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Harmonic Cutoffs in Lorentzian Group Averaging: Non-Descent and Regularity-Restricted Stability

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Abstract

For four scalar principal-series representations of $\mathrm{PSL}(2,\mathbb{C})$, we analyze the rectangular spherical-harmonic cutoff P_L before a single Haar average I. A generic fixed boost produces a null test whose cutoff image is nonzero for every finite L, for any nonzero smooth compactly supported spectral bump. Its leakage nevertheless has an upper bound of order $(L+2)\tanh(r/2)^{L+1}$. The worst auxiliary-unit-null leakage diverges. A diagonal-rotation kernel estimate gives a cubic upper bound on $\|IP_L\|$ and the exact invariant angular point-evaluation norm. Uniform derivative tails, including at coincident sphere points, imply operator-norm convergence from angular H^s for $s > 3$. Under specified Gram-domain and coefficient hypotheses, two lifts of the same uncut image with H^{10} growth $O(N^{10})$ and $L = N^3$ have state and centered first-graph differences $O(N^{-11})$ and $O(N^{-5})$. We do not identify the mathematical quotient with a physical GFT carrier or claim an optimal regularity threshold for averaging.

1 Introduction

Noncompact group averaging relates a test space to an invariant dual or a positive-form quotient, rather than generally giving an auxiliary orthogonal projector [1, 2, 3, 4]. Lorentzian spherical coefficients, boost matrix elements and group integrals are familiar [5, 6, 7, 8]; numerical truncations of internal spin sums are also standard [9]. Our question is different: does cutting the *external auxiliary angular representative* before averaging define an operation on its equivalence class, and if not, when can its representative dependence be controlled?

We fix a four-leg scalar model, Haar normalization and rectangular cutoff. The concrete boost witnesses and the quantitative diagonal-rotation estimates are the focus of this work; the dense-graph mechanism is standard functional analysis. Symmetry-improved Sobolev estimates and equivariant spectral theory are established subjects [10, 11, 12, 13]. For instance, Saintier's Theorem 0.1 treats the second-order invariant Sobolev inequality with effective dimension $n - \min_x \dim(Gx)$ and an orbit-volume-dependent best leading constant. In our angular model this dimension is six, corresponding to the critical H^2 to L^6 embedding. Our problem instead concerns uniform differentiated rectangular-projection tails and their transfer through a noncompact Haar average. Hebey and Vaugon are cited for historical context, not as a source of a theorem used below; the original full text has not been checked. The estimates below are derived directly. Ramacher's fixed-point equivariant Weyl law and its orbit-period corollary concern a spectral-energy cutoff. For $-\Delta_Q$, the invariant rectangular projector is between invariant energy projectors at energies $L(L+1)$ and $4L(L+1)$. Thus its pointwise growth is closely related to the existing orbit-dimension principle: the minimum diagonal-rotation orbit dimension here is two, rather than the generic dimension three. This does not itself supply the exact rectangular maximum, the uniform mixed-derivative constants, or the subsequent Haar and Gram estimates. We do not claim a new general embedding or Weyl law. Sections 2–4 establish the model and obstruction; Sections 5–6 give regularity-restricted recovery.

A related manuscript by the same author [15], now available as a Hexagon preprint, studies a selected angular lift and state-side cutoff. The technical comparison here uses the retained local manuscript; byte identity with the public server source has not been established. Our question is the dependence on two representatives with equal uncut images. The estimates below are self-contained subject to their stated domain and coefficient assumptions; they neither improve the unaveraged angular tail of that manuscript nor repeat its shape-family existence theorem. We do not identify the external angular cutoff with an internal channel projection.

2 Model and orbital averaging

Let $G = \mathrm{PSL}(2,\mathbb{C})$, $Q = (S^2)^4$, and let dm be the product of the probability round measures. Set $\Omega = (0,\infty)^4$, $d\Pi(\rho) = \prod_{i=1}^4 4\rho_i^2\, d\rho_i$. Restricting to positive labels suffices for all witnesses below. Write $\mathcal{H} = L^2(\Omega \times Q, d\Pi\, dm)$ and $\mathcal{D} = C_c^\infty(\Omega; C^\infty(Q))$ (compact spectral support). The angular harmonic projection P_L is the product of the four projections onto degrees $0 \le j_i \le L$. It acts on $\mathcal{H}$, not on an already constructed quotient.

For a projective point z, define the positive cocycle

$$\delta(g,z) = \frac{\|g(z,1)\|^2}{1+|z|^2}, \qquad \delta(gh,z) = \delta(g,hz)\delta(h,z),$$

using the continuous extension at infinity. The scalar compact-picture representation is

$$(T_\rho(h)u)(q) = \prod_{i=1}^4 \delta(h^{-1},q_i)^{-1+i\rho_i} u(h^{-1}q). \tag{1}$$

Since the round Jacobian is $\delta(g,z)^{-2}$, this is unitary for real ρ. The Haar convention is the Cartan radial factor $(2/\pi)\sinh^2 t\, dt$ with compact factors of mass one, where $h_t = \mathrm{diag}(e^{t/2}, e^{-t/2})$ modulo the center. This normalization is part of the model, not a convention inferred from other Lorentzian integrals.

Orbital coordinates. On the full-measure set of ordered distinct quadruples, there is a unique group element carrying $(0,1,\infty)$ to the first three entries. The fourth entry is specified by $z \in Z = \mathbb{CP}^1 \setminus \{0,1,\infty\}$. Thus $q = gs(z)$, where $s(z) = (0,1,\infty,z)$, gives smooth coordinates on this open set. Collisions are measure zero. Fix a positive smooth shape density $d\nu$ and write $dm = J(g,z) dg\, d\nu(z)$, absorbing any change of shape density in J. Invariance of dg and the round Jacobian give $J(hg,z) = J(g,z)\prod_i \delta(h, gs_i(z))^{-2}$. The averaging target is $\mathcal{Y} = L^2(\Omega \times Z, d\Pi\, d\nu)$. The transformation

$$W_\rho u(g,z) = \sqrt{J(g,z)} \prod_{i=1}^4 \delta(g, s_i(z))^{i\rho_i} u(gs(z))$$

is unitary onto the corresponding $L^2(G \times Z)$ space. By the cocycle and the Jacobian, $W_\rho T_\rho(h) u(g,z) = W_\rho u(h^{-1}g, z)$. Inverse images of smooth compactly supported functions of (ρ, g, z) belong to $\mathcal{D}$: their supports avoid collisions, so smooth zero extension is valid.

Lemma 2.1 (Absolute convergence and positive form)**.** *With $C_H = \pi^3/15$, the map*

$$I_\rho u(z) = \int_G W_\rho u(g,z)\, dg$$

is defined almost everywhere for smooth angular tests, with $\|I_\rho u\|_{L^2(d\nu)} \le \sqrt{C_H}\,\|u\|_\infty$. *In the direct integral this defines* $I : \mathcal{D} \to \mathcal{Y}$ *and*

$$\eta(u,v) = \langle Iu, Iv\rangle_{\mathcal{Y}} = \int_\Omega \int_G \langle u_\rho, T_\rho(h) v_\rho\rangle_{L^2(dm)}\, dh\, d\Pi(\rho). \tag{2}$$

Moreover $I_\rho T_\rho(h) = I_\rho$.

Proof. In Cartan coordinates, integration of $(\cosh t - x \sinh t)^{-1}$ against $dx/2$ gives $t/\sinh t$. The absolute matrix-element majorant integrated over the group is

$$\frac{2}{\pi}\int_0^\infty \frac{t^4}{\sinh^2 t}\, dt = \frac{\pi^3}{15},$$

times the two sup norms. The equality follows from $\sinh^{-2} t = 4\sum_{k\ge 1} k e^{-2kt}$ and $\zeta(4) = \pi^4/90$. To justify the orbital integral before using complex Fubini, take $\rho = 0$ and both angular tests equal to one. The transformed function is $\sqrt{J} \ge 0$. Tonelli and unimodularity give

$$\int_Z \left(\int_G \sqrt{J(g,z)} dg \right)^2 d\nu(z) = C_H.$$

The orbital integral for bounded tests is dominated by this inner nonnegative integral. This gives the norm bound; absolute integrability then permits Fubini in the complex pairing and proves (2). Compact spectral support permits the direct integral. Left translation of the Haar integral gives invariance. □

Set $\mathcal{N} = \ker(I|_{\mathcal{D}})$. The mathematical quotient carries the norm $\|u\|_\eta = \|Iu\|_{\mathcal{Y}}$. A fixed P_L descends to it only if $P_L\mathcal{N} \subseteq \mathcal{N}$. Let $K = \mathrm{SO}(3)$ be the compact rotation subgroup, $B = \int_K T(k)dk$ its normalized diagonal average. This is an auxiliary orthogonal projection; in the scalar compact picture the cocycle is one on K. We have $IB = I$, $[B, P_L] = 0$, and $B\mathcal{D} \subseteq \mathcal{D}$.

3 Auxiliary topology and the worst cutoff norm

Proposition 3.1 (Dense graph and null space)**.** *The graph of* I *restricted to inverse images of compact smooth orbital tests is dense in* $\mathcal{H} \times \mathcal{Y}$. *In particular* I *is not closable in the auxiliary* L^2 *topology, and* $\mathcal{N}$ *is dense in* $\mathcal{H}$.

Proof. Choose $\alpha \in C_c^\infty(G)$ with integral one. Noncompactness allows m disjoint left translates of its support C: at each step avoid the previous translates of CC^{-1}. Their mean α_m has integral one and norm $\|\alpha_m\|_2 = m^{-1/2}\|\alpha\|_2$. For compact smooth spectral-shape y, the lift $v_m(y) = W^{-1}(\alpha_m y)$ satisfies $Iv_m(y) = y$ and $\|v_m(y)\|_{\mathcal{H}} = m^{-1/2}\|\alpha\|_2 \|y\|_{\mathcal{Y}}$. Given (x, y_0), take compact orbital $x_k \to x$ and compact smooth $y_k \to y_0$; Ix_k is compact smooth in the spectral-shape variables. Correct x_k by $v_{m_k}(y_k - Ix_k)$ of norm at most $1/k$. The images equal y_k. Choosing $y_k = 0$ proves null density; the full graph closure proves nonclosability. □

Proposition 3.2 (Worst cutoff leakage)**.** *Let* $A_L = IP_L$ *denote its bounded extension from* $\mathcal{D}$ *and define*

$$\mathfrak{A}_L = \sup_{n \in \mathcal{N},\, \|n\|_{\mathcal{H}} = 1} \|IP_L n\|_{\mathcal{Y}}.$$

Then $\mathfrak{A}_L = \|A_L\|_{\mathcal{H}\to\mathcal{Y}}$, *this sequence is nondecreasing, and* $\mathfrak{A}_L \to \infty$. *The same supremum is obtained with* $n \in \mathcal{N} \cap B\mathcal{H}$. *More generally, every bounded operator sequence* $U_j : \mathcal{H} \to \mathcal{Y}$ *with* $U_j u \to Iu$ *for each fixed* $u \in \mathcal{D}$ *has* $\|U_j\| \to \infty$.

Proof. The product harmonic reproducing kernel has diagonal $(L+1)^8$, so $\|P_L f\|_\infty \le (L+1)^4 \|f\|_2$ and Lemma 2.1 gives boundedness. By Proposition 3.1, the unit null vectors are dense in the unit sphere, yielding the norm equality. Nested projections give $A_M P_L = A_L$ for $M \ge L$. On $\mathcal{D}$, the ordinary addition theorem and weighted Cauchy–Schwarz over dyadic degree shells give $\|(1-P_L)u_\rho\|_\infty \le C(L+1)^{-2}\|u_\rho\|_{H^6}$. Compact spectral support and Lemma 2.1 give $A_L u \to Iu$. For every positive R, the vectors $v_m(y)$ from the previous proof give a fixed test with $\|Iv_m\| / \|v_m\| > 2R$ for sufficiently large m. Pointwise convergence then implies $\|U_j\| > R$ for all sufficiently large j, including the case $U_j = A_j$. Finally $B\mathcal{N} \subset \mathcal{N} \cap B\mathcal{H}$ is dense in $B\mathcal{H}$ and $A_L = A_L B$, which yields the invariant-sector equality. □

This is a worst-case result in the auxiliary norm. It does not say a fixed smooth null vector diverges under truncation; every such vector has $IP_L n \to 0$. It is also distinct from nonclosability of any observable on the quotient.

Lemma 3.3 (Zero-cutoff normalization)**.** *One has $\mathfrak{A}_0 = \sqrt{C_H}$, with no unit vector attaining the norm. For real labels,*

$$0 \le C_H - \|I_\rho 1\|^2 \le \frac{\pi^5}{126} \sum_i \rho_i^2. \tag{3}$$

Proof. The one-leg constant matrix coefficient is $\sin(\rho_i t)/(\rho_i \sinh t)$, with the continuous limit at zero [5]. Hence $a(\rho) = \|I_\rho 1\|^2 = (2/\pi)\int t^4 \sinh^{-2} t \prod_i \operatorname{sinc}(\rho_i t) dt$. The identity $\operatorname{sinc} x = \int_0^1 \cos(ux) du$ gives $|1 - \operatorname{sinc} x| \le x^2/6$. Telescope the four factors, use their absolute bound one, and evaluate $(2/\pi)\int t^6 \sinh^{-2} t\, dt = \pi^5/21$ to obtain (3). Since P_0 projects onto constants in each fiber, $\|A_0\|^2 = \operatorname{ess\,sup}_\rho a(\rho) = C_H$ by positive-measure small boxes near zero. For every nonzero ρ the integrand product is strictly less than one for $t > 0$, so $a(\rho) < C_H$. This strict inequality almost everywhere excludes norm attainment by any nonzero vector. □

4 One boost for all finite cutoffs

For $r > 0$ let $f_{\rho,r}(x) = (\cosh r - x \sinh r)^{-1+i\rho}$, $x \in [-1,1]$, and $F_{\rho,r}(q) = \prod_i f_{\rho_i,r}(x_i)$, where x_i is the cosine of the polar angle relative to a fixed boost axis. This is the action of a boost (with the axis chosen according to the inverse-action convention) on the constant angular test. For a nonzero $\chi \in C_c^\infty(\Omega)$, define $n_{r,\chi} = \chi(\rho)(F_{\rho,r} - 1)$. Haar invariance makes it an exact null test. In what follows χ is only a spectral bump, not an orbital coordinate.

Theorem 4.1 (Explicit variable witness and universal generic fixed witness)**.** *For every integer $L \ge 0$, set $r_L = 5 + 4\log(L+1)$. On the fixed box $U = (0,1/4)^4$, every nonzero bump supported there obeys*

$$\|IP_L n_{r_L,\chi}\|_{\mathcal{Y}} \ge (\sqrt{7C_H/10} - 1/100)\,\|\chi\|_{L^2(d\Pi)} > 0. \tag{4}$$

There also exists a countable exceptional set $\mathcal{E} \subset (0,\infty)$, independent of χ, such that for every $r \notin \mathcal{E}$ and every nonzero $\chi \in C_c^\infty(\Omega)$,

$$\|IP_L n_{r,\chi}\|_{\mathcal{Y}} > 0 \qquad \text{for every integer } L \ge 0. \tag{5}$$

The good set is dense G_δ and full Lebesgue measure in every nonempty open interval of positive rapidities. No particular numerical rapidity is asserted to lie in this set.

Proof. The jth zonal coefficient of one factor is bounded by $(2j+1)r/\sinh r$ because $|P_j^{\mathrm{Leg}}(x)| \le 1$, where P_j^{Leg} denotes the Legendre polynomial. The tensor cutoff therefore satisfies

$$\|P_L F_{\rho,r}\|_\infty \le (L+1)^8 (r/\sinh r)^4.$$

For fixed L this tends to zero uniformly in real labels as $r \to \infty$. With $m = L+1 \geq 1$ and $r = 5+4\log m$, use $\sinh r \geq e^r/3$. The expression is at most $81e^{-20}(5+4\log m)^4 m^{-8} \leq 50625e^{-20}$, since the last factor is decreasing in m. Multiplying by $\sqrt{C_H}$ gives a number smaller than $1/100$: indeed $\pi < 22/7$ gives $\sqrt{C_H} < 3/2$, and $e > 8/3$ gives $(3/2)50625(3/8)^{20} < 1/100$. Equation (3) implies $a(\rho) > 7C_H/10$ on U, since $5\pi^2/168 < 3/10$. Apply Lemma 2.1 and the reverse triangle inequality fiberwise, then integrate $|\chi|^2$, proving (4).

For the generic statement define $G_L(r,\rho) = \|I_\rho(P_L F_{\rho,r} - 1)\|^2$. It is jointly real analytic on $(0,\infty)\times\mathbb{R}^4$. Indeed, for complex $r = a+ib$ the real part of $\cosh r - x\sinh r$ is $\cos b\,(\cosh a - x\sinh a) > 0$ whenever $|b| < \pi/2$, uniformly away from zero on compact subsets of this strip and $x \in [-1,1]$. The principal logarithm and the finite angular coefficients thus admit local holomorphic extensions. Write $V(r,\rho) = P_L F_{\rho,r} - 1$ and $V^\sharp(r,\rho;q) = \overline{V(\bar r,\bar\rho;q)}$. The holomorphic scalar integrand extending the squared norm in (2) is $V^\sharp(r,\rho;q)(T_\rho(h)V(r,\rho))(q)$, not a positive complex norm. On any compact parameter neighborhood with $\sum_i |\operatorname{Im}\rho_i| \leq \sigma < 2$, both finite angular functions have bounded sup norms. Since each conformal factor is between e^{-t} and e^t, the absolute matrix-element integral over q is at most a constant times $e^{\sigma t}(t/\sinh t)^4$. With the Haar radial density, the integrable majorant is a constant times $e^{\sigma t}t^4/\sinh^2 t$. Dominated holomorphic integration (or Morera's theorem on local polydiscs) gives a holomorphic extension of the scalar pairing, hence real analyticity of its restriction.

Fix $\rho^\circ = (1/8,1/8,1/8,1/8)$. By (4) (or its fiberwise proof), $G_L(r_L,\rho^\circ) > 0$. Thus each one-variable analytic function $r \mapsto G_L(r,\rho^\circ)$ has isolated zeros, and their union over finite L is countable. For r outside that union, $G_L(r,\cdot)$ is a nontrivial analytic function on connected $\mathbb{R}^4$, whose zero set has measure zero. For completeness, on each open product box $U'\times V$, the identity theorem supplies a point (x_0',t_0) at which the function is nonzero. The slice $x' \mapsto G_L(r,x',t_0)$ is then nontrivial; by induction its zero set in U' is null. Outside that set the one-variable function $t \mapsto G_L(r,x',t)$ is nontrivial and has isolated zeros. Fubini and a countable box cover prove the assertion. A countable union over L is still null, and $d\Pi$ is absolutely continuous. Thus integration against any nonzero $|\chi|^2$ gives a positive squared norm for each finite L. The countable union of closed (relative), discrete zero sets in the boost variable has residual, full-measure complement. The exceptional set used here need not be the exact set of bad boosts. □

Theorem 4.2 (Geometric recovery for the fixed witness)**.** *Fix $r > 0$ and a compact spectral bound $|\rho_i| \leq R$. For a constant $C(r,R)$ independent of L and χ supported in this region,*

$$\|IP_L n_{r,\chi}\|_{\mathcal{Y}} \leq C(r,R)(L+2)\tanh(r/2)^{L+1}\,\|\chi\|_{L^2(d\Pi)}\,. \tag{6}$$

Proof. Put $\tau = \tanh(r/2)$ and $c = \cosh(r/2)$. With $x = \cos\theta$,

$$\cosh r - x\sinh r = c^2(1-\tau e^{i\theta})(1-\tau e^{-i\theta}).$$

Both factors have positive real part and conjugate arguments, so the logarithms sum to the logarithm of their positive product. The binomial coefficients $a_k = (1-i\rho)_k/k!$ satisfy $|a_k|^2 = \prod_{j=1}^k(1+\rho^2/j^2) \leq e^{\pi^2R^2/6}$. The absolutely convergent two-factor expansion has symmetric Fourier coefficients $d_n = d_{-n}$ bounded by $e^{\pi^2R^2/6}\tau^{|n|}/(1-\tau^2)$ (the prefactor has modulus $c^{-2} \leq 1$). Truncation at $|n| \leq L$ gives a polynomial in x, since $\cos(n\theta)$ is a Chebyshev polynomial, and its sup-norm error is at most $C_0\tau^{L+1}$ with $C_0 = 2e^{\pi^2R^2/6}/[(1-\tau^2)(1-\tau)]$. The actual one-sphere harmonic projector fixes that polynomial and has sup-to-sup norm at most $L+1$ by the addition theorem. Thus its error is bounded by $(L+2)C_0\tau^{L+1}$. For an explicit constant, use $\|f_{\rho,r}\|_\infty \leq e^r$ and $(L+2)\tau^{L+1} \leq \sum_{k\geq 0}(k+2)\tau^{k+1} = \tau(2-\tau)/(1-\tau)^2$. Thus both the original and projected one-leg factors are bounded by $D = e^r + C_0\tau(2-\tau)/(1-\tau)^2$. Telescoping four products and using $I_\rho(F_{\rho,r}-1) = 0$ and Lemma 2.1 proves (6) with $C(r,R) = 4\sqrt{C_H}\,D^3C_0$. This is a finite upper constant for fixed r,R, not a matching lower bound or a uniform estimate as $r \to \infty$. □

Theorems 4.1 and 4.2 exhibit the distinction between a fixed witness with nonzero finite-cutoff leakage and the cutoff-dependent worst-case divergence of Proposition 3.2.

5 A uniform diagonal-rotation kernel estimate

Let $M = L+1$, $E_M = P_{M-1}B$, and denote the single-sphere kernel by $K_M(x,y) = \sum_{j<M}(2j+1)P_j^{\mathrm{Leg}}(x\cdot y)$. The common rotation is essential: the kernel of E_M is

$$E_M(q,p) = \int_K \prod_{i=1}^{4} K_M(q_i, kp_i)dk. \tag{7}$$

No replacement by four independent rotation integrals is made.

Proposition 5.1 (Exact angular maximum)**.** *Write $K_M(t) = \sum_{j<M}(2j+1)P_j^{\mathrm{Leg}}(t)$ and $J_M = \frac{1}{2}\int_{-1}^{1} K_M(t)^4 dt$. Then*

$$\|E_M\|^2_{L^2(Q)\to C^0(Q)} = J_M \le M^6, \tag{8}$$

with strict inequality for $M \ge 2$. Consequently $\mathfrak{A}_L \le \sqrt{C_H J_{L+1}} \le \sqrt{C_H}(L+1)^3$.

Proof. An orthogonal projection's point-evaluation norm squared equals its diagonal kernel. For each q_i, the random variable $q_i \cdot kq_i$ under normalized rotation Haar has uniform law on $[-1,1]$. Hölder applied to the four factors in (7) bounds the diagonal by J_M. Equality occurs when all four sphere points coincide. Legendre orthogonality gives $\frac{1}{2}\int K_M^2 = M^2$, while $|K_M| \le M^2$; hence $J_M \le M^6$. For $M \ge 2$ the kernel is a nonconstant polynomial, yielding strict inequality. Finally $IP_L = IP_L B$ and Lemma 2.1 transfer the angular upper bound to $\mathfrak{A}_L$. This is not an equality for the norm of the averaging map. For comparison, the classical derivative bound $\left\|(P_j^{\mathrm{Leg}})'\right\|_\infty \le j(j+1)/2 \le M^2/2$ shows $K_M(t) \ge M^2/2$ on $1 - M^{-2} \le t \le 1$. Integrating over this cap gives $J_M \ge M^6/32$. Thus the angular evaluation norm has exact growth order M^3; this lower bound is not a lower bound of the same order for IP_L. □

The exact angular maximum also has a positive finite channel formula. Write $K_M^2 = \sum_{k=0}^{2M-2} c_{M,k} P_k^{\mathrm{Leg}}$. The standard Legendre product formula and orthogonality [14] give

$$c_{M,k} = (2k+1)\sum_{a,b<M}(2a+1)(2b+1)\begin{pmatrix} a & b & k \\ 0 & 0 & 0 \end{pmatrix}^2, \qquad J_M = \sum_{k=0}^{2M-2}\frac{c_{M,k}^2}{2k+1}.$$

All terms are nonnegative rationals by the factorial formula for the squared $3j$ symbol. In particular $c_{M,0} = M^2$ and $\sum_k c_{M,k} = M^4$. These identities give a cancellation-free exact diagnostic in addition to the polynomial integral, not a new recoupling theorem. They apply to any finite consecutive degree band by restricting the two sums; the band lower bound below is analytic and not inferred from finite tests.

Theorem 5.2 (Derivative and tail estimates)**.** *Let $H^s(Q)$ use the spectral norm of $(1-\Delta_Q)^{s/2}$, with Δ_Q the nonpositive product Laplacian. For $d = 0,1,2$ and $s > d+3$,*

$$\left\|\nabla^d(1-P_L)Bf\right\|_\infty \le \frac{8^{d/2}2^{d+3}}{\sqrt{1-2^{6+2d-2s}}}(L+1)^{d+3-s}\left\|Bf\right\|_{H^s}. \tag{9}$$

The derivative norm is the full tensor norm in the product round metric. For $d = 0$ the power of $L+1$ is sharp in the invariant angular point-evaluation problem, and $s = 3$ is its endpoint of failure.

Proof. On a sphere, the degree-j evaluation vector has squared norm $d_j = 2j+1$. For directional derivatives of order $a = 1, 2$ it is bounded by $d_j\lambda_j$ and $d_j\lambda_j(\lambda_j - 1)$, respectively, with $\lambda_j = j(j+1)$. Indeed, the sums of squared full tensor norms are rotation invariant; their integrals follow from the eigenvalue equation and the integrated Bochner identity with $\mathrm{Ric} = g$. For $j < M$, both bounds are at most $d_j M^{2a}$; for $j = 0$ derivative vectors vanish. The same holds for $a = 0$.

Fix unit product-frame components. A differentiated kernel of orders a, b is a sum of rotation matrix coefficients between fixed evaluation vectors (covariant differentiation commutes with isometries). The two vector norms have product at most $d_j M^{a+b}$. More explicitly, put $w_0(j) = 1$, $w_1(j) = \lambda_j$, and $w_2(j) = \lambda_j(\lambda_j - 1)$ (zero for $j = 0$). Schur orthogonality of the distinct irreducible degree spaces gives the squared $L^2(K)$ bound $\sum_{j<M} d_j w_a(j) w_b(j) \le M^{2a+2b+2}$: within one degree, the squared norm of a matrix coefficient is the product of the squared vector norms divided by d_j. The triangle inequality gives the sup bound. Thus

$$\left\| K_M^{a,b} \right\|_{L^\infty(K)} \le M^{a+b+2}, \qquad \left\| K_M^{a,b} \right\|_{L^2(K)} \le M^{a+b+1}.$$

Differentiating (7) in a fixed product-frame component of order d on each side distributes the orders among the four legs, with $\sum a_i = \sum b_i = d$. Apply two $L^2(K)$ bounds and two sup bounds to the same rotation integral. For an orthonormal invariant box basis e_ν, this proves $\sum_\nu |\ell_d e_\nu(q)|^2 \le M^{6+2d}$ for each chosen component ℓ_d; summing over the at most 8^d components gives the tensor bound. It holds uniformly even at collisions. Since B commutes with every leg Laplacian, choose the basis in joint degree eigenspaces. On the shell $2^h(L+1) \le \max_i j_i < 2^{h+1}(L+1)$, the spectral weight $1 + \sum_i \lambda_{j_i} \ge (2^h(L+1))^2$. Weighted Cauchy–Schwarz in the invariant basis and summation of the geometric series of squared bounds give (9). The argument first holds on finite combinations and extends by completion and uniform convergence of the derivative series.

For the lower order at $d = 0$, restrict all four degrees to $M \le j_i < 2M$ and take $q_i = x$ for all i. Its diagonal kernel is

$$\frac{1}{2}\int_{-1}^{1} \left[\sum_{j=M}^{2M-1} (2j+1) P_j^{\mathrm{Leg}}(t) \right]^4 dt.$$

The derivative bound $\left\| (P_j^{\mathrm{Leg}})' \right\|_\infty \le j(j+1)/2 \le 4M^2$ follows by telescoping the Legendre derivative identity. Therefore all the summands' polynomials are at least $1/2$ on $1 - 1/(8M^2) \le t \le 1$. The degree multiplicity sum is $3M^2$, giving a kernel lower bound $(81/256)M^6$. On this band the spectral weight is at most $17M^2$. For $s \ge 0$ the squared dual H^s evaluation norm is at least $(81/256)17^{-s}M^{6-2s}$. Set $M = L+1$ for sharpness of the power if $s > 3$. For $s < 3$ high bands give unbounded evaluation (also for negative s, by comparison with H^0); for $s = 3$ each disjoint dyadic band contributes at least $81/(256 \cdot 17^3)$ to the squared dual norm at the same aligned point. The squared norms add over mutually H^3-orthogonal bands, so a union of h such bands has squared evaluation norm at least $81h/(256 \cdot 17^3)$, which diverges. This sharpness statement concerns angular evaluation, not I. □

6 Regularity-restricted stability

Corollary 6.1 (Operator-norm recovery)**.** *For $s > 3$, I extends uniquely to a bounded $I_s : X_s = L^2(d\Pi; H^s(Q)) \to \mathcal{Y}$. With IP_L denoting its previous extension,*

$$\|I_s - IP_L\|_{X_s \to \mathcal{Y}} \le \frac{8\sqrt{C_H}}{\sqrt{1 - 2^{6-2s}}} (L+1)^{3-s}. \tag{10}$$

The kernel of I_s is a proper closed subspace.

Proof. On $\mathcal{D}$, use $IB = I$, the constant part P_0Bf, and the tail estimate at $d = 0, L = 0$ to bound I by a constant times the X_s norm. Finite angular harmonics and smooth compact spectral tests are dense in X_s, so the extension is unique. For the difference use $I - IP_L = I(1 - P_L)B$ and (9). Nonzero constant tests in the small spectral box of Lemma 3.3 show the kernel is proper; continuity makes it closed. The finite-cutoff counterexamples belong to $\mathcal{D}$ and remain counterexamples in X_s. □

This result is a sufficient condition for the actual averaging map. The angular evaluation endpoint of Theorem 5.2 does not determine its minimal regularity. The divergent L^2 operator norm in Proposition 3.2 and the convergent Sobolev operator norm in (10) refer to different source spaces. In particular, for any fixed $R_0 < \infty$ the bound is uniform over pairs $u, v \in X_s$ with $I_s u = I_s v$ and $\|u - v\|_{X_s} \le R_0$: $\|IP_L(u - v)\|_{\mathcal{Y}} \le C_s R_0 (L + 1)^{3-s}$. This is a bound on the difference, not a descent statement for a finite projector or a bound on unrestricted auxiliary unit vectors.

Conditional first-graph estimates. To connect the cutoff to the specified Gram comparison we state separately the assumptions on the observables. Let O_a (a indexing six Gram entries) be the original scalar-label Gram operators, $O_{ij} = K_i \cdot K_j - J_i \cdot J_j$ for $i < j$ and $O_{ii} = \rho_i^2$ (with the source diagonal Duflo convention). Here J_i, K_i are the rotation and boost generators in the source convention. Assume they act on smooth tests, preserve the common domain and its fiberwise averaging null space, have no spectral derivatives, and commute with the diagonal rotation average B on that domain. The common domain is stable under B and includes all packet-weighted smooth, projected and complementary tests used below. They are symmetric on the mathematical quotient. These properties and the coefficient bounds below are the source hypotheses of this application, not derived from the cutoff alone. Fix a compact shape set and the same normalized spectral packet $F_{N,A}(\rho)$ supported where $\rho_i = N a_i(A) + r_i$, $|r_i| \le 1/4$, with $a_i(A)$ bounded. The packet satisfies $\int |F_{N,A}|^2 d\Pi = 1$. Assume the uniform original Gram bounds

$$\|O_a h\|_\infty \le C_O(N^2 \|h\|_{C^0} + N \|h\|_{C^1} + \|h\|_{C^2}), \qquad |A_a| \le C_A. \tag{11}$$

Set $D_a = O_a - N^2 A_a$. Norms and operators are interpreted on the original tests before I; on images, the descended action is used only on the common quotient domain. In particular no commutation of O_a with P_L is assumed.

Theorem 6.2 (Representative comparison)**.** *Suppose n is the difference of two smooth lifts with the same uncut image fiberwise on the packet. Assume their packet-weighted tests lie in the common smooth domain (including its spectral support); all projected and complementary tests used below are in that domain. Put $H_s = (\int |F_{N,A}|^2 \|Bn_\rho\|_{H^s}^2 \, d\Pi)^{1/2}$. Then, for $s > 3$,*

$$\|F_{N,A} I_\rho P_L n\|_{\mathcal{Y}} \le C_s H_s (L + 1)^{3-s}; \tag{12}$$

and under (11)*, for $s > 5$,*

$$\max_a \|F_{N,A} I_\rho D_a P_L n\|_{\mathcal{Y}} \le C_s' H_s \{N^2 (L + 1)^{3-s} + N(L + 1)^{4-s} + (L + 1)^{5-s}\}. \tag{13}$$

If $L = N^3$ for positive integers N and $H_s = O(N^s)$, the state order is $O(N^{9-2s})$ for $s > 3$ and the graph order is $O(N^{15-2s})$ for $s > 5$, with the graph hypotheses above. All moving-family constants must be uniform in N and in the chosen compact shape set. In particular the H^{10} hypothesis yields $O(N^{-11})$ and $O(N^{-5})$.

Proof. Let $Q_L = 1 - P_L$. The equality $I_\rho n = 0$ on the packet, null preservation, $IB = I$, and commutation of B with P_L and D_a give

$$I_\rho P_L n = -I_\rho Q_L B n, \qquad I_\rho D_a P_L n = -I_\rho D_a Q_L B n.$$

These are identities with the original order of operators, not $[D_a, P_L] = 0$. Apply Lemma 2.1, Theorem 5.2 at $d = 0, 1, 2$, and the spectral L^2 triangle inequality with the common packet. For explicit constants, take C^k to be the sum of the sup norms of full tensors through order k, and write $T_{s,d}$ for the coefficient in (9). Since $N \geq 1$, the coefficients of the $d = 0, 1, 2$ seminorms in the D_a bound are at most $(3C_O + C_A)N^2, 2C_O N, C_O$, respectively. We may therefore take $C_s = \sqrt{C_H}\, T_{s,0}$ and

$$C'_s = \sqrt{C_H} \max\{(3C_O + C_A)T_{s,0}, 2C_O T_{s,1}, C_O T_{s,2}\}.$$

For $s = 10$ one has $T_{10,0} < 9$, $T_{10,1} < 46$, $T_{10,2} < 257$, by squaring and comparing rational numbers. Substitution gives the stated powers. More generally, for fixed $\alpha \geq 0$, $L = \lceil N^\alpha \rceil$ and $H_s = O(N^p)$, the powers are $p + \alpha(3 - s)$ and $p + \alpha(3 - s) + 2\max(1, \alpha)$. □

For comparison, the unrestricted eight-dimensional addition-theorem bound gives $O(N^{-8})$ and $O(N^{-2})$ at $s = 10$, by replacing the exponent 3 by 4 in the tails. Thus the common rotation is a proof tool yielding a three-power improvement in the specified moving family. The sufficient graph-decay condition is $s > 15/2$; an existing $O(N^{3/2})$ centered residual upper bound is preserved at $s \geq 27/4$ when each lift satisfies the stated Sobolev bound. These are sufficient conditions, not optimal thresholds or a change in the original H^{10} input requirement.

7 Discussion and limitations

A fixed smooth null representative converges under truncation even though every finite cutoff fails to descend and the worst auxiliary norm diverges. The diagonal kernel estimate is uniform through collisions; its sharp exponent for angular evaluation need not be sharp for Haar averaging. We leave open the optimal growth of $\mathfrak{A}_L$, the matching fixed-boost decay rate, a numerical rapidity certified for all L, and the minimum averaging regularity. The first-graph result is conditional on the stated common-domain and coefficient hypotheses. No physical GFT carrier identification, general self-adjointness, joint spectral measure of six Gram operators, or priority claim is made.

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