective complete derived δ-algebra obtained at the preceding stage. Its image in $D \otimes_{W(P^\flat)} P$ is an element $\bar{f} \in \pi_0(D \otimes_{W(P^\flat)} P)$. Connectivity gives $\pi_0(D \otimes_{W(P^\flat)} P) = (\pi_0 D)/(d)$, so choose a lift $\widetilde{f} \in \pi_0 D$. The complete derived δ-algebra representing a nullhomotopy of $\bar{f}$ is

$$\widehat{D\{y\}_\delta \otimes_{D\{z\}_\delta} D}_{(p,d)}, \qquad z \longmapsto dy - \widetilde{f}, \quad z \longmapsto 0.$$

Here $D\{y\}_\delta$ and $D\{z\}_\delta$ are free derived δ-D-algebras on degree-zero generators. This is the nullhomotopy calculation in lemma 3.16; the displayed pushout is connective. The initial stage is connective by $\Delta_{C/W(P^\flat)} = W(C^\flat)$ from lemma 3.6. The presentation therefore proves connectivity of $\Delta_{R/W(P^\flat)}$.

Completed Frobenius perfection preserves connectivity here. Its reduction modulo d is therefore connective, so lemma 3.17 identifies it with $R_{\mathrm{perfd}} \cong \Gamma(h_R, \mathcal{O})$. □

Theorem 8.11 (Prismatic computation of functions)**.** *Let P be perfectoid, let d generate $\ker(\theta_P : W(P^\flat) \to P)$, and let R be a connective p-complete derived commutative P-algebra. There is a natural isomorphism*

$$\Gamma(h_R, \mathcal{O}) \cong \operatorname*{colim}_{\varphi} \widehat{\Delta_{R/W(P^\flat)}}_{(p,d)} \otimes_{W(P^\flat)} P.$$

The Frobenius transition maps are semilinear with respect to $\varphi_{W(P^\flat)}$; the $W(P^\flat)$-algebra structures in the colimit are twisted accordingly.

Proof. For each perfectoid P-algebra T equipped with $R \to T$, the adjunction and invertibility of Frobenius on $W(T^\flat)$ give

$$\operatorname*{colim}_{\varphi} \widehat{\Delta_{R/W(P^\flat)}}_{(p,d)} \longrightarrow W(T^\flat).$$

Reduction modulo d gives a map to T. These maps form a cone over the perfectoid R-algebras, hence give the comparison with $\Gamma(h_R, \mathcal{O}) = \lim_{T \in \mathrm{PerfAlg}_R} T$.

Choose elements $a_i \in \pi_0 R$ whose images generate $(\pi_0 R)/p$ over P/p, together with lifts to a map $P\langle x_i \mid i \in I\rangle \to R$. Form

$$R' = R \widehat{\otimes}_{P\langle x_i | i \in I\rangle} P\langle x_i^{1/p^\infty} \mid i \in I\rangle.$$

The map $h_{R'} \to h_R$ of sheaves on $\mathscr{P}_p$ is an effective epimorphism. Indeed, corollary 4.1 gives each perfectoid R-algebra a p-completely faithfully flat perfectoid extension in which one can choose compatible p-power roots of all the a_i. For each $q \geq 0$, Frobenius on $\pi_0((R')^{\widehat{\otimes}_R(q+1)})/p$ is surjective. This ring is generated over P/p by the root sequences from the $q+1$ factors. By lemma 8.10, each term has a universal discrete perfectoid algebra. Descent of functions on this nerve gives

$$\Gamma(h_R, \mathcal{O}) \;\cong\; \mathrm{Tot}_{[q]\in\Delta}\big((R')^{\widehat{\otimes}_R(q+1)}\big)_{\mathrm{perfd}}. \tag{8.4}$$

When I is finite, theorem 3.15 computes perfected prismatic cohomology by the same nerve. The module $P \cong W(P^\flat)/(d)$ is perfect over $W(P^\flat)$, so tensoring with P preserves this totalization. The termwise comparison of lemma 8.10 proves the theorem in this case.

For general R, choose the a_i to generate $\pi_0 R$ as a P-algebra. For each finite subset $E \subset I$, form

$$R_E = R \times_{\pi_0 R} \widehat{P[a_i \mid i \in E]}_p, \qquad P[a_i \mid i \in E] \subseteq \pi_0 R.$$

Before completion this pullback has degree-zero ring $P[a_i \mid i \in E]$ and maps isomorphically to R on π_j for $j > 0$. Hence $R \cong \widehat{\mathrm{colim}_E R_{Ep}}$. Reduction modulo p gives $(\pi_0 R_E)/p \cong P[a_i \mid i \in E]/p$. The chosen lifts of the a_i give compatible polynomial maps for these finite sets. Use them in (8.4) for R_E.

In each Čech degree the root extensions for R_E have completed filtered colimit the corresponding term for R. Their universal perfectoid algebras have the same property by the universal property and lemma 1.14. Tensoring any of these discrete rings over $\mathbf{Z}_p$ with $\mathbf{Z}/p^n$ gives homotopy groups only in degrees 0 and 1. Thus lemma 3.7 interchanges filtered colimit and totalization modulo p^n, and tensoring with the perfect $\mathbf{Z}_p$-module $\mathbf{Z}/p^n$ preserves totalizations. Taking the inverse limit over n gives

$$\widehat{\mathrm{colim}_E \Gamma(h_{R_E}, \mathcal{O})}_p \;\cong\; \Gamma(h_R, \mathcal{O}).$$

By proposition 3.4, perfected prismatic cohomology followed by tensoring with P preserves completed filtered colimits. The finite-set comparisons now give the assertion for R. □

Compare [1, Proposition 8.5] and [4, Corollary 4.4.4]. The polynomial de Rham comparison is proved in theorem 3.12.

Corollary 8.12 (Completed filtered colimits over a perfectoid ring)**.** *For a perfectoid ring P and a filtered diagram of connective p-complete derived commutative P-algebras R_i, there is a natural isomorphism*

$$\widehat{\mathrm{colim}_i \Gamma(h_{R_i}, \mathcal{O})}_p \;\cong\; \Gamma(h_{\widehat{\mathrm{colim}_i R_{ip}}}, \mathcal{O}).$$

Proof. Apply theorem 8.11 and proposition 3.4. □

Proposition 8.13 (Completed tensor products over a perfectoid ring)**.** *Let P be perfectoid and let $R_1 \leftarrow R_0 \to R_2$ be a diagram of connective p-complete P-algebras. Then*

$$\Gamma(h_{R_1}, \mathcal{O}) \widehat{\otimes}_{\Gamma(h_{R_0}, \mathcal{O})} \Gamma(h_{R_2}, \mathcal{O}) \cong \Gamma(h_{R_1 \widehat{\otimes}_{R_0} R_2}, \mathcal{O}).$$

Proof. For derived commutative P-algebras, this follows from proposition 3.4 and theorem 8.11: Frobenius perfection, completion, and extension of scalars to P preserve pushouts. For connective E_∞-P-algebras, lemma 8.2 reduces the assertion to the discrete rings $\pi_0 R_i$. Their tensor products have the same π_0 in E_∞-rings and in derived commutative rings. □

9 0-affineness of perfectoidizations

In this section all sheaves and stacks are on the auxiliary sites $\mathscr{A}_p$ or $\mathscr{P}_p$, as specified in each statement. The notation $\mathrm{QC}(F)$ denotes their compatible families of complete module coefficients, and $\Gamma(F,\mathcal{O})$ denotes their complete ring of functions. For a representable on $\mathscr{A}_p$, $\mathrm{QC}(\operatorname{Spf} A) = \mathrm{Mod}_A^{\wedge p}$. The recognition theorems at the end apply lemma 8.1 to obtain perfectoid formal schemes on the affine test site.

Definition 9.1 (p-adic 0-affineness)**.** Let F be a stack on $\mathscr{A}_p$ with quasi-affine diagonal and a p-completely faithfully flat cover $u : \operatorname{Spf} B \to F$. Assume that $\mathrm{QC}(F)$ is compactly generated and rigid over $\mathrm{Mod}_{\mathbf{Z}_p}^{\wedge p}$, and that $u_*\mathcal{O}_{\operatorname{Spf} B} \in \mathrm{CAlg}(\mathrm{QC}(F))$ is descendable. We say that F is *p-adically* 0*-affine* if the canonical functor

$$\mathrm{Mod}_{\Gamma(F,\mathcal{O})}^{\wedge p} \longrightarrow \mathrm{QC}(F)$$

is an isomorphism of symmetric monoidal $\mathrm{Mod}_{\mathbf{Z}_p}^{\wedge p}$-linear categories.

The definition of 0-affineness in greater generality lies outside the scope of these notes.

Proposition 9.2 (A descendable affine atlas proves p-adic 0-affineness)**.** *Let $F \in \mathrm{Sh}(\mathscr{A}_p; \mathrm{Spaces})$. Suppose $\operatorname{Spf} B \to F$ is an affine atlas whose Čech nerve has coordinate rings $B^{\widehat{\otimes}_{\Gamma(F,\mathcal{O})}(q+1)}$. If $\Gamma(F,\mathcal{O}) \to B$ is descendable, then F is p-adically* 0*-affine.*

Proof. Lemma 2.12 and (5.1) give

$$\mathrm{QC}(F) \cong \lim_{[q]\in\Delta} \mathrm{Mod}^{\wedge p}_{B^{\widehat{\otimes}_{\Gamma(F,\mathcal{O})}(q+1)}} \cong \mathrm{Mod}^{\wedge p}_{\Gamma(F,\mathcal{O})}\,.$$

Under this isomorphism, the atlas algebra is B, so it is descendable. The affine atlas gives F affine diagonal. The category $\mathrm{Mod}_{\Gamma(F,\mathcal{O})}^{\wedge p}$ is compactly generated by

$$\mathrm{cofib}\big(p : \Gamma(F,\mathcal{O}) \longrightarrow \Gamma(F,\mathcal{O})\big)$$

and is rigid over $\mathrm{Mod}_{\mathbf{Z}_p}^{\wedge p}$. □

9.1 Perfectoidization over complete discrete valuation rings

Let $\underline{\mathbf{Z}_p} = \lim_n \underline{\mathbf{Z}/p^n}$ be the inverse limit of constant finite group sheaves on $\mathscr{P}_p$. Over the representable $\underline{h_S}$ on $\mathscr{P}_p$, it is represented by $h_{\mathrm{Cont}(\mathbf{Z}_p,S)}$, where S has its p-adic topology. For a continuous action on h_S, the quotient is taken in sheaves on this site, and

$$S^{h\underline{\mathbf{Z}_p}} := \Gamma((h_S)/\underline{\mathbf{Z}_p}, \mathcal{O}) = \mathrm{Tot}\big(\mathrm{Cont}(\mathbf{Z}_p^\bullet, \mathbf{Z}_p)\widehat{\otimes}_{\mathbf{Z}_p} S\big),$$

with the maps of the continuous action nerve.

Lemma 9.3 (Continuous $\mathbf{Z}_p$-invariants are the fiber of $\gamma - 1$)**.** *For a $\underline{\mathbf{Z}_p}$-equivariant object $M \in \mathrm{Mod}_{\mathbf{Z}_p}^{\wedge p}$, including an unbounded object, let γ be the action of $1 \in \mathbf{Z}_p$. Then*

$$M^{h\underline{\mathbf{Z}_p}} \cong \mathrm{fib}(\gamma - 1 : M \to M).$$

For $N \in \mathrm{Mod}_{\mathbf{Z}_p}^{\wedge p}$ with trivial action,

$$(M\widehat{\otimes}_{\mathbf{Z}_p} N)^{h\underline{\mathbf{Z}_p}} \cong M^{h\underline{\mathbf{Z}_p}}\widehat{\otimes}_{\mathbf{Z}_p} N.$$

Proof. There is an equivariant fiber sequence

$$M \longrightarrow \mathrm{Cont}(\mathbf{Z}_p, \mathbf{Z}_p)\widehat{\otimes}_{\mathbf{Z}_p} M \xrightarrow{\text{translation}-1} \mathrm{Cont}(\mathbf{Z}_p, \mathbf{Z}_p)\widehat{\otimes}_{\mathbf{Z}_p} M,$$

where the middle terms have the diagonal action. Apply $\Lambda_{\mathbf{Z}_p}$ and tensor with $\mathbf{F}_p$. The resulting assertion is checked on each homotopy group using locally constant functions. Their translation-invariant functions are constant. A function on $\mathbf{Z}/p^n$ is a difference on a sufficiently larger cyclic quotient: its sum over a cycle becomes zero after repeating that cycle p times, and successive partial sums solve the difference equation. This proves surjectivity. The assertion before reduction follows from completeness.

Each middle term is coinduced; the diagonal action is identified with the coinduced action by the original action on M. Its invariants are M by adjunction. The difference operator on these invariants is $\gamma - 1$, with the corresponding choice of translation. Taking invariants, which preserves fibers, proves the first isomorphism. Tensoring the displayed fiber sequence with N proves the second isomorphism. □

Proposition 9.4 (p-complete faithful flatness of $R \to (R[x]/(f))_{\mathrm{perfd}}$ for monic f)**.** *For perfectoid R and monic $f \in R[x]$ of positive degree, $R \to (R[x]/f)_{\mathrm{perfd}}$ is p-completely faithfully flat. The same holds for a filtered tower obtained by adjoining roots of monic polynomials and then applying perfectoidization.*

Proof. The perfectoid algebra $(R[x]/f)_{\mathrm{perfd}}$ exists by proposition 4.7. Let $b \in (R[x]/f)_{\mathrm{perfd}}$ be the image of x. The universal properties give

$$C = \left(R[x^{1/p^\infty}]\widehat{\ }/(f(x))_p\right)_{\mathrm{perfd}} \cong \left((R[x]/f)_{\mathrm{perfd}}\widehat{[y^{1/p^\infty}]}/(y-b)_p\right)_{\mathrm{perfd}}.$$

Indeed, a map from either ring to a perfectoid R-algebra specifies a root of f and compatible p-power roots of that root. By proposition 3.18, both $R \to C$ and $(R[x]/f)_{\mathrm{perfd}} \to C$ are p-completely faithfully flat. On modules with p-power torsion, extension of scalars from $(R[x]/f)_{\mathrm{perfd}}$ to C detects both halves of the torsion t-structure and detects zero objects. Its composite with extension from R to $(R[x]/f)_{\mathrm{perfd}}$ is t-exact and conservative. Thus $R \to (R[x]/f)_{\mathrm{perfd}}$ is p-completely faithfully flat. The assertion for towers follows by composition and proposition 4.7 and lemma 2.7. □

Let $\mathcal{O}_K$ be a complete discrete valuation ring of mixed characteristic $(0, p)$ with perfect residue field k. Suppose K_∞/K is a totally ramified Galois extension whose completion L is a perfectoid field. Write $\mathfrak{m}_{\mathcal{O}_L} \subset \mathcal{O}_L$ for its maximal ideal, so $\mathcal{O}_L/\mathfrak{m}_{\mathcal{O}_L} \cong k$. The restriction $h_{\mathcal{O}_K} = i^*(\mathrm{Spf}\,\mathcal{O}_K)$ is a sheaf on $\mathscr{P}_p$. Continuous functions into $\mathcal{O}_L$ use its p-adic topology; continuous functions into k use the discrete topology.

Proposition 9.5 (A perfectoid atlas for $h_{\mathcal{O}_K}$)**.** *The map $g : h_{\mathcal{O}_L} \to h_{\mathcal{O}_K}$ is an affine atlas. Its Čech nerve has coordinate rings*

$$\mathrm{Cont}(\mathrm{Gal}(K_\infty/K)^q, \mathcal{O}_L) \times_{\mathrm{Cont}(\mathrm{Gal}(K_\infty/K)^q, k)} k \qquad (q \geq 0).$$

For $q = 1$, the two structure maps from $\mathcal{O}_L$ are $a \mapsto (\sigma \mapsto a)$ and $a \mapsto (\sigma \mapsto \sigma(a))$. Furthermore,

$$\mathrm{QC}(h_{\mathcal{O}_K}) = \mathrm{QC}([h_{\mathcal{O}_L}/\,\mathrm{Gal}(K_\infty/K)]) \times_{\mathrm{QC}([h_k/\,\mathrm{Gal}(K_\infty/K)])} \mathrm{Mod}_k\,. \tag{9.1}$$

The functor $\mathrm{Mod}_k \to \mathrm{QC}([h_k/\,\mathrm{Gal}(K_\infty/K)])$ gives a module its trivial continuous $\mathrm{Gal}(K_\infty/K)$-action.

See Bhatt [4, Proposition 4.5.1 and Example 4.5.2].

Proof. Every finite layer of K_∞/K is generated over $\mathcal{O}_K$ by a uniformizer. Over $\mathcal{O}_L$, its Eisenstein polynomial splits into its conjugate linear factors. A difference of two distinct conjugates has positive rational valuation; a power of the difference is a power of p times a unit. Formula (4.1) therefore identifies the perfectoidization of the split algebra with copies of $\mathcal{O}_L$ glued along $(\mathcal{O}_L/p)_{\mathrm{perfd}} = k$. Passing to the finite layers and completing gives the displayed ring for $q = 1$. Repeating these factorizations for q chosen embeddings gives the displayed ring for every q. Evaluation on the embeddings identifies all maps of the nerve.

For a perfectoid $\mathcal{O}_K$-algebra T, the pullback of g is represented by $(T\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_L)_{\mathrm{perfd}}$. Write $\mathcal{O}_L$ as the p-completed filtered colimit of the integer rings in the finite layers of K_∞/K. Each layer is finite free over $\mathcal{O}_K$. Proposition 9.4, followed by lemmas 1.14 and 2.7, shows that

$$T \longrightarrow (T\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_L)_{\mathrm{perfd}}$$

is p-completely faithfully flat. Thus g is an affine atlas.

For each q, all four coordinate rings in the displayed pullback are perfectoid. The projection to k is surjective, so k is idempotent over the pullback ring by propositions 1.11 and 4.3. The completed tensor product of k with $\mathrm{Cont}(\mathrm{Gal}(K_\infty/K)^q, \mathcal{O}_L)$ over the pullback ring is $\mathrm{Cont}(\mathrm{Gal}(K_\infty/K)^q, k)$: the ordinary pushout is this quotient, which is already perfect, and proposition 1.11 computes its universal perfectoid algebra. Thus the implication (iii) $\Rightarrow$ (v) in theorem 5.7 gives (5.7) in every degree. Taking the limit gives (9.1). □

Assume now $\mathrm{Gal}(K_\infty/K) \cong \mathbf{Z}_p$, choose a topological generator γ, and normalize the valuation by $v(\pi_K) = 1$. The category

$$\mathrm{Mod}_{\mathcal{O}_K/p}(\mathrm{QC}(h_{\mathcal{O}_K})) \cong \lim_{[q]\in\Delta} \mathrm{Mod}_{\left(\mathrm{Cont}(\mathrm{Gal}(K_\infty/K)^q,\mathcal{O}_L)\times_{\mathrm{Cont}(\mathrm{Gal}(K_\infty/K)^q,k)}k\right)/p}$$

has the t-structure detected by pullback to $\mathrm{Mod}_{\mathcal{O}_L/p}$. Here $\mathcal{O}_K/p$ denotes the commutative algebra object induced by $\mathcal{O}_K \to \mathcal{O}_K/p$, and global sections on this module category take values in $\mathrm{Mod}_{\mathcal{O}_K/p}$. We write $\pi_j F$ for its homotopy objects.

Lemma 9.6 (Global sections on $h_{\mathcal{O}_K}$ as a fiber)**.** *An object of* $\mathrm{QC}(h_{\mathcal{O}_K})$ *is a triple* $(\mathcal{E}, N, \alpha)$*, where* $\mathcal{E} \in \mathrm{Mod}_{\mathcal{O}_L}^{\wedge p}$ *has continuous semilinear* $\underline{\mathbf{Z}_p}$*-action and* $N \in \mathrm{Mod}_k$*. Put* $E = \Lambda_{\mathcal{O}_L}\mathcal{E}$*. The map* $\alpha : E\widehat{\otimes}_{\mathcal{O}_L}k \cong N$ *is an equivariant isomorphism for the trivial action on* N*. For the corresponding object* F*,*

$$\Gamma(h_{\mathcal{O}_K}, F) = \mathrm{fib}(d_F : E \longrightarrow \mathfrak{m}_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} E). \tag{9.2}$$

The map d_F *is* $\mathcal{O}_K$*-linear and is determined by* $\gamma - 1$ *together with* α*. If* $E = M$ *is discrete, the target is discrete and*

$$\iota d_F(x) = \gamma(x) - x, \qquad d_F(ax) = \gamma(a)d_F(x) + (\gamma(a) - a) \otimes x. \tag{9.3}$$

Here $a \in \mathcal{O}_L$*,* $x \in M$*, and* $\iota : \mathfrak{m}_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} M \to M$ *is multiplication. The same formulas hold for objects of* $\mathrm{Mod}_{\mathcal{O}_K/p}(\mathrm{QC}(h_{\mathcal{O}_K}))$*.*

Proof. Equation (9.1) gives the triples and

$$\Gamma(h_{\mathcal{O}_K}, F) = E^{h\underline{\mathbf{Z}_p}} \times_{N^{h\underline{\mathbf{Z}_p}}} N.$$

Tensoring $\mathfrak{m}_{\mathcal{O}_L} \to \mathcal{O}_L \to k$ with E identifies $\mathrm{fib}(E \to N)$ with $\mathfrak{m}_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} E$. Both E and $E \otimes_{\mathcal{O}_L} k$ are p-complete, hence so is $\mathfrak{m}_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} E$. The trivialization of the action on N

supplied by α gives a nullhomotopy of $E \xrightarrow{\gamma-1} E \to N$, hence a lift $d_F : E \to \mathfrak{m}_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} E$ of $\gamma - 1$. Applying lemma 9.3 to E and N identifies the displayed pullback with $\mathrm{fib}(d_F)$. Flatness of $\mathfrak{m}_{\mathcal{O}_L}$ gives discreteness for $E = M$. The identity

$$(\gamma - 1)(ax) = \gamma(a)(\gamma - 1)x + (\gamma(a) - a)x$$

and the k-linear nullhomotopy specified by α give (9.3). Forming $\mathcal{O}_K/p$-module objects preserves the categorical pullback, proving the formulas with $\mathcal{O}_K/p$ coefficients. □

Lemma 9.7 (Existence of a with $0 < v(a) < c < v(\gamma(a) - a)$)**.** *For every real $c > 0$, there exists $a \in \mathfrak{m}_{\mathcal{O}_L}$ such that*

$$0 < v(a) < c < v(\gamma(a) - a).$$

Proof. In the layer K_r/K of degree p^r, choose a uniformizer π. Write $\delta = v(\pi) = p^{-r}$ and choose r large enough that $N = \lfloor c/\delta \rfloor \geq p+2$. The Galois p-group acts on $(\pi)/(\pi^2)$ through a character with values in $k^\times$, which has no p-torsion. Thus $\gamma(\pi) \equiv \pi \bmod \pi^2$. Modulo π^2 each element of $\mathcal{O}_{K_r}$ is $u_0 + u_1\pi$ with $u_0, u_1 \in \mathcal{O}_K$, so γ acts trivially on $\mathcal{O}_{K_r}/\pi^2$. The binomial formula gives $\gamma(\pi)^p \equiv \pi^p \bmod \pi^{p+2}$. Consequently $(\gamma - 1)(\pi^p u) \in (\pi^{p+2})$ for all $u \in \mathcal{O}_{K_r}$.

On the $\mathcal{O}_K$-module $(\pi^p)/(\pi^{N+1})$, the image of $\gamma - 1$ lies in $(\pi^{p+2})/(\pi^{N+1})$. Each successive quotient $(\pi^i)/(\pi^{i+1})$ has $\mathcal{O}_K$-length one. The kernel therefore has length at least two. Choose a class in the kernel outside $(\pi^N)/(\pi^{N+1})$ and lift it to $a \in (\pi^p)$. Then

$$0 < v(a) < N\delta \leq c < (N+1)\delta \leq v(\gamma(a) - a).$$

□

Proposition 9.8 (Global sections detect nonzero discrete $\mathcal{O}_K/p$-modules on $h_{\mathcal{O}_K}$)**.** *For $0 \neq F \in \mathrm{Mod}_{\mathcal{O}_K/p}(\mathrm{QC}(h_{\mathcal{O}_K}))^\heartsuit$, one has $\pi_0\Gamma(h_{\mathcal{O}_K}, F) \neq 0$.*

Proof. Let M be the pullback of F to $\mathcal{O}_L/p$, and let $d : M \to \mathfrak{m}_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} M$ be the map of lemma 9.6. Define

$$M[\mathfrak{m}_{\mathcal{O}_L}] = \{x \in M : \mathfrak{m}_{\mathcal{O}_L} x = 0\}, \qquad K_M = \ker(\mathfrak{m}_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} M \to M).$$

The fiber sequence $\mathfrak{m}_{\mathcal{O}_L} \to \mathcal{O}_L \to k$ gives $K_M = \pi_1(k \otimes_{\mathcal{O}_L} M)$ and $\mathfrak{m}_{\mathcal{O}_L} K_M = 0$. If $M = M[\mathfrak{m}_{\mathcal{O}_L}]$, then $\mathfrak{m}_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} M = (\mathfrak{m}_{\mathcal{O}_L}/\mathfrak{m}_{\mathcal{O}_L}^2) \otimes_k M = 0$, so $\ker d = M \neq 0$.

Otherwise the smooth $\underline{\mathbf{Z}_p}$-action on $M/M[\mathfrak{m}_{\mathcal{O}_L}]$ makes $\gamma - 1$ locally nilpotent. Indeed an element fixed by γ^{p^r} is killed by $(\gamma-1)^{p^r}$. Choose $x \notin M[\mathfrak{m}_{\mathcal{O}_L}]$ with $(\gamma-1)x \in M[\mathfrak{m}_{\mathcal{O}_L}]$. Equation (9.3) gives $\mathfrak{m}_{\mathcal{O}_L} d(x) \subset K_M$; since $\mathfrak{m}_{\mathcal{O}_L}^2 = \mathfrak{m}_{\mathcal{O}_L}$ and $\mathfrak{m}_{\mathcal{O}_L} K_M = 0$, it follows that $\mathfrak{m}_{\mathcal{O}_L} d(x) = 0$.

The ideal $J = \mathrm{Ann}_{\mathcal{O}_L}(x)$ contains p, is proper, and is strictly smaller than $\mathfrak{m}_{\mathcal{O}_L}$. Set $c = \inf\{v(b) : 0 \neq b \in J\} > 0$. Then $\mathfrak{m}_{\mathcal{O}_L} J = \{b \in \mathcal{O}_L : v(b) > c\}$. Choose a as in lemma 9.7. The inequality $v(a) < c$ gives $ax \neq 0$. In

$$d(ax) = \gamma(a)d(x) + (\gamma(a) - a) \otimes x$$

the first term is zero because $\mathfrak{m}_{\mathcal{O}_L} d(x) = 0$; the second is zero because $\gamma(a) - a \in \mathfrak{m}_{\mathcal{O}_L} J$ and $Jx = 0$. Thus $0 \neq ax \in \ker d = \pi_0\Gamma(h_{\mathcal{O}_K}, F)$. □

Theorem 9.9 (p-adic 0-affineness of $i_! h_{\mathcal{O}_K}$ from a totally ramified perfectoid tower)**.** *Suppose K_∞/K is totally ramified with group $\mathbf{Z}_p$ and perfectoid completion. Then $i_! h_{\mathcal{O}_K}$ is p-adically 0-affine. The map $\Gamma(h_{\mathcal{O}_K}, \mathcal{O}) \to \mathcal{O}_L$ is descendable, and the tensor square of its fiber map in $\mathrm{Mod}^{\wedge p}_{\Gamma(h_{\mathcal{O}_K}, \mathcal{O})}$ is null. For every $F \in \mathrm{Mod}_{\mathcal{O}_K/p}(\mathrm{QC}(h_{\mathcal{O}_K}))$ and $j \in \mathbf{Z}$,*

$$\pi_j\Gamma(h_{\mathcal{O}_K}, F) = 0 \quad \Longrightarrow \quad \pi_j F = 0. \tag{9.4}$$

Proof. On $\mathrm{Mod}_{\mathcal{O}_K/p}(\mathrm{QC}(h_{\mathcal{O}_K}))$, the two functors $F \mapsto E$ and $F \mapsto \mathfrak{m}_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} E$ in (9.2) are t-exact. Their fiber gives, for every $j \in \mathbf{Z}$, an exact sequence

$$0 \to \pi_{-1}\Gamma(h_{\mathcal{O}_K}, \pi_{j+1}F) \to \pi_j\Gamma(h_{\mathcal{O}_K}, F) \to \pi_0\Gamma(h_{\mathcal{O}_K}, \pi_j F) \to 0.$$

By proposition 9.8, the rightmost term detects whether $\pi_j F$ vanishes. This proves (9.4) and conservativity on $\mathrm{Mod}_{\mathcal{O}_K/p}(\mathrm{QC}(h_{\mathcal{O}_K}))$.

For $F \in \mathrm{QC}(h_{\mathcal{O}_K})$, write $F/p = F \otimes_{\mathbf{Z}_p} \mathbf{F}_p$, regarded as an object of $\mathrm{Mod}_{\mathcal{O}_K/p}(\mathrm{QC}(h_{\mathcal{O}_K}))$. Formula (9.2) commutes with reduction modulo p, so

$$\Gamma(h_{\mathcal{O}_K}, F/p) \cong \Gamma(h_{\mathcal{O}_K}, F) \otimes_{\mathbf{Z}_p} \mathbf{F}_p.$$

Consequently $\Gamma(h_{\mathcal{O}_K}, F) = 0$ implies $F/p = 0$, and then $F = 0$ by p-completeness, which can be checked after pullback to $\mathcal{O}_L$. Equation (9.2) also shows that $\Gamma(h_{\mathcal{O}_K}, -) : \mathrm{QC}(h_{\mathcal{O}_K}) \to \mathrm{Mod}_{\mathbf{Z}_p}^{\wedge p}$ preserves colimits and gives the projection formula

$$\Gamma(h_{\mathcal{O}_K}, F \mathbin{\widehat{\otimes}}_{\mathbf{Z}_p} M) \cong \Gamma(h_{\mathcal{O}_K}, F) \mathbin{\widehat{\otimes}}_{\mathbf{Z}_p} M \qquad (M \in \mathrm{Mod}_{\mathbf{Z}_p}^{\wedge p}).$$

Monadicity applied to pullback and global sections gives a symmetric monoidal isomorphism

$$\mathrm{QC}(h_{\mathcal{O}_K}) \cong \mathrm{Mod}^{\wedge p}_{\Gamma(h_{\mathcal{O}_K}, \mathcal{O})}.$$

Put $X = h_{\mathcal{O}_K}$ and $A = \Gamma(X, \mathcal{O}) = \mathrm{Hom}_{\mathrm{QC}(X)}(\mathcal{O}_X, \mathcal{O}_X)$. Evaluating the natural fiber sequence of exact functors in lemma 9.6 at $\mathcal{O}_X$ gives a fiber sequence of A-modules

$$\mathfrak{m}_{\mathcal{O}_L}[-1] \longrightarrow A \longrightarrow \mathcal{O}_L.$$

Indeed, evaluation gives each term the action of the endomorphism spectrum of $\mathcal{O}_X$. On the functors $F \mapsto E$ and $F \mapsto \mathfrak{m}_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} E$ this action factors through $A \to \mathcal{O}_L$ by pullback to the atlas. Tensoring the fiber map over A with either outer term therefore gives a null map. The assertion on tensor powers in fiber sequences in proposition 5.3 gives a null tensor square over $\Gamma(h_{\mathcal{O}_K}, \mathcal{O})$ and proves descendability. The symmetric monoidal identification just proved identifies the atlas algebra $g_* \mathcal{O}_{h_{\mathcal{O}_L}}$ with $\mathcal{O}_L$, and its nerve with $\mathcal{O}_L^{\widehat{\otimes}_{\Gamma(h_{\mathcal{O}_K}, \mathcal{O})}(q+1)}$. By lemma 8.4, $\mathrm{Spf}\, \mathcal{O}_L \to i_! h_{\mathcal{O}_K}$ is an affine atlas with this nerve. Equation (8.2) and proposition 9.2 prove that $i_! h_{\mathcal{O}_K}$ is p-adically 0-affine. □

Corollary 9.10 (Cyclotomic formula for global sections of $h_{\mathbf{Z}_p}$)**.** *The stack $i_! h_{\mathbf{Z}_p}$ is p-adically 0-affine. Let L be the completion of the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}_p$. Then*

$$\Gamma(h_{\mathbf{Z}_p}, \mathcal{O}) = \mathcal{O}_L^{h\underline{\mathbf{Z}_p}} \times_{\mathbf{F}_p^{h\underline{\mathbf{Z}_p}}} \mathbf{F}_p.$$

Proof. Let $\Delta \subset \mathbf{Z}_p^\times$ be the finite torsion subgroup and take $K_\infty = \mathbf{Q}_p(\mu_{p^\infty})^\Delta$. This is totally ramified with Galois group $\mathbf{Z}_p$. For compatible primitive roots ζ_{p^m}, the elements $\tau_m = \prod_{\delta \in \Delta}(\zeta_{p^m}^\delta - 1)$ are uniformizers of its sufficiently large finite layers, whose integer rings are $\mathbf{Z}_p[\tau_m]$. The congruences $\tau_{m+1}^p \equiv \tau_m \pmod p$ make Frobenius on $\mathcal{O}_L/p$ surjective. Normalize the valuation by $v(p) = 1$. In a layer of degree p^r, with $r \geq 1$, its uniformizer τ_m has valuation p^{-r}, so $\varpi = \tau_m^{p^{r-1}}$ satisfies $(\varpi^p) = (p)$ in $\mathcal{O}_L$. Corollary 6.11 makes $\mathcal{O}_L$ perfectoid. Apply theorem 9.9 and lemma 9.6 to the structure sheaf. □

9.2 Completed tensor products

Theorem 9.11 (Completed tensor products)**.** *For connective p-complete rings in a diagram $R_1 \leftarrow R_0 \rightarrow R_2$, the canonical map*

$$\Gamma(h_{R_1}, \mathcal{O}) \widehat{\otimes}_{\Gamma(h_{R_0}, \mathcal{O})} \Gamma(h_{R_2}, \mathcal{O}) \longrightarrow \Gamma(h_{R_1 \widehat{\otimes}_{R_0} R_2}, \mathcal{O})$$

is an isomorphism.

Over a perfectoid base, this is Bhatt–Scholze [1, Proposition 8.13]; see also Bhatt [4, Theorem 4.4.5].

Proof. By lemma 8.2, it suffices to treat discrete $\mathbf{Z}_p$-algebras R_i. The case over a perfectoid base is proposition 8.13. Put $Y = h_{\mathbf{Z}_p}$. For each such algebra R, the assignments

$$h_T \longmapsto \Gamma(h_{R \widehat{\otimes}_{\mathbf{Z}_p} T}, \mathcal{O}), \qquad T \in \mathrm{PerfAlg}_{\mathbf{Z}_p},$$

form a commutative algebra in $\mathrm{QC}(Y)$ by the case over a perfectoid base. That case also computes their pushouts on every perfectoid test. By corollary 9.10 and (8.2), global sections identify $\mathrm{QC}(Y)$ symmetric monoidally with $\mathrm{Mod}^{\wedge p}_{\Gamma(Y,\mathcal{O})}$ and preserve these pushouts. This proves the general formula. □

Corollary 9.12 (Surjections on π_0 induce idempotent algebras of functions on perfectoidization)**.** *Let $R \rightarrow S$ be a map of connective p-complete rings inducing a surjection on π_0. Then*

$$\Gamma(h_S, \mathcal{O}) \widehat{\otimes}_{\Gamma(h_R, \mathcal{O})} \Gamma(h_S, \mathcal{O}) \cong \Gamma(h_S, \mathcal{O}). \tag{9.5}$$

Restriction of scalars along $\Gamma(h_R, \mathcal{O}) \rightarrow \Gamma(h_S, \mathcal{O})$ is fully faithful.

Proof. A surjection on π_0 makes maps from S to a discrete perfectoid ring a subspace of the maps from R. Hence $h_{S \widehat{\otimes}_R S} \cong h_S$. Applying theorem 9.11 to $S \leftarrow R \rightarrow S$ gives (9.5), and the extension–restriction counit is an isomorphism. □

9.3 Proof of p-adic 0-affineness by adjoining compatible p-power roots

Theorem 9.13 (p-adic 0-affineness of $i_! h_R$ when Frobenius is finite)**.** *Let R be a connective p-complete E_∞-$\mathbf{Z}_p$-algebra and let $D = (\pi_0 R)_{\mathrm{red}}$. If D/p is finite as a module over $(D/p)^p$, then $i_! h_R$ is p-adically 0-affine. In particular, finiteness of $(\pi_0 R)/p$ over its Frobenius image suffices.*

For algebras topologically of finite type over a perfectoid ring, see Bhatt [4, Proposition 4.6.2 and Theorem 4.6.3].

Proof. Put $X = h_R$ and $Y = h_{\mathbf{Z}_p}$. Every perfectoid ring is discrete, reduced and p-adically complete, so $X = h_D$. Choose $a_1, \ldots, a_r \in D$ whose reductions generate D/p as an algebra over $(D/p)^p$.

Let $h_P \rightarrow Y$ be the cyclotomic atlas of corollary 9.10, and put $E = D \otimes^{\mathrm{ord}}_{\mathbf{Z}_p} P$. On perfectoid tests, $X \times_Y h_P = h_E$. To construct an affine atlas of h_E, put

$$Q = P\langle x_1, \ldots, x_r \rangle, \qquad Q_\infty = P\langle x_1^{1/p^\infty}, \ldots, x_r^{1/p^\infty} \rangle,$$

with $x_i \mapsto a_i$ in $\widehat{E}_p$. Write E_∞ for the ordinary E-algebra adjoining compatible p-power roots of every a_i. Frobenius is surjective on E_∞/p: its image contains P/p, the image of $(D/p)^p$, and every adjoined root as well as each a_i. Since D is reduced, $D[p^\infty] = D[p]$. The

algebra E_∞ is flat over D: P is flat over $\mathbf{Z}_p$, and each successive adjunction is defined by a monic polynomial. Hence $E_\infty[p^\infty] = E_\infty[p]$. This boundedness of p-power torsion gives $\widehat{E_{\infty p}}/p = E_\infty/p$. Thus lemma 8.7 gives a perfectoid ring T representing

$$h_{E_\infty} = h_E \times_{h_Q} h_{Q_\infty}.$$

Over a perfectoid test of h_E, this map adjoins compatible roots of the images of $a_1, \dots, a_r$ and applies perfectoidization. Successive applications of proposition 3.18 to the corresponding linear polynomials show that $h_T \to h_E$ is an affine atlas. Its composite with the pullback $h_E \to X$ of the cyclotomic atlas is an affine atlas of X.

Write $A = \Gamma(X, \mathcal{O})$ and $C = \Gamma(Y, \mathcal{O})$. By theorem 9.11, the ring map of the composite atlas factors as

$$A \longrightarrow A\widehat{\otimes}_C P = \Gamma(h_E, \mathcal{O}) \longrightarrow \Gamma(h_E, \mathcal{O})\widehat{\otimes}_{\Gamma(h_Q, \mathcal{O})} Q_\infty = T.$$

The first arrow is a base change of the descendable map $C \to P$ of theorem 9.9. The second is a base change of the descendable map $\Gamma(h_Q, \mathcal{O}) \to Q_\infty$: by theorem 8.11, this is the perfected prismatic map of theorem 3.15 extended along $W(P^\flat) \to P$. Thus $A \to T$ is descendable. For each $q \geq 0$, the Čech term of $h_T \to X$ is represented by a perfectoid ring, and theorem 9.11 identifies its coordinate ring with $T^{\widehat{\otimes}_A(q+1)}$. Now lemma 8.4 and proposition 9.2 and (8.2) give the assertion. The final sufficient condition follows because D/p is a quotient of $(\pi_0 R)/p$. □

Corollary 9.14 (p-adic 0-affineness of $i_! h_R$ for algebras topologically of finite type over complete discrete valuation rings)**.** *Let $\mathcal{O}_K$ be a complete discrete valuation ring of mixed characteristic $(0, p)$ with perfect residue field. Let R be a p-adically complete $\mathcal{O}_K$-algebra topologically of finite type. Then $i_! h_R$ is p-adically 0-affine.*

This case is stated in Bhatt [4, Remark 4.6.6].

Proof. Choose a uniformizer π of $\mathcal{O}_K$. The maximal ideal $(\bar{\pi})$ of $\mathcal{O}_K/p$ has nilpotence exponent $e = v_K(p)$. For $p^n \geq e$, reduction identifies the subring $(\mathcal{O}_K/p)^{p^n}$ with k: Frobenius kills the maximal ideal, and its map on k is an isomorphism. Successive division by $\bar{\pi}$ shows that $1, \bar{\pi}, \dots, \bar{\pi}^{e-1}$ generate $\mathcal{O}_K/p$ over $(\mathcal{O}_K/p)^{p^n}$. Hence $\mathcal{O}_K/p$ is finite over $(\mathcal{O}_K/p)^p$. Finitely generated algebras and their quotients preserve this property: if a ring is generated over its Frobenius subring by finitely many elements, the monomials with exponents less than p generate it as a module. Thus R/p is finite over $(R/p)^p$, and theorem 9.13 applies. □

9.4 Recognition of perfectoid formal affines

Proposition 9.15 (Perfectoidness of complete functions on a stack with a perfectoid atlas)**.** *Let X be a stack on $\mathscr{A}_p$ with a perfectoid atlas, and suppose that $\Gamma(X, -) : \mathrm{QC}(X) \to \mathrm{Mod}_{\mathbb{Z}_p}^{\wedge p}$ preserves colimits. If $\Gamma(X, \mathcal{O})$ is connective, it is a discrete perfectoid ring.*

More generally, let F be a sheaf of spaces on $\mathscr{P}_p$ with connective $\Gamma(F, \mathcal{O})$. Let $h_P \to h_{\mathbf{Z}_p}$ be the cyclotomic atlas of corollary 9.10. For h_T equal to h_P or to $h_P \times_{h_{\mathbf{Z}_p}} h_P$, consider the canonical map

$$\Gamma(F, \mathcal{O})\widehat{\otimes}_{\Gamma(h_{\mathbf{Z}_p}, \mathcal{O})} T \longrightarrow \Gamma\Big(F \times_{h_{\mathbf{Z}_p}} h_T, \mathcal{O}\Big). \tag{9.6}$$

If both maps are isomorphisms, then $\Gamma(F, \mathcal{O})$ is a discrete perfectoid ring and

$$\Gamma(F, \mathcal{O}) \longrightarrow \Gamma(F, \mathcal{O})\widehat{\otimes}_{\Gamma(h_{\mathbf{Z}_p}, \mathcal{O})} P$$

is p-completely faithfully flat. Both maps in (9.6) are isomorphisms whenever $\Gamma(F, -) : \mathrm{QC}(F) \to \mathrm{Mod}_{\mathbf{Z}_p}^{\wedge p}$ preserves colimits.

Proof. Put $F = i^*X$. The perfectoid atlas and (8.2) identify the complete coefficient categories and global sections of F and X. It suffices to prove the general assertions for F; their last assertion then supplies the two required base-change isomorphisms.

In the proof write $A = \Gamma(F, \mathcal{O})$, $Y = h_{\mathbf{Z}_p}$ and $C = \Gamma(Y, \mathcal{O})$. The description of A as a limit over all perfectoid ring tests mapping to F gives $\pi_i A = 0$ for $i > 0$. Thus A is discrete, and $A[p^\infty] = A[p]$ by lemma 1.3 applied to the terms of the ordinary limit.

We prove that $A \to A\widehat{\otimes}_C P$ is p-completely faithfully flat. For a discrete A/p-module M, regard M as a C-module and use $\mathrm{QC}(Y) \cong \mathrm{Mod}_C^{\wedge p}$ to obtain $\mathcal{M} \in \mathrm{Mod}_{\mathbf{F}_p}(\mathrm{QC}(Y))$ with $\Gamma(Y, \mathcal{M}) \cong M$. Equation (9.4) gives $\pi_i \mathcal{M} = 0$ for $i \neq 0$. On the flat atlas of Y the pullback of $\mathcal{M}$ is therefore discrete. Its module of sections is $M\widehat{\otimes}_C P \cong M\widehat{\otimes}_A (A\widehat{\otimes}_C P)$. Apply lemma 2.2. Scalar extension is conservative by base change of the descendable map $C \to P$ of theorem 9.9.

Consequently $A\widehat{\otimes}_C P$ and its tensor square over A are p-completely flat over A and discrete by lemma 2.8. For each of the two representables h_T on $\mathscr{P}_p$ in (9.6), T is perfectoid by theorem 9.9. The right side of that comparison is a limit of perfectoid T-algebras. Its left side is respectively $A\widehat{\otimes}_C P$ or $(A\widehat{\otimes}_C P)\widehat{\otimes}_A (A\widehat{\otimes}_C P)$. Lemma 8.6 therefore makes both rings perfectoid. Theorem 2.15 now applies to $A \to A\widehat{\otimes}_C P$.

For the last assertion write $f : F \to Y$. For each of the two affine maps $h_T \to Y$, the right side of (9.6) is $\Gamma(F, f^*T)$, where T corresponds under $\mathrm{QC}(Y) \cong \mathrm{Mod}_C^{\wedge p}$ to the direct image of $\mathcal{O}_{h_T}$. For $N \in \mathrm{Mod}_C^{\wedge p}$ the natural map

$$A\widehat{\otimes}_C N \longrightarrow \Gamma(F, f^*N)$$

is an isomorphism for $N = C$. If $\Gamma(F, -)$ preserves colimits, both sides preserve colimits in N. Since C generates $\mathrm{Mod}_C^{\wedge p}$ under colimits and shifts, the map is an isomorphism for all N. □

Corollary 9.16 (A connective ring of complete functions gives a perfectoid formal affine)**.** *Let R be a connective p-complete ring and suppose that*

$$A = \Gamma(h_R, \mathcal{O})$$

is connective. Then A is a discrete perfectoid ring, $h_R \cong h_A$ on $\mathscr{P}_p$, and

$$(\mathrm{Spf}\, R)^{\mathrm{pfd}} \cong \mathrm{Spf}\, A.$$

This last object is a perfectoid formal scheme on the affine test site. For every perfectoid ring T,

$$\mathrm{Map}_{\mathrm{CAlg}}(A, T) \cong \mathrm{Map}_{\mathrm{CAlg}}(R, T).$$

On the auxiliary site of all complete algebras one also has $i_! h_R \cong \mathrm{Spf}\, A$.

Proof. Theorem 9.11 gives the two cyclotomic base-change isomorphisms (9.6) for $F = h_R$. Proposition 9.15 makes A perfectoid. Formula (8.1) then makes A the initial perfectoid R-algebra, giving $h_R \cong h_A$ and the displayed mapping property. Lemma 8.1 gives the perfectoid formal scheme, and the definition of $i_!$ gives the auxiliary assertion. □

The assertion for h_R is also [16, Corollary 4.2].

Theorem 9.17 (A criterion giving a perfectoid formal affine)**.** *Let* $\mathrm{Spf}\, B \to X$ *be a perfectoid atlas on $\mathscr{A}_p$, and suppose $A = \Gamma(X, \mathcal{O})$ is connective. The auxiliary sheaf X is represented by* $\mathrm{Spf}\, A$ *with A perfectoid if and only if*
1. $A \to B$ is p-completely faithfully flat;

2. the canonical map

$$B\widehat{\otimes}_A B \longrightarrow \Gamma(\operatorname{Spf} B \times_X \operatorname{Spf} B, \mathcal{O})$$

is an isomorphism.

Whenever these conditions hold, the realization is the perfectoid formal scheme

$$X^{\mathrm{pfd}} \cong \operatorname{Spf} A.$$

If X is p-adically 0-affine for the complete coefficient category on $\mathscr{A}_p$, condition (2) holds and extension of scalars along $A \to B$ is conservative. In that case the two conditions are equivalent to t-exactness of $\mathrm{Mod}_A^{\wedge p} \to \mathrm{QC}(X)$. The t-structures here are the torsion t-structures; the one on $\mathrm{QC}(X)$ is detected by pullback to the perfectoid atlas.

Proof. The rings of the atlas nerve are discrete, so their limit A has no homotopy groups in degrees greater than zero. Connectivity makes it discrete. Under (2), the two nerves of $\operatorname{Spf} B$ over X and over $\operatorname{Spf} A$ agree: every higher term is an iterated pullback of their common first intersection over $\operatorname{Spf} B$. Under (1) the second nerve presents $\operatorname{Spf} A$ on $\mathscr{A}_p$, and the first presents X. Perfectoid descent theorem 2.15 makes A perfectoid. Conversely, an isomorphism $X \cong \operatorname{Spf} A$ identifies the atlas and its first intersection with the indicated faithfully flat map and tensor square. This proves the criterion on auxiliary test sheaves. Applying lemma 8.1 proves the asserted representability.

If X is p-adically 0-affine, its atlas algebra corresponds to $B \in \mathrm{CAlg}(\mathrm{Mod}_A^{\wedge p})$. Affine base change identifies its pullback to the atlas with $B\widehat{\otimes}_A B$, proving (2). Extension to B is the composite of the coefficient isomorphism $\mathrm{Mod}_A^{\wedge p} \cong \mathrm{QC}(X)$ and the conservative pullback to the atlas. This pullback detects both halves of the t-structure by flat descent. Consequently t-exactness of the coefficient isomorphism is equivalent to p-complete flatness of $A \to B$; its conservativity supplies faithfulness. □

Corollary 9.18 (A cyclotomic test for a perfectoid formal affine)**.** *Let $h_P \to h_{\mathbb{Z}_p}$ be the cyclotomic atlas of corollary 9.10. Let F be a sheaf on $\mathscr{P}_p$ with connective $\Gamma(F, \mathcal{O})$. Then F is represented by a perfectoid ring if and only if $F \times_{h_{\mathbb{Z}_p}} h_P$ is represented by a perfectoid ring. In that case, with $A = \Gamma(F, \mathcal{O})$,*

$$F \cong h_A, \qquad \mathrm{Real}_{\mathrm{perf}}(F) \cong \operatorname{Spf} A.$$

Proof. Write $Y = h_{\mathbf{Z}_p}$ and $A = \Gamma(F, \mathcal{O})$. If F is represented by a perfectoid ring, the assertion about its pullback is part of the definition of the atlas $h_P \to Y$. Conversely, write $F \times_Y h_P = h_B$ with B perfectoid, and put $C = \Gamma(Y, \mathcal{O})$. The Čech terms of $h_P \to Y$ have rings $P^q = P^{\widehat{\otimes} C(q+1)}$ by theorem 9.9. The pullback of F to each term is affine with ring $B\widehat{\otimes}_P P^q$, using any vertex $h_{P^q} \to h_P$. These rings with their pullback maps define an algebra $\mathcal{B} \in \mathrm{CAlg}(\mathrm{QC}(Y))$ by faithfully flat descent. Taking the limit over the nerve gives $\Gamma(Y, \mathcal{B}) \cong \Gamma(F, \mathcal{O}) = A$. Under $\mathrm{QC}(Y) \cong \mathrm{Mod}_C^{\wedge p}$, pullback of $\mathcal{B}$ to h_{P^q} is therefore represented by $A\widehat{\otimes}_C P^q$. This proves the two comparisons in (9.6).

By proposition 9.15, A is perfectoid and $A \to B \cong A\widehat{\otimes}_C P$ is p-completely faithfully flat. Moreover, $h_B \times_F h_B$ has coordinate ring $A\widehat{\otimes}_C P^1 \cong B\widehat{\otimes}_A B$. The Čech nerve of $h_B \to F$ therefore also presents h_A. Lemma 8.1 gives the stated realization. □

Corollary 9.19. *Let X be a p-adically 0-affine sheaf on $\mathscr{A}_p$ and suppose $A = \Gamma(X, \mathcal{O})$ is connective. Suppose it has a perfectoid atlas $u : \operatorname{Spf} B \to X$ for which $u_* \mathcal{O}_{\operatorname{Spf} B}$ is a filtered colimit of finite locally free $\mathcal{O}_X$-modules in $\mathrm{QC}(X)$. Then A is perfectoid, $X \cong \operatorname{Spf} A$, and*

$$X^{\mathrm{pfd}} \cong \operatorname{Spf} A.$$

This applies in particular when the atlas is finite locally free.

Proof. Put $A = \Gamma(X, \mathcal{O})$. By theorem 9.17 it suffices to prove that B is p-completely flat over A. Global sections identify B with the indicated filtered colimit in $\mathrm{Mod}_A^{\wedge p}$. We prove that for each finite locally free $\mathcal{E}$ on X, the A-module $\Gamma(X, \mathcal{E})$ is p-completely flat. It is dualizable, so its reduction along $A \to A/p$ is perfect.

Let k be a residue field of A/p. Conservativity of extension to B makes $B \otimes_A k$ nonzero. This algebra is connective, so $\pi_0(B \otimes_A k)$ has a residue field k'. Pulling $\mathcal{E}$ back along $\operatorname{Spf} k' \to \operatorname{Spf} B \to X$ gives the discrete vector space $\Gamma(X, \mathcal{E}) \otimes_A k'$. Faithful flatness of $k \to k'$ gives discreteness of $\Gamma(X, \mathcal{E}) \otimes_A k$.

For the residue-field criterion, localize the perfect module over A/p at any prime. A nonzero perfect module over a local ring is bounded below and its lowest nonzero homotopy group is finitely presented: mapping to discrete modules shifted to that degree commutes with filtered colimits by compactness. Nakayama's lemma shows that this lowest group survives in the residue fiber. A discrete residue fiber therefore makes the localized module connective. Lift a basis of the fiber to degree zero. The induced map from a finite free module has perfect cofiber with zero residue fiber, hence is an isomorphism. Thus the reduction of $\Gamma(X, \mathcal{E})$ over A/p is finite projective. Reduction preserves filtered colimits, so $B \otimes_A (A/p)$ is flat and discrete. Apply lemma 2.2. Lemma 8.1 gives the stated realization. □

Remark 9.20. On the auxiliary site $\mathscr{A}_p$, let $k = \mathbf{F}_p$ and let G be a cyclic group of prime order $\ell \neq p$. The stack $B\underline{G}$ over $\operatorname{Spf} k$ has the finite étale perfectoid atlas $\operatorname{Spf} k \to B\underline{G}$; its first intersection is $\operatorname{Spf} k^G$. Its derived p-complete coefficients are k-modules with G-action. The regular representation is a compact generator and is dualizable. The constants in the atlas algebra k^G split off by averaging, so the atlas algebra is descendable. Averaging over G splits the invariants functor off the forgetful functor, so global sections preserve colimits and $\Gamma(B\underline{G}, \mathcal{O}) \cong k$. The trivial torsor over k has automorphism group G, whereas $\operatorname{Spf} k$ has a single k-point with trivial automorphism group. Here condition (2) of theorem 9.17 would require the diagonal map $k \to k^G$ to be an isomorphism.

9.5 Functions on the restriction of an infinite torus

Proposition 9.21 (The cyclotomic Čech nerve of adjoining compatible p-power roots)**.** *Let T be a perfectoid algebra over $\mathbf{Z}_p[\mu_{p^\infty}]^\wedge_p$ with its chosen compatible primitive roots of unity. For $a \in T$, let $T_a = \left(T[\widehat{b^{1/p^\infty}]/(b-a)}_p\right)_{\mathrm{perfd}}$. The map $T \to T_a$ is p-completely faithfully flat. Its completed Čech nerve has terms*

$$T_a^{\widehat{\otimes}_T(q+1)} \cong \mathrm{Cont}(\mathbf{Z}_p^q, T_a) \times_{\mathrm{Cont}(\mathbf{Z}_p^q, (\widehat{T_a/(pa)}_p)_{\mathrm{perfd}})} (\widehat{T_a/(pa)}_p)_{\mathrm{perfd}}. \tag{9.7}$$

Continuous functions use the p-adic topology on the targets. The nerve maps use the action multiplying the chosen roots by roots of unity. This action is trivial on $(\widehat{T_a/(pa)}_p)_{\mathrm{perfd}}$.

Proof. Apply proposition 3.18 to the monic polynomial $b - a$ for existence of T_a and p-complete faithful flatness. After choosing compatible roots b_n of a, the other root at level n satisfies

$$y^{p^n} - a = \prod_{\lambda \in \mu_{p^n}} (y - \lambda b_n).$$

For distinct λ, λ', a positive integral power of $\lambda - \lambda'$ is p^k times a unit in $\mathbf{Z}_p[\mu_{p^\infty}]^\wedge_p$, for some $k > 0$. Since $b_n^{p^n} = a$, the ordinary ideals generated by $(\lambda - \lambda')b_n$ and by pa have the same radical. Their perfectoid quotients are consequently the same. Repeated use of (4.1) glues all these branches along $(\widehat{T_a/(pa)}_p)_{\mathrm{perfd}}$. Passing to the limit over n gives the continuous functions in (9.7). Repeating these factorizations for q additional compatible

sequences of roots proves (9.7) in degree q. Evaluations at roots respect composition, so these identifications also identify the nerve maps. The differences of the roots vanish in $(\widehat{T_a/(pa)}_p)_{\mathrm{perfd}}$, proving triviality of the action there. □

Proposition 9.22 (Global sections fail to preserve countable coproducts for the torus with countably many coordinates)**.** *Let*

$$X = h_{(\mathbf{Z}_p[\mu_{p^\infty}]^\wedge_p)\langle t_i^{\pm 1} | i \geq 1\rangle}.$$

Global sections on $\mathrm{QC}(X)$ *fail to preserve countable coproducts. Consequently the canonical functor* $\mathrm{Mod}^{\wedge p}_{\Gamma(X,\mathcal{O})} \to \mathrm{QC}(X)$ *is not an isomorphism.*

Proof. Put $P = \mathbf{Z}_p[\mu_{p^\infty}]^\wedge_p$. Write $\mathfrak{m} \subset P$ for the maximal ideal and $k = P/\mathfrak{m}$. Choose compatible primitive roots of unity and put

$$S = P\langle t_i^{\pm 1/p^\infty} \mid i \geq 1\rangle, \qquad G = \prod_{i \geq 1} \mathbf{Z}_p, \qquad \overline{S} = k[t_i^{\pm 1/p^\infty} \mid i \geq 1].$$

Apply proposition 9.21 to finitely many variables and take completed filtered colimits. The nerve on perfectoid tests has rings

$$C_q = \mathrm{Cont}(G^q, S) \times_{\mathrm{Cont}(G^q, \overline{S})} \overline{S}.$$

The map $h_S \to X$ is p-completely faithfully flat by proposition 3.18 and lemma 2.7. Faithfully flat descent gives

$$\mathrm{QC}(X) \cong \lim_{[q] \in \Delta} \mathrm{Mod}^{\wedge p}_{C_q}.$$

Let $M = \mathfrak{m}/p\mathfrak{m}$, a nonzero discrete P/p-module, and let $F \in \mathrm{QC}(X)$ be its pullback. Flatness and idempotence of $\mathfrak{m}$ give $M \otimes_P k = 0$. Tensoring the nerve with M therefore gives

$$\Gamma(X, F) = (M \otimes_P S)^{hG}.$$

The module $M \otimes_P S$ is the direct sum of its Laurent monomials. The constant monomial is a G-equivariant direct summand equal to M with trivial action. Projection $G \to \mathbf{Z}_p^n$ has a section. Inflation consequently makes $M^{h\mathbf{Z}_p^n}$ a retract of M^{hG}. Iterating lemma 9.3 gives

$$M^{h\mathbf{Z}_p^n} = \bigoplus_{j=0}^{n} M[-j]^{\oplus \binom{n}{j}}, \qquad \pi_{-n}\Gamma(X, F) \neq 0 \quad (n \geq 1).$$

For every perfectoid test T, the module $F(T) = M \otimes_P T$ is connective and killed by p. Hence

$$\bigoplus_{n \geq 1} F(T)[n] = \prod_{n \geq 1} F(T)[n].$$

These products commute with pullback because they agree with the coproducts. They therefore compute the product in $\mathrm{QC}(X)$, giving $\bigoplus_{n\geq 1} F[n] = \prod_{n \geq 1} F[n]$. Global sections preserve products. If they preserved this coproduct, the canonical inclusion

$$\bigoplus_{n \geq 1} \pi_{-n}\Gamma(X, F) \longrightarrow \prod_{n \geq 1} \pi_{-n}\Gamma(X, F)$$

would be surjective. Each factor is nonzero, so the inclusion is proper. This proves the assertion about coproducts. Restriction of scalars $\mathrm{Mod}^{\wedge p}_B \to \mathrm{Mod}^{\wedge p}_A$ along a map of p-complete rings $A \to B$ preserves coproducts, which proves that the displayed canonical functor is not an isomorphism. □

10 Integralization

10.1 Integralization of maps of almost perfectoid formal schemes

Let $\mathscr{T} = \mathrm{Shv}_{\mathrm{Zar}}(\mathscr{A}_{\mathrm{perf}}, \mathrm{Spaces})$ be the category of sheaves on the affine site of definition 7.3. Perfectoid formal schemes and almost perfectoid formal schemes have the meanings fixed in definition 7.5.

Definition 10.1 (p-adically integral morphisms)**.** A morphism $H \to X$ of almost perfectoid formal schemes is *p-adically integral* if there is a Zariski open cover $\{W_i \to X\}_{i \in L}$ with the following presentations. For every i, there are a perfectoid ring R_i, an ideal $J_i \subset R_i$, a Zariski open immersion

$$W_i \longrightarrow (\mathrm{Spf}\, R_i)^{a_{J_i}},$$

and a p-adically integral map $R_i \to B_i$ of perfectoid rings such that

$$H \times_X W_i \cong W_i \times_{(\mathrm{Spf}\, R_i)^{a_{J_i}}} (\mathrm{Spf}\, B_i)^{a_{J_i B_i}} \quad \text{over } W_i.$$

Here p-adic integrality means that the ordinary map $R_i/pR_i \to B_i/pB_i$ is integral, as in definition 7.8. Existence of these presentations is the condition; no presentation is part of the data of the morphism. Write $\mathrm{Integral}_p(X)$ for the full subcategory of $\mathscr{T}_{/X}$ consisting of these morphisms.

Proposition 10.2 (Zariski descent of integral morphisms)**.** *For a Zariski open cover $\{X_i \to X\}_{i \in L}$ of an almost perfectoid formal scheme, write $X_{i_0 \dots i_n} = X_{i_0} \times_X \cdots \times_X X_{i_n}$. Restriction induces an isomorphism of categories*

$$\mathrm{Integral}_p(X) \xrightarrow{\sim} \lim_{[n] \in \Delta} \prod_{(i_0, \dots, i_n) \in L^{n+1}} \mathrm{Integral}_p(X_{i_0 \dots i_n}).$$

Proof. The cover is an effective epimorphism in $\mathscr{T}$, so the slice categories of sheaves satisfy the displayed descent statement without the integrality condition. The condition in definition 10.1 is preserved by restriction to an open: restrict each displayed presentation to its intersection with that open. Conversely, presentations on the members of a cover form presentations on their composite refinement. An object obtained by descent from almost perfectoid formal schemes is again an almost perfectoid formal scheme, since their open atlases combine to an open atlas of the descended sheaf. Slice descent therefore restricts to the stated full subcategories, including their mapping Spaces. □

Definition 10.3 (p-adic integralization)**.** Let X be an almost perfectoid formal scheme and let $Y \to X$ be a morphism in $\mathscr{T}$. A *p-adic integralization* is a factorization

$$Y \xrightarrow{\eta} \mathrm{Int}_X(Y) \longrightarrow X$$

with $\mathrm{Int}_X(Y) \in \mathrm{Integral}_p(X)$ such that composition with η induces an isomorphism in Spaces

$$\mathrm{Map}_{\mathscr{T}_{/X}}(\mathrm{Int}_X(Y), T) \xrightarrow{\sim} \mathrm{Map}_{\mathscr{T}_{/X}}(Y, T) \qquad (T \in \mathrm{Integral}_p(X)).$$

Equivalently, this is an initial object of the category of such factorizations, whose morphisms commute with both arrows.

For fixed X, let $\mathscr{T}^{\mathrm{int}}_{/X}$ be the full subcategory of objects for which this integralization exists. The definition gives an adjunction

$$\mathrm{Int}_X : \mathscr{T}^{\mathrm{int}}_{/X} \rightleftarrows \mathrm{Integral}_p(X),$$

with right adjoint the inclusion. In particular, integralization is functorial, and

$$\mathrm{Int}_X(T) \cong T \quad (T \in \mathrm{Integral}_p(X)), \qquad \mathrm{Int}_X(\mathrm{Int}_X(Y)) \cong \mathrm{Int}_X(Y).$$

Remark 10.4 (Ordinary complements and formal integralization). For every almost perfectoid formal scheme X, the morphism $\varnothing \to X$ is p-adically integral: locally it has the presentation $R \to 0$. Thus

$$\mathrm{Int}_X(\varnothing) \cong \varnothing.$$

For a perfectoid ring R, the formal principal open obtained by inverting p is

$$(\operatorname{Spf} R)[1/p] \cong \operatorname{Spf}(\widehat{R[1/p]}_p) \cong \operatorname{Spf} 0 \cong \varnothing.$$

The ordinary open $\operatorname{Spec} R[1/p]$ has the same restriction to the perfectoid site: $h_{R[1/p]} \cong h_0$, since a unital map from $R[1/p]$ to a p-complete perfectoid ring T forces $T = 0$. Its realization is therefore also $\varnothing$ by lemma 8.1. For $R = \mathcal{O}_{\mathbf{C}_p}$, however, $X = (\operatorname{Spf} R)^{a_{(p)}}$ is nonempty: its category of quasicoherent modules has nonzero unit $\sqrt{pR}$ by remark 7.2. Hence

$$\mathrm{Int}_X\big(\mathrm{Real}_{\mathrm{perf}}(h_{R[1/p]})\big) \cong \varnothing \ncong X.$$

For an arbitrary perfectoid ring R and a finitely generated ideal $J \subset R$, define the ordinary ring

$$D = \{b \in \Gamma(\operatorname{Spec} R \setminus V(J), \mathcal{O}_{\operatorname{Spec} R}) : b \text{ is integral over the image of } R\}.$$

Corollary 7.21 gives

$$\Big(\operatorname{Spf}\big((\widehat{D}_p)_{\mathrm{perfd}}\big)\Big)^{a_{J(\widehat{D}_p)_{\mathrm{perfd}}}} \xrightarrow{\sim} (\operatorname{Spf} R)^{a_J}.$$

This uses the ordinary integral model before completion, as in construction 7.20.

10.2 The factorization on full perfectoid formal affines

Theorem 10.5 (Integral factorization on full affines)**.** *Let $R \to B$ be a map of perfectoid rings. Define the ordinary subring*

$$D = \Big\{b \in B/\sqrt{pB} : b \text{ is integral over the image of } R/\sqrt{pR}\Big\}$$

and the ordinary pullback

$$A = B \times_{B/\sqrt{pB}} D.$$

Then A is perfectoid, $R \to A$ is p-adically integral, and $A \to B$ is injective. For every perfectoid R-algebra C such that C/pC is integral over R/pR, composition induces an isomorphism in Spaces

$$\mathrm{Map}_{\mathrm{CAlg}_R}(C, A) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{CAlg}_R}(C, B).$$

Thus

$$\operatorname{Spf} B \longrightarrow \operatorname{Spf} A \longrightarrow \operatorname{Spf} R$$

is initial among factorizations whose intermediate objects are full perfectoid formal affines with p-adically integral coordinate maps from R.

Proof. The ordinary rings $R/\sqrt{pR}$ and $B/\sqrt{pB}$ are perfect $\mathbf{F}_p$-algebras: Frobenius on the special fibre of a perfectoid ring is surjective, and its reduction has injective Frobenius. The subring D is perfect. Indeed, if $d \in D$ and $e \in B/\sqrt{pB}$ is its unique p-th root, then $e^p = d$ makes e integral over D; transitivity of integrality gives $e \in D$.

The three rings defining A are therefore perfectoid R-algebras. The map $B \to B/\sqrt{pB}$ is surjective, so corollary 1.8 makes A perfectoid. Since $D \subset B/\sqrt{pB}$, projection identifies

A with a subring of B containing $\sqrt{pB}$. Moreover, $A \to D$ is surjective with kernel $\sqrt{pB}$. We claim that

$$\sqrt{pA} = \sqrt{pB} \quad \text{as subsets of } A.$$

One inclusion follows from $A \subset B$. For the other, if $a \in \sqrt{pB}$ and $a^n = pb$ with $b \in B$, then $ba \in \sqrt{pB} \subset A$, so $a^{n+1} = p(ba) \in pA$. Consequently

$$A/\sqrt{pA} \cong D.$$

The right side is integral over $R/\sqrt{pR}$. The reduced special-fibre criterion in proposition 7.9 therefore proves that $R \to A$ is p-adically integral.

For a perfectoid R-algebra C as in the statement, every R-algebra map $C \to B$ induces a map $C/\sqrt{pC} \to B/\sqrt{pB}$ with image in D, since its source is integral over $R/\sqrt{pR}$. The pullback property gives a factorization through A. The injection $A \subset B$ makes it unique. All these mapping spaces are discrete, because the target rings have only their degree-zero homotopy groups. This proves the isomorphism in Spaces and hence the asserted initiality against the displayed affine targets. □

Corollary 10.6 (Characteristic p and almost coefficients)**.** *Use the notation of theorem 10.5. If R has characteristic p, then $A = D \subset B$ is the ordinary integral closure of the image of R in B. For arbitrary R, the maps*

$$A[1/p] \longrightarrow B[1/p], \qquad (\operatorname{Spf} B)^{a_{(p)}} \longrightarrow (\operatorname{Spf} A)^{a_{(p)}}$$

are isomorphisms, the first in ordinary rings and the second in $\mathscr{T}$.

Proof. In characteristic p, perfectoid rings are perfect and reduced, so the formula for A reduces to $A = D$. In general $pB \subset \sqrt{pB} \subset A \subset B$, which proves the assertion after inverting p. Apply corollary 7.15 to $A \to B$ with the ideal $(p) \subset A$ to obtain the almost isomorphism. □

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