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Sharp radial moments and even inverse coefficients of exterior univalent maps

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Abstract

The unit slit maximizes radial boundary moments of centered exterior univalent maps for every real exponent $7/4 \le p \le 97/10$. We prove a relative deficit of at least $(1-|b_1|^2)/10000$, where b_1 is the first forward Laurent coefficient, and characterize equality. At $p=1$ we modify Goel's circular arc by a short cut in the source and obtain an explicit counterexample to Pommerenke's inequality with the origin in the omitted set. The cut family gives failures for $0<p<8/7$ with the origin omitted, and the uncut family gives failures for $0<p<4/3$ in the full exterior class. We transfer the radial estimates to zeros of fixed-order derivatives of polynomials with connected lemniscates. For the inverse Laurent coefficients, we prove that the supremum of $|B_{2m}|$ is asymptotic to $\binom{2m}{m}/(2m)$.

Keywords. Exterior univalent maps, radial moments, polynomial lemniscates, derivative zeros, inverse coefficients.

1 Introduction

Let Σ_0 consist of the injective holomorphic maps $F(z)=z+\sum_{j\ge1} b_j z^{-j}$ on $\Delta=\{|z|>1\}$. The omitted continuum is $K=\mathbb{C}\setminus F(\Delta)$. Its logarithmic capacity is one and its conformal centroid is zero. The radial trace $F^*(e^{it})=\lim_{R\downarrow1}F(Re^{it})$ exists for almost every t. Every finite radial limit belongs to K. If it belonged to $F(\Delta)$, continuity of F^{-1} there would give $e^{it}\in\Delta$. Setting the two points equal in the omitted-set inequality [3, (7)] gives $|w|\le 2$ on K. For $\rho>1$ the map $F(\rho z)/\rho$ belongs to Σ_0 and extends across the unit circle. Applying the same bound to its radial values gives

$$|F(\rho e^{it})|\le 2\rho \qquad (\rho>1,\ t\in\mathbb{R}). \tag{1}$$

With $\mathbb{E}$ denoting normalized angular average, for $p>0$ write

$$R_p(F)=\frac{1}{2\pi}\int_{-\pi}^{\pi}|F^*(e^{it})|^p\,dt, \qquad S(p)=\frac{\Gamma(p+1)}{\Gamma(1+p/2)^2}=\mathbb{E}|2\cos t|^p, \qquad D=1-|b_1|^2.$$

The map $z+z^{-1}$ omits $[-2,2]$ and has moment $S(p)$. The area theorem gives $D\ge0$. In the inverse problem of Springer [4], we use the convention of [3]

$$G(w)=F^{-1}(w)=w-\sum_{n\ge1}B_n w^{-n} \qquad (w \text{ near infinity}).$$

Baernstein, Laugesen and Pritsker proved slit maximality at $p = 2$ in 2011 and established the convex comparison for real projections that we use below [1]. Li proved the sharp radial comparison for real $p \in [4, 8]$ [3, Theorems 9 and 12] and its consequences for polynomial zeros [3, Proposition 13]. Goel proved the odd-order inverse coefficient bound in every order, including a disk estimate with prescribed b_1, and obtained an even-order lower bound of order $2^{2m}(2m)^{-3/2}$ using a circular-arc family [2]. We prove the matching upper bound and determine the leading constant.

Section 2 uses that arc family to obtain the strict mean-modulus counterexample with $0 \in K$. Section 3 extends the sharp radial comparison to $[7/4, 97/10]$. Its proof uses a single deficit formula to account for the convex projection gain and the cost of transverse displacement. Section 4 derives both strict bounds and counterexamples for derivative-zero moments. Section 5 proves the even inverse coefficient asymptotic by comparing a global radial bound with the phases at the two peaks of an arc perturbation.

2 The mean modulus and the omitted origin

The unit slit has two angles where its boundary trace passes through the origin. In Goel's circle-arc family [2, Section 8], a transverse displacement of size κ lifts the modulus at both angles. At angular distance q from either angle, with $\kappa \ll q \ll 1$, the gain is of order κ^2/q. Integration gives a logarithm at each end.

A short cut in the source plane removes the preimage of the point that becomes the origin after centering. The cut changes the angular density near one end. Its loss has the same logarithmic scale, with half the coefficient of the combined gain. The surviving term gives a counterexample to Pommerenke's mean-modulus inequality with $0 \in K$ [1, Conjecture 3].

Theorem 1. *For $0 < \kappa < 1/2$, put*

$$A_\kappa(z) = z - \frac{1-\kappa^2}{z-\kappa}, \quad r = 1 + \frac{\kappa}{2}, \quad d = \frac{r + r^{-1} - 2}{4} = \frac{\kappa^2}{16r}, \quad a = 1 + d, \quad b = 2d,$$

let $J(z) = z + z^{-1}$ on Δ, and use the branch J^{-1} with values in Δ. Define

$$\Psi(z) = J^{-1}(aJ(z) + b), \qquad F_\kappa(z) = \frac{A_\kappa(\Psi(z)) - b}{a}.$$

Then $F_\kappa \in \Sigma_0$, $0 \notin F_\kappa(\Delta)$, and, as $\kappa \downarrow 0$,

$$R_1(F_\kappa) = \frac{4}{\pi} + \frac{\kappa^2}{4\pi} \log \frac{1}{\kappa} + O(\kappa^2).$$

In particular $R_1(F_\kappa) > 4/\pi$ for all sufficiently small positive κ, and the explicit choice $\kappa = 2^{-100}$ also satisfies the strict inequality.

Proof. The map A_κ is injective on Δ. Equal images at distinct points z, w imply $w = (\kappa z - 1)/(z - \kappa)$, which has modulus less than one if $|z| > 1$, since

$$|z - \kappa|^2 - |\kappa z - 1|^2 = (1 - \kappa^2)(|z|^2 - 1).$$

Also $aJ + b$ maps Δ bijectively onto $\mathbb{C} \setminus [-2, 2 + 4d]$. The identity $J(r) = 2 + 4d$ shows that Ψ maps Δ onto $\Delta \setminus (1, r]$. At infinity $\Psi(z) = az + b + O(1/z)$, giving the required normalization of F_κ. Finally

$$A_\kappa(1) = -\kappa < b < \frac{3\kappa^2}{4(1 - \kappa/2)} = A_\kappa(r).$$

The preimage of b under A_κ in Δ is therefore unique and lies in $(1, r)$. It has been removed from the image of Ψ.

For the upper semicircle put $t_0 = \arccos((1-d)/(1+d))$. For $t_0 < t < \pi$, let $\theta \in (0, \pi)$ satisfy $(1+d)\cos t + d = \cos\theta$. Write $w_R = \Psi(Re^{it})$ and $w_\pm = e^{\pm i\theta}$. The quadratic equation for J gives

$$(w_R - w_+)(w_R - w_-) = (J(w_R) - 2\cos\theta)w_R.$$

For $R > 1$, the sign of $\operatorname{Im} J(w_R)$ is positive because $\operatorname{Im} J(Re^{it}) = (R - R^{-1})\sin t$. Since $|w_R| > 1$, the sign of $\operatorname{Im} w_R$ is positive as well. Thus $|w_R - w_-| \geq \sin\theta$ and $|w_R| \leq |J(w_R)| + 1$. The product identity gives $w_R \to w_+$. The lower trace follows by conjugation.

On the circular part of the upper trace we have

$$t(\theta) = 2\arccos\frac{\cos(\theta/2)}{\sqrt{1+d}}, \qquad w(\theta) = \frac{dt}{d\theta} = \frac{\sin(\theta/2)}{\sqrt{\sin^2(\theta/2) + d}}.$$

For $0 < t < t_0$, the argument of J^{-1} tends to a real value above two. The radial value of Ψ is $\psi(t) = J^{-1}(2(a\cos t + d)) \in [1, r]$. With $x + iy = A_\kappa(e^{i\theta}) - b$, conjugation gives the exact decomposition

$$R_p(F_\kappa) = \frac{a^{-p}}{\pi}\left(\int_0^\pi |x+iy|^p w(\theta)\,d\theta + \int_0^{t_0} |A_\kappa(\psi(t)) - b|^p\,dt\right). \tag{2}$$

The slit integral is $O_p(\kappa^{p+1})$. Indeed, $\sin^2(t_0/2) = d/(1+d)$ gives $t_0 = O(\kappa)$, and monotonicity and the endpoint values give $A_\kappa - b = O(\kappa)$ on $[1, r]$.

On the circle, uniformly in θ, the expansion of A_κ begins with

$$A_\kappa(e^{i\theta}) = 2i\sin\theta - \kappa e^{-2i\theta} + \kappa^2(e^{-i\theta} - e^{-3i\theta}) + O(\kappa^3). \tag{3}$$

The imaginary part has the exact form

$$y = 2\sin\theta\frac{1 - \kappa\cos\theta}{1 - 2\kappa\cos\theta + \kappa^2} = 2\sin\theta\big(1 + \kappa\cos\theta + O(\kappa^2)\big).$$

Thus $y \geq 0$ on the upper semicircle and $\int_0^\pi y\,d\theta = 4 + O(\kappa^2)$.

Write $g = |x+iy| - y$. On $[\kappa, \pi - \kappa]$, $x^2 = \kappa^2 + O(\kappa^2\sin^2\theta + \kappa^3)$ by (3). Rationalizing the square root gives $g = x^2/(2y) + O(x^4/y^3)$ and hence

$$g = \frac{\kappa^2}{4\sin\theta} + O\left(\kappa^2\sin\theta + \frac{\kappa^3}{\sin\theta} + \frac{\kappa^4}{\sin^3\theta}\right).$$

The error integrates to $O(\kappa^2)$ on that interval. Each excluded end has length κ and there $0 \leq g \leq |x| = O(\kappa)$. Also $1 - w \leq d/(2\sin^2(\theta/2))$. On the middle interval, $g \ll \kappa^2/\sin\theta$, so the integral of $(1-w)g$ is $O(\kappa^2)$ by integrating θ^{-3} near zero and $(\pi - \theta)^{-1}$ near π. This proves

$$\int_0^\pi wg\,d\theta = \frac{\kappa^2}{2}\log(1/\kappa) + O(\kappa^2).$$

For the loss in the imaginary part, substitute $u = \sin(\theta/2)$ and use $\sin\theta\,d\theta = 4u\,du$. The exact formula for y shows that

$$\int_0^\pi (1-w)y\,d\theta = (8 + O(\kappa))\int_0^1 u\big[1 - (1 + d/u^2)^{-1/2}\big]\,du.$$

The integral on $u \le \sqrt{d}$ is $O(d)$. On $[\sqrt{d}, 1]$, the bracket is $d/(2u^2) + O(d^2/u^4)$. This gives $4d\log(1/\sqrt{d}) + O(\kappa^2) = \frac{1}{4}\kappa^2\log(1/\kappa) + O(\kappa^2)$ for the loss. In (2), the difference between the gain and this loss leaves $\frac{1}{4}\kappa^2\log(1/\kappa)$ in the unnormalized upper semicircle integral. The normalization contributes an error of order κ^2. Appendix A proves the strict inequality at the explicit parameter. □

The counterexample also works with the origin in the interior of a smooth omitted domain. Fix a counterexample and dilate to $F_\kappa(Rz)/R$ for $R > 1$ sufficiently close to one. The strict inequality persists by bounded convergence of the radial traces. The image of the closed exterior $|z| \ge R$, including infinity, is compact in the sphere and avoids zero, so the omitted set now contains a neighborhood of zero. The radial test e^x also contradicts [1, Conjecture 2] and hence its equivalent Conjecture 1.

Corollary 2. *For every fixed $0 < p < 8/7$, the same family satisfies $R_p(F_\kappa) > S(p)$ for all sufficiently small positive κ.*

The endpoint rescaling and the second variation are given in Appendix A.

Corollary 3. *Over the full class Σ_0, with no restriction on the omitted origin, the uncut family A_κ gives counterexamples for every fixed $0 < p < 4/3$.*

3 Sharp radial moments on a closed interval

Theorem 4. *For $7/4 \le p \le 97/10$ and $F \in \Sigma_0$,*

$$R_p(F) \le S(p)(1 - D/10000).$$

Equality in $R_p(F) \le S(p)$ occurs exactly for $F(z) = z + b_1/z$ with $|b_1| = 1$.

Rotate the map so that $b_1 = r \ge 0$. Write $U = \cos t$, $X + iY = F^*(e^{it})/2$, $T = |F^*(e^{it})|/2$, $c_p = \mathbb{E}|U|^p = 2^{-p}S(p)$, and $\Delta_p = c_p - \mathbb{E}T^p$. The area theorem and Parseval give

$$\mathbb{E}Y^2 \le D\left(\frac{1}{16} + \frac{1-r}{8(1+r)}\right), \qquad c_2 - \mathbb{E}X^2 \le \frac{3D}{8},$$
$$c_4 - \mathbb{E}T^4 \ge \frac{3D}{16}, \qquad c_8 - \mathbb{E}T^8 \ge \frac{51D}{1280}.$$

Indeed $8\mathbb{E}Y^2 = (1-r)^2 + \sum_{j\ge 2}|b_j|^2$ and $8\mathbb{E}X^2 \ge (1+r)^2$. The endpoints $R_2 \le 2(1 - D/4)$ and $R_4 \le 6(1 - D/2)$ will also be used. The omission bound gives $0 \le |X| \le T \le 1$.

3.1 Projection and transverse displacement

For $-1 \le x \le 1$ let $P(x) = \mathbb{E}(U - x)_+ - \mathbb{E}(X - x)_+$. The projection comparison of [1, Theorem 2] applied to the hinge function gives $P(x) \ge 0$. Comparison with the affine tests x and $-x$ gives $\mathbb{E}X = \mathbb{E}U = 0$.

Lemma 5. *For $p > 1$,*

$$\Delta_p = p(p-1)\int_{-1}^{1} |x|^{p-2}P(x)\,dx - \frac{p}{2}\,\mathbb{E}\int_0^{Y^2} (X^2 + s)^{p/2-1}\,ds. \tag{4}$$

For exponents $q_j > 1$ and real coefficients A_j, put $h(u) = u^p - \sum_j A_j u^{q_j}$, with a finite sum. Suppose $h(|x|) + Mx^2$ is convex on $[-1,1]$ and $(d/ds)h(\sqrt{s}) \le C$ for $0 < s \le 1$. Then

$$\Delta_p \ge \sum_j A_j \Delta_{q_j} - M(c_2 - \mathbb{E}X^2) - C\,\mathbb{E}Y^2. \tag{5}$$

The convexity hypothesis can be replaced by the integrated condition

$$\int_{-1}^{1} \left(\frac{d^2}{dx^2} h(|x|) + 2M \right) P(x)\,dx \ge 0.$$

Proof. For $-1 \le v \le 1$, integration of the first derivative of $f(v) = |v|^p$ gives

$$f(v) = f(-1) + f'(-1)(v+1) + \int_{-1}^{1} f''(x)(v-x)_+\,dx.$$

Here f' is absolutely continuous and the value assigned to f'' at zero does not affect the integral. The identity follows by integrating f'' on $[-1, v]$ and using Fubini. The hinge is bounded by two and $|x|^{p-2}$ is integrable because $p > 1$. Taking the difference of the expectations at U and X cancels the affine terms.

Since $T^2 = X^2 + Y^2$, integration of the derivative of $s^{p/2}$ gives the second term of (4). When $X = 0$ and $p < 2$, the integral means the limit from positive s. Its value is $(Y^2)^{p/2}/(p/2)$ because $p/2 > 0$. It is zero when $Y = 0$. The moment difference is bounded on the unit disk, so its angular expectation is finite.

Apply the same identities to h. The projection contribution is bounded below by $-M(c_2 - \mathbb{E}X^2)$ because $2\int_{-1}^{1} P = c_2 - \mathbb{E}X^2$. Pointwise convexity implies the integrated curvature condition by $P \ge 0$. The transverse contribution costs at most $C\,\mathbb{E}Y^2$ by integrating the derivative bound from X^2 to T^2. Adding back the known moments gives (5). □

The remaining estimates control one of two terms in (4). Below $p = 2$, the tangent comparison leaves a Fourier error with an integrable singular weight. Just above $p = 2$, the Green trace supplies the integrated curvature bound. At higher exponents the reference moments absorb the negative curvature of the residual test.

Exponent range	Input to the radial comparison
$[7/4, 2]$	Weighted Fourier bound, Appendix B
$[2, 12/5]$	Green trace on a diameter, (7)
$[12/5, 4]$	Second- and fourth-moment compensation, (8)
$[4, 8]$	Earlier real-exponent comparison [3, Theorem 12]
$[8, 97/10]$	Strengthened eighth moment, (9)

3.2 The weighted Fourier input

Set $\epsilon = 2 - p$, $W = F^*/2 - U = \sum_{j \ge 1} a_j e^{-ijt}$ and $\ell = (1 - r)/2$. Thus $a_1 = -\ell$, and the area theorem gives $\sum_j j|a_j|^2 \le \ell$. The case $D = 0$ is the slit, so suppose $D > 0$. For $1 < p \le 2$, the even function $|x|^p - p(p-1)x^2/2$ is convex on $[-1, 1]$ and extends by its endpoint tangents. Apply the projection inequality and then its tangent inequality at U. Since $\mathbb{E}U \operatorname{Re} W = -\ell/2$, this gives

$$-p\mathbb{E}\big[|U|^{p-2} U \operatorname{Re} W\big] \ge \frac{p(p-1)}{2}\ell.$$

Take the tangent to $s^{p/2}$ at U^2 for almost every t. The first variation is $p|U|^{p-2}U\operatorname{Re}W$. The transverse and longitudinal squares together give the weighted remainder $(p/2)|U|^{p-2}|W|^2$. The weight is integrable at the two zeros of U. Combining the bounds gives

$$c_p - \mathbb{E}T^p \geq \frac{p}{2}\big[(p-1)\ell - \mathbb{E}|U|^{p-2}|W|^2\big]. \tag{6}$$

For $r \geq 9/10$ the weighted Fourier estimate of Appendix B gives

$$\mathbb{E}(|U|^{p-2}|W|^2) \leq (3/4 - 1/60)\ell.$$

Thus the linear gain in (6) dominates the weighted error by a uniform multiple of ℓ, and $D = 2(1+r)\ell \leq 4\ell$.

For $r \leq 9/10$, $\mathbb{E}T^2 \leq Q = 381/800$ and Jensen gives $\mathbb{E}T^p \leq Q^{1-\epsilon/2}$. The tangent at $p = 2$ gives $c_p \geq (1/2)(1 + 19\epsilon/100)$ because $\log 2 > 69/100$. The last bound exceeds $Q^{1-\epsilon/2}$ at $\epsilon = 0$ and $1/4$, and hence between them by convexity. At the second endpoint this follows from

$$\log(2Q^{7/8}) \leq 7/80 - (7/8)(19/400) < 19/400 - (19/400)^2/2 \leq \log(1 + 19/400).$$

These estimates also give $c_p - \mathbb{E}T^p \geq D/10000$. In the near case the margin in the Fourier estimate is at least $1/60$. In the far case the log gap at $\epsilon = 1/4$ is at least $139/320000$, giving an absolute moment gap at least $139/1280000$ by $1 - e^{-x} \geq x/2$ on $[0, 1]$. The gap at $\epsilon = 0$ exceeds the same value, so convexity gives the stated bound throughout.

3.3 The Green trace input

For this range the first term of (4) needs a lower bound after integration against P. A Green trace inequality controls how much of the projection discrepancy can concentrate near the origin. If $V \geq 0$ is continuous and superharmonic in a neighborhood of the closed disk of radius R, then, for $\eta \geq 0$,

$$\int_{-R}^{R} |x|^\eta V(x)\,dx \geq \frac{R^\eta}{(1+\eta)^2}\int_{-R}^{R} V(x)\,dx. \tag{7}$$

Scale to $R = 1$. First take the unit disk Green kernel with real pole $a \in [0, 1)$ and add its traces at x and $-x$, for $0 < x < 1$. This gives

$$G_a(x) = \log\frac{1 - a^2x^2}{|x^2 - a^2|}, \qquad G_0(x) = -2\log x.$$

Its integral C_a is at most $2 = C_0$. The integral of the numerator logarithm is nonpositive and the integral of $-\log|x-a|$ over $[-1, 1]$ is $2-(1-a)\log(1-a)-(1+a)\log(1+a) \leq 2$. For $x \geq a$, $G_a \geq G_0$. On $0 < x < a$, G_a increases and G_0 decreases, so the normalized difference $G_a/C_a - G_0/2$ changes sign at most once, from negative to positive. Its integral is zero. The moment of any increasing test therefore increases, giving (7) since $\frac{1}{2}\int_0^1 x^\eta G_0(x)\,dx = (1+\eta)^{-2}$.

For a complex pole z, the difference of the two sides of (7) for its Green trace is harmonic in z off the diameter, and continuous with zero outer boundary values. Continuity at the diameter follows from the integrable majorant for translated logarithmic singularities. Vanishing at the outer boundary also follows from $G(x, z) = \frac{1}{2}\log(1 + (1-x^2)(1-|z|^2)/|x-z|^2)$ and $\log(1+u) \leq 4u^{1/4}$. The harmonic minimum principle in each half disk extends the inequality to every pole. Taking the inward derivative at the circle gives the same inequality for Poisson kernels. The limit can be taken uniformly in x for boundary pole different from $1, -1$, two boundary points of measure

zero. Finally, approximate V by nonnegative smooth superharmonic convolutions in the closed disk and apply the Green representation with its nonnegative source and nonnegative boundary values. Uniform convergence on the diameter proves (7).

Let w be the auxiliary subharmonic function of [1, Section 5] on the upper half plane in the comparison for K and $[-2,2]$. It is nonpositive and continuous up to the real line. Its real trace satisfies that $-w(2x)$ is a positive constant times $\mathbb{E}(U-x)_+ - \mathbb{E}(X-x)_+$. This follows from the vertical Green integral formula, or the twice-integrated projected measure identity in [1, Section 2]. At every x_0 in the interior of the projection of K the normal difference $w(x_0+iy)-w(x_0)$ is $o(y)$. In fact, equation (24) of [1] expresses the difference using integrals over intervals or sets of length $2y$. The slit Green function vanishes at x_0, and the minimizing integral of the Green function of K is between zero and its integral on an interval of length $2y$ centered at a point above x_0 in K. Continuity makes both integrals $o(y)$. Even reflection of w across this interval is subharmonic. To see this by harmonic comparison, a positive interior maximum of the reflected w minus a harmonic majorant would have to lie on the real diameter. The Hopf boundary lemma on the upper and lower halves would give opposite strict signs for the normal derivative of the harmonic function there, using the zero one-sided difference quotients. This is impossible.

Suppose $r \ge 9/10$. Parseval gives $\mathbb{E}(X-U)^2 = \mathbb{E}Y^2 \le 21/1600$. If $X \le 1/2$ almost everywhere, the arc $|t| \le \pi/6$ alone contributes at least $(2-\sqrt{3})/12 > 1/48$, a contradiction. The opposite arc deals with $X \ge -1/2$. The projection interval of $K/2$ therefore strictly contains $[-1/2,1/2]$, and $V(z) = -w(2\operatorname{Re} z + 2i|\operatorname{Im} z|)$ satisfies the hypotheses of (7) at $R = 1/2$. On the rest of the real trace use $V \ge 0$ and $|x|^\eta \ge (1/2)^\eta$. For $\delta_p^X = c_p - \mathbb{E}|X|^p$, the projection term of (4) gives

$$\delta_p^X \ge \frac{p}{2}\frac{2^{-(p-2)}}{p-1}\delta_2^X \ge \frac{p}{4}\delta_2^X \qquad (2 \le p \le 12/5).$$

The last inequality uses $2^{2/5} < 4/3$. Since $T \le 1$, $T^p - |X|^p \le (p/2)Y^2$. Also $\delta_2^X \ge D/8 + \mathbb{E}Y^2$ and $\mathbb{E}Y^2 \le (21/304)D$ for $r \ge 9/10$. This gives $c_p - \mathbb{E}T^p \ge (p/4)(1/8 - 21/304)D$. For $r \le 9/10$, use the Hölder bound $\log(R_p/S(p)) \le (p-2)/20 - D(1+(p-2)/2)/4$, which is at most $-D/10$ for $D \ge 19/100$ and $p-2 \le 2/5$.

3.4 From 12/5 to 8

For $12/5 \le p \le 4$, put $\vartheta = (p-2)/2$, $\beta = (1+2\vartheta)/3$, and $\alpha = \max_{[0,1]}(v^\vartheta - \beta v)$. The function

$$h(u) = u^p - \frac{p\alpha}{2}u^2 - \frac{p\beta}{4}u^4$$

is decreasing on $[0,1]$. The even function $h(|x|) + p\alpha x^2/2$ is convex on $[-1,1]$, since its second derivative at $x > 0$ is $3p\beta(x^{2\vartheta} - x^2) \ge 0$. Use (5) with $M = p\alpha/2$ and $C = 0$. The removable quadratic part enters through Δ_2 and cancels the projection correction except for the transverse energy $\mathbb{E}Y^2$. This gives

$$c_p - \mathbb{E}T^p \ge \frac{3p\beta D}{64} - \frac{p\alpha}{2}\mathbb{E}Y^2. \tag{8}$$

When $1/5 \le \vartheta \le 1/4$, one has $\beta \ge 7/15$ and $\alpha \le (4/5)(3/7)^{1/4} < 13/20$. Use $r \ge 19/20$ to give $\mathbb{E}Y^2 \le 41D/624$. When $1/4 \le \vartheta \le 1$, one has $\beta \ge 1/2$ and $\alpha \le (3/4)2^{-1/3} < 3/5$. Use $r \ge 4/5$ to give $\mathbb{E}Y^2 \le 11D/144$. The bounds on α follow by maximizing at the left endpoint in ϑ, since the expression decreases pointwise in ϑ. In both cases (8) is at least $pD/1920 \ge D/800$.

For the complementary ranges of r, Hölder interpolation and the tangent bound $S(p) \ge 2e^\vartheta$ give

$$\log\frac{R_p}{S(p)} \le \vartheta(\log 3 - 1) - \frac{D(1+\vartheta)}{4}.$$

The tangent bound follows from log-convexity of the cosine moment and $(\log S)'(2) = 1/2$, by the gamma recurrence. Since $e^{11/10} > 3$ by the Taylor polynomial through degree five, $\log 3 - 1 < 1/10$. If $\vartheta \le 1/4$, use $D \ge 39/400$ and $\vartheta/[10(1+\vartheta)] \le 1/50$. Otherwise use $D \ge 9/25$ and $\vartheta/[10(1+\vartheta)] \le 1/20$. In both cases the logarithmic ratio is at most $-7D/156$. The bounds $e^{-x} \le 1 - x/2$ for $0 \le x \le 1$ and $c_p \le 1/2$ for $p \ge 2$ finish the lower extension with relative deficit $D/400$.

The interval $[4, 8]$ follows from the boundary-moment theorem in [3].

3.5 A stronger eighth moment and the interval from 8 to 97/10

The eighth-moment estimate used here is

$$R_8(F) \le 70 - (54 - 14D)D. \tag{9}$$

Appendix C keeps the losses in the second and third Grunsky rows and compares them against the common area budget.

For $0 \le \delta = p - 8 \le 17/10$, let $b = 37/50$ and $h(t) = t^p - (pb/8)t^8$. The derivative with respect to t^2 is at most $(p/2)(1-b)$ on $[0,1]$, so $h(T) - h(|X|) \le (p/2)(1-b)Y^2$. Let $M(\delta) = \max_{0\le t\le 1} t^6(7b - (7+\delta)t^\delta)_+$. This is nondecreasing in δ. For fixed $0 < t \le 1$, the logarithm of $(7+\delta)t^\delta$ is concave in δ and starts at $\log 7$. Once the expression is below $7b$, it is decreasing. Maximizing in t at $\delta = 17/10$ gives

$$M(\delta) \le M(17/10) = \frac{4403}{3850}\left(\frac{148}{319}\right)^{60/17} < \frac{4403}{3850}\left(\frac{148}{319}\right)^3 \frac{2}{3} < \frac{763}{10000}.$$

The power comparison follows from $148^9 3^{17} < 319^9 2^{17}$. The loss M compensates for the negative curvature of the residual test near zero. Apply (5) with the quadratic compensation $pM/2$, the derivative bound $(p/2)(1-b)$, and the strengthened eighth moment. This gives

$$\frac{c_p - \mathbb{E}T^p}{p} \ge \frac{b(54-14D)D}{2048} - \frac{M}{2}(c_2 - \mathbb{E}X^2) - \frac{1-b}{2}\mathbb{E}Y^2.$$

For $r \ge 11/12$, the last two moments divided by D are bounded by $47/184$ and $25/368$ respectively, by Parseval. The coefficient of D is strictly larger than

$$\frac{37}{50}\frac{54 - 14(23/144)}{2048} - \frac{763}{20000}\frac{47}{184} - \frac{13}{100}\frac{25}{368} > \frac{1}{10000}.$$

For $r \le 11/12$ use $\mathbb{E}T^p \le \mathbb{E}T^8$, and $c_p \ge c_8 \exp(-\delta/15)$ by log-convexity and $(\log c_p)'(8) = \sum_{j=5}^{8} 1/j - \log 2 > -1/15$. Thus

$$\log(R_p/S(p)) \le \frac{17}{150} - \frac{(54-14D)D}{70} \le -\frac{D}{50}.$$

For the last step, the function $(54 - 14D)/70 - 17/(150D)$ is concave on $[23/144, 1]$ and its values at both endpoints exceed $1/50$. This proves the upper range.

In every range the relative deficit is at least $D/10000$. Equality in the sharp bound forces $D = 0$, at which point the area theorem gives exactly the stated slit maps. They attain $S(p)$ by the cosine integral.

4 Zeros of polynomials with connected lemniscates

For a degree-n polynomial P with leading coefficient a_n and root centroid c, the capacity of $K_t = \{|P| \le t\}$ is $\gamma = (t/|a_n|)^{1/n}$. The normalized polynomial $Q(w) = P(c + \gamma w)/(a_n \gamma^n)$ is monic and centered. When K_t is connected, its normalized exterior map belongs to Σ_0 and satisfies $Q(F(z)) = z^n$. The comparison below passes from the angular measure on the level curves to the root measure through a logarithmic potential. Convexity of $|z|^p$ then controls the change under differentiation.

Corollary 6. *Let P have degree at least two and let $\{|P| \le t\}$ be connected, where $t > 0$. Center by the root centroid and scale by the capacity of the lemniscate. For $7/4 \le p \le 97/10$ and each $0 \le j < \deg P$, the mean of the pth powers of the normalized moduli of the zeros of $P^{(j)}$, counted with multiplicity, is strictly less than $S(p)$. For each fixed $p \in [7/4, 97/10]$ and $j \ge 0$, the supremum over all such pairs (P, t) with $\deg P > j$ is $S(p)$ and is unattained.*

Proof. Use the exterior map F just described. The identity $Q(F(z)) = z^n$ follows by comparing the exterior Green function of the lemniscate, $n^{-1} \log |Q|$, with $\log |F^{-1}|$ and using normalization at infinity. For $\rho > 1$, the points $F(\rho e^{i(\theta + 2\pi k/n)})$ for $0 \le k < n$ give all the roots of $Q(w) - \rho^n e^{in\theta}$. Factorization and the angular mean of the logarithm give

$$\mathbb{E}_\theta \log |F(\rho e^{i\theta}) - x| = \frac{1}{n} \log \max(\rho^n, |Q(x)|) \ge \frac{1}{n} \log |Q(x)|.$$

Average over $x = s e^{i\alpha}$ at any $s > 0$, using $\mathbb{E}_\alpha \log |y - s e^{i\alpha}| = \log s + \log^+(|y|/s)$. Fubini applies because the positive parts are bounded and the integral over α is bounded below by $\log s$. If q_i are the roots of Q, we obtain

$$\frac{1}{n} \sum_i \log^+(|q_i|/s) \le \mathbb{E}_\theta \log^+(|F(\rho e^{i\theta})|/s).$$

Integrate at $s = e^u$ against $p^2 e^{pu}\, du$ on $\mathbb{R}$. The integral of $\log^+(a e^{-u})$ with that weight is a^p for $a \ge 0$. Letting $\rho \downarrow 1$ gives the root mean at most $R_p(F)$. Dominated convergence uses (1).

For each derivative step, compress the diagonal matrix of the current roots to the orthogonal complement of the normalized all-ones vector. Its characteristic polynomial is the monic derivative by expansion of the corresponding diagonal cofactor. In a Schur basis of the compression, each derivative root is a convex combination of the original roots, with weights $|v_j(i)|^2$ from the basis vectors. The sum of the weights in each row is $(N-1)/N$ when the original degree is N. Jensen's inequality therefore bounds the derivative mean by the original mean for $p \ge 1$, and iteration gives the same bound at every indicated order. This is the de Bruijn–Springer comparison in the form used in [3, Section 6].

Equality with $S(p)$ would force a unit slit $F(z) = z + b_1/z$. The Laurent identity $Q(F(z)) = z^n$ would then have coefficient b_1^n at z^{-n} on the left and zero on the right, which is impossible. For sharpness use the monic polynomials $2T_n(w/2)$ at level two, with T_n the Chebyshev polynomial. Each component of the open sublevel contains a zero. The interval $[-2, 2]$ lies in the closed sublevel and meets all those components. The closure of their union with the interval is the closed sublevel, so it is connected. The cosine grid of zeros tends to the arcsine law. Repeated Rolle interlacing gives the same limit after any fixed number of differentiations. The capacity $2^{1/n}$ tends to one, so the normalized moments tend to $S(p)$. □

Corollary 7. *For every $0 < p < 4/3$ and fixed integer $j \ge 0$, there are monic polynomials with connected lemniscates for which the normalized pth mean of the zeros of the jth derivative exceeds $S(p)$.*

Proof. Take a counterexample F from the uncut arc family. Choose $\rho > 1$ close enough to one that $\rho^{-p}\mathbb{E}|F(\rho e^{it})|^p > S(p)$. Put $\Gamma = F(\rho\partial\mathbb{D})$, let ω_i range over the nth roots of unity and put $P_n(w) = \prod_i (w - F(\rho\omega_i))$. Set $t_n = 2\max_\Gamma |P_n|$. The open sublevel has every root in one component because it contains the connected curve Γ. Every component of a polynomial sublevel contains a zero by the minimum modulus principle. The closed sublevel is the closure of the open one by the open mapping theorem. Thus the lemniscate is connected.

For $|z| = |u| = 1$, the divided difference $L(z,u) = (F(\rho z) - F(\rho u))/(\rho(z-u))$ extends continuously with diagonal value $F'(\rho z)$ and has no zeros. For fixed z it is zero-free and analytic on the closed exterior in u, with value one at infinity, so $\mathbb{E}_u \log|L(z,u)| = 0$. Uniform Riemann sums on the product of circles now give

$$|P_n(F(\rho z))| = \rho^n |z^n - 1| \exp(o(n))$$

uniformly in z. Evaluating where $z^n = -1$ gives a matching lower bound for the maximum. Consequently $t_n^{1/n} \to \rho$. The root centroid tends to zero by the centered Laurent expansion and the root pth mean after normalization tends to $\rho^{-p}\mathbb{E}|F(\rho e^{it})|^p$.

For derivative order j, take P_n^M at level t_n^M. The lemniscate, centroid and normalized scale are unchanged. Each root of P_n is a root of the jth derivative of multiplicity at least $M - j$. For fixed n with root mean already greater than $S(p)$, the lower bound on the derivative mean tends to that root mean as $M \to \infty$. The remaining zeros contribute nonnegatively. This gives the claimed examples. □

5 Sharp asymptotics for the even inverse coefficients

Theorem 8. *Over the full class Σ_0 one has*

$$\sup_{F\in\Sigma_0} |B_{2m}(F)| \sim \frac{1}{2m}\binom{2m}{m} \sim \frac{4^m}{2\sqrt{\pi}\, m^{3/2}}.$$

5.1 The radial bound and the residue

The upper limit follows from a radial estimate uniform in F. We use the two-point inequality [3, (7)]

$$4|w - P| + |w + P|^2 \le 16 \qquad (w, P \in \operatorname{conv} K), \tag{10}$$

and the projection comparison of Baernstein, Laugesen and Pritsker [1, Theorem 2]. The latter bounds the average of a convex test of any unit-direction projection by its average on $2\cos t$. Choose P_1 of largest modulus in $\operatorname{conv} K$. If $|P_1| \le 39/20$, the trivial bound $R_p \le (39/20)^p$ suffices. Otherwise choose P_2 of largest modulus in the closed half-plane of that convex hull opposite P_1 if the half-plane contains a point of modulus at least $39/20$.

For a point $|w| = 2 - e \ge 39/20$ and $|P_i| \ge |w|$, putting $u = |w - P_i|$ in (10) and using the parallelogram identity gives $u(4-u) \le 16e$. Since $u \le 4$ and $e \le 1/20$, this implies $u \le 17e/4$ or $u \ge 4 - 17e/4$. The product $u(4-u)$ on the intervening interval is at least $(17e/4)(4-17e/4) \ge 16e$, with strict inequality in the interior. In the forward half-plane of P_1, $|w - P_1|^2 \le 8$, so the small alternative holds. In the opposite half-plane, the small alternative relative to P_1 is impossible for both w and P_2, since distance to P_1 there is at least $|P_1|$. Equation (10) gives $|w + P_1|^2 \le 17e$ and $|P_2 + P_1|^2 \le 17e$. Thus $|w - P_2| \le 2\sqrt{17e} < 4 - 17e/4$, giving the small alternative relative to P_2. For each assigned high-radius point, put $\xi_i = P_i/|P_i|$. The cosine rule gives

$$2 - \operatorname{Re}(\bar{\xi}_i w) = e + \frac{|w - P_i|^2 - (|P_i| - |w|)^2}{2|P_i|} \le e + 5e^2.$$

Use the convex truncation of [3, (14)–(15)]. Set $c = 39/20$ and

$$h(x) = \max\left\{c,\, 2 - \frac{\sqrt{1 + 20(2-x)} - 1}{10}\right\} \qquad (x \le 2),$$

extended by $h(x) = x$ on $x \ge 2$. This is positive, increasing and convex. The preceding estimate implies $|w| \le h(\mathrm{Re}(\bar{\xi}_i w))$ on each assigned side. If a side has no point above c, every point there has modulus at most the constant part of the test for the other side. The projection comparison for h^p, $p \ge 1$, therefore bounds R_p by at most two copies of $\mathbb{E}h(2\cos t)^p$.

Write $e(v) = 2 - 2\cos v$. On the active part of the test, the substitution $2 - 2\cos t = e(v) + 5e(v)^2$ gives

$$v \in [0, \arccos(39/40)], \qquad t \in [0, \arccos(31/32)].$$

The endpoints correspond to $e = 0$ and $e = 1/20$. Then $h(2\cos t) = 2\cos v$ and

$$\frac{dt}{dv} = \frac{1 + 10e(v)}{\sqrt{(1 + 5e(v))(1 - 5e(v)^2/(4 - e(v)))}} \le 1 + \frac{800}{79}\sin^2 v \qquad (0 \le e(v) \le 1/20).$$

Here the denominator is at least one and $e(v) \le (80/79)\sin^2 v$. Integrating the active part and bounding the flat part by c^p gives

$$R_p(F) \le 2c^p + S(p)\left(1 + \frac{800}{79(p+2)}\right) \qquad (p \ge 1), \tag{11}$$

using $\int_0^{\pi/2} \cos^p v \sin^2 v\, dv = (p+2)^{-1} \int_0^{\pi/2} \cos^p v\, dv$.

For $n \ge 1$, the coefficient of w^{-1} in $G(w)w^{n-1}$ is $-B_n$ in our convention. On a sufficiently large circle in z, substitute $w = F(z)$ and integrate by parts to obtain

$$nB_n = \frac{1}{2\pi i}\int_{|z|=R} F(z)^n\, dz, \qquad R > 1.$$

The extension to every $R > 1$ follows by contour deformation. Let $R \downarrow 1$ and use (1) to get $nB_n = \mathbb{E}(F^*(e^{it}))^n e^{it}$. In particular, $n|B_n| \le R_n$. Since $S(2m) = \binom{2m}{m}$ and $c < 2$, Stirling's formula and (11) give the upper limit for the ratio in Theorem 8.

5.2 The arc and the two peak phases

For the lower limit take the uncut arc $A_{\pi/n}$ for even n sufficiently large. On $|\sin t| \ge 3/4$, (3) gives uniform convergence

$$\left(\frac{A_{\pi/n}(e^{it})}{2i\sin t}\right)^n \longrightarrow Q(t) = \exp\left(-\frac{\pi e^{-2it}}{2i\sin t}\right).$$

The base is $1 + Q_1(t)/n + O(n^{-2})$ uniformly on this region. Here Q_1 is the exponent in Q. The logarithm of the base, multiplied by n, converges uniformly to Q_1. For $|z| = 1$, the identity

$$A_\kappa(z) - (z - z^{-1}) = \frac{\kappa(\kappa z - 1)}{z(z - \kappa)}$$

gives a displacement of modulus exactly κ, since $|\kappa z - 1| = |z - \kappa|$. The complementary part of the residue is $o(S(n))$ by $|A_{\pi/n}(e^{it})| \le 3/2 + \pi/n$ there. The probability measure $|2\sin t|^n\, dt/(2\pi S(n))$ tends weakly to two point masses of weight $1/2$ at $\pi/2$ and $-\pi/2$. Outside fixed neighborhoods of

these points, $|\sin t| \le r_0 < 1$, so the normalized mass is $O(\sqrt{n}\, r_0^n)$ by Stirling's formula. Symmetry gives equal weights at the two peaks. The factors $Q(\pi/2) = e^{-i\pi/2}$ and $Q(-\pi/2) = e^{i\pi/2}$ cancel the corresponding factors e^{it} at both peaks. Since $(2i\sin t)^n = (-1)^{n/2}|2\sin t|^n$ for even n, the normalized residue tends to one after multiplication by $(-1)^{n/2}$. The lower and upper limits agree, proving Theorem 8.

Odd-order coefficients. Goel's disk theorem [2, Theorem 1.3] uses a plus sign in the inverse Laurent series. In our minus-sign convention its disk reads

$$\left|B_{2m+1} - 2^{-m} C_m b_1^{m+1}\right| \le (1 - 2^{-m}) C_m, \qquad C_m = \frac{1}{m+1}\binom{2m}{m}.$$

The triangle inequality gives $|B_{2m+1}| \le C_m(1 - 2^{-m}(1 - r^{m+1}))$ for $r = |b_1|$. Since $1 - r^{m+1} \ge 1 - r \ge (1-r)^{m+1}$, a consequence of Goel's theorem is

$$|B_{2m+1}| \le C_m \left(1 - \frac{(1-|b_1|)^{m+1}}{4^m}\right), \qquad m \ge 1.$$

6 Open questions

This paper leaves slit maximality undecided for $p \in [4/3, 7/4)$ in Σ_0, for $p \in [8/7, 7/4)$ in the subclass $0 \in K$, and for $p > 97/10$ in both classes. The exact maxima at general fixed even orders $2m \ge 4$ and the second asymptotic term in Theorem 8 also remain open.

A Explicit constants and low-exponent expansions

A.1 The explicit parameter

One may take $\kappa = 2^{-100}$ without using the asymptotic remainder. For $\kappa \le 1/100$, the geometric series in (3) gives $|x + \kappa\cos 2\theta| \le 25\kappa^2/8$. The exact formula for y gives $y \le (21/10)\sin\theta$ and

$$\int_0^\pi y\, d\theta = 2 + \frac{1-\kappa^2}{\kappa}\log\frac{1+\kappa}{1-\kappa} \ge 4 - 2\kappa^2.$$

For $4\kappa \le q = \min(\theta, \pi - \theta) \le 1/10$, these estimates give $|x| \ge 47\kappa/50$, $|x| \le 26\kappa/25$, $w \ge 49/50$ and $|x + iy| + y \le 223q/50$. The bound on w follows from $d \le \kappa^2/16$, $\sin(\theta/2) \ge \theta/\pi$ and $\pi^2 < 10$. Therefore $wg \ge 9\kappa^2/(50q)$ on both ends. For the density loss put $u = \sin(\theta/2)$ and split at $\kappa/4$. The pointwise bound $1 - w \le \min(1, d/(2u^2))$ and $y \le (21/10)\sin\theta$ give

$$\int_0^\pi (1-w) y\, d\theta \le \frac{42}{5}\left(\int_0^{\kappa/4} u\, du + \frac{\kappa^2}{32}\int_{\kappa/4}^1 \frac{du}{u}\right) = \frac{21\kappa^2}{80}\left(1 + \log\frac{4}{\kappa}\right).$$

Since $e < 11/4$, this is at most $(21/80)\kappa^2\log(11/\kappa)$. Dropping the nonnegative slit term in (2) and using $4a \le 4 + \kappa^2/4$ gives

$$a\big(\pi R_1(F_\kappa) - 4\big) \ge \kappa^2\left(\frac{39}{400}\log(1/\kappa) - \frac{9}{4} - \frac{9}{25}\log 40 - \frac{21}{80}\log 11\right) > 0$$

at $\kappa = 2^{-100}$, by $\log 2 > 1/2$, $\log 40 < 4$ and $\log 11 < 3$.

A.2 Low-exponent asymptotics

Proof of Corollary 2. For $1 < p < 8/7$, use $U = 2i \sin\theta$ in (3). The linear term in the expansion of the unweighted angular mean vanishes, since it is a constant times $\int_0^\pi \sin^p\theta \cos\theta \, d\theta$. The translation by b is real and has zero first variation at U. The three quadratic terms divided by $p\kappa^2$ are

$$|U|^p(1 - 2\sin^2\theta), \qquad \tfrac{1}{2}|U|^{p-2}, \qquad \tfrac{p-2}{2}|U|^p \cos^2\theta.$$

Integration is justified by Taylor's formula and domination by a constant times $\sin^{p-2}\theta$ where $\sin\theta \geq 4\kappa$. On the complement the integral after subtraction of the constant and linear terms is $O_p(\kappa^{p+1}) = o(\kappa^2)$. The cosine integral recurrence therefore gives the unweighted quadratic coefficient as $pS(p)(4-3p)/(8(p-1))$. For the density correction, $d^{-1}(1-w) \to [2\sin^2(\theta/2)]^{-1}$. The same cutoff gives an integrable majorant on its complement and $O_p(\kappa^{p+1})$ for the actual loss on the excluded ends. Symmetrizing in θ and $\pi - \theta$ and using the recurrence gives a loss of $dS(p)p/(p-1) + o(\kappa^2)$. Multiplying by a^{-p} gives one more loss of $pdS(p) + o(\kappa^2)$. With $d = \kappa^2/16 + O(\kappa^3)$ this yields

$$R_p(F_\kappa) = S(p) + pS(p)\frac{8-7p}{16(p-1)}\kappa^2 + o(\kappa^2).$$

For $0 < p < 1$, the exact formula for y gives $\int_0^\pi y^p \, d\theta = \pi S(p) + O_p(\kappa^2)$ by cancellation of the linear cosine term. The remaining contributions come from the two ends and the added slit. Write $s = \min(\theta, \pi - \theta)$. On either end, under $s = \kappa v/2$, one has $x/\kappa \to -1$ and $y/\kappa \to v$. The weight tends to $v/\sqrt{1+v^2}$ at zero and to one at π. The rescaled transverse difference $\kappa^{-p}(|x+iy|^p - y^p)$ is bounded by $C_p \min(1, v^{p-2})$, by the concavity of the $p/2$ power on squares and the estimates $|x| \leq C\kappa$, $y \asymp s$. The rescaled loss $\kappa^{-p}(1-w)y^p$ near zero is bounded by $C_p \min(v^p, v^{p-2})$. On the half near π that loss is $O_p(\kappa^2)$ before rescaling the integral. These bounds justify dominated convergence.

In the slit term of (2), put $t = t_0 u$ with $0 \leq u \leq 1$. The identity $(\psi-1)^2/\psi = 4d - 2a(1-\cos t)$ and $t_0 \sim \kappa/2$ give $(\psi - 1)/\kappa \to \frac{1}{2}\sqrt{1-u^2}$. Since $A_\kappa(1) = -\kappa$ and its real derivative on $[1, r]$ tends uniformly to two, the rescaled slit integrand tends to $(1 - \sqrt{1-u^2})^p$. This convergence is dominated by the bound $|A_\kappa - b| \leq C\kappa$ on $[1, r]$. Combining the three terms gives

$$\frac{2\pi}{\kappa^{p+1}}(R_p(F_\kappa) - S(p)) \longrightarrow I - \frac{1}{p+1} + B,$$

where $I = \int_0^\infty ((1+v^2)^{p/2} - v^p)\, dv$ is the contribution at π and $B = \int_0^1 (1 - \sqrt{1-u^2})^p \, du$ comes from the slit. At zero the integral is $\int_0^\infty (v(1+v^2)^{(p-1)/2} - v^p)\, dv = -1/(p+1)$ by evaluating the antiderivative. The normalization $a^{-p} = 1 + O_p(\kappa^2)$ contributes zero to this limit.

Integration by parts gives $I = \frac{p}{p+1}\int_0^\infty (1+v^2)^{p/2-1}\, dv > \frac{p\pi}{2(p+1)}$. On the other hand the layer measure of $1 - \sqrt{1-u^2}$ gives

$$\frac{1}{p+1} - B = p\left(\int_0^1 u^p \sqrt{\frac{2-u}{u}}\, du - \frac{1}{p+1}\right) < \frac{p\pi}{2(p+1)}.$$

For the last step, u^p is increasing and $\sqrt{(2-u)/u}$ is decreasing, so the average of their product is strictly below the product of their averages. The second average is $1 + \pi/2$, by substituting $u = 1 - \cos t$. This proves positivity for $0 < p < 1$. The case $p = 1$ is Theorem 1. □

Proof of Corollary 3. With $a = 1$, $b = 0$ and $w = 1$, the unweighted expansion in the preceding proof gives the quadratic coefficient $pS(p)(4-3p)/(8(p-1))$ for $p > 1$. At $p = 1$ both endpoint contributions remain, giving $\kappa^2 \log(1/\kappa)/(2\pi) + O(\kappa^2)$ above $S(1)$. At $0 < p < 1$ both rescaled endpoints contribute the positive integral I, with no loss from the boundary density. These assertions use $b = d = 0$ in the integral estimates for the two endpoints, independently of the auxiliary slit construction. □

B The weighted Fourier estimate

The weighted estimate holds for $0 \le \epsilon = 2 - p \le 1/4$. Shift the angular variable so the weight is $|\sin t|^{-\epsilon}$. Its constant Fourier coefficient m is at most $5/4$. Indeed

$$\sin(\pi x/2) \ge (\pi x/2)(1 - 5x^2/12), \qquad (1 - 5x^2/12)^{-1/4} \le 1 + 3x^2/20 \quad (0 \le x \le 1),$$

where the second inequality follows from the secant bound for a convex function of x^2. Integration with $(2/\pi)^{1/4} < 9/10$ gives $m \le (9/10)(4/3 + 3/55) < 5/4$. The nonzero-frequency coefficients are $m\alpha_k$ at frequencies $\pm 2k$, where

$$\alpha_k = \prod_{j=1}^{k} \frac{2j - 2 + \epsilon}{2j - \epsilon} \le \frac{1}{7\sqrt{k}} \quad (k \ge 1), \qquad \alpha_0 = 1.$$

To obtain the recurrence, express the sum and difference of consecutive cosine coefficients using products with $\cos t$ and $\sin t$, and integrate the derivative of $\sin^{1-\epsilon} t$ against $\cos((2k-1)t)$ on $[0, \pi]$. The bound follows from $\alpha_1 \le 1/7$ and $(8k+1)^2(k+1) \le (8k+7)^2 k$ for $k \ge 1$.

In the energy coordinates $\sqrt{j}a_j$, the weighted square norm is the quadratic form with matrix entries $m\alpha_{|j-n|/2}/\sqrt{jn}$ when j, n have the same parity, and zero otherwise. The shift of phase does not affect the estimates. We use three elementary bounds for this form. Here $H_k = 1 + \cdots + 1/k$.

- The even block is $(m/2)B$ with $B_{jk} = \alpha_{|j-k|}/\sqrt{jk}$ for $j, k \ge 1$. The first diagonal entry is one, and the first row away from that entry has square norm at most $1/49$ by $\sum_{k\ge1} 1/(k(k+1)) = 1$. The tail has diagonal norm at most $1/2$ and off-diagonal square Hilbert–Schmidt norm at most $(2/49)\sum_{k\ge1}(H_{k+1}-1)/k^2 \le 2\zeta(2)/49 < (3/10)^2$. The sum bound follows from interchanging sums and using $\sum_{k\ge j-1} k^{-2} \le j/(j-1)^2$. The 2×2 quadratic bound with diagonal $1, 4/5$ and off-diagonal $1/7$ has largest eigenvalue less than $27/25$, since $(27/25-1)(27/25-4/5) > 1/49$. The even block therefore has norm at most $27/40$.

- The odd block on $j \ge 3$ has diagonal norm at most $5/12$. Its off-diagonal norm is at most $(5/8)\sqrt{(2/49)\sum_{k\ge1} H_k/k^2} < 5/24$. In the sum, the contribution from $j = 1$ after interchanging sums is $\zeta(2)$, and the rest is at most $\zeta(2) + \zeta(3) - 2$ by the integral bound for $\sum_{k\ge j} k^{-2}$. The model sum is thus less than $31/12$ using $\zeta(2) < 5/3$ and $\zeta(3) - 1 \le 1/4$. The odd tail norm is at most $5/8$.

- The first diagonal entry is at most $5/4$. The row from $j = 1$ to the odd tail has square norm bounded by
$$\frac{25}{16 \cdot 49} \sum_{k\ge1} \frac{1}{k(2k+1)} < \frac{1}{36}.$$

All bounds extend from finite sums by completion in the energy norm and ordinary L^2 convergence.

If $r \geq 9/10$, let $s = |a_1|^2/\ell = \ell \leq 1/20$ and let v be the fraction of the energy bound in the odd tail. The preceding estimates give

$$\frac{\mathbb{E}(|U|^{-\epsilon}|W|^2)}{\ell} \leq \frac{5}{4}s + \frac{27}{40}(1-s) - \frac{v}{20} + \frac{1}{3}\sqrt{sv} \leq \frac{27}{40} + s\left(\frac{23}{40} + \frac{5}{9}\right) < \frac{3}{4} - \frac{1}{60}.$$

The second step completes the square in $\sqrt{v}$. Thus (6) is strictly positive throughout $7/4 \leq p \leq 2$.

C The eighth-moment algebra

We prove (9) by retaining the losses in the Grunsky estimates used in the proof of [3, Theorem 9]. Put $s = r^2 = 1 - D$, $u = c_{23} = b_4 + b_1 b_2$, $v = b_1 b_2$ and $g = b_3 + b_1^2$. With $A = 4b_3 + 6b_1^2$, $w = b_5 + 3b_1 b_3 + \frac{3}{2}b_2^2 + b_1^3$ and $y = b_6 + 3b_1 b_4 + 3g b_2$, the finite area estimate there gives

$$R_8 \leq 2 + 24s + 20|b_2|^2 + |A|^2 + 12|u + 2v|^2 + 8|w|^2 + 4|y|^2. \tag{12}$$

The strong Grunsky inequality in row three and the expansion

$$c_{34} = b_6 + b_1 b_4 + 2b_2 b_3 + b_1^2 b_2$$

give $|c_{33}| \leq 1/3 - |b_3|^2/2 - U_0$, with $U_0 = |u|^2 + 2|c_{34}|^2 \leq 1/6$, by the tangent inequality for the square root. The second row gives

$$|b_2|^2 + 2|c_{22}|^2 + 3|u|^2 \leq \frac{1}{2}, \qquad c_{22} = b_3 + \frac{1}{2}b_1^2.$$

Put $L = |b_2|^2 + 3|u|^2$. Substitution of $|c_{22}|^2 \leq 1/4 - L/2$ into the square gives

$$\begin{aligned}\left|2b_1 b_3 + \frac{2}{3}b_1^3\right|^2 &= \frac{4}{3}s|b_3|^2 + \frac{8}{3}s|c_{22}|^2 - \frac{2}{9}s^3 \\ &\leq \frac{4}{3}s|b_3|^2 + \frac{2}{3}s - \frac{2}{9}s^3 - \frac{4}{3}sL.\end{aligned}$$

The difference between $\left(2/3 + s|b_3|^2\right)^2$ and the first three terms of this upper bound is

$$\frac{2}{9}(1-s)^2(s+2) + s^2|b_3|^4 \geq 0.$$

Thus the square is bounded by $\left(2/3 + s|b_3|^2\right)^2 - \frac{4}{3}sL$. Since $2/3 + s|b_3|^2 \leq 1$, the concavity of the square root gives $|w| \leq 1 + A_0 - U_0 - V_0$, with $A_0 = s|b_3|^2 + |b_2|^2/2 \leq D/3$ and $V_0 = (2/3)sL \leq 1/3$. First subtract U_0 when expanding the square, then subtract V_0. Using $2(1 + A_0) - U_0 \geq 11/6$ and $2(1 + A_0 - U_0) - V_0 \geq 4/3$ gives

$$8|w|^2 \leq 8 + 8(2 + D/3)A_0 - (44/3)U_0 - (64/9)sL.$$

The first block of (12), using the full row-two inequality, is bounded by

$$12 + 24s^2 - 4|b_2|^2 - 32|b_3|^2 + 12|u + 2v|^2 - 72|u|^2.$$

For the last block use $y = c_{34} + 2b_1 u + b_3 b_2$. Completing a square and then using $|A + B|^2 \leq (13/12)|A|^2 + 13|B|^2$ gives

$$4|y|^2 - (88/3)|c_{34}|^2 \leq \frac{88}{19}|2b_1 u + b_3 b_2|^2 \leq \frac{1144}{57}s|u|^2 + \frac{1144}{19}|b_3|^2|b_2|^2.$$

The resulting coefficient of $|b_3|^2$ is at most $-32+8(2+D/3)s+(572/19)D \le -1$, by $2|b_2|^2 \le D$ and $0 \le D \le 1$. Expand $12|u+2v|^2$ and use $260/3-1144/57-12 \ge 109/2$. Thus (12) gives $R_8 \le 22+24s+24s^2+Q$ with $a_2=|b_2|$ and

$$Q=\left(\frac{16}{3}+\frac{356}{9}s\right)a_2^2+48\,\mathrm{Re}(u\bar{v})-\left(\frac{109}{2}+\frac{64}{3}s\right)|u|^2.$$

Set $\lambda=8+10s$. The inequality $Q \le \lambda(2a_2^2+4|u-v|^2)$ follows by comparing the quadratic form in a_2 and $|u|$. Its diagonal entries are $32/3+(112/9)s+40s^2$ and $173/2+(184/3)s$, and the magnitude of its off-diagonal entry is at most $(56+40s)\sqrt{s}$. Its determinant is bounded below by $(4/27)P(s)$, where

$$P(s)=5760s^3-1733s^2-9486s+6228.$$

Taylor expansion at $s_0=5/6$ gives $P(s) \ge P(s_0)+P'(s_0)(s-s_0) \ge 14057/36$ for $0 \le s \le 1$. The discarded quadratic term has coefficient $12667+5760(s-s_0) \ge 7867$. Now $u-v=b_4$ and $2|b_2|^2+4|b_4|^2 \le D$. Substituting $Q \le (8+10s)D$ proves the strengthened eighth moment.

Lean verification

The Lean 4.34.1 replay verifies the strict counterexample of Theorem 1 at $\kappa=2^{-100}$ on the actual radial trace, including the omitted-origin condition. Its proof uses the two inverse-branch limits, the angular substitution and the explicit integral lower bound, with every integral estimate discharged inside the proof. The $\kappa \downarrow 0$ asymptotic in the same theorem is proved analytically. Lean also verifies the actual-trace identity (4) for all $p>1$. The finite-moment implication (5) is formalized with the integrated curvature and transverse bounds as explicit hypotheses. The deduction of both bounds from the general pointwise tests is given in the analytic proof.

The closed interval and equality classification in Theorem 4, the strengthened eighth moment (9), and the strictness, supremum and nonattainment for connected polynomial lemniscates are verified. For the lower exponent ranges, Lean uses earlier weighted Stokes, Green, projection and moment estimates. The paper uses the weighted Fourier matrix and the diameter Green trace above.

For the low-exponent expansions, Lean verifies the uncut arc's actual-trace integral and the circular contribution as a lower bound for the cut arc. The full decomposition (2), scaled cut-segment limits and endpoint asymptotics are analytic. For the transfer to low-exponent derivative-zero counterexamples, Lean verifies the power-sublevel and root-multiplicity steps. The root-sampling limit and the normalized derivative moments in that transfer are analytic. For Theorem 8, Lean verifies the actual-trace residue bound, both local geometric cases, the scalar Jacobian bound and the two peak phases, including a bound on the actual off-peak trace. The global radial estimate and the two-peak limit of the residue are analytic. The formal odd-order corollary assumes Goel's disk theorem as stated. The full audit was run in the private Lean project. The public verification materials contain the audited types, axiom reports and source excerpts. The fixed public Lean release for the earlier paper corresponds to [3].

Declaration of AI use

The manuscript text was primarily AI-generated. OpenAI's ChatGPT and Codex were used for proof exploration, Lean 4 formalization, and drafting and revising the exposition. The author has checked and understood parts of the contribution.

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