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The PI = WI conjecture for connected complex reductive groups

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Abstract

The nonabelian Hodge homeomorphism compares the coarse moduli of polystable Higgs bundles on a smooth projective complex curve with its character variety. For every connected complex reductive structure group and every flat topological component, we prove that the normalized perverse Leray filtration of the Hitchin map on rational intersection cohomology agrees, in every degree, with the even and the adjacent odd steps of the Betti weight filtration. In higher genus a coroot translation on a moving full-flag family produces one lowering operator for both filtrations on the scalar intersection-complex summand. The main result of this paper was obtained using the Pharos system.

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1 Introduction

Let C be a smooth projective connected complex curve and G a connected complex reductive group. A semisimple representation of $\pi_1(C)$ in G determines, through its harmonic metric, a polystable

G-Higgs bundle. In each flat topological type this correspondence gives a homeomorphism $\Psi : M_{\mathrm{Dol}} \longrightarrow M_{\mathrm{B}}$, where M_{B} is the corresponding component of $\mathrm{Hom}(\pi_1(C), G)//G$ [Sim94]. Its two algebraic structures give different ways to filter rational intersection cohomology. On M_{Dol} the perverse Leray filtration P records the perverse truncations of the direct image of the intersection complex under the Hitchin map $h : M_{\mathrm{Dol}} \to \mathcal{A}_G$. On M_{B} the filtration W is the weight filtration of the algebraic mixed Hodge structure. The question is whether the harmonic homeomorphism identifies the subspaces singled out by these constructions.

De Cataldo, Hausel and Migliorini established the comparison for the smooth twisted rank-two moduli [dCHM10]. For general linear groups, Maulik and Shen proved the coprime-degree theorem by controlling the perversities of tautological classes. Hausel, Mellit, Minets and Schiffmann obtained it from Hecke operators [MS24, HMMS22]. Felisetti and Mauri proved the intersection-cohomology comparison for the untwisted general and special linear moduli that admit a symplectic resolution, including genus two in rank two [FM22]. The geometric formulation of Mauri, Mazzon and Stevenson makes both weight cutoffs explicit [MMS18]. The decomposition of the general linear Hitchin complex on the reduced spectral locus is known in arbitrary degree [MM22]. For elliptic curves the residual-torus descriptions in [FGN19] and the comparison on the neutral component in [BFG+26] describe how the Weyl group enters.

A fixed-determinant manuscript uses a moving flag to construct a common lowering operator in smooth, coprime degree and composite matrix rank [Ope26]. In a singular component, ordinary cup product still acts on intersection cohomology, but extracting this cohomology from a flagged space requires an intersection-complex summand. Its inclusion and projection must be compared at both endpoints of the Hodge family. For groups with several root lengths one must also determine the effect of a coroot translation on the surface class.

We prove the comparison for the untwisted coarse spaces in every flat topological type. If $n = \dim M_{\mathrm{Dol}}$, the normalization on that component uses the Hitchin defect $d_h = \dim(M_{\mathrm{Dol}} \times_{\mathcal{A}_G} M_{\mathrm{Dol}}) - n$ and truncates $Rh_*\mathrm{IC}_{M_{\mathrm{Dol}}}$ at perverse degree $k - d_h$.

Theorem 1.1. *Let C be a smooth projective connected complex curve and let G be a connected complex reductive group. In each flat topological type, for every $m, k \in \mathbb{Z}$,*

$$P_k\mathrm{IH}^m(M_{\mathrm{Dol}}) = \Psi^* W_{2k}\mathrm{IH}^m(M_{\mathrm{B}}) = \Psi^* W_{2k+1}\mathrm{IH}^m(M_{\mathrm{B}}).$$

Theorem 1.1 is Theorem 10.2, proved in Section 10. The component convention already matters for PGL_2 on an elliptic curve. Its two flat components have dimensions two and zero and defects one and zero. With their respective normalizations, their degree-zero units both enter the perverse filtration at zero. The equality of the two Betti steps in the theorem also shows that all odd weight gradeds vanish.

In genus at least two the theorem has a formulation on a projective compactification of the character variety. Write $\dim M_{\mathrm{B}} = 2d$ and $d_h = d$, fix an affine open immersion $j : M_{\mathrm{B}} \hookrightarrow \overline{B}$ with $\overline{B}$ projective, and let $E = \mathrm{IC}_{M_{\mathrm{B}}}(d)$ be the weight-zero intersection Hodge module. Under the harmonic homeomorphism, the degree-m Hitchin step with index k, after the Tate twist by d, is the image of

$$H^{m-2d}\big(\overline{B}, W_{2k-m}(j_*E)\big) \longrightarrow \mathrm{IH}^m(M_{\mathrm{B}})(d).$$

The next weight cutoff has the same image. The formula is independent of the compactification by Corollary 10.3.

For the proof in higher genus we work with a simply connected simple factor and fix the central value of the product of commutators. We introduce a moving marked point with a full

Borel flag. For generic regular puncture monodromy, a root-group stratification reduces the weight calculation to tori. Mellit's intersection pairing for marked surfaces [Mel25] identifies the restriction of the symplectic class on these tori. At the central value, full flags over the regularly stable locus give a multiplicity-one scalar summand of a projective semismall direct image. Its two maps compare the Higgs and Betti realizations, including at singular points.

An integral coroot acts on the generic flagged space by changing the logarithmic lattice. This change preserves the Hitchin coefficients and acts on the Betti realization by monodromy of the ordered residues. We square the logarithm of a unipotent power of the monodromy on the flagged space, project to the scalar summand and integrate over the marking curve. After normalization the resulting operator f lowers P by two and W by four. With e the adjoint determinant cup, integration on the full flag computes $[[f,e],e] = -2e$. The relative and weight Lefschetz theorems then identify both filtrations with the same sums of eigenspaces of $[e,f]$.

Sections 3 and 4 treat the regular puncture. Sections 5 and 6 construct the scalar summand and establish its Lefschetz and rigidity properties. Sections 7 and 8 construct the common operator. Section 9 treats low genus and finite central descent. Section 10 combines these results and proves the compactification formula.

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2 Notation and conventions

Set-up 2.1. Let C be a smooth projective connected complex curve of genus g and let G be a connected complex reductive group. In a fixed flat topological type write M_{B} for the corresponding component of $\mathrm{Hom}(\pi_1 C, G)//G$, and M_{Dol} for the coarse space of polystable G-Higgs bundles in the matched type. We write $\Psi : M_{\mathrm{Dol}} \longrightarrow M_{\mathrm{B}}$ for the harmonic homeomorphism and $h : M_{\mathrm{Dol}} \longrightarrow \mathcal{A}_G$ for the Hitchin map. All cohomology has rational coefficients unless specified otherwise.

For an irreducible equidimensional complex variety X of dimension n, write IC_X for its middle-perversity intersection complex, normalized to $\mathbb{Q}[n]$ on the regular locus. Put $I_X = \mathrm{IC}_X[-n]$, so $\mathrm{IH}^m(X) = H^m(X, I_X)$. For the Hitchin map on one component of dimension n, put $d_h = \dim(M_{\mathrm{Dol}} \times_{\mathcal{A}_G} M_{\mathrm{Dol}}) - n$. Our normalization is

$$P_k\mathrm{IH}^m(M_{\mathrm{Dol}}) = \mathrm{im}\big[H^{m-n}\big(\mathcal{A}_G, {}^{\mathrm{p}}\tau_{\le k-d_h} Rh_*\mathrm{IC}_{M_{\mathrm{Dol}}}\big) \longrightarrow \mathrm{IH}^m(M_{\mathrm{Dol}})\big]\,.$$

On $\mathrm{IH}^m(M_{\mathrm{B}})$, W denotes the increasing weight filtration of its natural mixed Hodge structure. Our Tate convention is that $\mathbb{Q}(-1)$ has weight two. An isomorphism of stratified spaces induces the usual topological identification of the intersection complexes. Comparisons of filtrations in this paper use Ψ^*.

In the sections concerning a simple factor, G is simply connected and simple, $T \subset B_+ \subset G$ is a maximal torus in a Borel subgroup, $r = \dim T$, $b = \dim(G/B_+)$, $\mathcal{W} = N_G(T)/T$, and

$R = (g-1)\dim G$. Fix a point $p \in C$. For a central element $z \in G$ of order n_z, let B_z be the quotient by simultaneous conjugation of the tuples in G^{2g} satisfying $\prod_{i=1}^{g}[A_i, B_i] = z$. Let X_z be its Higgs partner on the gerbe of n_zth roots of $\mathcal{O}_C(p)$, with the central inertia determined by z. Choose the generator by the positive peripheral-loop convention. We use the same notation for the harmonic homeomorphism and the Hitchin map on this pair. Central twists act through $H^1(C, Z(G))$. We take complex coefficients when separating individual characters. For regular semisimple $t \in T$, let $D_t \subset G^{2g}$ be the locus of tuples with commutator product t and let $Y_t = D_t//T$ be its ordered punctured character variety. The two-parameter notation $Y_{\lambda,u}$ below denotes a space of flagged logarithmic connections.

3 Root-group coordinates at a regular puncture

We use the fixed pair $T \subset B_+$ and write U for the unipotent radical of B_+. Let $K(\mathbf{A}, \mathbf{B}) = \prod_{i=1}^{g}[A_i, B_i]$, with $[A, B] = ABA^{-1}B^{-1}$. We compute the cohomology of Y_t by adjoining one variable $u \in U$. This lets us express the boundary equation as $K(\mathbf{A}, \mathbf{B})u = t$ and read its torus component one simple reflection at a time. We use the following root subgroup and Bruhat coordinates [Dem70, Exp. XXII, 5.5.1–5.5.2, 5.6.5, 5.6.8 and 5.7.3–5.7.4].

Proposition 3.1. *In a split connected reductive group, products of the root subgroups give ordered affine coordinates on a unipotent radical. Commutators of root subgroups for nonopposite roots involve only positive integral combinations of the two roots. Bruhat cells admit unipotent and torus coordinates relative to representatives in the normalizer.*

Let U^- be the opposite unipotent radical. For $q \in \mathcal{W}$ and a representative $\dot{q}$, put

$$U_{\mathrm{L}}(q) = U \cap \dot{q}U^-\dot{q}^{-1}, \qquad U_{\mathrm{R}}(q) = U \cap \dot{q}^{-1}U^-\dot{q}.$$

In the cell $B_+\dot{q}B_+$ we can use either of the unique factorizations

$$U_{\mathrm{L}}(q)\,\dot{q}\,T\,U, \qquad U\,\dot{q}\,T\,U_{\mathrm{R}}(q).$$

Both record the same torus element, which we call the *Bruhat diagonal*. The two inversion subgroups have dimension $\ell(q)$, the Coxeter length.

Definition 3.2. Fix a pinning with root maps $x_\alpha : \mathbb{A}^1 \to U_\alpha$. For a simple reflection $s = s_\alpha$ let $\dot{s}$ be the image of $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ under the pinned homomorphism $\mathrm{SL}_2 \to G$, and put $f_s(x) = \dot{s}x_\alpha(x)$. For each $q \in \mathcal{W}$ choose a reduced expression $q = s_1 \cdots s_k$. Its *word coordinates* are given by

$$f_q(x_1, \ldots, x_k) = f_{s_1}(x_1) \cdots f_{s_k}(x_k), \qquad \dot{q} = \dot{s}_1 \cdots \dot{s}_k.$$

For $q = 1$ both products are the identity.

Moving all the simple lifts to the left in f_q gives the root factors for $(s_{j+1} \cdots s_k)^{-1}\alpha_j$, where $s_j = s_{\alpha_j}$. These are the positive roots carried to negative roots by q, each occurring once. Thus Proposition 3.1 identifies f_q with an affine chart on $\dot{q}U_{\mathrm{R}}(q)$.

Lemma 3.3. *Let G be simply connected and simple and let $g \geq 1$. Fix $w_i, v_i \in \mathcal{W}$ for $1 \leq i \leq g$. Concatenate the words of Definition 3.2 for $w_i, v_i, w_i^{-1}, v_i^{-1}$ in that order over the handles, and write $s_1, \ldots, s_m$ for their letters, with $m = 2\sum_i(\ell(w_i) + \ell(v_i))$. On the locus $A_i \in B_+\dot{w}_iB_+$, $B_i \in B_+\dot{v}_iB_+$, $u \in U$, there are algebraic product coordinates*

$$(u_*, \mathbf{x}, \mathbf{h}, \xi) \in U \times \mathbb{A}^m \times T^{2g} \times \mathbb{A}^{2gb-m/2}$$

and a translate $\eta : T^{2g} \to T$ of a homomorphism such that

$$K(\mathbf{A}, \mathbf{B})u = u_* f_{s_1}(x_1) \cdots f_{s_m}(x_m)\eta(\mathbf{h}).$$

Simultaneous conjugation by T conjugates u_*, acts by characters on the entries of $\mathbf{x}, \xi$, and translates $\mathbf{h}$ by a homomorphism.

The free U at the left supplies the extra variable for the preceding handle. The affine coordinates ξ record the fibers of the two interior multiplications on each handle.

Proof. We change the variables of one commutator, retain the coordinates of its four Bruhat factors, and then iterate from the last handle.

Independent unipotent variables. Write a single handle of labels w, v in opposite choices of chart,

$$A = n_{\mathrm{L}} \dot{w} h_1 u_1, \qquad B = u_2 \dot{v} h_2 n_{\mathrm{R}}, \qquad n_{\mathrm{L}} \in U_{\mathrm{L}}(w), \quad n_{\mathrm{R}} \in U_{\mathrm{R}}(v),$$

with $h_1, h_2 \in T$ and $u_1, u_2 \in U$. Set $u' = u_1 u_2$, $u'' = n_{\mathrm{R}} u_1^{-1}$, and $u_0 = u_2^{-1} u$. For each n_{R} these are three independent U-coordinates, with inverse

$$u_1 = (u'')^{-1} n_{\mathrm{R}}, \qquad u_2 = u_1^{-1} u', \qquad u = u_2 u_0.$$

In these coordinates the product $[A, B]u$ is

$$n_{\mathrm{L}} \dot{w} h_1 u' \dot{v} h_2 u'' h_1^{-1} \dot{w}^{-1} n_{\mathrm{L}}^{-1} n_{\mathrm{R}}^{-1} h_2^{-1} \dot{v}^{-1} u_0.$$

For each label q write $(\dot{q})^{-1} = \kappa_q \dot{q^{-1}}$ with $\kappa_q \in T$. Since $U_{\mathrm{R}}(v) = U_{\mathrm{L}}(v^{-1})$, the two charts of the same Bruhat cell give the unique change

$$n_{\mathrm{R}}^{-1} h_2^{-1} \dot{v}^{-1} u_0 = u_3 h_2^{-1} \kappa_v f_{v^{-1}}(\mathbf{x}_{v^{-1}}).$$

For fixed h_2 this exchanges (n_{R}, u_0) with the free pair $(u_3, \mathbf{x}_{v^{-1}})$. The change and its inverse are regular in h_2. We use $u_4 = n_{\mathrm{L}}^{-1} u_3$ as the free variable. Its inverse is $u_3 = n_{\mathrm{L}} u_4$ since we retained n_{L}.

The four word charts. For $d_0 \in T$ consider a product $u_{\mathrm{l}} d_0 \dot{q} u_{\mathrm{r}}$ with both unipotents free. In root coordinates factor $u_{\mathrm{l}} = nj$ with $n \in U_{\mathrm{L}}(q)$ and $j \in U \cap \dot{q} U \dot{q}^{-1}$. Keep j and multiply $(d_0 \dot{q})^{-1} j (d_0 \dot{q})$ into u_{r}. The latter change is a translation with inverse for each j, d_0. The remaining pair consists of the left Bruhat coordinates at fixed diagonal and hence can be exchanged for the right ones in $U d_0 f_q$. This leaves $\dim(U \cap \dot{q} U \dot{q}^{-1}) = b - \ell(q)$ independent affine coordinates out of the product.

Apply this change first to $u'' h_1^{-1} \dot{w}^{-1} u_4$ and then to $u' \dot{v} h_2 U$, using the free left U from the first change on the right of the second. The two retained factors have dimensions $b - \ell(w)$ and $b - \ell(v)$. The initial $n_{\mathrm{L}} \dot{w} h_1 U$ already uses $U_{\mathrm{L}}(w)$ on the left. Exchanging its two charts gives the outgoing u_* and the coordinates of f_w. With the arguments of the four words suppressed, the product is now

$$u_* f_w \, h_1 f_v \, h_2 \, h_1^{-1} \kappa_w f_{w^{-1}} \, h_2^{-1} \kappa_v f_{v^{-1}}.$$

The four sets of word coordinates, both handle tori, the outgoing U, and the two retained affine factors vary independently.

Iteration and equivariance. For $\delta_0 \in T$ and $s = s_\alpha$ we have

$$\delta_0 f_s(x) = f_s(\alpha(s^{-1}(\delta_0))x) \, s^{-1}(\delta_0).$$

We move all torus elements to the right by this rule. Reusing the outgoing U on each preceding handle gives the stated $\eta(\mathbf{h})$. The two retained factors per handle account for $2gb - m/2$ coordinates.

To track conjugation by $k \in T$, consider a Bruhat factor transformed by left and right multipliers $k_{\mathrm{l}}, k_{\mathrm{r}}^{-1} \in T$. In either chart its left and right unipotents are conjugated by k_{l} and k_{r}, respectively, and its diagonal is multiplied by $q^{-1}(k_{\mathrm{l}})k_{\mathrm{r}}^{-1}$. Initially $k_{\mathrm{l}} = k_{\mathrm{r}} = k$. At each change of chart the intermediate multipliers are Weyl transforms or products of the previous ones, hence depend homomorphically on k. Adjacent unipotents that we multiply carry the same conjugation. Splitting nj commutes with this conjugation by uniqueness of the root coordinates. Moving a variable torus element past a root adds the character of its translation to the weight of that root coordinate. This gives the claimed actions on all the retained coordinates and on the word. □

Lemma 3.4. *Let G be simply connected and simple and let $g \geq 1$. For each choice of handle labels in Lemma 3.3, read its simple word of length m from right to left, starting at $q = 1$ in $\mathcal{W}$. At a letter s pass to sq, with the additional choice to stay at q if $\ell(sq) < \ell(q)$. For each path ending at 1, let d count its ascents and j its stays, and put*

$$a = 2gb - m/2 + d, \qquad E = T^{2g} \times (\mathbb{C}^{\times})^{j}.$$

Then the locus $K(\mathbf{A}, \mathbf{B})u \in T$ in $G^{2g} \times U$ has a finite T-invariant closed filtration with one difference $\mathbb{A}^a \times E$ for each such choice of labels and path. On this difference the map to T is $(\xi', e) \mapsto \tau\Phi(e)$ for a constant $\tau \in T$ and a homomorphism $\Phi : E \to T$. The conjugation action is diagonal on $\mathbb{A}^a$ and translates E through a homomorphism $\rho : T \to (\ker \Phi)^{\circ}$.

Only a stay contributes a new invertible root coordinate to the Bruhat diagonal.

Proof. We carry a Bruhat factorization of the processed suffix, then impose the torus condition on the last such factorization.

The suffix recurrence. Write the current suffix as $u_{\mathrm{l}}\dot{q}\delta u_{\mathrm{r}}$, starting with $\delta = \eta(\mathbf{h})$, $q = 1$, and both unipotents equal to 1. For the next letter $f_s(x)$, let $s = s_\alpha$ and write $U^{(\alpha)}$ for the subgroup generated by the positive root groups other than U_α. Proposition 3.1 gives the simple-root projection $\pi_\alpha : U \to U/U^{(\alpha)} \simeq \mathbb{G}_a$, normalized by $\pi_\alpha x_\alpha = \mathrm{id}$. Thus

$$x_\alpha(x)u_{\mathrm{l}} = n' x_\alpha(y), \qquad n' \in U^{(\alpha)}, \qquad y = x + \pi_\alpha(u_{\mathrm{l}}).$$

The reflection s permutes the other positive roots, so n' moves across $\dot{s}$ into the left unipotent. The translation $x \mapsto y$ has a regular inverse on the current piece since u_{l} uses only suffix coordinates. We carry the resulting unipotents into the next step.

Put $\gamma_{s,q} = (\dot{s}\dot{q})^{-1}\dot{s}\dot{q} \in T$. If $q^{-1}\alpha > 0$, the root $x_\alpha(y)$ moves across $\dot{q}\delta$ into the right unipotent. This is an ascent to sq with diagonal $\gamma_{s,q}\delta$ and a free affine coordinate y. If $q^{-1}\alpha < 0$ and $y = 0$, the product of the two lifts gives a descent to sq with the same diagonal rule. If $q^{-1}\alpha < 0$ and $y \neq 0$, multiplication in the pinned SL_2 gives

$$\dot{s}x_\alpha(y) = x_\alpha(-y^{-1})\,\alpha^{\vee}(y^{-1})\,x_{-\alpha}(y^{-1}).$$

Now $-q^{-1}\alpha > 0$, so the last root moves to the right unipotent. The Weyl element stays at q, the coordinate y is invertible, and the diagonal becomes $(q^{-1}\alpha^{\vee})(y^{-1})\delta$. In all three cases the factors are regular on the indicated piece. For equivariance, let k_{l} be the torus multiplier entering $f_s(x)$ from the left in Lemma 3.3 and put $k' = s^{-1}(k_{\mathrm{l}})$. Then $k_{\mathrm{l}} f_s(x)(k')^{-1} = f_s(\alpha(k')x)$, and the

suffix unipotent u_{l} is conjugated by the same k'. Consequently y has character $\alpha(k')$. In the stay case y^{-1} has the inverse character, as required by the two root factors in the rank-one identity. Moving these factors across the lifts thus preserves the multiplier rule for the next suffix.

The terminal equation and the filtration. The value can lie in T only for $q = 1$. There the value has diagonal δ, and its unipotent equation uniquely fixes the still free left factor of Lemma 3.3 as

$$u_* = (u_{\mathrm{l}}\delta u_{\mathrm{r}}\delta^{-1})^{-1}.$$

The diagonal rules give $\delta = \tau\Phi(e)$ in the handle tori and the j invertible coordinates. Each ascent supplies one affine coordinate in addition to the $2gb - m/2$ retained on the handles. All affine coordinates carry characters and the action on E is by translations. Since conjugation by T fixes the terminal diagonal, these translations lie in $\ker\Phi$, and their image is connected.

Order the initial products of Bruhat cells by a linear extension of the product Bruhat order, taking smaller cells first. The successive unions are closed. On each piece with a choice at $q^{-1}\alpha < 0$, the test $y = 0$ is closed and $y \neq 0$ is its open complement. Adjoining the closed child to the previous closed union again gives a closed union. Inserting successive refinements in this order, then restricting to $K(\mathbf{A}, \mathbf{B})u \in T$, gives the required filtration. □

A mixed Hodge structure is Hodge–Tate if its odd weight gradeds vanish and its weight $2k$ graded has type (k, k) for each k.

Theorem 3.5. *Let $g \geq 1$, let G be simply connected and simple, and let $t \in T$ be regular semisimple and lie outside $[P, P]$ for every proper parabolic P containing T. Let U be the unipotent radical of B_+. The variety $D_t//T$ has Hodge–Tate ordinary and compactly supported cohomology. On a nonempty open in T obtained by removing finitely many translates of proper subtori, the ordinary and compactly supported direct images of this family are local systems with fiberwise base change. On that open $(D_t \times U)//T$ is cohomologically equivalent to $D_t//T$ and admits a finite closed filtration whose successive differences are disjoint unions of*

$$(\mathbb{C}^\times)^c \times (\mathbb{A}^a/F), \qquad a + c/2 = 2gb + (g-1)r,$$

where F is finite and acts diagonally on $\mathbb{A}^a$.

The extra exclusions for the family come from the terminal maps $\tau\Phi$ in Lemma 3.4 whose images have dimension less than r.

Proof. Write T° for the locus of parameters satisfying the hypotheses on t. We identify the auxiliary fiber with the level of the word, take the quotient of its terminal pieces, and use the resulting filtration and duality to compute the two direct images.

The auxiliary level. Put $\widetilde{Y}_t = (D_t \times U)//T$, with simultaneous conjugation on all factors. In ordered root coordinates $U \simeq \mathbb{A}^b$ and T acts linearly. Scalar multiplication in these coordinates contracts $\widetilde{Y}_t$ algebraically onto the unit section of $\widetilde{Y}_t \to Y_t$. The projection and section therefore induce inverse isomorphisms on ordinary cohomology as mixed Hodge structures. The same contraction works over T°.

For $(\mathbf{A}, \mathbf{B}) \in D_t$ and $n \in U$, conjugate all the entries by n and append $u = nt^{-1}n^{-1}t$. The new commutator product is ntn^{-1}, so its product with u is t. In order of increasing root height, the α-coordinate of u is $(1 - \alpha(t)^{-1})$ times that of n, plus a polynomial in lower coordinates. This follows from the commutator formula of Proposition 3.1. Regularity of t makes the diagonal coefficients invertible. Solving in that order recovers n from u, and conjugation by n^{-1} recovers

the original tuple. This is a T-equivariant isomorphism from $D_t \times U$ to $\{K(\mathbf{A},\mathbf{B})u = t\}$, algebraic also in $t \in T^\circ$.

Closed orbits and the quotient of a piece. If a nonzero cocharacter λ of T has a conjugation limit on a tuple in D_t, every entry belongs to its parabolic

$$P_\lambda = \{x \in G \mid \lim_{\epsilon \to 0} \lambda(\epsilon) x \lambda(\epsilon)^{-1} \text{ exists in } G\}.$$

This is a proper parabolic containing T, since G has finite center. The commutator product would then put t in $[P_\lambda, P_\lambda]$. Thus no such cocharacter has a limit, and the T-stabilizer is finite. To see that the orbit is closed, embed the affine tuple space equivariantly in a T-module. The weights of the coordinates nonzero at the tuple span the whole real character space as a cone, since a separating integral cocharacter would give a limit. There is consequently a relation among these weights with all coefficients positive integers. The corresponding invariant monomial keeps all those coordinates nonzero on the orbit closure. In that coordinate torus the orbit is a translate of a subtorus, hence closed. The same limit test applies on $D_t \times U$ by projection. Both affine quotients are geometric. Invariant closed sets have closed images, and the restrictions to their invariant differences are geometric quotients. These assertions also hold over T°, since the action preserves each parameter.

Restrict the filtration of Lemma 3.4 to t, and fix a nonempty piece with equation $\tau\Phi(e) = t$. Each connected component of its torus fiber is a torsor for $J = (\ker \Phi)^\circ$. The action translates it by $\rho : T \to J$. Its kernel F is finite, since F is exactly the stabilizer at the origin of the affine coordinates over any point of this torsor. Put $S = \rho(T)$. Choose an origin in the torsor and split $J \to J/S$ on the cocharacter lattices, so $J \simeq S \times (J/S)$. Dividing first by F leaves the factor $\mathbb{A}^a/F$ with its induced T/F-action. The residual torus $T/F \simeq S$ translates the S-coordinate freely, so setting this coordinate equal to 1 identifies the quotient with

$$(J/S) \times (\mathbb{A}^a/F).$$

There are finitely many components and T preserves each one. Geometricity now descends the closed filtration to $\widetilde{Y}_t$.

Smooth levels and orientations. We verify that the quotients and their fibers have the rational orientations needed for duality. For one handle put $J_A = ABA^{-1}$ and $K_1 = [A, B]$. With $X_A = dA\, A^{-1}$, $X_B = dB\, B^{-1}$, differentiation gives

$$dK_1\, K_1^{-1} = (1 - \mathrm{Ad}_{J_A})X_A + (\mathrm{Ad}_A - \mathrm{Ad}_{K_1})X_B.$$

Use a nondegenerate invariant form on $\mathrm{Lie}\, G$. If Z annihilates both summands, then J_A fixes Z and $\mathrm{Ad}_{A^{-1}} Z = \mathrm{Ad}_{K_1^{-1}} Z = \mathrm{Ad}_B Z$. The left side is fixed by B because J_A fixes Z. Hence B fixes Z, and then so does A. In the differential of a product the test vector for the next handle is transported by the inverse preceding commutator. That commutator fixes the test vector, so induction gives the common fixed Lie algebra as the annihilator of $dK\, K^{-1}$. On D_t this algebra centralizes t, hence lies in $\mathrm{Lie}\, T$, and the finite T-stabilizer makes it zero. Thus K is submersive there and $K^{-1}(T^\circ) \to T^\circ$ is smooth. The same holds after taking its product with U.

At any tuple take r coordinate functions nonzero there with independent weights $\chi_1, \dots, \chi_r$. The homomorphism $(\chi_1, \dots, \chi_r) : T \to (\mathbb{C}^\times)^r$ is an isogeny. On the common nonvanishing locus, denote the tuple of functions by f. In the pullback of the isogeny, a pair (k, x) with $(\chi_1(k), \dots, \chi_r(k)) = f(x)$ maps to $(k, k^{-1}x)$. This identifies the pullback with T times the slice $f^{-1}(1)$. The geometric quotient on this open is the slice modulo the finite kernel of the isogeny.

The slice is smooth over T°, since its product with T is an étale cover of the original smooth family. This construction applies to both families. Locally the quotient of a smooth complex space by a finite group has link a sphere modulo the point stabilizer. Finite averaging gives the invariant cohomology of the sphere, and the complex action preserves its orientation. Thus the total spaces and fibers are oriented rational homology manifolds. In a slice chart the map to T° is a submersion and the stabilizer acts trivially on the base. Keeping the base coordinates and averaging complementary local coordinates equivariantly gives the usual complex orientation on each fiber by restriction of the relative orientation.

Cohomology of an individual fiber. The finite quotient map and rational averaging identify $H_c^*(\mathbb{A}^a/F)$ with $H_c^*(\mathbb{A}^a)^F$, which is $\mathbb{Q}(-a)$ in degree $2a$ and zero otherwise. A torus has Hodge–Tate compactly supported cohomology by localization for $\mathbb{C}^\times \subset \mathbb{A}^1$ and the Künneth formula. The compactly supported localization sequences of the closed filtration give the same property for $\widetilde{Y}_t$, since Hodge–Tate mixed Hodge structures are closed under subquotients and extensions. Poincaré duality gives Hodge–Tate ordinary cohomology on $\widetilde{Y}_t$. The contraction transfers it to Y_t, and duality there gives Hodge–Tate compactly supported cohomology. This proves both assertions for every $t \in T^\circ$.

The open with a common dimension formula. The complement of T° consists of the root kernels $\{\alpha(t) = 1\}$ and the sets $T \cap [P, P]$ for the maximal proper parabolics containing T. These parabolics are the finitely many Weyl translates of the standard maximal parabolics. In each case $T \cap [P, P]$ is the coroot torus of its Levi containing T, of rank $r - 1$. Index the finitely many terminal pieces of Lemma 3.4 by ν, with diagonals $\tau_\nu \Phi_\nu$, and take

$$V = T^\circ \setminus \bigcup_{\dim \operatorname{im} \Phi_\nu < r} \tau_\nu \operatorname{im} \Phi_\nu.$$

Every excluded set is a finite union of translates of proper subtori. Their complement is nonempty because T is irreducible. On V every contributing Φ_ν has rank r. A terminal path with d ascents also has d descents since it begins and ends in length zero. It has $m - 2d$ stays. Consequently

$$a = 2gb - m/2 + d, \qquad c = \dim(J/S) = 2gr + (m - 2d) - r - r.$$

The first subtraction of r fixes the diagonal and the second divides by the orbit torus S. Since m is even, c is even, and these formulas give $a + c/2 = 2gb + (g - 1)r$ on every component over V.

Compact supports over the parameter torus. For every full-rank $\Phi_\nu : E_\nu \to T$, its map on cocharacter lattices has finite cokernel. Clearing denominators in rational right inverses gives one integer $N > 0$ and homomorphisms $\sigma_\nu : T \to E_\nu$ with $\Phi_\nu \sigma_\nu = [N]$, the N-th power map. Choose $e_\nu \in E_\nu$ with $\Phi_\nu(e_\nu) = \tau_\nu^{-1}$. On the finite étale cover $[N]^{-1}V \to V$, the parameter $t = s_0^N$ has torus fiber

$$e_\nu \sigma_\nu(s_0) \ker \Phi_\nu.$$

Translation by $e_\nu \sigma_\nu(s_0)$ commutes with $\rho(T)$. The affine characters and F are fixed on the piece, so this trivializes each quotient stratum over the cover, including all its components.

Write $\widetilde{\pi}$ and π for the families over V with fibers $\widetilde{Y}_t$ and Y_t, respectively. For a product stratum the compactly supported direct images are constant on the cover. Localization for the relative closed filtration gives locally constant finite-rank $R^i\widetilde{\pi}_!\mathbb{Q}$ there, since kernels, cokernels and extensions of local systems are local systems. This property descends to V. Compactly supported base change identifies their stalks with $H_c^i(\widetilde{Y}_t)$.

Ordinary base change. The relative complex dimension of $\widetilde{\pi}$ is $n_{\mathrm{aux}} = (2g-1)\dim G + b - r$. The orientation shifts on source and base give by Verdier duality

$$R\widetilde{\pi}_* \mathbb{Q} \simeq R\mathcal{H}om(R\widetilde{\pi}_! \mathbb{Q}, \mathbb{Q}_V)[-2n_{\mathrm{aux}}].$$

Since the complex on the compactly supported side has finite-rank local systems as cohomology sheaves, forming its dual commutes with taking a stalk. In degree i this identifies the stalk on the left with $H_c^{2n_{\mathrm{aux}}-i}(\widetilde{Y}_t)^\vee$. The relative orientation trace restricts in the slice charts to the trace on $\widetilde{Y}_t$. Consequently the ordinary restriction to $H^i(\widetilde{Y}_t)$ is the fiberwise Poincaré duality isomorphism under this identification. This gives both local constancy and ordinary base change for $\widetilde{\pi}$.

The projection and the unit section are inverse on ordinary cohomology over every open subset of V and on each fiber, by the same scalar contraction. They identify $R^i\pi_*\mathbb{Q}$ and its fiber restriction with the corresponding local system and restriction for $\widetilde{\pi}$. Duality for the oriented total space of π gives local constancy of $R^i\pi_!\mathbb{Q}$ as well, with stalks $H_c^i(Y_t)$ by compactly supported base change. □

4 The symplectic class on the punctured space

Let U be the unipotent radical of B_+. Write T_{gen} for the nonempty open of Theorem 3.5 on which the cell dimension formula holds. It lies in the regular locus avoiding the derived groups of proper parabolics containing T. For $t \in T_{\mathrm{gen}}$ let $\pi_t : \widetilde{Y}_t = (D_t \times U)//T \longrightarrow Y_t$ be the projection. We compute the surface class on the torus factors of the strata of $\widetilde{Y}_t$. Their common weight center will then give the Lefschetz isomorphisms on Y_t.

Remove an open disk about p from C. For the adjoint local system on this surface, parabolic first cohomology is the kernel of restriction to the boundary.

Proposition 4.1. *For an oriented surface with prescribed boundary conjugacy classes and a nondegenerate invariant symmetric form, the corrected closed surface form descends on parabolic first cohomology to the cup pairing evaluated on the relative fundamental class. Pullback to fixed framed boundary holonomy annihilates the conjugacy-class correction.*

On D_t the boundary generator has fixed value t^{-1}, so the full surface relator has value one. We write ω for the resulting class on Y_t with coefficient pairing the adjoint trace form [GHJ+97, Theorems 7.1 and 8.3 and Key Lemma 8.4].

In the marked permutation model, an oriented surface Σ has alternating black and white marks on its boundary. The black marks form a labelled set A. The torus of the model has cocharacter lattice $H_1(\Sigma, A; \mathbb{Z})$, and its moment is induced by the boundary of a relative cycle. Matching labelled boundary intervals specifies the gluing, called convolution on the torus data. Reduction at a fixed moment uses the subtorus preserving that value.

Proposition 4.2. *In the marked permutation surface model, concatenating handle words gives convolution of the associated torus data. At a moment in the image the alternating pairing on the kernel of the moment differential modulo the infinitesimal torus action is the intersection form of the capped surface and is nondegenerate.*

We use the marked disks for transpositions and the marked covers of a once-punctured torus for handles. The surface in the reduced pairing is obtained by gluing these pieces and capping each boundary circle by a disk [Mel25, Example 6.4 and Propositions 6.5, 6.6 and 7.4].

Proposition 4.3. *Morphisms of rational mixed Hodge structures are strict for the weight filtration, and hence taking a weight graded preserves exact sequences.*

In particular we can apply a fixed weight graded to the compact-support localization sequence at each closed term of the filtration [Del71, Theorem 2.3.5(iii)].

Lemma 4.4. *Let G be simply connected and simple, let $g \geq 1$, and let $t \in T_{\mathrm{gen}}$. Let $S = (\mathbb{C}^\times)^c \times (\mathbb{A}^a/F)$ be a connected piece in the filtration of $\widetilde{Y}_t$ in Theorem 3.5, where F is a finite group acting diagonally on $\mathbb{A}^a$. Then $\pi_t^*\omega|_S$ is pulled back from the class of a nondegenerate translation-invariant two-form on $(\mathbb{C}^\times)^c$.*

Proof. We compute the surface form on a handle, follow its torus factors through the Bruhat reduction, and identify the resulting pairing inside the intersection form of a marked surface.

The surface form on the enlarged level. For group-valued maps f_i, write

$$(f_1 \mid f_2) = \langle f_1^{-1} df_1 \wedge df_2\, f_2^{-1} \rangle, \qquad (f_1 \mid \cdots \mid f_N) = \sum_{i=1}^{N-1} (f_1 \cdots f_i \mid f_{i+1}),$$

where $\langle\, , \rangle$ is the adjoint trace form. In right trivializations $X_i = df_i\, f_i^{-1}$, the cup cochain on a triangle with successive edges f_1, f_2 pairs X_1 with $\mathrm{Ad}_{f_1} X_2$. Its antisymmetrization is $(f_1 \mid f_2)$ up to the fixed normalization of the alternating product. Triangulating the fundamental polygon from one vertex therefore gives the word form

$$(A_1 \mid B_1 \mid A_1^{-1} \mid B_1^{-1} \mid \cdots \mid A_g \mid B_g \mid A_g^{-1} \mid B_g^{-1}).$$

The constant boundary edge contributes zero. Proposition 4.1 identifies this form on D_t with the pullback of the surface form, up to the same nonzero scalar. We keep this normalization throughout the calculation.

The product rule and invariance give

$$(f_1 \mid f_2) + (f_1 f_2 \mid f_3) = (f_2 \mid f_3) + (f_1 \mid f_2 f_3).$$

Thus splitting a letter into factors adds its internal word form. The pair $(f \mid f^{-1})$ vanishes by symmetry, so reversing a word and inverting all its letters negates its form. A letter in U and one in B_+ pair to zero in either order, since a positive root space pairs only with the opposite root space.

Put $K = \prod_i [A_i, B_i]$. In the coordinates of Theorem 3.5, simultaneous conjugation of a tuple in D_t by $n \in U$ changes its product to ntn^{-1}. Appending the unique $u \in U$ with $(ntn^{-1})u = t$ gives the enlarged level $Ku = t$ in the new handle coordinates. To compute its word form, expand each conjugated letter as $n \mid f \mid n^{-1}$. Its internal form cancels that of the inverse letter. Adjacent $n^{-1} \mid n$ cancel, and the remaining outer term $(n \mid t \mid n^{-1})$ vanishes in B_+. The appended letter contributes $(K \mid u) = 0$ since $K \in B_+$. Hence the enlarged word computes the pullback of the same class on every stratum.

The torus factors of a handle. Use the pinned root maps x_α and simple lifts $\dot{s}_\alpha$ in the proof of Theorem 3.5. Its word f_w for a reduced expression of $w \in \mathcal{W}$ is a product of simple letters $\dot{s}_\alpha x_\alpha(\zeta)$ with independent root coordinates. Moving the constant lifts to the left puts the remaining letters in positive inversion root groups. Thus f_w has zero internal form. For a Bruhat factorization $n_\ell \dot{q}\, \theta\, n_r$, with $n_\ell, n_r \in U$ and $\theta \in T$, the internal form $(n_\ell \mid \dot{q} \mid \theta \mid n_r)$ is independent of the two unipotent choices at fixed diagonal. Indeed, changing them transfers an element of $U \cap \dot{q} U \dot{q}^{-1}$

across $\dot{q}\,\theta$. On either side it lies in U, and its pairings with the diagonal and with the unipotent factor on that side vanish.

For one handle of Bruhat labels w, v, write its entries as $n_1\dot{w}t_1n_2$ and $n_3\dot{v}t_2n_4$ in the left and right Bruhat charts, respectively. Expanding the commutator and its incoming unipotent u gives

$$n_1\dot{w}t_1(n_2n_3)\dot{v}t_2(n_4n_2^{-1})t_1^{-1}\dot{w}^{-1}(n_1^{-1}n_4^{-1})t_2^{-1}\dot{v}^{-1}(n_3^{-1}u).$$

The internal forms of each entry and its inverse cancel, and multiplying adjacent unipotents costs zero. Apply the two Bruhat factorizations from the right, keeping each diagonal at its indicated end of the corresponding word. After the outgoing U factor this leaves

$$f_w \mid t_1 \mid f_v \mid t_2 \mid t_1^{-1} \mid f_{w^{-1}} \mid t_2^{-1} \mid f_{v^{-1}}.$$

The preceding invariance of the internal form justifies these changes with bars in place. Moving a torus letter across a simple letter acts on the torus by the reflection and rescales the root coordinate. Both presentations have zero internal pairing of torus and root. Moving t_1 to the right across f_v and t_2^{-1} to the left across $f_{w^{-1}}$ puts four consecutive torus letters between f_v and $f_{w^{-1}}$. Up to constant factors from the lifts, their word is

$$v^{-1}(t_1) \mid t_2 \mid t_1^{-1} \mid w^{-1}(t_2^{-1}). \tag{4.1}$$

Their product is the torus moment of this part of the word. Its internal form retains all six pairings of the four letters. Sliding a diagonal further to the end multiplies the intervening suffixes at their ends by torus elements. Their Bruhat labels, and thus the strata of Theorem 3.5, are the same in these coordinates.

Write $V = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. For a collection of torus coordinates with real cocharacter space E, denote its moment differential by $L : E \to V$, its action differential by $J : V \to E$, and its form in logarithmic coordinates by η. The twist $\tau \in \mathcal{W}$ means that acting by $k \in T$ multiplies the word on the left by $\tau(k)$ and on the right by k^{-1}. On a torus moment this gives $LJ = \tau - 1$. For (4.1) we have $E = V \oplus V$. Define four linear maps by

$$(\ell_1, \ell_2, \ell_3, \ell_4)(x, y) = (v^{-1}x, y, -x, -w^{-1}y).$$

The data of the four letters are

$$\begin{aligned} L(x,y) &= (v^{-1}-1)x + (1-w^{-1})y, \\ J(\xi) &= \big((w^{-1}-1)vw\,\xi,\ (v^{-1}-1)vw\,\xi\big), \\ \tau &= v^{-1}w^{-1}vw, \qquad \eta = \sum_{1\le i<j\le 4} \langle \ell_i \wedge \ell_j \rangle. \end{aligned}$$

Here $\langle \ell_i \wedge \ell_j \rangle$ antisymmetrizes the pairing of the two maps. To see the action, simultaneous conjugation changes t_1 by the multiplier $w^{-1}(k)k^{-1}$ and t_2 by $v^{-1}(k)k^{-1}$. The suffix $f_{w^{-1}}f_{v^{-1}}$ carries twist $w^{-1}v^{-1}$, so the conjugating parameter has differential $vw\,\xi$ when the parameter immediately to the right of the four letters is ξ.

For two consecutive torus factors the product rule gives data on $E_1 \oplus E_2$ by

$$\begin{aligned} L(e_1,e_2) &= L_1e_1 + L_2e_2, \quad J(\xi) = (J_1\tau_2\xi, J_2\xi), \\ \tau &= \tau_1\tau_2, \qquad\qquad \eta = \eta_1 + \eta_2 + \langle L_1 \wedge L_2 \rangle. \end{aligned} \tag{4.2}$$

The last term is the pairing of the logarithmic differentials of the two moments. The action on the first factor uses the left multiplier of the second.

The coroot contribution at a stay. Read the simple letters and the intervening four-letter torus factors from right to left. At a suffix with Bruhat expression $n_\ell \dot{q}\, \theta\, n_r$, subtract its internal Bruhat form $(n_\ell \mid \dot{q} \mid \theta \mid n_r)$ from the accumulated word form. We claim that the remainder is the logarithmic form obtained by (4.2), with θ as its moment. It is zero for the empty suffix. A torus factor on the left passes through n_ℓ and across $\dot{q}$ to the left of θ. Its data become $(q^{-1}L, Jq, q^{-1}\tau q, \eta)$, and pairing its new moment with θ gives exactly the cross term in (4.2).

For a simple letter $\dot{s}_\alpha x_\alpha(\zeta)$, write $x_\alpha(\zeta) n_\ell = n' x_\alpha(\zeta')$, where ζ' is ζ plus the simple-root coordinate of n_ℓ. The positive root groups other than U_α generate the kernel of this coordinate, so $\dot{s}_\alpha n' \dot{s}_\alpha^{-1} \in U$. Put $\beta = q^{-1}\alpha$. If $\beta > 0$, the root $x_\alpha(\zeta')$ passes across $\dot{q}$ into U on the right. If $\beta < 0$ and $\zeta' = 0$, that root factor is the identity. In both cases the new Bruhat label is $s_\alpha q$, the diagonal changes at most by a constant from the lifts, and subtracting the new Bruhat form leaves the same remainder.

If $\beta < 0$ and $\zeta' \neq 0$, the Bruhat label stays q. In the root SL_2, with simple lift $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, we have

$$\dot{s}_\alpha x_\alpha(\zeta') = x_\alpha(-1/\zeta')\, \alpha^\vee(1/\zeta')\, x_{-\alpha}(1/\zeta').$$

These three factors have zero internal form since they depend on one parameter. The left root joins the left U, and the right root crosses $\dot{q}$ into $U_{-\beta} \subset U$ and passes through θ. Moving the middle factor across $\dot{q}$ inserts $\delta = \beta^\vee(1/\zeta')$ just before θ. The new diagonal is $\delta\theta$. After subtracting the new Bruhat form, the added term is $(\delta \mid \theta)$. In particular the extra torus coordinate has zero intrinsic two-form.

Using its moment as cocharacter coordinate, this factor has space $E = (s_\beta - 1)V = \mathbb{R}\,\beta^\vee$, moment L the inclusion, action $J = s_\beta - 1$, and twist $s_\beta = q^{-1} s_\alpha q$. Indeed, if the unreduced suffix has twist γ, its diagonal has twist $\sigma = q^{-1}\gamma$. At a stay the new diagonal has twist $s_\beta \sigma$. Dividing its transformation law by that of θ gives the multiplier $s_\beta(\sigma(k))\sigma(k)^{-1}$ on δ, as prescribed by (4.2). This proves the claim for every suffix.

At a terminal cell $q = 1$, $\theta = t$, and the unipotent equation fixes the free left U factor. The outer Bruhat form vanishes since all its letters are in B_+ and only one is toral. Each whole handle has twist $wv(v^{-1}w^{-1}vw)w^{-1}v^{-1} = 1$. Each component of the torus moment fiber is a translate of the connected kernel, with real cocharacter space $\ker L$. The residual T acts on it by translations with differential J. Thus the torus factor after quotient has real cocharacter space $\ker L / \operatorname{im} J$, also in the presence of finite stabilizers. The calculation gives the restriction of the basic surface form there as the form induced by η, independently of the affine coordinates.

The marked handle and disk pieces. The adjoint trace form restricts to $\langle x, y \rangle = \sum_\alpha \alpha(x)\alpha(y)$ on V, with the sum over all roots. This is a positive $\mathcal{W}$-invariant inner product. Embed V into $\Pi = \mathbb{R}[\mathcal{W}]$ with its orthonormal permutation basis by sending x to $(\lambda_0(\rho^{-1}x))_{\rho \in \mathcal{W}}$ for a nonzero $\lambda_0 \in V^\vee$. The map is equivariant and its kernel is zero by irreducibility of the reflection representation. The positive self-adjoint operator comparing the two inner products has invariant eigenspaces. Irreducibility makes it scalar, so we can rescale the embedding to obtain an orthogonal decomposition $\Pi = V \oplus V^\perp$ with the given metric on V.

Write e_ρ for the basis vector of $\mathbb{Z}[\mathcal{W}]$ at label ρ. On a marked surface Σ with black marks $\mathcal{A}$, let rot move each mark clockwise to the next mark in a boundary collar. The cocharacter space is $E = H_1(\Sigma, \mathcal{A}; \mathbb{R})$, with $L = \partial$ and $\tau = \mathrm{rot}^2 |_{\mathcal{A}}$. The action $J(e_\rho)$ follows the boundary from ρ to $\tau(\rho)$. For relative cycles κ, ν, put $\Omega_\Sigma(\kappa, \nu) = \mathrm{rot}(\kappa) \cdot \nu$, using oriented intersection. The first cycle now has white endpoints. With the word-form convention, $\eta_\Sigma(\kappa, \nu)$ is $\Omega_\Sigma(\kappa, \nu) - \Omega_\Sigma(\nu, \kappa)$. The boundary arc satisfies $J(e_\rho) \cdot \mathrm{rot}(\nu) = (L\nu)_\rho$, the signed endpoint count at ρ [Mel25, §6.1, (6.2)–(6.4)].

For the handle labelled w, v, the lifted arcs of the punctured-torus cover run from ρ to $v^{-1}\rho$ and from $w^{-1}\rho$ to ρ. Together these arcs give a basis of relative homology, identifying it with $\Pi \oplus \Pi$. The boundary is $(v^{-1}-1)x + (1-w^{-1})y$. Their boundary action is the J of the handle above, and their alternating intersection coefficients are the six pairings of (4.1) [Mel25, Proposition 7.4]. For a transposition exchanging ρ and ρ', take a disk with these two black marks and a one-black-mark disk for each remaining label. The arc ϵ from ρ to ρ' has boundary $e_{\rho'} - e_\rho$. The two boundary paths have relative classes $J(e_\rho) = \epsilon$ and $J(e_{\rho'}) = -\epsilon$ [Mel25, Example 6.4]. The two-form vanishes on this one-dimensional space. For s_β the disjoint transpositions have orthogonal difference lines and fix one another's lines. Together their data have space $(s_\beta - 1)\Pi$, inclusion moment, action $s_\beta - 1$, and zero two-form. Weyl conjugation relabels the marks.

The pairing across a glued interval. Consider two successive pieces Σ_1, Σ_2 and denote their black marks of label ρ by $a_{1,\rho}, a_{2,\rho}$. At every label glue the intervals

$$[a_{1,\rho}, \mathrm{rot}_1(a_{1,\rho})] \quad \text{and} \quad [a_{2,\rho}, \mathrm{rot}_2^{-1}(a_{2,\rho})]$$

by matching black to black and white to white. Each common interval has zero homology relative to its black endpoint. Relative Mayer–Vietoris therefore identifies the new cocharacter space with $E_1 \oplus E_2$, and identifying the two black endpoints adds their boundary coefficients, giving $L = L_1 + L_2$ on this direct sum. Follow the boundary path $J_2(e_\rho)$ by $J_1(e_{\tau_2(\rho)})$. Their adjacent half-arcs traverse the same glued interval in opposite directions. Cancelling them gives the new boundary path from ρ to $\tau_1\tau_2(\rho)$. Its class is $(J_1(e_{\tau_2(\rho)}), J_2(e_\rho))$, so the new twist is $\tau_1\tau_2$ and $J(\xi) = (J_1\tau_2\xi, J_2\xi)$.

For $\kappa = (\kappa_1, \kappa_2)$ and $\nu = (\nu_1, \nu_2)$, consider one endpoint of κ_1 at label ρ. Its old rotated endpoint is now the white mark $\mathrm{rot}_2^{-1}(a_{2,\rho})$. To reach the new rotated endpoint $\mathrm{rot}_2(a_{2,\rho})$, append the arc $\mathrm{rot}_2^{-1} J_2(e_\rho)$ in the second collar, with multiplicity $(L_1\kappa_1)_\rho$. The rotated endpoints of κ_2 already lie at their new marks. The appended arc has intersection number

$$(\mathrm{rot}_2^{-1} J_2(e_\rho)) \cdot \nu_2 = J_2(e_\rho) \cdot \mathrm{rot}_2(\nu_2) = (L_2\nu_2)_\rho.$$

Summing over the intervals gives the rotated intersection on the glued surface,

$$\Omega_\Sigma(\kappa, \nu) = \Omega_{\Sigma_1}(\kappa_1, \nu_1) + \Omega_{\Sigma_2}(\kappa_2, \nu_2) + \langle L_1\kappa_1, L_2\nu_2 \rangle.$$

Antisymmetrizing gives the mixed pairing $\langle L_1\kappa_1, L_2\nu_2\rangle - \langle L_1\nu_1, L_2\kappa_2\rangle$, the term $\langle L_1 \wedge L_2\rangle$ in (4.2). For the left disk arc ϵ and a right cycle ν_2 this term is $(L_2\nu_2)_{\rho'} - (L_2\nu_2)_\rho$. Thus the disk joins at stays and the handle joins both realize the cell's convolution by oriented gluing [Mel25, §6.2, (6.5)–(6.6), and Proposition 6.5].

The reduced intersection form. After all joins the boundary permutation is the identity. We may reduce the permutation model at the moment of its identity point, since the differentials and logarithmic forms are translation invariant. On the glued surface Σ, the relative homology sequence identifies $\ker L_\Pi$ with $H_1(\Sigma; \mathbb{R})$. Each boundary circle has one black mark, so its full boundary class is $J_\Pi(e_\rho)$ for that label. Capping the circles by disks therefore identifies $\ker L_\Pi / \operatorname{im} J_\Pi$ with the first homology of the capped surface. Absolute cycles can be represented outside the collar where the rotation acts. On these cycles Ω_Σ is oriented intersection, and its unhalved antisymmetrization is twice that pairing. With the normalization of Proposition 4.2 the reduced form is the intersection form itself, nondegenerate on each capped component [Mel25, (6.8)–(6.9) and Proposition 6.6].

All the maps for a handle preserve $V \oplus V^\perp$. For a stay the space $(s_\beta - 1)\Pi$ splits as $(s_\beta - 1)V \oplus (s_\beta - 1)V^\perp$. Its first term is precisely the geometric coroot line in its moment

coordinate. Both the six handle pairings and the cross term in (4.2) pair the two summands to zero. The moments, the actions with their suffix twists, and hence the reductions split. In particular

$$\frac{\ker L_\Pi}{\operatorname{im} J_\Pi} = \frac{\ker L_V}{\operatorname{im} J_V} \ \oplus^\perp \ \frac{\ker L_{V^\perp}}{\operatorname{im} J_{V^\perp}}.$$

A vector in the radical of the geometric summand would annihilate the entire capped-surface homology. The form on the torus of S is therefore nondegenerate. □

Theorem 4.5. *Let G be simply connected and simple and $g \geq 1$. On a nonempty open of regular semisimple $t \in T$, put $Y_t = D_t//T$ and $D = (2g-2)\dim G + 2b$. Let ω be the surface symplectic class for the adjoint trace form. For $j \geq 0$ and $m \in \mathbb{Z}$, cup product induces isomorphisms*

$$\omega^j : \operatorname{Gr}^W_{D-2j} H^m(Y_t, \mathbb{C}) \longrightarrow \operatorname{Gr}^W_{D+2j} H^{m+2j}(Y_t, \mathbb{C}).$$

Proof. We first compute the compactly supported weight gradeds of a stratum. Localization assembles their Lefschetz isomorphisms on $\widetilde{Y}_t$. The Thom isomorphism and duality then determine the center on Y_t.

The center on a stratum. Take $t \in T_{\mathrm{gen}}$ and a stratum component $S = H \times (\mathbb{A}^a/F)$ with $H = (\mathbb{C}^\times)^c$. The algebra $H^*(H)$ is exterior on $H^1(H) = \mathbb{Q}(-1)^c$. The periods of a logarithmic two-form on products of cocharacter circles are $(2\pi i)^2$ times its cocharacter pairings. Lemma 4.4 thus gives a symplectic element of $\bigwedge^2 H^1(H, \mathbb{C})$. In a symplectic basis, wedge by this element and contraction by its inverse bivector give the exterior $\mathfrak{sl}_2$ action, with grading $k - c/2$ on $\bigwedge^k$. Its jth Lefschetz power identifies exterior degrees $c/2 - j$ and $c/2 + j$.

Torus duality gives

$$H_c^{c+k}(H) = \left(\textstyle\bigwedge^{c-k} H^1(H)\right)^\vee(-c), \qquad 0 \leq k \leq c.$$

This group has weight $2k$, and cup product on it is transpose to cup product on ordinary cohomology. Finite averaging and complex orientation give $H_c^{2a}(\mathbb{A}^a/F) = \mathbb{Q}(-a)$ as the only affine contribution. Tensoring shifts degree and weight by $2a$. Thus on $H_c^*(S, \mathbb{C})$ the jth power identifies weights $2M - 2j$ and $2M + 2j$ with degree shift $2j$, where Theorem 3.5 gives the same half-weight center $M = a + c/2 = 2gb + (g-1)r$ on every piece.

Localization with a common center. By Theorem 3.5, $H^2(Y_t)$ is Hodge–Tate. Its third Hodge step vanishes by the form-degree filtration in degree two, so all its diagonal types (k, k) have $k \leq 2$. Thus $\omega \in W_4 H^2(Y_t, \mathbb{C})$. Cup product on compact supports respects the weight filtrations. Cupping with $(\pi_t^*\omega)^j$ gives maps from grade $2M - 2j$ to grade $2M + 2j$, depending only on the weight-four symbol of $\pi_t^*\omega$.

Let $Z_{i-1} \subset Z_i$ be successive closed terms in the filtration and put $S_i = Z_i \setminus Z_{i-1}$. Apply Proposition 4.3 to the exact sequence

$$H_c^{m-1}(Z_{i-1}) \longrightarrow H_c^m(S_i) \longrightarrow H_c^m(Z_i) \longrightarrow H_c^m(Z_{i-1}) \longrightarrow H_c^{m+1}(S_i).$$

Take grade $2M - 2j$ in this row and grade $2M + 2j$ in the row with degrees increased by $2j$, after complexification. Cup by the global class commutes with all arrows, including the boundary maps since its degree is even. The two maps on S_i are isomorphisms by the stratum calculation and finite direct sums. The two on Z_{i-1} are isomorphisms by induction from the empty set. The five lemma gives the middle isomorphism, and induction gives compact-support Lefschetz on $\widetilde{Y}_t$.

The Thom twist and ordinary duality. In the quotient charts in the proof of Theorem 3.5, D_t is a smooth level of the commutator product and the T-orbits are closed with finite stabilizers. The level has codimension $\dim G$ in G^{2g}, so $\dim Y_t = (2g-1)\dim G - r = D$. Exponentiation identifies U with $\operatorname{Lie} U$ T-equivariantly. Over the quotient stack $[D_t/T]$ the enlargement is consequently a vector bundle of rank b. Its Thom class is invariant in each finite quotient chart because all stabilizers act complex linearly. Taking rational invariants gives the Thom isomorphisms on the coarse spaces, which agree on overlaps by the complex orientation. They give

$$H_c^{m+2b}(\widetilde{Y}_t) \simeq H_c^m(Y_t)(-b)$$

and intertwine cup by $\pi_t^*\omega$ and by ω. The half-weight center on $H_c^*(Y_t)$ is therefore

$$M - b = (2g-1)b + (g-1)r = D/2.$$

Finally the smooth finite quotient charts give the rational orbifold duality pairing $H^m(Y_t) \otimes H_c^{2D-m}(Y_t) \to \mathbb{Q}(-D)$. It pairs ordinary weight w with compact-support weight $2D - w$ and transposes cup powers. After complexification, the compact-support isomorphism from degree $2D - m - 2j$ and weight $D - 2j$ to degree $2D - m$ and weight $D + 2j$ transposes to the asserted ordinary isomorphism. □

5 Specialization of the flagged Hodge family

We retain the notation for a simple factor. When the Cartan projection of a logarithmic residue specializes to the scalar value, its derived part can still be nilpotent. The choice of a Borel whose nilradical contains this part gives a projective map. We use it to realize scalar intersection cohomology in ordinary cohomology and to compare this realization at the Higgs and Betti endpoints.

Proposition 5.1. *For a smooth projective family of complex curves with relative effective marking divisor and split connected reductive structure group of fixed degree, the stack of semistable meromorphic λ-connections has a quasi-projective adequate moduli space. The semistability test uses invariant weighted parabolic reductions. For a principal meromorphic Higgs bundle, semistability implies semistability of the associated adjoint Higgs bundle. A family in the full stack over a discrete valuation ring with semistable generic fiber admits semistable reduction after finite extension.*

The extension over a valuation ring will be applied to the unweighted scaling family [HZ25, Theorem 2.27, Corollary 2.33 and Proposition 4.48].

Proposition 5.2. *Let a noetherian algebraic stack over an algebraically closed field have affine diagonal and a good moduli space. At a closed point with linearly reductive stabilizer there is an étale local presentation over the good moduli space by the quotient of an affine scheme by that stabilizer.*

These Cartesian charts give an invariant section algebra for the flag polarization [AHR20, Theorem 4.12].

Proposition 5.3. *Let $g \geq 2$, let G be connected and reductive over $\mathbb{C}$, and let D be a fixed line bundle on C of degree at least $2g-2$. The coarse semistable D-valued Hitchin morphism in each topological type is proper and surjective. For $D = K_C$ every fiber has pure dimension $(g-1)\dim G + \dim Z(G)$ and the total dimension is twice this number.*

We will embed the moving twist $K_C(x)$ in a fixed line of high degree to apply the properness assertion [CFH+25, Propositions 3.27(ii) and 3.28(i)].

Proposition 5.4. *Let Y be smooth quasi-projective with a $\mathbb{C}^\times$-action whose fixed locus is proper and in which every point has a zero limit. For a smooth surjective equivariant morphism $Y \to \mathbb{A}^1$ of positive weight, restriction to any parameter fiber induces an isomorphism on ordinary rational cohomology.*

The smooth one-parameter restriction theorem uses the attracting strata of the scaling action [HR15, Theorem 1.3.1 and Corollary 1.3.3].

Proposition 5.5. *For a simple simply connected group over a smooth projective complex curve of genus greater than one with one parahoric point, the stack of parahoric bundles is very good and its cotangent stack is irreducible.*

The parahoric at a full flag is an Iwahori. The irreducibility assertion concerns its entire cotangent stack [BKV19, Theorem 7 and proof of Theorem 10(ii)].

Proposition 5.6. *A choice of representatives of logarithmic exponents modulo $\mathbb{Z}$ determines the regular-singular extension of a flat vector bundle across a normal crossings divisor. At a puncture with no nonzero integral difference between distinct residue eigenvalues, the local monodromy is conjugate to the exponential of $-2\pi i$ times the residue.*

We use the exponent convention of the logarithmic extension, including its nilpotent part [Del70, Chapter II, Proposition 5.4, Remark 5.5(ii), Corollary 5.6 and Theorem 5.9].

Proposition 5.7. *The incidence of an element in a parabolic of prescribed type with prescribed Levi conjugacy class carries the complex quasi-Hamiltonian structure whose group-valued moment is that element. In the nonresonant logarithmic case, the corresponding local monodromy is conjugate to the exponential of the exponent with the connection sign convention.*

For a Borel the Levi conjugacy class is a point of the abstract Cartan [Boa11, Theorems A, B and 9].

Proposition 5.8. *For quasi-Hamiltonian spaces of a compact group, inversion of the moment map negates the two-form, and fusion multiplies the moment maps. At a regular level, reduction with finite stabilizers yields a symplectic orbifold. The fused doubles yield the handle commutator moment.*

The inversion formula fixes the sign of the puncture factor relative to the handle word [AMM98, Proposition 4.4 and Theorems 5.1, 6.1 and 9.3].

Proposition 5.9. *For semisimple local systems on a smooth projective complex variety, the harmonic metric defines preferred sections of the twistor family. On the coarse general linear moduli they identify the Hodge family topologically over the parameter line with the product with its Higgs fiber.*

We use the product given by the preferred sections to identify the two restrictions on scalar cohomology [Sim97, Section 4].

Proposition 5.10. *On a connected smooth proper complex Deligne–Mumford stack the harmonic correspondence between semisimple local systems and polystable Higgs objects with vanishing rational characteristic classes induces a homeomorphism of the corresponding coarse spaces. The correspondence is compatible with tensor operations and extension of structure group.*

Tensor compatibility retains the specified central inertia on the root gerbe [Sim11, Theorems 6.10, 8.8 and 9.7].

Proposition 5.11. *For $g \geq 2$ and a complex almost simple group, each component of a nonempty central-relator character variety is a normal symplectic singularity of dimension $(2g-2)\dim G$. The regularly stable locus is smooth and dense.*

This supplies the dimension and the dense open on which we normalize the scalar maps [Shu25, Corollaries 4.5, 5.10, 5.11 and Proposition 6.1].

Proposition 5.12. *For a projective morphism, the direct image of a polarizable pure Hodge module decomposes into shifted polarizable pure perverse cohomology objects, with the corresponding shifts in weight. An ample relative class induces hard Lefschetz between their opposite perverse degrees.*

For a semismall map from a rational homology manifold the support bounds put this decomposition in a single perverse degree [Sai88, Theorem 5.3.1].

Proposition 5.13. *A polarizable variation of pure Hodge structure on a smooth dense open extends uniquely as a polarizable pure Hodge module with the given strict support. On a smooth algebraic base a mixed Hodge module with shifted local system realization is a graded-polarizable admissible variation of mixed Hodge structure.*

We use the strict-support extension for the scalar summand and the variation on the Cartan for the ordinary Betti cohomology [Sai90, Theorems 3.21 and 3.27].

Proposition 5.14. *For a connected complex reductive group, the internally fused double has the commutator as moment. Fusion multiplies the moments and adds the invariant Maurer–Cartan cross term to the two-form. Smooth reduction at the identity has a holomorphic symplectic form.*

The double and its fusion formulas use algebraic invariant forms on the complex group [Boa14, Section 2, Example 2.4 and Theorems 2.6–2.7].

Proposition 5.15. *For a good moduli morphism from a locally noetherian Artin stack, a vector bundle descends if and only if its geometric stabilizer acts trivially at every closed orbit. A closed substack has a good moduli morphism to its scheme-theoretic image.*

In particular a closed condition of divisibility on Higgs fields can be imposed on a good moduli space [Alp13, Lemma 4.14 and Theorem 10.3].

Proposition 5.16. *The enhanced mixed Hodge module formalism on complex schemes is functorial for diagrams. Rational realization is compatible with pullbacks and, on bounded-below constructible objects, ordinary direct images by morphisms of finite type.*

The diagram functoriality will retain the complexes and their restriction maps on multiple intersections of marking charts [Tub25, Section 2.1 and Proposition 4.2.6].

Proposition 5.17. *Let $Y \to Z$ be a projective symplectic resolution of a complex symplectic singularity. The morphism is semismall, and the reduced point fibers have no odd cohomology. Their even cohomology is pure of Hodge type (a,a) in degree $2a$.*

This gives the fiber bound for smooth symplectic resolutions [Kal06, Lemmas 2.9 and 2.11 and Theorem 2.12]. We will obtain the bound on the flag quotient by comparing the cotangent forms on two lifts of the same unweighted point.

Construction 5.18. For $g \geq 2$ and the simple scalar pair (X_z, B_z) let $\epsilon = \exp(2\pi i/n_z)$ and $H = (G \times \mathbb{G}_m)/\langle (z, \epsilon^{-1}) \rangle$. The central inclusion is $\iota : \mathbb{G}_m \to H$, and the abelianization is induced by the n_zth power. Let $x : S \to C$ be a connected component of the finite étale cover obtained by taking n_zth roots of $\mathcal{O}_C(x - p)$ in $\mathrm{Pic}^0(C)$. Allow x also to be a single fixed mark when no integration is used. Write $Y_{\lambda,u}$ for the tiny generic rational-weight, full-Borel coarse moduli of logarithmic H-valued λ-connections at x with prescribed abelianization $(\mathcal{O}_C(x), \lambda d_x)$ and ordered residue $-\lambda\, d\iota(1)/n_z + u$, $u \in \mathrm{Lie}\, T$. Here d_x makes the canonical meromorphic section horizontal. The notation for a moving mark includes S in the total space.

Put $\Lambda = \mathbb{A}^1 \times \mathrm{Lie}\, T$. A parameter fiber fixes $(\lambda, u) \in \Lambda$ and retains S. We will also take the whole construction over a single $s \in S$, in which case all restriction maps use that same marking. The ordered residue is the image of the residue in the Cartan quotient of the chosen Borel. The abelianization includes an isomorphism to the specified line with its operator.

We first identify the scalar locus, where the whole derived residue vanishes. Write $\chi_H : H \to \mathbb{G}_m$ for abelianization. We have $\chi_H \iota(a) = a^{n_z}$ and $\iota(\epsilon) = z$. At p, pull an H-bundle with its specified abelianization back to the gerbe of roots of $\mathcal{O}_C(p)$. Tensor it centrally by the inverse of the tautological root. The resulting abelianization is canonically trivial and gives a G-bundle with the prescribed central inertia. Tensoring back gives a bundle with trivial inertia, which descends to C. These operations preserve reductions to parabolics since the center acts trivially on their flag varieties. The root carries the divided λ-connection with residue $-\lambda/n_z$. Removing it on the scalar locus gives a holomorphic operator on the gerbe. Off p use the trivial root whose unit raises to the canonical section of $\mathcal{O}_C(p)$. This unit is horizontal, so tensoring preserves each lifted handle holonomy. At $\lambda = 1$ the positive peripheral loop has holonomy $\exp(2\pi i\, d\iota(1)/n_z) = z$.

Let $\Delta \subset C \times S$ be the moving graph and $\Delta_p = \{p\} \times S$. The Poincaré line for the chosen Picard root gives a line $\mathcal{D}$ on $C \times S$ with $\mathcal{D}^{n_z} = \mathcal{O}(\Delta - \Delta_p)$ up to a line from S. It carries the relative divided divisor connection, with residues $-1/n_z$ at Δ and $1/n_z$ at Δ_p. Tensor by $\iota(\mathcal{D})$, with this operator multiplied by λ, to move a scalar object from p to $x(s)$. At a coincidence of the two points the divisor terms cancel. Locally trivializing the line from S defines this operation and its inverse with the prescribed abelianizations. On overlaps a change of trivialization is a base unit and has an étale local n_zth root, giving a central isomorphism. The inverse coarse maps therefore glue. Denote the fixed-mark scalar Hodge family by $\mathcal{X} \to \mathbb{A}^1$. The moving scalar locus is $S \times \mathcal{X}$. For a fixed scalar object, its fixed-pole holonomies are constant on isotopies of lifted handles avoiding the poles. The correcting line contributes holonomies in μ_{n_z}, locally constant in a marking chart. Thus the scalar product coordinates are flat in the Betti marking charts.

Lemma 5.19. *Let $g \geq 2$ and take a simple scalar pair with the marking and abelianization of Construction 5.18. For a sufficiently small generic rational weight in the positive chamber, the morphism forgetting the flag and taking the unweighted S-equivalence class is projective over the ordered parameters. Weighted semistability equals stability, with finite stabilizers at fixed abelianization. Over a regularly stable scalar object its fiber is the full variety G/B_+.*

Proof. We construct the projective quotient and identify its numerical criterion at a closed unweighted orbit.

The projective quotient. Proposition 5.1 applies to H and the relative effective divisor Δ. In characteristic zero the unweighted moduli morphism is good. One can fix the abelianization étale locally on the rank-one moduli. Indeed, tensoring by a central line changes the abelianization by its n_zth power. On the Picard variety the power map is étale and surjective. On the differences of rank-one operators with prescribed residue it multiplies holomorphic one-forms by n_z. Thus

the rank-one variations split off locally. At the specified line, prescribing its isomorphism removes the scalar automorphisms up to the finite kernel of the power map. The scalar torus acts trivially on the underlying moduli and on flags. This gives the same unweighted coarse fiber with the specified abelianization.

Over the unweighted semistable stack times $\operatorname{Lie} T$ take the flag bundle at the graph. Preservation of the Borel and the prescribed Cartan projection are closed conditions on it. Choose a regular dominant character θ in the root lattice and let $\mathcal{L}$ be the ample flag line with fiber character $-\theta$ at the reference Borel. It is trivial in the central ι-direction. The diagonal of the unweighted stack is affine. In a faithful representation the equations on a pair of bundle maps over the proper curve say that they are mutual inverses and preserve the tensors, operator and abelianization. In an affine chart of Proposition 5.2 write the stack as $[V/K]$, with K reductive, and the closed flag incidence upstairs as D_V. The quotient over $V/\!/K$ is

$$\operatorname{Proj}_{V/\!/K} \bigoplus_{a\geq 0} \Gamma(D_V, \mathcal{L}^a)^K.$$

The section algebra is of finite type after passing to a Veronese subalgebra, as is its reductive invariant algebra. Its degree-zero part is finite over the coordinate ring of $V/\!/K$. Indeed $\Gamma(D_V, \mathcal{O})$ is a finite module over $\Gamma(V, \mathcal{O})$ by projectivity, and taking reductive invariants of a finite equivariant module gives a finite module over the invariant ring. A further Veronese is generated in degree one over its degree-zero part. These are the local algebras obtained by pushing the flag section algebra to the good moduli space. Invariants commute with étale base change, so the relative Proj glues and is projective, including over strictly polystable unweighted points.

The weight inequalities. Use the sign in which the ordinary degrees of compatible maximal parabolic character lines of an unweighted semistable object are nonpositive. For each type take an integral positive multiple of the fundamental character trivial on $\iota(\mathbb{G}_m)$. Each nonzero degree in these tests has absolute value at least one. There are finitely many tests and relative Weyl positions. Choose θ off all the hyperplanes $\langle w\theta, \nu_i\rangle = 0$, where ν_i ranges over the fundamental maximal-parabolic cocharacter rays and $w \in \mathcal{W}$. Scale θ by a positive rational number so that the absolute correction in every integral degree test is less than one. All strict ordinary inequalities retain their signs.

A filtration in the unweighted stack is a compatible parabolic reduction with a cocharacter in its Levi center. Its graded object lies in the semistable stack precisely when its active maximal parabolic degrees vanish. In one direction the graded object has compatible opposite parabolics, which force both signs of each degree to be nonpositive. In the other direction, a destabilizing character for the Levi would also destabilize a refinement of the original reduction. Adding a large multiple of the original parabolic character makes the refining character dominant without changing its positive degree. Thus the Levi is semistable of central slope zero. A canonical destabilizing reduction of the graded object would be preserved by its grading torus. Restriction of its character to the centralizer Levi then gives a Levi parabolic character of the same positive degree, since the central degrees are zero. This proves semistability of the graded object. These zero-degree filtrations are exactly the ones tested by relative Hilbert–Mumford on the semistable quotient charts. In fact a filtration has constant image in the good moduli space and lifts after the Cartesian étale base change. The flag extends over its limit by projectivity. The weight on the limiting flag line is $-\langle w\theta, \nu\rangle$. Semistability requires this weight to be nonnegative, giving the parabolic inequality $\langle w\theta, \nu\rangle \leq 0$ at ordinary degree zero. Passing to a power of $\mathcal{L}$ has no effect on its sign. In a common chamber ν is a positive combination of the active maximal rays. Semistability gives the required weak sign on each ray, and genericity makes each of these signs

strict. Thus every semistable point is stable with finite stabilizer at fixed abelianization.

The geometric points of a fiber of the Proj consist of all stable flagged objects whose underlying graded is the given polystable object, up to isomorphism. This includes flags on extensions in its S-equivalence class. When the unweighted object is regularly stable its class is already a single orbit and all compatible flags pass the weight test. For scalar residue every flag is compatible, and the central stabilizer fixes each flag. □

Lemma 5.20. *For $g \geq 2$ and a simple scalar pair in Construction 5.18, the stable stack is smooth over $S \times \Lambda$ of relative dimension $2R + 2b$. With S retained or with a single fixed mark, restriction from the whole coarse family to the inverse image of any point of Λ or any line through its origin is an isomorphism on rational cohomology.*

Proof. We compute the deformation complex at zero, establish the existence of scaling limits, and compare the attracting strata with their intersections with each parameter fiber.

Smoothness at zero. At a fixed mark $x(s)$ write $\mathcal{P}$ for sections of the derived adjoint bundle with value in the flag Borel and $\mathcal{N}$ for those with value in its nilradical. For a Higgs field φ at $(\lambda, u) = (0, 0)$ the relative deformation complex is

$$\mathcal{P} \xrightarrow{[\varphi, \cdot]} \mathcal{N} \otimes K_C(x(s)) = \mathcal{P}^\vee \otimes K_C.$$

The equality uses the adjoint trace form and the fact that the annihilator of a Borel is its nilradical. The central summand has been removed by the specified abelianization. Infinitesimal automorphisms vanish by Lemma 5.19. Duality of the complex and Serre duality give vanishing of its second hypercohomology. To lift over a parameter thickening, lift the marked bundle and flag on the curve and choose local operators of the prescribed symbol and diagonal residue in adapted frames. Their differences on overlaps give an error in $H^1(\mathcal{P}^\vee \otimes K_C)$, which is killed by a change of gluing in $H^1(\mathcal{P})$ by the vanishing just proved. Since $\chi(\mathcal{P}) = (1 - g) \dim G - b$, the relative dimension is $-2\chi(\mathcal{P}) = 2R + 2b$.

There are points over zero for every mark and z. A Coxeter element w has no fixed vector on the real root space, so $a \mapsto aw(a)^{-1}$ is surjective on the compact maximal torus. Choosing a lift of w gives a single commutator equal to z. Complete this tuple with identity handles. Proposition 5.11 now gives regularly stable scalar tuples, and Proposition 5.10 gives their regularly stable Higgs partners. Every flag above such a partner supplies a point by Lemma 5.19.

Scaling and the fixed locus. Scale the operator, λ and u together, keeping the mark fixed. The unweighted scaling orbit extends in the full stack by scaling on the original bundle. Its generic fiber is semistable, so Proposition 5.1 gives semistable reduction after finite extension. The abelianization persists by separatedness. Projectivity in Lemma 5.19 then extends the flagged orbit in the coarse space. In a projective embedding the original orbit and its ramified base change have the same limiting point. The complement of the relative smooth locus is closed and invariant. It has closed coarse image since the stack has finite inertia and the stable geometric quotient has the orbit topology. Any point there would have a limit over zero in that image. This proves smoothness everywhere, and the open image at zero and scaling give surjectivity in the parameter directions.

A fixed point lies over $(0, 0)$ in the nilpotent logarithmic Hitchin fiber. To see properness as the mark varies, choose a fixed line D_0 of sufficiently high degree on C. Locally on S there is an inclusion $K_C(\Delta) \hookrightarrow D_0$ on the family. Base change for the sections of $D_0 \otimes K_C^{-1}(-\Delta)$ gives a section nonzero on each curve fiber after shrinking. Its zero divisor is finite flat over this open. Multiplication identifies logarithmic Higgs fields with D_0-valued fields vanishing along that divisor,

a closed condition. Division preserves compatibility of saturated parabolic reductions, since it can be checked on the complement of the divisor. The degrees of the reductions agree. Hence this gives a closed good moduli image by Proposition 5.15. Multiplication on invariant coefficients uses the corresponding powers of the section. The nilpotent fiber is a closed part of the fixed-D_0 nilpotent fiber times this open, with the closed abelianization constraint. Proposition 5.3 and the projective flag quotient give properness over S. Since S is proper, the fixed locus is proper as well.

Restriction on an attracting stratum. We give the orbifold and several-parameter argument for the restriction principle in Proposition 5.4. The attracting strata admit a finite filtration by open unions, each new stratum closed in the next union. Indeed one can order the first nonzero weight coordinates in an equivariant projective embedding using a linearized ample power, and separate the fixed components of each weight. Existence of the limits keeps these strata in the quasi-projective space. At a fixed point the identity component of the group of lifts of scaling is isogenous to the torus and centralizes the finite inertia. Its action lifts to a smooth orbifold chart at the point. Averaging coordinates over its circle and the inertia linearizes their action at the point. Holomorphic continuation gives equivariance for scaling wherever the chart is defined. Write its weight spaces as V^-, V^0, V^+. Over a small fixed neighborhood in V^0 the attracting coordinates are precisely those in V^+. Their entire spaces are obtained by scaling a small positive vector into the chart and then scaling back. A negative coordinate cannot have a zero limit in the chart.

On V^+ the map to Λ is homogeneous of a common positive weight k after the torus cover. Its Taylor series is a polynomial linear in the weight-k variables, plus terms in lower positive weights. Smoothness gives a surjection from the weight-k summand to Λ at every point of the fixed neighborhood. Choose a local right inverse and average it over inertia, which acts trivially on Λ. Solving for these variables exhibits each level on the stratum as affine spaces in the other variables and the kernel, modulo finite inertia. The inverse image of a line has the same description with one free parameter. These local products contract over the fixed neighborhood. Thus pullback from the fixed component identifies the ordinary cohomology of the stratum with that of each restriction. The stratum is transverse to the level or line by the same surjection. Its complex normal bundle has the same rank after restriction. The Thom isomorphism and transverse base change on finite charts pass to rational invariants. The Gysin sequences for the successive open unions therefore identify the restriction maps on the whole spaces by induction. The argument applies over S and after fixing s. □

For the Betti construction write $K = \prod_{i=1}^{g} [A_i, B_i]$ for the peripheral word. A point of the incidence consists of a handle tuple and a Borel P with $K \in P$ and Cartan projection $t \in T$. Take the quotient for the flag polarization of Lemma 5.19 by simultaneous conjugation. Denote this family at a fixed mark by $p_{\mathrm{B}} : \mathcal{B} \to T$. The quotient is algebraic since the incidence is closed in the product of the affine tuple and parameter space with G/B_+. A cocharacter ν has a limit on the tuple exactly when all handles belong to its parabolic $P(\nu)$. The flag always has a limit, with line weight $-\langle w\theta, \nu\rangle$ in the relative position w. Testing the maximal rays in a common chamber shows again that semistability equals stability.

Lemma 5.21. *Let $g \geq 2$, fix a simple scalar pair and a mark, and set $\lambda = 1$ in Construction 5.18. On a neighborhood of $u = 0$ identified with a neighborhood of z by $t = z\exp(-2\pi i u)$, logarithmic monodromy gives an analytic isomorphism to the weighted Betti incidence. It commutes with forgetting the flag and taking the unweighted class. The ordinary cohomology sheaves of p_{B} are local systems there, and the canonical ordinary base-change maps are isomorphisms.*

Proof. We construct the comparison in families, check the same weight inequalities on both sides, and use relative duality to compute the ordinary stalk maps.

The logarithmic lattice. Choose a faithful direct sum of irreducible representations of H. Shrink the Cartan polydisk so that the derived weights lie in a logarithm window about zero and all their differences within each summand have absolute value less than $1/2$. The fixed central term acts by a scalar on each summand. In a local coordinate v and an adapted frame write the operator as $d + A(v)\,dv/v$, where $A(v) = A_0 + \sum_{j\geq 1} A_j v^j$. The gauge $U(v) = 1 + \sum_{m\geq 1} U_m v^m$ to constant residue satisfies

$$(m + \operatorname{ad}(A_0))U_m = -\sum_{j=1}^{m} A_j U_{m-j}, \qquad U_0 = 1.$$

All these operators on the endomorphism blocks are invertible. The nilpotent part of A_0 changes none of their eigenvalues. Locally in the connection parameters choose positive constants c, ρ with $\|A_j\| \leq c\rho^{-j}$. The inverses have norm at most c'/m for a common $c' > 0$. The Neumann series gives this bound for $m > 2\sup\|\operatorname{ad}(A_0)\|$, and holomorphic invertibility gives it for the finitely many smaller m after shrinking. The coefficients of $(1 - v/\rho)^{-cc'}$ majorize $\|U_m\|$ by the recursion. The gauge therefore converges holomorphically in v and in parameters even when the Jordan type changes. To see that its values lie in H, use successive formal gauges $\exp(v^m X_m)$ to cancel the order-m term with $X_m \in \operatorname{Lie} H$, by invertibility of the same operator on $\operatorname{Lie} H$. Uniqueness of the matrix solution identifies its Taylor series with this formal group-valued gauge. At $v = 0$ it is the identity on the chosen flag.

For the inverse let M denote peripheral monodromy. In the chosen window the residue in H is

$$-\frac{d\iota(1)}{n_z} - \frac{1}{2\pi i}\log(z^{-1}M).$$

The logarithm with $\log(1) = 0$ is holomorphic by functional calculus on the entire generalized eigenspaces. In any parabolic containing M, take the Jordan decomposition of $z^{-1}M$. The chosen toral logarithm of its semisimple part and the finite nilpotent logarithm of its unipotent part both lie in the parabolic Lie algebra. Their sum is the required logarithm. The Cartan projection in the chosen Borel is consequently the prescribed one. Gluing this constant-residue model to the punctured flat bundle on an annulus gives the inverse lattice in families. Isomorphisms intertwine the logarithms and extend. Every flat parabolic reduction on the punctured curve extends in this model by its logarithm-invariant parabolic at the pole, and restriction recovers the flat reduction. A coordinate centered on the moving graph gives the same construction locally over a marking chart. This proves the principal, flagged family comparison with the exponential convention of Propositions 5.6 and 5.7.

The common stability test. For an existing invariant maximal parabolic reduction let η be its integral test character trivial on the central line. The induced logarithmic line connection has degree equal to minus the sum of its residues, hence

$$\deg E_\eta = -\langle \eta, wu \rangle$$

with $w \in \mathcal{W}$ representing the relative position of the parabolic and flag. Indeed its character factors through the Levi abelianization and kills the nilpotent residue. Shrink the disk so that the absolute value on the right is less than one for all the finitely many characters and positions. Integrality forces degree zero for each reduction that exists. Thus both stability tests use exactly

its flag line weight, also at $t = z$. The unweighted filtrations and their gradeds correspond as well. Equivariance and the inverse construction give the asserted isomorphisms of the coarse diagrams.

The canonical stalk map. Put $d = 2R + 2b$. By Lemma 5.20, a basis of the degree-i cohomology of the fixed-mark total Hodge family restricts to a basis on every $(1, u)$ fiber. The analytic isomorphism gives classes $\beta_1, \dots, \beta_{a_i}$ on the whole Betti family over the disk with this property. Cup product and the complex orientation trace give

$$R^{2d-i}p_{\mathrm{B}!}\mathbb{Q} \longrightarrow \underline{\mathbb{Q}}^{a_i}, \qquad \xi \longmapsto \big(\mathrm{Tr}(\xi \smile \beta_j)\big)_j.$$

Compact-support base change identifies each stalk with compactly supported fiber cohomology, and Poincaré duality identifies this pairing with the dual of the chosen basis. The trace and duality hold on a smooth orbifold submersion by complex orientation on uniformizing charts and finite rational invariants. The displayed map is therefore an isomorphism in every degree. Verdier duality on the smooth disk identifies $R^i p_{\mathrm{B}*}\mathbb{Q}$ with the dual of $R^{2d-i}p_{\mathrm{B}!}\mathbb{Q}$, a local system of rank a_i. The canonical map from its stalk to $H^i(\mathcal{B}_t, \mathbb{Q})$ sends the germs of the classes β_j to a basis. It is surjective, and both spaces have rank a_i, proving ordinary base change. □

Lemma 5.22. *For $g \geq 2$ and a simple scalar pair, let q_λ be the projective morphism of Lemma 5.19 at a fixed mark and $(\lambda, 0)$. Then*

$$\dim\big(Y_{\lambda,0} \times_{\mathrm{im}\, q_\lambda} Y_{\lambda,0}\big) = 2R + 2b.$$

Proof. We bound the Higgs fiber product using its two cotangent forms. Scaling transfers this bound to the other fibers.

The cotangent form on the full fiber product. At $\lambda = 0$ the stack of strongly parabolic Higgs fields with fixed abelianization is the cotangent stack of bundles with the specified abelianization and flag. For the flag-adapted sheaf $\mathcal{P}$ in Lemma 5.20, the tangent space to the bundle stack in degree zero is $H^1(\mathcal{P})$, whose dual is the space of these Higgs fields. The tautological one-form pairs a field with the deformation of the flagged bundle by Serre duality. It is defined also at unstable objects. On the stable orbifold its differential is the nondegenerate cotangent form, equivalently the pairing of the self-dual deformation complex.

Let E be an irreducible component of the reduced fiber product at zero. At its general point lift both factors to smooth finite uniformizing charts and take a component above E. Passing to a further generically finite cover of this component and shrinking its smooth locus gives a smooth parameter scheme U carrying the two families. We show that their tautological one-forms agree on U. The two unweighted objects have the same polystable graded at the geometric generic point. Take their filtrations to that graded and identify the two limits. More explicitly, in an affine chart of Proposition 5.2, one can specialize each orbit to the closed orbit in its closure by a cocharacter, after conjugation. This gives the Rees family of the filtration. All choices, including the isomorphism of the closed objects, are defined over a finite extension of the function field and spread out after shrinking U. Extend the two flags along these families by properness of the flag bundle. The incidence is closed, so the limits are strongly parabolic flags on the same unweighted Higgs field. Their weighted stability at the limit is immaterial for evaluation of the tautological one-form.

Over the nonzero Rees parameter each flagged family is isomorphic to its original family pulled back from U. Its tautological form in the U directions therefore specializes to that of its limiting family, by regularity on the cotangent stack. It remains to compare the two limiting flags. Shrink to a fixed relative position of the two flags. A deformation of a bundle with this pair of flags is

represented in $H^1(\mathcal{P}_{12})$, where $\mathcal{P}_{12}$ has value in the intersection of their two Borels at the mark. This follows by using local frames for the pair, whose fixed-position orbit is homogeneous with that intersection as stabilizer. For the individual flags the maps on deformations are induced by $\mathcal{P}_{12} \subset \mathcal{P}_1, \mathcal{P}_2$. Pulling the two cotangent vectors back to $\mathcal{P}_{12}^{\vee} \otimes K_C$ gives the same meromorphic Higgs field in both cases. Their Serre pairings with every deformation of the pair agree. This proves the equality of the one-forms and hence of their differentials on U.

Equality of rational differential forms descends along the generically finite separable cover. Thus at its general smooth point the lifted component above E is isotropic for the difference of the two symplectic forms on the product of the charts. The latter product has dimension $2d$ for $d = 2R + 2b$, so $\dim E \leq d$. This tests components above strictly polystable classes, including pairs with different underlying extensions. The diagonal attains dimension d.

Specialization of the dimension bound. Form the relative fiber product along $u = 0$ over the entire λ-line at this mark. Division by λ identifies its restriction over $\mathbb{G}_m$ with the product of its fiber at one and $\mathbb{G}_m$. For a component of dimension e at one take the closure of this product component. Each member of a pair has a scaling limit by Lemma 5.20, and the two limits have the same image by separatedness of the unweighted target. Hence the closure meets the fiber at zero. On this irreducible closure of dimension $e + 1$, the zero parameter cuts a nonempty effective Cartier divisor on the reduced structure. Its dimension is e, so the Higgs bound gives $e \leq d$. The diagonal again gives equality. This proves the assertion for all λ. □

Lemma 5.23. *Let $g \geq 2$, fix a simple scalar pair (X_z, B_z) and fix the mark p. For the scalar Hodge family $\mathcal{X}$ of Construction 5.18, preferred harmonic sections give a homeomorphism $X_z \times \mathbb{C} \longrightarrow \mathcal{X}^{\mathrm{an}}$ over $\mathbb{C}$. Its restriction at zero is the identity. Its restriction at one followed by logarithmic monodromy is Ψ.*

Proof. We construct the two directions on harmonic bundles with their descent isomorphisms. We then check the topology of the quotient by taking limits in unitary frames.

Harmonic representatives with descent. Remove the central divided operator as in Construction 5.18. Take a connected étale cover of C of degree n_z. The pullback of $\mathcal{O}_C(p)$ has a root since it has degree n_z and multiplication by n_z on Pic^0 is surjective. This gives a finite étale curve atlas of the root gerbe. Take a Galois refinement $\widetilde{C}$ with deck group Γ. A scalar object on the gerbe consists of an object on $\widetilde{C}$ and isomorphisms from its deck pullbacks satisfying the cocycle identity and the prescribed inertia condition.

Fix a maximal compact subgroup $K_G \subset G$ and a faithful representation on a Hermitian space V with $K_G = G \cap U(V)$. On $\widetilde{C}$ write $\partial_A + \bar{\partial}_A$ for the unitary connection of a harmonic representative, ϕ for its Higgs field and ϕ^* for the metric adjoint. The preferred operators of Proposition 5.9 are

$$\bar{\partial}_\lambda = \bar{\partial}_A + \lambda \phi^*, \qquad D_\lambda = \lambda \partial_A + \phi.$$

Here $\bar{\partial}_A \phi = 0$ and $F_A + [\phi, \phi^*] = 0$ are the harmonic bundle equations. For a metric k realizing these operators let $\partial_{\lambda,k}$ denote the $(1,0)$ Chern operator of $(\bar{\partial}_\lambda, k)$. It equals $\partial_A - \bar{\lambda}\phi$, so the inverse on metric representatives is

$$\phi = \frac{D_\lambda - \lambda \partial_{\lambda,k}}{1 + |\lambda|^2}, \qquad \bar{\partial}_A = \bar{\partial}_\lambda - \lambda \phi^*, \qquad \partial_A = \partial_{\lambda,k} + \bar{\lambda}\phi.$$

The formulas preserve direct sums. If a stable vector Higgs summand had strictly semistable value at λ, take the preferred sections through the stable factors of its graded there. Their direct sum would have the same coarse value at λ. Injectivity in Proposition 5.9 would then decompose

the original stable summand at zero. Thus the formulas take polystable vector representatives to polystable representatives.

To recover the extra structure, take a smooth section σ of an induced tensor or Hom bundle, using its induced operators and L^2 norms. Integration by parts on the compact curve gives

$$\|\bar\partial_\lambda\sigma\|^2 + \|D_\lambda\sigma\|^2 = (1+|\lambda|^2)(\|\bar\partial_A\sigma\|^2 + \|\phi\sigma\|^2),$$
$$\|\bar\partial_A\sigma\|^2 + \|\phi\sigma\|^2 = \|\partial_A\sigma\|^2 + \|\phi^*\sigma\|^2.$$

The mixed terms in the first line cancel by $\bar\partial_A\phi = 0$. In the second line the commutator of the two covariant derivatives contributes F_A, which cancels the commutator of the Higgs terms. Thus every holomorphic λ-horizontal tensor or morphism is A-parallel and intertwines both Higgs operators. This applies to each defining tensor of G and to each map from a deck pullback, with the pulled-back harmonic data on its domain. These same maps give the descent isomorphisms at every parameter, with the same cocycle and inertia action.

In particular, start with a polystable scalar λ-object. Its associated vector object on $\widetilde{C}$ is polystable. At zero this follows by harmonic pullback. At nonzero λ it follows by restricting reductive holonomy to a finite-index subgroup and applying the representation. Use Proposition 5.9 on this vector object to obtain a metric representative of its preferred section. By the preceding identities its flat connection $\partial_A + \bar\partial_A + \phi + \phi^*$ at one descends with the specified G-tensors. The cover holonomy has finite index in the gerbe holonomy, so their Zariski closures have the same reductive identity component. Proposition 5.10 supplies a harmonic K_G-reduction for this descended flat object. On pulling its metric back, the identity between the two flat realizations at one intertwines their unitary connections and Higgs operators by the Hom calculation. Thus the new metric realizes the original λ-object and makes the deck maps isometries. Conversely an isomorphism at any λ intertwines all the recovered data. Its positive factor in the polar decomposition in G is a parallel automorphism commuting with both Higgs terms and the descent data. Its unitary factor therefore gives the same identification. This identifies coarse points with orbits of K_G-valued gauges also at reducible objects. After quotienting by gauges fixed at a reference frame, the remaining frame changes form the compact group K_G.

The coarse topology at varying parameter. Smooth convergence of the operators, tensors and deck maps in local frames gives convergence in the analytic coarse space. A partition of unity in the frames realizes the vector bundles as images of convergent smooth projections in a fixed trivial bundle, so nearby smooth bundles can be identified. Choose a common sufficiently positive twist so that the limiting vector bundles used for the data are generated by sections and have vanishing H^1. Nearby twisted Dolbeault operators are still surjective on $(0,1)$-forms. A bounded right inverse at the limit gives continuous projections onto their kernels and hence convergent bases of sections. Evaluation gives convergent Quot coordinates, by taking sections of the relations after a further twist. The λ-operator is a bundle map from the first jet bundle to the bundle of one-form values with symbol λ. Its matrices on twisted sections converge, as do those of the tensor and deck morphisms. These coordinates give convergence in the parameter space for the bundles with these data and hence in its coarse quotient.

For the converse compactness check, consider a convergent sequence of coarse points of $\mathcal{X}$, allowing λ to vary. Pullback and the representation give a morphism to the general linear Hodge moduli of $\widetilde{C}$. Indeed they preserve the polystable representatives just described, and the induced filtration of a semistable object has that vector graded as its polystable limit. By the coarse homeomorphism in Proposition 5.9 the vector Higgs points on the preferred sections stay in a compact set. In particular the characteristic coefficients of their ϕ are bounded. The same bound applies to a convergent sequence in X_z at zero.

Fix a metric on $\widetilde{C}$ and denote the associated vector bundle of a harmonic representative by $\mathcal{V}$. At a maximum of $|\phi|^2$ the Bochner formula for the holomorphic section ϕ of $\mathrm{End}(\mathcal{V}) \otimes K_{\widetilde{C}}$ gives $|[\phi, \phi^*]|^2 \leq c|\phi|^2$. Here the curvature of $\mathcal{V}$ is $-[\phi, \phi^*]$ and the curvature of $K_{\widetilde{C}}$ supplies the fixed bound c. If the maxima tended to infinity, divide the field at these points by its norm in unitary frames. A subsequential limit would have norm one, all eigenvalues zero by the bounded characteristic coefficients, and zero commutator with its adjoint. A normal nilpotent matrix is zero. This proves uniform bounds for ϕ and hence for F_A.

In radial K_G-frames on coordinate disks, integrating curvature along radii bounds the connection matrices. The equation $\bar{\partial}_A \phi = 0$ and interior $\bar{\partial}$ estimates bound ϕ in $W^{1,q}$ for $q > 2$, hence in a Hölder norm on smaller disks. The radial formula applied to $F_A = -[\phi, \phi^*]$ gives the same Hölder bound for the connection matrices. Iterating the interior estimates gives bounds for all derivatives on smaller disks. On overlaps, transition matrices and deck isometries take values in K_G. Their derivatives are determined by the connection matrices they intertwine. Compactness of K_G and these bounds give a subsequence converging smoothly on a finite Γ-invariant disk cover, including all its transition and deck maps. The limiting data satisfy the harmonic equations, the defining tensor identities and the descent identities.

Let $\mathcal{H}$ be the space of these descended harmonic data with the C^∞ topology modulo unitary gauge. The maps $\mathcal{H} \to X_z$ at zero and $\mathcal{H} \times \mathbb{C} \to \mathcal{X}^{\mathrm{an}}$ at varying parameter are continuous bijections by the operator and orbit descriptions. For a convergent sequence in either coarse space, every subsequence has a further subsequence converging in gauges as just proved. Its coarse class is the prescribed limit by separatedness, and the polar decomposition identifies its unitary orbit uniquely. Thus the inverse to each bijection is sequentially continuous. The coarse spaces are first countable, so both maps are homeomorphisms. At zero the operators are the original Higgs pair. At one their sum is its harmonic flat connection. Restoring the central divided operator and using the horizontal root off p gives precisely the lifted handles defining Ψ. □

Theorem 5.24. *For $g \geq 2$ and every simple scalar pair, the flagged stack of Construction 5.18 is a smooth orbifold over the marking, λ, and ordered residue parameters. Restriction from its coarse family to parameter fibers and lines through zero induces rational cohomology isomorphisms. For fixed marking the ordinary Betti direct images of the flagged incidence have fiberwise base change and are local systems near z and on a dense open of generic regular residues.*

At $u = 0$ the projective forgetful morphism to the unweighted image is relatively semismall and its fixed-mark Betti source is an irreducible symplectic orbifold, birational to that image. The scalar intersection complex has multiplicity one. Its normalized inclusion J_λ and projection L_λ change ordinary degree by $2b$ and $-2b$, respectively, and satisfy $L_\lambda J_\lambda = (-1)^b |\mathcal{W}| \mathrm{id}$. They agree at the two endpoints under the specified harmonic comparison. On the fixed-mark Betti complexes they are Hodge-module maps with opposite Tate shifts and are natural in the diagram of marking charts, whose finite derived limit computes cohomology with moving marking.

Proof. We identify the regular Betti fibers and the irreducible scalar incidence, normalize the scalar summand on its transverse flag fiber, and compare the maps along the Hodge line and on marking charts.

Restriction and the regular Betti family. Lemma 5.20 gives smoothness and the restriction isomorphisms, and Lemma 5.21 gives ordinary base change near z. For regular t, fixing the Cartan order selects one Borel for each peripheral element conjugate to t. After conjugating that element to t the residual group is T. If the tuple were contained in a proper parabolic, that parabolic would contain the centralizer torus of its regular peripheral element and t would belong to its derived subgroup. Outside these finitely many proper torus loci the underlying tuple

is stable. Thus $\mathcal{B}_t = Y_t$ there, also in families, and Theorem 3.5 supplies the dense open with ordinary base change. Its fibers have Hodge–Tate cohomology. By Propositions 5.16 and 5.13 the algebraic ordinary direct image on its lisse locus yields variations. Their fibers carry the native cohomology structures because the Hodge-module base-change map realizes the ordinary stalk map and rational realization detects isomorphisms. The weight-graded Hodge numbers are locally constant. The central disk meets the dense open, so the fixed-mark cohomology at z is Hodge–Tate too.

Irreducibility at the scalar parameter. At a fixed mark choose a local coordinate w and let $\mathcal{I}$ be the Iwahori of H, the integral loops reducing to the flag Borel. We identify its flagged bundle stack with prescribed abelianization $\mathcal{O}_C(x(s))$ with the Iwahori bundle stack for G. Since the kernel torus T is connected, there is a cocharacter of the torus of H of χ_H-degree one. Compose its translation with an element of the affine Weyl group taking its image alcove back to the fundamental alcove. Coroot translations and finite Weyl elements have χ_H-degree zero. A loop representative of the resulting element preserves the integral torus and permutes the positive affine root groups, hence normalizes $\mathcal{I}$. Its image under χ_H is w up to a unit. Modify the gluing frame at the mark by this loop, choosing the direction that changes $\mathcal{O}_C(x(s))$ to $\mathcal{O}_C$. Changes of adapted frame lie in $\mathcal{I}$, so the modified bundle and flag are well defined and the inverse loop undoes the construction. In a faithful representation the modified lattices are bounded between fixed powers of w times the original lattice, so this modification is algebraic in finite jets and respects the defining tensors. A trivialized abelianization gives precisely a G-bundle.

By Proposition 5.5 the cotangent stack of this bundle stack is irreducible. At $(0,0)$ its Higgs fields have nilpotent radical residue, with unrestricted Hitchin coefficients. Weighted stability cuts out a nonempty open by Lemma 5.20. The fixed-mark $Y_{0,0}$ is consequently irreducible. Restriction in H^0 gives connectedness at $(1,0)$ and the logarithmic comparison gives connectedness of $\mathcal{B}_z$. Its finite smooth quotient charts are locally irreducible, so $\mathcal{B}_z$ is irreducible.

The symplectic incidence and its image. The Higgs orbifold has the cotangent form used in Lemma 5.22. On the Betti side take the puncture space of Proposition 5.7 with moment M and Borel Cartan label z. Invert its moment and negate its form. Fuse it with the handle doubles in the order of the surface word. The resulting identity level is $KM^{-1} = 1$, so the incidence has exactly the prescribed peripheral element, also when z has order greater than two. The inversion and fusion identities of Proposition 5.8 are identities of invariant forms. They apply with the complex puncture form and the complex doubles of Proposition 5.14. All the Maurer–Cartan forms and group operations in the formulas are algebraic. At the identity the moment identity pairs its differential with the infinitesimal action by the invariant form. The annihilator of its image is the infinitesimal stabilizer, which vanishes on the stable locus. Here the level is regular and the radical of the restricted form is the orbit tangent. It descends to a closed nondegenerate algebraic form on the finite quotient charts of the stable orbifold. Division by λ and the logarithmic comparison identify the fixed nonzero Hodge fibers analytically with this incidence.

Write q for flag-forgetting to its unweighted image at $u = 0$, including the specified parameters. Lemma 5.22 gives the dimension $d = 2R + 2b$ for the fiber product at every (s, λ). Its relative fiber product over the whole image has dimension at most $d + \dim S + 1$, the dimension of the source, and the diagonal attains this value. This proves relative semismallness. The same bound holds on the fixed-mark Betti image by Lemma 5.21.

Near a regularly stable scalar object the transverse diagram is the Springer map

$$T^*(G/B_+) = \{(P, N) : N \in \operatorname{Lie} R_u(P)\} \longrightarrow \{N \in \operatorname{Lie} G : N \text{ nilpotent}\}, \qquad (P, N) \longmapsto N,$$

where $R_u(P)$ is the unipotent radical. For a framed tuple ρ at $K = z$, the derivative of the handle relator is the last cellular coboundary for its adjoint local system. Its cokernel $H^2(C, \mathrm{ad}\rho)$ is dual to the infinitesimal centralizer, which vanishes at a regularly stable tuple. Thus the peripheral map is submersive, and the logarithm of $z^{-1}K$ pulls back the displayed diagram smoothly. On the Higgs side the corresponding residue boundary map takes values in the ordinary obstruction space, dual to the same space of infinitesimal automorphisms. More explicitly, allowing a logarithmic pole enlarges the last term of the ordinary deformation complex by its residue skyscraper $\mathrm{ad}E_{x(s)}$. The connecting map lands in the vanishing second hypercohomology. With a frame this gives a submersion in the residue variables. These descriptions descend through the framing since the scalar stabilizer acts trivially. They hold relatively along the regularly stable scalar locus. In the Betti diagram regular nilpotent logarithms therefore occur with stable underlying tuple. A regular nilpotent belongs to a unique Borel, and its flag varies algebraically on its orbit because the centralizer preserves that flag. This gives an inverse to q on a nonempty open. Irreducibility of the source gives the claimed birationality. The normal Betti source also maps to the normalization of its image. This factor is projective by taking its graph in the base change of q to the normalization. Its fiber product is contained in the fiber product over the image, so the same semismall bound holds there.

The transverse Springer model also identifies the point fibers on this regularly stable neighborhood. Its map factors through the normalization of the nilpotent cone as a symplectic resolution. Indeed the cotangent moment map identifies its form on the regular orbit with the orbit symplectic form. The other nilpotent orbits have even dimension and codimension at least two. The orbit form and its nonvanishing volume thus extend on the smooth locus of the normalization, and the cotangent form extends them on the resolution. Proposition 5.17 gives even, pure diagonal cohomology for the fibers in this neighborhood.

The normalized scalar maps. The scalar locus is closed in the unweighted image. The condition of scalar residue is closed in the unweighted stack, and Proposition 5.15 identifies its good moduli image. The flag image contains the dense regularly stable scalar locus and is closed by projectivity of q, so it contains the whole scalar locus. Each scalar component has fiber dimension $2R$ over the marking and λ, by Proposition 5.11 and the harmonic comparison. Over its regularly stable open the flag fiber is G/B_+, of dimension b, exactly half the codimension $2b$ of the scalar support.

Apply Proposition 5.12 on the total Hodge line at $u = 0$, and also at fixed marking to the Betti morphism. The orbifold source is a rational homology manifold, as follows from finite invariants and complex orientation on its charts. Its shifted constant Hodge module is its pure intersection complex. Write d_q for the dimension of the source of q. For a stratum of dimension a, semismallness bounds the dimension of its fiber by $(d_q - a)/2$. Proper base change therefore makes the stalk of the shifted direct image vanish in degrees greater than $-a$. Duality gives the costalk bound, so the direct image is perverse. At the dense scalar stratum its top fiber cohomology is the constant line $\mathbb{Q}(-b)$, with the flag orientation. In perverse degree zero the strict stalk inequality of middle extension on the boundary makes this top group the multiplicity system on that stratum. Thus the scalar summand is the strict-support extension of this one line, with multiplicity one on each scalar component.

On the regularly stable scalar open write π for the flag bundle and j for its inclusion in the flagged source over the corresponding open of the target. Normalize the two maps there by $j_*\pi^*$ (Gysin) and $\pi_* j^*$ (integration). The Springer description identifies the normal bundle of j with T^*_π. Thus the scalar by which their composite acts is

$$\int_{G/B_+} c_b(T^*(G/B_+)) = (-1)^b \chi(G/B_+) = (-1)^b |\mathcal{W}|,$$

the last equality coming from the Bruhat cells. Purity, strict support and multiplicity one extend each map uniquely in the semisimple direct image. Write I^H and $\mathbb{Q}^H$ for the Hodge-module complexes with the indicated rational realizations. On the fixed-mark Betti target with scalar inclusion $i : B_z \hookrightarrow \operatorname{im} q$, the maps in ordinary normalization are

$$i_* I^H_{B_z}(-b)[-2b] \xrightarrow{J_1} Rq_* \mathbb{Q}^H_{\mathcal{B}_z} \xrightarrow{L_1} i_* I^H_{B_z}(-b)[-2b].$$

In particular J_1 maps $\mathrm{IH}^m(B_z)$ into $H^{m+2b}(\mathcal{B}_z)(b)$, and L_1 maps $H^m(\mathcal{B}_z)$ into $\mathrm{IH}^{m-2b}(B_z)(-b)$. At every scalar point ξ, proper base change takes this pair to maps between $(I^H_{B_z})_\xi(-b)[-2b]$ and $R\Gamma(q^{-1}(\xi)_{\mathrm{red}}, \mathbb{Q}^H)$. Their composite is the same nonzero scalar, so the shifted stalk is a summand of the cohomology complex of the full fiber, also for a strictly polystable ξ. The same dimension shifts on the total Hodge line give J_λ, L_λ after restriction. The computed scalar determines $L_\lambda J_\lambda$ on the whole intersection complex.

Comparison along the Hodge line. Lemma 5.23 identifies the fixed scalar family with the product over λ and its endpoint map with Ψ. The identification preserves stabilizers and stable-factor decompositions by the calculation on Hom bundles. In the product the ordinary-normalized intersection complex is $I_{X_z} \boxtimes \mathbb{Q}_{\mathbb{C}}$. For a moving mark, tensoring by $\iota(\mathcal{D})$ gives the same central corrections at both endpoints. Thus the restriction comparison on scalar intersection cohomology is Ψ^* at fixed mark and $(\mathrm{id}_S \times \Psi)^*$ over S.

Proper base change restricts the total $Rq_*\mathbb{Q}$ to the actual direct image at each λ. The scalar product just obtained restricts its I to the actual fiber I in ordinary normalization. On the regularly stable open transverse base change for j and proper base change for π restrict the total maps to the flag Gysin and flag integration. Multiplicity one then identifies their restrictions on every stratum with the normalized fiber maps. At one the analytic logarithmic isomorphism also preserves both complex orientations and the forgetful diagram. If Φ_Y is the comparison from the Betti fiber to the Higgs fiber obtained by logarithmic pullback and the two restriction isomorphisms of Lemma 5.20 on the $u = 0$ line, this gives at a fixed mark

$$\Phi_Y J_1 = J_0 \Psi^*, \qquad \Psi^* L_1 = L_0 \Phi_Y.$$

The same identities hold with the product scalar comparison when the mark varies.

Descent on marking charts. Take a finite good cover of S by surface marking charts, with contractible nonempty intersections U_I. Index them by their nonempty index sets, with an arrow $I \to J$ for $I \subset J$. An isotopy carrying the reference puncture to $x(s)$ trivializes the marked surface over U_I. Allowing all incident Borels gives a product with a fixed-puncture algebraic family $p_I : \mathcal{B}_I \to T$. On an overlap let γ_{JI} denote the change from the J model to the I model. It substitutes words on the handles. If the peripheral word changes to cKc^{-1} it sends the Borel to cPc^{-1}. It also multiplies the lifted handles by the central holonomies prescribed by the root trivializations. These are locally constant algebraic isomorphisms, valid also for a nonsemisimple peripheral element. An inner change disappears in the simultaneous quotient. The centralizer of the peripheral word in the free group is cyclic by translation on its axis in the Cayley tree. Each positive handle letter occurs once in the cyclically reduced word, so the word generates its centralizer. Two choices of peripheral conjugator therefore differ by a peripheral power, which preserves the incident Borel. The central corrections compose by tensor product. This gives the cocycle identities on the coarse spaces.

Assign to I the complex $R(p_I)_* \mathbb{Q}^H_{\mathcal{B}_I}$ over T, with arrows given by pullback under γ_{JI}. At z first keep the complexes on the unweighted target Z_I of the chart. With $q_I : \mathcal{B}_{I,z} \to Z_I$ and scalar

inclusion $i_I : B_{z,I} \to Z_I$, put

$$\mathcal{C}_I = (i_I)_* I^H_{B_{z,I}}(-b)[-2b], \qquad \mathcal{K}_I = R(q_I)_* \mathbb{Q}^H_{\mathcal{B}_{I,z}}.$$

On Z_J the pullback identifications for every arrow give the commutative diagram

$$\begin{array}{ccccc} \gamma_{JI}^* \mathcal{C}_I & \xrightarrow{\gamma_{JI}^* J_I} & \gamma_{JI}^* \mathcal{K}_I & \xrightarrow{\gamma_{JI}^* L_I} & \gamma_{JI}^* \mathcal{C}_I \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ \mathcal{C}_J & \xrightarrow{J_J} & \mathcal{K}_J & \xrightarrow{L_J} & \mathcal{C}_J. \end{array}$$

Indeed the word maps and central twists preserve flag pullback, Gysin and integration on the regularly stable locus. Strict support and multiplicity one give both equalities everywhere. After the common dimension shift these complexes are in the perverse heart. Negative Ext groups between them vanish, so their mapping spaces are discrete and the compatible squares give coherent maps of diagrams. Taking ordinary direct image to a point preserves these maps by Proposition 5.16.

Finally, ordinary base change on the disk and the generic open identifies the realization of the pullback of $R(p_I)_* \mathbb{Q}^H$ at t with $R\Gamma(\mathcal{B}_{I,t}, \mathbb{Q})$. Write $\mathcal{B}_t^{\mathrm{mov}}$ for the total Betti space over S at fixed t. Since each U_I is contractible, descent gives

$$R\Gamma(\mathcal{B}_t^{\mathrm{mov}}, \mathbb{Q}) \simeq \operatorname*{holim}_I R\Gamma(\mathcal{B}_{I,t}, \mathbb{Q}).$$

In Čech degree a one takes the product of the complexes on the nonempty $(a+1)$-fold intersections, with the alternating pullbacks as differential. Rational realization commutes with this finite totalization. For the product scalar space put

$$A_S = C^\bullet(\mathrm{Nerve}\{U_i\}, \mathbb{Q}) \otimes R\Gamma(B_z, I^H_{B_z}).$$

Its rational realization computes the intrinsic, ordinary-normalized intersection cohomology of $S \times B_z$. Each marking cochain has weight zero in the diagram, since each chart contributes $\mathbb{Q}(0)$ and its cochain degree comes from the nerve. The scalar term $\mathcal{C}_I$ of the flag diagram also carries $(-b)[-2b]$. This twist and shift commute with the finite derived limit, giving

$$\operatorname*{holim}_I R\Gamma(Z_I, \mathcal{C}_I) \simeq A_S(-b)[-2b].$$

Taking global sections and the derived limit of the two chart maps therefore gives

$$A_S(-b)[-2b] \xrightarrow{J_1} \operatorname*{holim}_I R\Gamma(Z_I, \mathcal{K}_I) \xrightarrow{L_1} A_S(-b)[-2b].$$

For a single mark, both scalar terms in degree $m + 2b$ are $\mathrm{IH}^m(B_z)(-b)$. For every marking cover, the composite is $(-1)^b |\mathcal{W}| \mathrm{id}$ by the chartwise normalization. □

6 Lefschetz and rigidity on the scalar summand

We fix the mark in Construction 5.18. On the generic regular open of Theorem 4.5, the half-weight Lefschetz center is $R + b$. We show that the top class of the flag accounts for b, and identify the operator on the scalar summand with the determinant of the adjoint bundle. We will also need to detect a degree-two morphism of the scalar intersection complex on the regularly stable locus.

The passage from differential forms on a symplectic orbifold to the cohomology of a fiber uses the filtered comparison of [Sai00, Theorems 0.1 and 0.2].

Proposition 6.1. *The filtered de Rham realization of the constant Hodge-module complex of a complex algebraic variety is identified with its Du Bois complex, compatibly with proper direct image.*

We use formal functions for the individual sheaves of forms [Gro61, III, Theorem 4.1.5] and formal de Rham comparison for the whole complex [Har75, Chapter IV, §1, Theorem 1.1].

Proposition 6.2. *For a proper morphism of noetherian schemes and a coherent sheaf, the completion of each higher direct image along a closed subset is computed by formal cohomology along its inverse image.*

Proposition 6.3. *For a complex algebraic scheme embedded as a closed subscheme of a smooth scheme, the de Rham complex of the completed ambient space computes the ordinary complex cohomology of its analytification.*

On the reduced fiber we use its intrinsic mixed Hodge structure [Del74, §8.2, Theorem 8.2.4(iii)].

Proposition 6.4. *Ordinary cohomology in degree m of a proper complex algebraic variety has weights at most m. Its Hodge filtration on a proper smooth simplicial resolution is computed by the degree of holomorphic forms.*

The Hodge type of the adjoint determinant will come from the evaluation class on the stack of local systems [She17, main theorem].

Proposition 6.5. *For a complex linear algebraic group, the surface slant of a degree-four characteristic class on the Betti local-system stack of a closed surface lies in weight at most four and in both Hodge filtration steps two.*

For restriction to an open set we use the strict inequalities for a middle extension. For the flag and Hitchin maps we use the projective direct-image theorem [BBD82, §1.4.23–1.4.25, Theorems 6.2.5 and 6.2.10].

Proposition 6.6. *On the boundary of a middle extension the ordinary stalk and costalk inequalities for middle perversity are strict. Projective direct images of the intersection complex decompose into shifted perverse summands and satisfy relative hard Lefschetz for a relatively ample class.*

The transverse singularity along the genus-two reducible stratum also has a description by a surface Ext quiver [DHM25, Theorems 5.1 and 13.4].

Proposition 6.7. *At a semisimple direct sum of a Σ-collection in the right two-Calabi–Yau setting with good moduli, the stack and coarse analytic germs agree with those of the associated preprojective quiver at zero. For a totally negative quiver and a dimension vector in its set Σ_Q, the intersection Poincaré polynomial contains only even powers.*

To recover an ordinary degree-two class from its smooth restriction we use the local fundamental group theorem [Bra21, Theorem 1].

Proposition 6.8. *For a complex Kawamata log terminal germ, the topological fundamental group of the regular part of a sufficiently small neighborhood is finite.*

The sign of the determinant polarization is fixed by the Plücker line on the sheaf moduli space [Sim94b, Theorems 1.19 and 1.21]. We call a Higgs bundle *semiharmonic* if it is potentially semistable and its rational characteristic classes vanish.

We use the spectral construction and finiteness under extension of structure group in the following form [Sim94, Lemmas 6.8 and 9.14, Corollaries 6.9 and 9.19].

Proposition 6.9. *Semistable Higgs vector bundles on a smooth projective curve admit the spectral sheaf construction with the Plücker linearization in the sheaf moduli construction. On coarse semiharmonic moduli extension of structure group by a homomorphism of reductive groups with finite kernel gives a finite morphism onto its image.*

Two comparisons apply directly to scalar SL_2. The coprime-degree comparison is [MS24, Theorems 0.2 and 0.3], and the comparison in the untwisted genus-two case is [FM22, Main Theorem].

Proposition 6.10. *In genus at least two, coprime-rank-and-degree general linear moduli satisfy the perverse–weight comparison. For coprime determinant degree and prime matrix rank the fixed-determinant comparison holds as well.*

Proposition 6.11. *For the untwisted general and special linear coarse moduli in genus one, and in genus two and matrix rank two, the normalized Hitchin and half-weight filtrations on intersection cohomology agree under the harmonic homeomorphism.*

For a smooth symplectic resolution the fiber calculation is Proposition 5.17 [Kal06, Theorem 2.12]. The flag source can have finite quotient singularities. In that case invariant differential forms give the required estimate.

Proposition 6.12. *Let $q : Y \to Z$ be projective and semismall with Y an irreducible algebraic symplectic orbifold. The reduced fibers of q have cohomology only in even degrees, and their degree $2a$ cohomology is a direct sum of $\mathbb{Q}(-a)$. If Z is normal and q is birational and surjective, $\mathrm{Gr}_4^W H^2(Z) \to \mathrm{Gr}_4^W H^2(Y)$ is surjective.*

Proof. We bound the direct images of orbifold forms, pass to the Hodge filtration of a proper fiber, and use its degree-two cohomology in the Leray sequence.

Vanishing for forms. Put $d = \dim Y$, and denote the sheaf of invariant a-forms on orbifold charts by $\Omega_Y^{[a]}$. A finite complex-linear group preserves the orientation of a sphere in its representation space. The rational cohomology of the quotient sphere is the invariant cohomology, by finite pushforward and averaging. Thus $\mathrm{IC}_Y = \mathbb{Q}_Y[d]$. On each finite smooth chart the invariant summand of the direct image of the constant Hodge module is the constant Hodge module on the quotient. Taking its filtered de Rham complex in Proposition 6.1 identifies the a-th Du Bois piece with $\Omega_Y^{[a]}$ in degree zero.

Semismallness gives the perverse support inequality for $Rq_*\mathbb{Q}_Y[d]$ by proper base change, and self-duality gives the cosupport inequality. Hence the projective direct image is a single perverse Hodge module M by Proposition 5.12 [Sai88, Theorem 5.3.1]. Filtered de Rham comparison gives

$$Rq_*\Omega_Y^{[a]}[d-a] \simeq \mathrm{Gr}_{-a}^F \mathrm{DR}(M).$$

In a smooth ambient space the complex on the right ends in ordinary degree zero, as one sees from the de Rham complex of the filtered $\mathcal{D}$-module. Thus $R^p q_*\Omega_Y^{[a]} = 0$ for $p + a > d$. The symplectic form identifies a-forms with $(d-a)$-forms equivariantly on the charts. Applying the same bound with $d - a$ gives vanishing also for $p > a$.

The Hodge filtration on the fiber. Fix a closed point $w \in Z$ and let $E = q^{-1}(w)_{\mathrm{red}}$. Take a finite proper hyperresolution $\varepsilon : Y_\bullet \to Y$ by smooth varieties. In each term complete the de Rham complex along the reduced inverse image $E_\bullet$ of w. Proposition 6.3 identifies its hypercohomology termwise with the cohomology of $E_\bullet$. Proper base change for cohomological descent identifies the

cohomology of the total complex with $H^*(E,\mathbb{C})$. The spectral sequence for its filtration by form degree has first page

$$E_1^{a,p} = (\widehat{R^p q_* \Omega_Y^{[a]}})_w \quad \Longrightarrow \quad H^{a+p}(E,\mathbb{C}).$$

Indeed, Proposition 6.2 applies to the forms on each smooth term. The ideals of the schematic and reduced fibers define the same completion. Completion is exact on the coherent direct-image stalks, so it commutes with taking cohomology of the finite diagram, in which $R\varepsilon_* \Omega^a_{Y_\bullet}$ is the a-th Du Bois piece.

This formal filtration maps into the intrinsic Hodge filtration. To see the inclusion, resolve the proper diagram $E_\bullet$ by a smooth proper hyperresolution. Restriction of completed forms to its terms preserves form degree and induces the preceding comparison on cohomology. Form degree on this last diagram computes the Hodge filtration by Proposition 6.4. Since the first page vanishes for $p > a$, we obtain $F^{\lceil k/2 \rceil} H^k(E,\mathbb{C}) = H^k(E,\mathbb{C})$. Properness gives weights at most k. On each weight graded both Hodge indices are at least $\lceil k/2 \rceil$, by conjugation. For odd k there is no such Hodge type. For $k = 2a$ the resulting pure structure of type (a,a) is a sum of $\mathbb{Q}(-a)$.

Descent in weight four. In the birational case with normal target, Stein factorization gives $q_* \mathbb{Q}_Y = \mathbb{Q}_Z$. The fiber calculation and proper base change give $R^1 q_* \mathbb{Q}_Y = 0$. Leray now identifies the kernel of the edge map

$$H^2(Y) \longrightarrow H^0(Z, R^2 q_* \mathbb{Q}_Y)$$

with $q^* H^2(Z)$. Evaluation at the point fibers separates sections. Finitely many fibers separate the finite-dimensional image of the edge map, since each additional evaluation can be chosen to decrease the dimension of the remaining kernel. The cokernel of q^* thus embeds by algebraic restrictions in a sum of copies of $\mathbb{Q}(-1)$. Weight strictness gives the asserted surjection on Gr_4^W by Proposition 4.3 [Del71, Theorem 2.3.5(iii)]. □

We next give two restriction tests on the Betti space. The first keeps track of the shifts on the boundary of a middle extension. The second will let us identify an ordinary class in the mixed Hodge structure of the coarse space.

Lemma 6.13. *Let $g \geq 2$, let G be simply connected and simple, and let $z \in Z(G)$. Write $j : V \hookrightarrow B_z$ for the regularly stable open and $i : B_z \setminus V \hookrightarrow B_z$ for its complement. Then*

$$i^* \mathrm{IC}_{B_z} \in {}^{\mathrm{p}}D^{\leq -2}, \qquad i^! \mathrm{IC}_{B_z} \in {}^{\mathrm{p}}D^{\geq 2}.$$

Moreover $j^* : H^2(B_z,\mathbb{Q}) \to H^2(V,\mathbb{Q})$ is injective.

Proof. We determine the parity of the stalks and of their support dimensions. We then use the local fundamental group for the assertion about ordinary cohomology.

Even stalks. Write $q : Y \to Z$ for the fixed-mark flag map at the scalar parameter. Theorem 5.24 puts $I_{B_z}[-2b](-b)$ in $Rq_* \mathbb{Q}_Y$ as a direct summand supported on $B_z \subset Z$. Over V the fiber is the full flag variety, and its complex orientation trivializes the top cohomology local system. By proper base change the shifted stalk at every scalar point is a summand of the cohomology of its fiber. Proposition 6.12 therefore gives even stalks. Since $\dim B_z = 2R$ by Proposition 5.11 [Shu25, Corollary 4.5 and Proposition 6.1], the perverse normalization also has even stalks.

Even dimensions on the boundary. We stratify by the stabilizer of a closed tuple. On the smooth space $M = G^{2g}$ of handles, let K be the commutator product and Ω the fused two-form of Proposition 5.14 [Boa14, Example 2.4 and Theorems 2.6–2.7]. At a tuple x with $K(x) = z$ the form is nondegenerate by the quasi-Hamiltonian kernel identity. Put $A = G_x$, $\mathfrak{a} = \mathrm{Lie}\, A$, and

$T_{\mathrm{orb}} = T_x(Gx)$. The moment identity at the central value identifies $d\log(z^{-1}K)_x$ with the dual of the infinitesimal orbit map under Ω_x and the trace form. Its image is $\mathfrak{a}^\circ \subset (\operatorname{Lie} G)^*$ and its kernel is $T_{\mathrm{orb}}^\perp$. The orbit lies in the level, so T_{orb} is isotropic.

The stratum of x is locally the locus with stabilizer A in $M^A \cap K^{-1}(z)$, modulo $N_G(A)/A$. This description follows from the étale quotient chart at the closed orbit in Proposition 5.2 [AHR20, Theorem 4.12], where points of the same type are the fixed points of the slice group. The fixed locus M^A is smooth with tangent space $(T_xM)^A$, as one sees by averaging local coordinates over a compact form of A. On it the logarithmic moment takes values locally in $(\mathfrak{a}^\circ)^A$. Indeed K commutes with A, and pairing the moment identity with $\xi \in \mathfrak{a}$ gives $d\langle \log(z^{-1}K), \xi\rangle = 0$ there. The differential surjects onto this space by exactness of A-invariants. Thus the fixed level is smooth near x. Dividing by the normalizer removes the tangent directions $T_{\mathrm{orb}}^A = (\operatorname{Lie} G/\mathfrak{a})^A$. The stratum has tangent space $(T_{\mathrm{orb}}^\perp/T_{\mathrm{orb}})^A$, which is symplectic since the pairing on a symplectic A-module stays nondegenerate on its invariant summand.

The cohomology sheaves of IC_{B_z} are locally constant along each such stratum. In fact, invariant functions on M have differentials spanning the invariant conormal to Gx at x. Indeed, an invariant conormal covector extends by translation to a section of $\mathcal{J}/\mathcal{J}^2$, where $\mathcal{J}$ is the ideal of the closed orbit. This section lifts to an invariant function by reductivity. The vector fields $\Omega^{-1}df$ for these functions are defined near the central level. Wherever $1 + \operatorname{Ad} K$ is invertible, the moment identity and invariance of f give $dK(\Omega^{-1}df) = 0$. The fields thus descend to local vector fields on the quotient. Their values span $(T_{\mathrm{orb}}^\perp/T_{\mathrm{orb}})^A$. Their local flows preserve orbit types. Composing flows for a basis gives a disk in the stratum through x and a family of local analytic isomorphisms carrying x to points of that disk. Pullback of the intrinsic intersection complex by this family identifies its restriction to the disk with a constant complex.

There are finitely many orbit types here. To see this, embed M equivariantly as a closed subvariety of a representation with a norm invariant under a maximal compact subgroup. On a closed orbit choose a point of minimum norm. In the polar decomposition of a stabilizer element $k\exp(i\xi)$, convexity of $\|\exp(ti\xi)x\|^2$ and minimality give $\xi x = 0$ and $kx = x$. The complex stabilizer is therefore the complexification of the compact stabilizer. A compact linear action has finitely many orbit types by induction on the real representation dimension. On its compact unit sphere take a finite cover by normal tubes to orbits. The isotropy types in each tube are those of its linear normal slice in the sphere, of smaller dimension. This proves the finiteness including the origin. In particular the support dimension of each nonzero ordinary cohomology sheaf on the boundary is even.

The two middle-extension bounds. On a boundary stratum of complex dimension s, Proposition 6.6 bounds the stalk degrees of IC_{B_z} above by $-s-1$. Since s and the stalk degrees are even, this gives $-s-2$. Verdier duality on the smooth stratum shifts the dual of a local system by $2s$, so the costalk degrees start at $-s+2$. These are the asserted perverse bounds on the whole boundary. Equivalently, the support dimension bound for $\mathcal{H}^k(i^*\mathrm{IC}_{B_z})$ improves from $-k-1$ to $-k-2$, and duality gives the shriek assertion.

For $g = 2$, $G = \mathrm{SL}_2$, $z = 1$, the stratum of $L \oplus L^{-1}$ with $L^2 \neq 1$ has dimension four in a six-dimensional space. Its parameters are the rank-one characters of C with the two summands unordered. In the local model of Proposition 6.7 the two off-diagonal extension spaces are dual of dimension two, by the surface Euler characteristic. The scalar self-extensions vary the two characters, with their product fixed in the determinant slice. Transversely the equation is $v(u) = 0$ in $\mathbb{C}^2 \oplus (\mathbb{C}^2)^*$, modulo the torus scaling the two vectors inversely. The matrix uv identifies the quotient with the cone of traceless matrices of rank at most one, the surface $\mathbb{C}^2/\{\pm 1\}$. Its link is a rational homology sphere. Along this codimension-two stratum the star and shriek restrictions

of IC_{B_z} are therefore concentrated in degrees -6 and -2, respectively. At the points $L^2 = 1$ the flag argument still gives even stalks.

Ordinary restriction. Each component of B_z is a normal symplectic singularity by Proposition 5.11. The symplectic volume trivializes the canonical divisor and extends without poles on a resolution. Thus all discrepancies are nonnegative and the singularities are Kawamata log terminal. Proposition 6.8 gives finite fundamental group for the regular part of a sufficiently small neighborhood. The even stratum dimensions and density of V give its complement complex codimension at least two, also in the regular part. Paths and homotopies in that manifold avoid this complement by transversality. Consequently $j_*\mathbb{Q}_V = \mathbb{Q}_{B_z}$ by normality and $R^1 j_*\mathbb{Q}_V = 0$. Leray gives the injection in ordinary degree two. □

To identify the class after specialization, we keep a frame at the boundary circle. Forgetting this frame can change a degree-two class by flag characters, whose first Chern classes have weight two.

Lemma 6.14. *Let $g \geq 2$, let G be simply connected and simple, and let $z \in Z(G)$. Let $\mathcal{Y} \to \Delta$ be the fixed-mark flagged Betti family of Theorem 5.24 on a sufficiently small polydisk about z in T, and let $q : \mathcal{Y}_z \to Z$ be the forgetful map to the unweighted image. Write $V \subset B_z$ for the regularly stable locus, $\pi : q^{-1}(V) \to V$ for its full flag bundle, and $\mathcal{E}_{\mathrm{ad}}$ for the universal family of adjoint local systems on C parametrized by V. There are a class $\sigma \in H^2(\mathcal{Y}, \mathbb{Q})$ and a scalar $c \in \mathbb{C}^\times$ with the following properties.*

(i) If $t \in \Delta$ is regular semisimple and lies outside $[P, P]$ for every proper parabolic P containing T, then the image of σ_t in $\mathrm{Gr}_4^W H^2(Y_t, \mathbb{C})$ is c times the image of the surface class for the adjoint trace form.

(ii) On $q^{-1}(V)$, the restriction of σ_z is $\pi^* \int_C \mathrm{ch}_2(\mathcal{E}_{\mathrm{ad}})$.

Proof. We extend the adjoint bundle over a disk filling the puncture and compute its surface slant in boundary frames.

A class over the parameter disk. Remove the interior of a small disk about p from C and call the resulting surface Σ. Base the handles at its boundary. The universal punctured local system over the stack of flagged tuples gives an adjoint vector bundle on Σ in families. On its boundary the gluing matrix in a frame is $\mathrm{Ad}\, K$, where K is the peripheral word. Its eigenvalues are the eigenvalues of $\mathrm{Ad}\, t$, since the flag puts K in a Borel with Cartan projection t. These eigenvalues are near one throughout Δ. The logarithm with $\log 1 = 0$ gives a continuous conjugation-equivariant matrix logarithm, including on unipotent Jordan blocks. The path $\exp(s \log \mathrm{Ad}\, K)$, $0 \leq s \leq 1$, identifies the boundary bundle with the pullback of its evaluation bundle at the basepoint. Glue in this evaluation bundle over the disk. Changes of frame conjugate the entire path, so the resulting bundle is defined over C times the flagged stack. Its ch_2 slanted with $[C]$ defines σ on the coarse family, since the stable flag stack has finite stabilizers and its rational cohomology equals that of its coarse space by finite-chart averaging. Over V the word is z, so the path is constant. The adjoint local system extends flatly across the disk and descends over V since the central stabilizer acts trivially. This gives (ii).

The calculation in a boundary frame. For t as in (i) every tuple is stable. A proper parabolic reduction would put t in its derived group, and a maximal torus of that parabolic through t equals T by regularity. Fixing the ordered boundary frame gives D_t, with remaining change of frame by T. On D_t the boundary connection in the above trivialization is fixed. Choose a smooth connection on the family over $C \times D_t$ extending the given flat surface connections on Σ, with a

fixed connection on the collar and filling disk. Such extensions are obtained by extending the horizontal components from the collar and using a partition of unity. Write $\mathcal{F}$ for its curvature. For a parameter tangent v, the mixed component $\beta_v = (\iota_v \mathcal{F})|_{T\Sigma}$ represents the infinitesimal flat deformation with fixed boundary holonomy. Indeed it is the variation of the surface connection up to a covariant differential, and it vanishes on the collar.

In the Chern–Weil representative $\operatorname{tr}(\mathcal{F} \wedge \mathcal{F})/(2(2\pi i)^2)$, the surface curvature vanishes on Σ, and all parameter components vanish on the filling disk. Its surface integral on two tangents v, w is consequently a fixed nonzero constant times $\int_\Sigma \operatorname{tr}(\beta_v \wedge \beta_w)$. This is the pulled-back surface pairing by Proposition 4.1 [GHJ+97, Theorems 7.1 and 8.3, Key Lemma 8.4]. The trace here is taken in the adjoint representation throughout.

Finally $D_t \to [D_t/T]$ is a principal torus bundle. In degree two the kernel of pullback is the transgression of $H^1(T)$ by its Leray sequence. Transgressing the character generators gives first Chern classes of the algebraic character lines, all in W_2. The finite stabilizers identify the rational mixed Hodge cohomology of $[D_t/T]$ with that of Y_t. Thus equality up to the computed constant on D_t gives (i) on Y_t. □

For a vector bundle on the curve we take the determinant of cohomology with the convention $\det R\Gamma = \det H^0 \otimes (\det H^1)^{-1}$.

Lemma 6.15. *Let C be a smooth projective connected complex curve and Z a smooth complex algebraic orbifold of finite type. Let $p_Z : C \times Z \to Z$ be projection and $\mathcal{B}$ a vector bundle on $C \times Z$ with $c_1(\mathcal{B}) = 0$ in rational cohomology. Let F be a line bundle on C, also denoting its pullback to $C \times Z$ by F. Then*

$$c_1\big(\det Rp_{Z*}(\mathcal{B} \otimes F)\big) = \int_C \operatorname{ch}_2(\mathcal{B}) \quad \text{in } H^2(Z, \mathbb{Q}).$$

Proof. We compute the determinant after a finite map to the projective line, then compare the two curve integrals on a cover where the sheets split.

The determinant on the projective line. A nonconstant rational function gives a finite flat map $\varphi : C \to \mathbb{P}^1$ of degree d. The bundle $\mathcal{U} = (\varphi \times \mathrm{id}_Z)_*(\mathcal{B} \otimes F)$ has the same derived pushforward to Z as $\mathcal{B} \otimes F$. Write $\rho : \mathbb{P}^1 \times Z \to Z$ for projection. The diagonal of two projective lines is cut out by a section of $\mathcal{O}(1,1)$. Tensor its defining sequence on the product with Z by $\mathcal{U}$ from the first copy and push over that copy. In the Grothendieck group of perfect complexes on $\mathbb{P}^1 \times Z$ this gives

$$[\mathcal{U}] = \rho^*[R\rho_*\mathcal{U}] - [\mathcal{O}(-1)]\,\rho^*[R\rho_*(\mathcal{U}(-1))].$$

Integration of the degree-four Chern character gives $\int_{\mathbb{P}^1} \operatorname{ch}_2(\mathcal{U}) = c_1(R\rho_*(\mathcal{U}(-1)))$, since $\operatorname{ch}(\mathcal{O}(-1)) = 1 - c_1(\mathcal{O}(1))$. At an unramified value of φ, the restriction of $\mathcal{U}$ is a sum of restrictions of $\mathcal{B}$ tensored with fixed lines, so its first Chern class vanishes. Restricting the same identity there gives $c_1(R\rho_*\mathcal{U}) = c_1(R\rho_*(\mathcal{U}(-1)))$. The first Chern class of a perfect complex is that of its determinant, by additivity on its terms.

Comparison of the integrals. Let $\nu : \widetilde{C} \to \mathbb{P}^1$ be the normalization in the Galois closure of the function-field extension of φ. There are d maps $\sigma_i : \widetilde{C} \to C$ over $\mathbb{P}^1$, each of degree $\deg(\nu)/d$. The fiber product $\widetilde{C} \times_{\mathbb{P}^1} C$ is the sum of their graphs as a Cartier divisor in $\widetilde{C} \times C$. Indeed it is the pullback of the diagonal, generically the disjoint union of these graphs, and is finite over $\widetilde{C}$. Put $D_0 = 0$ and $D_i = \sum_{j=1}^{i} \Gamma_{\sigma_j}$. The kernel of $\mathcal{O}_{D_i} \to \mathcal{O}_{D_{i-1}}$ is $\mathcal{O}_{\Gamma_{\sigma_i}}(-D_{i-1})$. Tensor with the pullback of $\mathcal{B} \otimes F$ and push along the finite projection to $\widetilde{C} \times Z$. By finite base change this expresses $[(\nu \times \mathrm{id}_Z)^*\mathcal{U}]$ as the sum of the classes of $(\sigma_i \times \mathrm{id}_Z)^*\mathcal{B}$ twisted by lines from $\widetilde{C}$. Each

twist leaves ch_2 unchanged because $c_1(\mathcal{B}) = 0$ and the square of a degree-two class on $\widetilde{C}$ vanishes. Hence

$$\deg(\nu) \int_{\mathbb{P}^1} \mathrm{ch}_2(\mathcal{U}) = \sum_{i=1}^{d} \deg(\sigma_i) \int_C \mathrm{ch}_2(\mathcal{B}) = \deg(\nu) \int_C \mathrm{ch}_2(\mathcal{B}).$$

Dividing by $\deg(\nu)$ gives the claimed determinant class. □

Theorem 6.16. *Let $g \geq 2$, let G be simply connected and simple, and let $z \in Z(G)$. For a line bundle F of degree $g-1$ on C let e denote cup product with the first Chern class of the determinant of cohomology of the adjoint bundle tensored by F, pulled back from the adjoint coarse space. On the Dolbeault side e satisfies relative hard Lefschetz about perverse index R. On the Betti side its harmonic transport has bidegree $(2,2)$, the scalar intersection cohomology is Hodge–Tate, and for $m \in \mathbb{Z}$, $j \geq 0$,*

$$e^j : \mathrm{Gr}^W_{2R-2j}\, \mathrm{IH}^m(B_z) \longrightarrow \mathrm{Gr}^W_{2R+2j}\, \mathrm{IH}^{m+2j}(B_z)$$

is an isomorphism. On the scalar spaces $X = X_z, B_z$, restriction to the regularly stable open V is injective on $\mathrm{Hom}(I_X, I_X[2]) \to H^2(V, \mathbb{Q})$ in the rational constructible derived category.

Proof. We first identify the determinant on the two harmonic partners and compute its perverse center. We then specialize the framed slant, descend its weight-four symbol through the flag map, and identify the resulting action on scalar intersection cohomology. Finally we apply the boundary bounds to derived morphisms.

The determinant polarization. Put $G_{\mathrm{ad}} = G/Z(G)$. The adjoint bundle of a scalar object has trivial inertia on the root gerbe and is a bundle on C. The maps to the matched G_{ad}-component are finite surjections. The canonical section of $\mathcal{O}_C(p)$ off p gives the marking convention for the handle lifts on the root gerbe. With the positive peripheral loop their commutator product is z. For a given adjoint tuple with this obstruction, all lifts differ by $Z(G)^{2g}$. The finite quotient by these twists commutes with simultaneous conjugation. On the Higgs side descent of the adjoint object defines the algebraic map. Its analytic map corresponds to the Betti map by extension of structure group under the harmonic homeomorphism on the root gerbe, Proposition 5.10 [Sim11, Theorems 6.10, 8.8 and 9.7]. It is therefore proper with finite fibers, hence finite.

For a family of adjoint bundles E on $C \times S_0$, let $\lambda_F(E) = \det Rp_{S_0*}(E \otimes F)$, where p_{S_0} projects to the base S_0. First form the line on the ordinary adjoint component. For a polystable degree-zero Higgs vector bundle $\bigoplus_i E_i \otimes M_i$ with distinct stable factors, an automorphism with actions g_i on the multiplicity spaces acts on the determinant by

$$\prod_i \det(g_i)^{\chi(E_i \otimes F)} = 1.$$

Indeed $\chi(E_i \otimes F) = \deg E_i = 0$. Proposition 5.15 [Alp13, Theorem 10.3] gives descent on the vector moduli, and the adjoint representation pulls the line back to the adjoint component. The same character calculation gives descent before adding flags on the unweighted logarithmic Higgs target. The associated adjoint Higgs bundle is semistable by Proposition 5.1 [HZ25, Proposition 2.32]. The eigenbundles of a semisimple stabilizer element and their complements are invariant, so each has degree zero. Each eigenvalue thus has exponent zero in the determinant character. Semisimple elements detect a character of a reductive stabilizer.

For the sign, take a positive line B on C, write $\lambda_k(E) = \det Rp_{S_0*}(E \otimes B^k)$, and let $P_{\mathrm{sp}}(k)$ be the spectral Hilbert polynomial with this twist. The Plücker polarization in Proposition 6.9 is

given by a positive power of

$$\lambda_{k_2}^{P_{\mathrm{sp}}(k_1)} \otimes \lambda_{k_1}^{-P_{\mathrm{sp}}(k_2)}, \qquad k_2 \gg k_1 \gg 0.$$

Indeed the quotient is framed at k_1 and the Grassmannian determinant at k_2 is corrected by the determinant of that frame. On the spectral support away from infinity the polarizing twist restricts to the pullback of B. For adjoint bundles $\det E$ is trivial. Adding a point p' to a twist contributes the factor $\det E|_{\{p'\}\times S_0}$ times a constant line, so all the $\lambda_k(E)$ have the class of $\lambda_F(E)$. Since $P_{\mathrm{sp}}(k_2) > P_{\mathrm{sp}}(k_1)$, a positive power of the inverse determinant is the pullback of an ample line. The adjoint representation is faithful on G_{ad}, so its map to vector moduli is finite onto its image by Proposition 6.9. The pullback of the inverse determinant to X_z is relatively ample.

Let $\eta_D = c_1(\lambda_F)$ denote the pulled-back determinant class on X_z, and let $V_D \subset X_z$ be the regularly stable Higgs locus. The universal adjoint bundle descends to a vector bundle E_D on $C \times V_D$. Indeed, the inertia of the curve gerbe acts through $\langle z \rangle$ and the stabilizer of a regularly stable object is $Z(G)$. Both act trivially in the adjoint representation. At $s \in V_D$, write E_s and ϕ_s for the adjoint bundle and Higgs field on the curve. The obstruction group of the complex $E_s \xrightarrow{[\phi_s,-]} E_s \otimes K_C$ is dual by the trace form and Serre duality to the Lie algebra of the finite stabilizer. It vanishes, and the center acts trivially on deformations, so V_D is smooth. Since the adjoint representation has trivial determinant, $c_1(E_D) = 0$ on $C \times V_D$. Lemma 6.15 on this smooth base gives

$$\eta_D|_{V_D} = c_1(\lambda_F(E_D)) = \int_C \mathrm{ch}_2(E_D) = -\int_C c_2(E_D).$$

The canonical adjoint Hitchin map has total dimension $2R$ and fibers of pure dimension R by Proposition 5.3 [CFH+25, Propositions 3.27(ii) and 3.28(i)]. The finite scalar map is over the same Hitchin base. Thus the base has dimension R, the fiber square has dimension $3R$, and $d_h = R$. The Hitchin map is projective by properness and relative ampleness. Relative hard Lefschetz in Proposition 5.12 applies to $-\eta_D$ on the perverse direct images of IC_{X_z}. The perverse Leray spectral sequence degenerates by the projective decomposition. Taking hypercohomology and multiplying the j-th power by $(-1)^j$ gives

$$e^j : \mathrm{Gr}^P_{R-j}\,\mathrm{IH}^m(X_z) \xrightarrow{\sim} \mathrm{Gr}^P_{R+j}\,\mathrm{IH}^{m+2j}(X_z).$$

The class on the actual Betti space. Put $\eta_B = (\Psi^{-1})^*\eta_D$. To identify its restriction to V, use the ordinary general linear Hodge family for the adjoint vector representation. The determinant of cohomology λ_F of its underlying bundle descends over the parameter line by the same stabilizer test. At a nonzero parameter an invariant direct summand carries a flat connection and has degree zero. The preferred harmonic sections identify the restrictions of this one line at the Higgs and de Rham endpoints by Proposition 5.9 [Sim97, §4]. These sections commute with passage to the adjoint bundle by Proposition 5.10.

Let V_{dR} be the regularly stable open of the unweighted scalar de Rham moduli at $\lambda = 1$, with the prescribed inertia on the same root gerbe. This is an open in the finite-type scalar support of Theorem 5.24. Riemann–Hilbert with the chosen handle lifts identifies its analytification with that of $V \subset B_z$. Since V is smooth by Proposition 5.11, V_{dR} is smooth as well. As on V_D, the universal adjoint bundle with connection descends to an algebraic bundle E_{dR} on $C \times V_{\mathrm{dR}}$ with trivial determinant. In a frame, parallel transport gives the adjoint of the scalar handle tuple, with peripheral monodromy $\mathrm{Ad}(z) = 1$. The framed tuples form a G_{ad}-torsor over V, and the identifications by parallel transport commute with changes of frame. Thus the underlying

topological bundle corresponds in families to $\mathcal{E}_{\mathrm{ad}}$ of Lemma 6.14. The comparison along the harmonic sections carries the determinant class to $c_1(\lambda_F(E_{\mathrm{dR}}))$ on this de Rham open. Applying Lemma 6.15 there and transporting the surface integral by Riemann–Hilbert gives

$$\eta_B|_V = \int_C \mathrm{ch}_2(\mathcal{E}_{\mathrm{ad}}).$$

The map from V to the adjoint Betti stack is algebraic, induced on the handle tuples by the adjoint representation. Proposition 6.5 puts the displayed rational class in $W_4 \cap F^2 \cap \overline{F}^2$. Ordinary restriction in Lemma 6.13 is an injective morphism of mixed Hodge structures, hence strict for these filtrations. We obtain a map $\mathbb{Q}(-2) \to H^2(B_z)$ with value η_B. The ordinary cup action on I_{B_z} is the action of the tensor unit on the intersection Hodge complex. Exterior product and restriction to the diagonal therefore give a morphism of mixed Hodge structures

$$\mathbb{Q}(-2) \otimes \mathrm{IH}^m(B_z) \longrightarrow \mathrm{IH}^{m+2}(B_z).$$

This is the harmonic transport of e.

Specialization at fixed marking. Use $\mathcal{Y} \to \Delta$ and σ of Lemma 6.14, and put $Y = \mathcal{Y}_z$. By Theorem 5.24 the ordinary direct images are local systems with base change to every fiber in Δ. Their algebraic mixed Hodge module direct images thus give variations with the native fiber mixed Hodge structures by Proposition 5.13 [Sai90, Theorems 3.21 and 3.27]. In particular their weight filtrations are flat and their graded Hodge numbers are locally constant. Fiber cup products are the restrictions of the product on the direct images. The disk meets the generic regular open, where Theorem 3.5 gives Hodge–Tate cohomology and Theorem 4.5 gives Lefschetz about half-weight $R + b$. Part (i) of Lemma 6.14 identifies the raising symbol there up to a nonzero constant. The restrictions of σ are horizontal since it is one ordinary class on the family. Thus $H^*(Y)$ is Hodge–Tate and $\omega_z = [\sigma_z] \in \mathrm{Gr}_4^W H^2(Y)$ induces isomorphisms

$$\omega_z^j : \mathrm{Gr}^W_{2(R+b)-2j} H^k(Y) \longrightarrow \mathrm{Gr}^W_{2(R+b)+2j} H^{k+2j}(Y)$$

for every k and $j \geq 0$. Isomorphy over $\mathbb{C}$ suffices here since σ is rational.

Descent to the scalar support. Factor the flag map q as $Y \xrightarrow{f} \widetilde{Z} \xrightarrow{\nu} Z$, with Z its unweighted image and ν the finite normalization. The factorization exists since Y is normal and the map is birational by Theorem 5.24. The graph of f is closed in $Y \times_Z \widetilde{Z}$, so f is projective and surjective. It is semismall since $Y \times_{\widetilde{Z}} Y$ is closed in $Y \times_Z Y$. Proposition 6.12 and weight strictness give $\gamma \in W_4 H^2(\widetilde{Z}, \mathbb{Q})$ such that

$$f^*\gamma - \sigma_z \in W_3 H^2(Y).$$

Over each point of V the flag fiber is connected. Its surjection onto the corresponding finite ν-fiber singles out one point. The reduced closure of this branch over each scalar component maps finitely and birationally to that component, since the generic extension in characteristic zero is separable. The scalar component is normal by Proposition 5.11, so this finite map is an isomorphism. Denote the restriction of γ to this copy of B_z by γ_B.

The scalar summand already occurs in $Rf_*\mathbb{Q}_Y$ on this branch. At a regularly stable scalar tuple the differential of the commutator word K is onto by the moment identity. The pairs consisting of a Borel and an element of z times its unipotent radical have dimension $2b$. Taking the inverse image by K and dividing by conjugation gives $\dim Y = (2g-2)\dim G + 2b$. Thus the b-dimensional flag realizes half the codimension of the scalar support, and its top cohomology has rank one. The semismall decomposition before finite pushforward therefore gives the oriented

top class and its middle extension. Finite pushforward identifies it with the multiplicity-one summand of Theorem 5.24. Its inclusion and projection give complementary subspaces $\operatorname{im} J_1$ and $\ker L_1$, since $L_1 J_1 = (-1)^b |\mathcal{W}| \mathrm{id}$. Cup by γ is natural on complexes over $\widetilde{Z}$. By the projection formula it acts upstairs by $f^*\gamma$, and on the scalar summand by cup with γ_B. It commutes with both maps over $\widetilde{Z}$ and hence after finite pushforward. Passing to weight gradeds, ω_z and all its powers preserve both complementary subspaces. Their isomorphisms on the direct sum restrict to isomorphisms on $\operatorname{im} J_1$.

In degree $m + 2b$ this image is $\mathrm{IH}^m(B_z)(-b)$ as a mixed Hodge structure. It follows that $\mathrm{IH}^m(B_z)$ is Hodge–Tate. Removing the shift of $2b$ in weight from the two indices above gives $2R - 2j$ and $2R + 2j$ on scalar intersection cohomology.

Identification on the flag over V. On the flag bundle $\pi : q^{-1}(V) \to V$, Lemma 6.14(ii) gives

$$\pi^*((\gamma_B - \eta_B)|_V) = (f^*\gamma - \sigma_z)|_{q^{-1}(V)} \in W_3 H^2(q^{-1}(V)).$$

Flag pullback is injective. In fact $\pi_*(c_b(T_\pi)\, \pi^*\alpha) = |\mathcal{W}|\alpha$ by integration of the Euler class of G/B_+. Composing this pullback with ordinary restriction in Lemma 6.13 gives an injective morphism of mixed Hodge structures. Strictness yields $\gamma_B - \eta_B \in W_3 H^2(B_z)$. Its cup action raises weights by at most three, so γ_B and η_B induce the same map of weight degree four on $\mathrm{IH}^*(B_z)$. This identifies the isomorphisms on the scalar summand with the asserted powers of e. In rank one, Proposition 6.10 also gives these isomorphisms directly from relative hard Lefschetz for $z = -1$ (odd determinant degree), and Proposition 6.11 does so for $g = 2,\ z = 1$. The framed calculation applies in particular to the neutral component in every higher genus.

Restriction of derived morphisms. Set $\mathcal{P} = \mathrm{IC}_{B_z}$. Write j_V for the open immersion of V and i_∂ for the closed immersion of its complement. For $a \le 3$ the bounds of Lemma 6.13 give

$$\mathrm{Hom}(\mathcal{P}, i_{\partial *} i_\partial^! \mathcal{P}[a]) = \mathrm{Hom}(i_\partial^* \mathcal{P}, i_\partial^! \mathcal{P}[a]) = 0,$$

since $2 - a > -2$. Apply $\mathrm{Hom}(\mathcal{P}, -)$ to the localization triangle

$$i_{\partial *} i_\partial^! \mathcal{P} \longrightarrow \mathcal{P} \longrightarrow R j_{V *} j_V^* \mathcal{P} \longrightarrow i_{\partial *} i_\partial^! \mathcal{P}[1].$$

The terms supported on the boundary vanish for $a = 2$ and $a + 1 = 3$. After shifting both copies of $\mathcal{P}$ to ordinary normalization, restriction identifies the degree-two morphism group with $\mathrm{Hom}(\mathbb{Q}_V, \mathbb{Q}_V[2]) = H^2(V, \mathbb{Q})$. The harmonic homeomorphism matches stabilizers of polystable partners and identifies their intersection complexes and the restrictions to the regularly stable opens. This gives the same injection on X_z. In rank one as well, testing equality of two morphisms uses this injection alone. □

7 Integral coroot modifications

We use the logarithmic family of Construction 5.18. Write $T_H \subset B_H$ for the images of $T \times \mathbb{G}_m$ and $B_+ \times \mathbb{G}_m$ in H. The roots Φ of (G, T) are characters of T_H, trivial on the central factor. We take the real cocharacter space to be $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ and identify cocharacters with their differentials. We shift the bundle along the marking graph by an integral coroot and compute the change in its adjoint characteristic class.

Proposition 7.1. *Let A be a ring and t a nonzerodivisor. Finite projective modules over $A[t^{-1}]$ and the t-adic completion of A, identified on the punctured completion, glue to a finite projective A-module uniquely up to compatible isomorphism.*

This form of the Beauville–Laszlo theorem allows us to change a bundle on the formal neighborhood of the graph while keeping its given identification on the complement [BL95, main theorem]. Write Δ for the marking graph and E for the universal H-bundle on the product of C with the flagged stack. The flag on Δ gives a T_H-torsor by quotienting by the unipotent radical of B_H. For $\alpha \in \Phi$ let L_α denote its associated line of weight α. We use the same convention for the character lines on a full flag bundle.

Theorem 7.2. *Let $g \geq 2$, fix a simple scalar pair and a nonzero integral coroot-lattice element ℓ. Choose v in the imaginary real cocharacter space outside the hyperplanes of roots and of Weyl translates of the fundamental weights. For $a \in [0,1]$ and $k \in \mathbb{Z}$ there are isomorphisms $Y_{a,v} \to Y_{a,v-ak\ell}$ preserving punctured data and abelianization. At $a = 0$ they act over the logarithmic Hitchin base, and at $a = 1$ they induce the identity on ordered punctured monodromy. Write T_ℓ for the Higgs translation at $k = 1$, $A = \int_C \mathrm{ch}_2(\mathrm{ad}E)$, $B_{\mathrm{ad}}(h,h') = \sum_\alpha \alpha(h)\alpha(h')$ with the sum over all roots, and $y_\ell = \sum_\alpha \alpha(\ell)c_1(L_\alpha)$ for the corresponding flag character on the moving graph. Then*

$$(T_\ell^k)^* A = A + k y_\ell + \tfrac{1}{2} k^2 B_{\mathrm{ad}}(\ell,\ell)\, x^* c_1(T_C).$$

For the full flag bundle π of any principal adjoint bundle E_0 the corresponding classes satisfy

$$\pi_*\big(c_{\mathrm{top}}(T_\pi) y_\ell^2\big) = \frac{2|\mathcal{W}|}{r}\, B_{\mathrm{ad}}(\ell,\ell)\ \mathrm{ch}_2(\mathrm{ad}E_0).$$

Proof. We construct the shift from a finite jet of a torus reduction and check stability, Higgs invariants and ordered monodromy. Comparing the lattices in the adjoint root lines then gives the change in A, and summing over the flags fixed by the torus gives the flag integral.

The flag determines a formal torus reduction. Locally in a family for the faithfully flat topology, choose a relative parameter τ for Δ and a formal frame adapted to the flag. Write the operator and its ordered residue as

$$a\,d + \Gamma(\tau)\frac{d\tau}{\tau}, \qquad \Gamma(0) \in \mathrm{Lie}\, B_H, \qquad \xi = -a\, d\iota(1)/n_z + u \in \mathrm{Lie}\, T_H.$$

We consider both $u = v$ and $u = v - ak\ell$. For every root α, the imaginary part of $\alpha(\xi)$ is the nonzero imaginary part of $\alpha(v)$. Use the positive root subgroups in increasing height to conjugate $\Gamma(0)$ to ξ (Proposition 3.1, [Dem70, Exp. XXII, §§ 5.5–5.7]). At root α the coefficient of the change is $\alpha(\xi)$, and the commutators affect only terms of greater height. The resulting frames at the mark differ by T_H.

Suppose the coefficients of orders less than j are toral, with $j \geq 1$. For $X \in \mathrm{Lie}\, H$ a change of frame by $1 + \tau^j X$ modulo τ^{j+1} adds $[\xi, X] + jaX$ to the coefficient of order j. On the α-root space this adds $(\alpha(\xi) + ja)X_\alpha$. The factor is nonzero, so we can kill every root component at this order, using the corresponding root subgroup elements. Induction gives a formal frame with coefficient in $(\mathrm{Lie}\, T_H)[[\tau]]$. For two such frames, remove the constant torus transition and then its torus terms successively. The first remaining root coefficient would satisfy $(\alpha(\xi) + ja)X_\alpha = 0$. Thus the two frames differ by a regular torus gauge. The ordered formal reduction preserved by the operator is unique, so it is independent of the choice of τ.

A finite jet determines the modified bundle and flag. In a reducing frame σ use the new frame $\sigma\tau^{-k\ell}$ in the same punctured bundle. The toral coefficient commutes with $\tau^{-k\ell}$, and the logarithmic derivative contributes $-ak\ell$. The new residue is therefore $\xi - ak\ell$ when we start at $u = v$. Take the standard B_H-flag in the new frame. A regular torus gauge commutes with the shift, and replacing τ by a unit times τ multiplies the new frame by a regular torus gauge.

Choose a faithful closed representation $H \hookrightarrow \mathrm{GL}(V)$ and set $M = \max_\chi |k\chi(\ell)|$ over its T_H-weights. On the weight χ the new lattice is $\tau^{-k\chi(\ell)}$ times the old one. Conjugation by $\tau^{k\ell}$ can lower the order of a matrix entry by at most $2M$. Hence two lifts of the same frame modulo τ^{2M+1} differ in the new frame by $1+O(\tau)$. This is a regular H-gauge since H is closed in $\mathrm{GL}(V)$, and it induces the identity on the new fiber. The same bound leaves every off-torus error of order $2M+1$ divisible by τ after the shift. Thus this jet determines the lattice and the flag and gives a logarithmic operator with the specified residue. Computing the jet only inverts the factors $\alpha(u)+j\lambda$ for $0 \le j \le 2M$. These factors and their counterparts at $u-\lambda k\ell$ are invertible on a Zariski open containing $(\lambda, u) = (a, v)$ for $0 \le a \le 1$. The finite jet depends algebraically on the family there, hence continuously along the path. The weight bound applies to arbitrary integral coroot shifts in every type, including the exceptional types.

For each representation, Proposition 7.1 glues the modified lattice to the original bundle off the Cartier divisor Δ by their punctured identification. All these gluings commute with tensor products, duals and equivariant maps, since they use the same H-valued loop. Their tensor frames form the modified principal H-bundle, with the original frames off Δ and the translated frames on its completion. The operator extends in the new frame, and independence of the choices gives descent in families. The loop lies in the simply connected derived group G, so the topological type and the prescribed abelianization with its operator are preserved. At the shifted residue the new torus frame is again a reducing frame with the given ordering. The opposite shift gives the inverse, and composition adds the cocharacters. In particular the shift by $k\ell$ on the Higgs fiber is T_ℓ^k.

The shift preserves stability, Higgs invariants and ordered monodromy. An invariant proper parabolic reduction would induce one to a maximal parabolic. At the mark its Lie algebra contains the regular semisimple residue ξ in the reducing frame. The Zariski closure of $\exp(\mathbb{C}\xi)$ lies in a maximal torus of that parabolic. This torus centralizes ξ, so it equals T_H. Choose a positive integral multiple η of the corresponding fundamental weight in the root lattice. It defines a parabolic character trivial on the center. The induced line has logarithmic a-connection with residue $(w\eta)(u)$ for some $w \in \mathcal{W}$. On a line of degree d with one pole this residue equals $-ad$. Indeed, in a meromorphic section the connection form acquires the additional residue a times the order of the section at each point. Summing all residues gives $(w\eta)(u)+ad=0$, also at $a=0$. At either value of u the imaginary part of $(w\eta)(u)$ is that of $(w\eta)(v)$, which is nonzero by hypothesis. Thus there are no invariant proper parabolic reductions. The shift and its inverse preserve the stable loci for the prescribed weights and descend to the asserted coarse isomorphisms.

At $a=0$ the invariant polynomials of the Higgs fields agree off Δ. They are regular sections of the same powers of $K_C(x)$, so they agree across the mark. This also gives the action over the logarithmic Hitchin base throughout the punctured Higgs line $u = sv$, $s \in \mathbb{C}^\times$. There the root denominators are $s\alpha(v)$, and the line residues excluded above are $s(w\eta)(v) \neq 0$.

At $a=1$ the punctured identification preserves the local system. By Proposition 5.7 ([Boa11, Theorem A]), its nonresonant local monodromy is conjugate to $\exp(-2\pi i\xi)$. Both residues give the same torus element $z\exp(-2\pi iv)$, since $\iota(\epsilon) = z$ and $\exp(2\pi ik\ell) = 1$. For each torus character γ its absolute value satisfies

$$\log|\gamma(z\exp(-2\pi iv))| = 2\pi\,\gamma(v/i).$$

The real cocharacter v/i lies in the interior of a Weyl chamber, so this torus element has trivial Weyl stabilizer. It is regular semisimple, and its images in the ordered Cartans of the Borels containing it are its distinct Weyl translates. The prescribed image therefore selects the same flag at both ends.

The graph gives the quadratic term. Let $\mathcal{Y}$ be the flagged stack over $Y_{0,v}$ and let $q : C \times \mathcal{Y} \to \mathcal{Y}$ be projection. The graph inclusion $\delta : \mathcal{Y} \hookrightarrow C \times \mathcal{Y}$ is a section of q, so its normal line is $N = \delta^*\mathcal{O}(\Delta) = x^*T_C$. For a line L on $\Delta \simeq \mathcal{Y}$ the divisor sequence, tensored by q^*L, gives

$$\operatorname{ch}(\delta_*L) = \exp(q^*c_1(L))(1 - \exp(-[\Delta])), \qquad q_* \operatorname{ch}_2(\delta_*L) = c_1(L) - \tfrac{1}{2}c_1(N).$$

Here $q_*[\Delta] = 1$ and $q_*([\Delta]^2) = c_1(N)$ by restriction to the graph.

In the formal torus reduction the adjoint root line for α changes by $\mathcal{O}(m\Delta)$ with $m = k\alpha(\ell)$. Compare the two adjoint lattices by taking their quotients by their intersection. For $m > 0$ the added layers have classes $\delta_*(L_\alpha \otimes N^j)$ for $1 \le j \le m$. For $m < 0$ we subtract the layers with $m < j \le 0$. The zero weights are fixed. With this signed convention the contribution of the root to $(T_\ell^k)^*A - A$ is

$$\sum_{j=1}^{m} (c_1(L_\alpha) + (j - \tfrac{1}{2})c_1(N)) = mc_1(L_\alpha) + \tfrac{1}{2}m^2c_1(N).$$

Summing the linear terms gives ky_ℓ, and summing the squares gives $\frac{1}{2}k^2B_{\mathrm{ad}}(\ell,\ell)c_1(N)$. This proves the first formula on the coarse space as well. Indeed the flagged stack is an orbifold by Theorem 5.24, and its finite stabilizers have zero higher rational cohomology.

The flag integral is a Weyl sum. The Euler characteristic of G/B_+ is $|\mathcal{W}|$ by its Bruhat cells. Thus $\pi_*c_{\mathrm{top}}(T_\pi) = |\mathcal{W}|$, and the projection formula makes π^* injective over $\mathbb{Q}$. After this pullback E_0 has a Borel reduction, hence a topological torus reduction because the quotient of a Borel by its torus is contractible. By naturality of fiber integration we can compute with such a torus reduction.

A regular torus Lie algebra element gives a vertical vector field on the flag bundle. Its zeros are the sections indexed by $\mathcal{W}$, one for each Borel containing the torus. At each zero the vertical derivative is invertible because its eigenvalues are root values. With the complex orientation each zero section contributes with multiplicity $+1$ to the relative Euler class. Restriction to the section indexed by w sends a flag character α to $w\alpha$. Identifying characters with first Chern classes, we obtain for the pushforward of $c_{\mathrm{top}}(T_\pi)y_\ell^2$ the polynomial

$$h \longmapsto \sum_{w \in \mathcal{W}} B_{\mathrm{ad}}(w\ell, h)^2$$

on the real cocharacter space. The roots span its dual and are permuted by $\mathcal{W}$, so B_{ad} is positive definite and Weyl invariant. The self-adjoint operator

$$h \longmapsto \sum_{w \in \mathcal{W}} B_{\mathrm{ad}}(w\ell, h)\, w\ell$$

commutes with $\mathcal{W}$. Its eigenspaces are invariant, so it is scalar by irreducibility of the reflection representation of the simple group G. Each summand has trace $B_{\mathrm{ad}}(\ell,\ell)$. The Weyl sum of squares is consequently $|\mathcal{W}|B_{\mathrm{ad}}(\ell,\ell)B_{\mathrm{ad}}(h,h)/r$. The nonzero weights of $\mathrm{ad}E_0$ are the roots, each with multiplicity one. Thus $B_{\mathrm{ad}}(h,h)$ is the trace of $\mathrm{ad}(h)^2$, and $\frac{1}{2}B_{\mathrm{ad}}(h,h)$ represents $\mathrm{ch}_2(\mathrm{ad}E_0)$. This gives the claimed coefficient $2|\mathcal{W}|B_{\mathrm{ad}}(\ell,\ell)/r$. □

8 A common lowering operator

We retain the moving mark $x : S \to C$ of Construction 5.18. Let $\theta \in H^2(X_z, \mathbb{Q})$ be the determinant class defining e in Theorem 6.16. We use m_β for cup product by an ordinary class β, including

its action on an intersection complex. One coroot translation will give both the logarithm that lowers Betti weights and a morphism of Hitchin direct images. Its double commutator with m_θ will determine the normalization.

We use the following construction of diagram weights. Its monodromy assertion concerns the Hodge–Tate case [Ope26, Proposition 4.1 and Lemma 4.3].

Proposition 8.1. *Let a flat bundle over a compact manifold be given, over a finite good cover, by algebraic fiber families over a common smooth base and locally constant algebraic changes of charts. Suppose their ordinary integral cohomology sheaves are local systems with fiberwise base change. The finite derived limit of their mixed Hodge module direct images gives graded-polarizable variations on the cohomology of the total fibers. If all fiber terms are Hodge–Tate, the total fibers are Hodge–Tate. If all fiber terms are Hodge–Tate and the monodromy of a loop preserves a full lattice, some positive power of that monodromy acts trivially on every weight graded.*

On the Higgs side we use specialization over the coefficient base. The restriction maps in the next lemma identify its action on ordinary cohomology [Ope26, Lemma 7.2].

Lemma 8.2. *Let $Y \to \mathbb{A}^1$ be smooth and $q : Y \to E \times \mathbb{A}^1$ proper over the line. Assume that over the punctured line the family is a product and that the restriction maps from $H^*(Y)$ to the fibers at zero and one are isomorphisms. An automorphism over $E \times \mathbb{G}_m$ induces an endomorphism of $Rq_{0*}\mathbb{Q}_{Y_0}$ whose action is that of its fiber at one transported by the restriction maps.*

We record how to differentiate a polynomial identity for cup products after this specialization. The logarithm of the unipotent part is defined even when the specialized automorphism has a semisimple part.

Lemma 8.3. *Let K be an object of a $\mathbb{Q}$-linear triangulated category with finite-dimensional morphism spaces. Let U be an automorphism of K and let D be the logarithm of its unipotent part in $\mathbb{Q}[U]$. Let $d \in \mathbb{Z}$, let $c : K \to K[d]$, and suppose that a polynomial $p(t)$ with coefficients in $\mathrm{Hom}(K, K[d])$ satisfies $U^k[d]\, c\, U^{-k} = p(k)$ for every integer $k \geq 0$. Then*

$$D[d]\, c - cD = p'(0).$$

Proof. Write $U = U_{\mathrm{s}}U_{\mathrm{u}}$ for its commuting semisimple and unipotent parts. These are rational polynomials in U by Jordan decomposition in the finite-dimensional algebra $\mathbb{Q}[U]$. On $\mathrm{Hom}(K, K[d])$ the operator $B(c') = U[d]\, c'\, U^{-1}$ has semisimple and unipotent parts given by the same formula for U_{s} and U_{u}. Indeed left and right multiplication commute, so the two semisimple multiplications are simultaneously diagonalizable over $\overline{\mathbb{Q}}$, and the two unipotent multiplications have unipotent product. A sufficiently high finite difference of p vanishes, so $(B-1)^j c = 0$ for some j and $B_{\mathrm{s}}c = c$. The nilpotent operator $c' \mapsto D[d]c' - c'D$ exponentiates to B_{u}. Its exponential with parameter t, applied to c, thus agrees with $p(t)$ at every nonnegative integer. The two polynomials have the same linear coefficient. □

The last input compares the filtered subspaces when the two graded Lefschetz actions admit a common lowering operator [Ope26, Proposition 2.4].

Lemma 8.4. *Let V be finite-dimensional over $\mathbb{Q}$ and let F, G be finite increasing exhaustive separated filtrations. Suppose a common e raises each filtration by at most two and induces hard Lefschetz with the same center R on their associated gradeds. If a common f lowers each filtration by at least two and $[[f, e], e] = -2e$ with the usual commutator, then $F_kV = G_kV$ for every integer k.*

Theorem 8.5. *Let $g \geq 2$, let G be simply connected and simple, and take any $z \in Z(G)$. Identify $\mathrm{IH}^*(X_z)$ and $\mathrm{IH}^*(B_z)$ via the harmonic homeomorphism. There is an $H^1(C, Z(G))$-equivariant rational operator f of degree -2 with*

$$fP_k \subset P_{k-2}, \qquad fW_j \subset W_{j-4}, \qquad [[f,e],e] = -2e.$$

On the Dolbeault side f is induced by a morphism $Rh_*I_{X_z} \to (Rh_*I_{X_z})[-2]$. For each m, k the subspaces $P_k\mathrm{IH}^m(X_z)$, $\Psi^*W_{2k}\mathrm{IH}^m(B_z)$, and $\Psi^*W_{2k+1}\mathrm{IH}^m(B_z)$ coincide, also on every central-character summand.

Proof. We identify the Higgs translation with monodromy of a loop in the ordered Betti torus and compute the weight shift of its logarithm. We specialize the same translation over the logarithmic Hitchin base. The determinant on the unweighted family then lets us compute the double commutator on the scalar intersection complex.

Transport of the coroot translation by restriction. Fix a nonzero integral coroot-lattice element ℓ. Let $V \subset T$ be the generic open of Theorem 3.5. Its construction removes the root and proper Levi walls and the images of the nondominant terminal torus maps. On V the ordinary direct images of the fixed-mark punctured family are local systems with fiberwise base change. For each deleted translate choose a nontrivial character equation $\chi(t) = c_\chi$ containing it. Choose v in the imaginary real cocharacter space outside the root and Weyl-translated fundamental-weight hyperplanes of Theorem 7.2. Take v small enough that av lies in the logarithm window about zero and $z\exp(-2\pi i a v)$ lies in the lissity neighborhood of z in Theorem 5.24 for $0 \leq a \leq 1$. We can also require $|\chi(t_*)| \neq |c_\chi|$ for each chosen equation, where $t_* = z\exp(-2\pi i v)$. Taking logarithms of absolute values excludes only finitely many proper real affine hyperplanes in the choice of v. The loop

$$\gamma(a) = t_* \exp(2\pi i a \ell), \qquad 0 \leq a \leq 1,$$

stays in V since the second factor has unit absolute value under every character. It closes because ℓ is integral.

Write $\mathcal{Y}$ for the full coarse flagged family, and $\rho_{\lambda,u} : H^*(\mathcal{Y}) \to H^*(Y_{\lambda,u})$ for restriction, an isomorphism by Theorem 5.24. The maps $\Phi_a : Y_{a,v} \to Y_{a,v-a\ell}$ of Theorem 7.2 vary continuously in a. Indeed the lattice modification in the formal torus reduction uses finitely many jets, whose root denominators are nonzero along these arcs since $\operatorname{Im}\alpha(v) \neq 0$ for every root α. The operator

$$\rho_{a,v}^{-1}\Phi_a^*\rho_{a,v-a\ell}$$

is constant in a. At any a_0 represent a rational homology basis of the source fiber by compact cycles. Near their supports lift the interval vector field in smooth orbifold charts, average over each finite chart group, and glue by a partition of unity. Its short-time flow transports the cycles in the source family. They remain a basis because their pairings with the restricted ambient classes stay constant. Their images under Φ_a also give homologies in $\mathcal{Y}$, so all matrix coefficients of the displayed pullback are constant, including at the endpoints.

At $a = 0$ the displayed operator is $\mathsf{T} = \rho_{0,v}^{-1}T_\ell^*\rho_{0,v}$.

Logarithmic normal form along the lifted loop. At $\lambda = 1$ write $\tau(u) = z\exp(-2\pi i u)$ and $\Xi(u) = -d\iota(1)/n_z + u$. The lift of γ is the segment $u(a) = v - a\ell$. Cover the whole segment by a finite chain of overlapping polydisks in $\operatorname{Lie} T$. On each polydisk τ is biholomorphic onto its image in V, and $\operatorname{Im}\alpha(u)$ is nonzero for every root α. Such polydisks exist since $\operatorname{Im}\alpha(u(a)) = \operatorname{Im}\alpha(v)$. We choose the ordered exponent $\Xi(u)$ on each of them. It satisfies $\exp_H(-2\pi i\Xi(u)) = \tau(u)$ since $\iota(\exp(2\pi i/n_z)) = z$.

Take a local coordinate w vanishing on the marking graph and a Borel-adapted frame. The residue differs from $\Xi(u)$ by an element of the nilradical. Use the root coordinates of Proposition 3.1 [Dem70, Exp. XXII, §§ 5.5–5.7] in order of increasing height. Conjugation in the α-root subgroup changes that coordinate with nonzero coefficient $\alpha(u)$. Every other coordinate it changes has greater height. This determines a unique holomorphic unipotent change of frame putting the residue in the ordered torus. Write the resulting connection as $d + D(w)\,dw/w$ with $D(w) = \Xi(u) + \sum_{i\geq 1} D_i w^i$. At order $\nu \geq 1$ a gauge $\exp(w^\nu X_\nu)$ changes the coefficient by $(\nu + \operatorname{ad}\Xi(u))X_\nu$. This operator is multiplication by ν on the Cartan, including the center, and by $\nu + \alpha(u)$ on each root space. It is invertible in every direction. Successive corrections therefore give a unique formal H-valued normalizing gauge with value 1 at the pole. Its coefficients depend holomorphically on the parameters.

To check convergence, use a faithful matrix representation of H and write the formal gauge as $Q(w) = \sum_{\nu\geq 0} Q_\nu w^\nu$, $Q_0 = 1$. The equation $wQ' = Q\Xi(u) - D(w)Q$ gives

$$(\nu + \operatorname{ad}\Xi(u))Q_\nu = -\sum_{i=1}^{\nu} D_i Q_{\nu-i}.$$

Locally in the family there are $c_0, r_0 > 0$ with $\|D_i\| \leq c_0 r_0^{-i}$. For $\nu > 2\sup\|\operatorname{ad}\Xi(u)\|$ on a smaller parameter chart, the inverse on all matrices has norm at most $2/\nu$ by the geometric series. The finitely many initial Q_ν are bounded there by the Lie algebra construction. Choose $K > 0$ so the coefficients of $K(1 - w/r_0)^{-2c_0}$ bound those initial norms. Its coefficients h_ν satisfy $\nu h_\nu = 2c_0 \sum_{i=1}^{\nu} r_0^{-i} h_{\nu-i}$, so the displayed recursion gives $\|Q_\nu\| \leq h_\nu$ for all ν. The gauge thus converges holomorphically in w and in the family parameters. It is invertible near $w = 0$ and satisfies the defining equations of H there since its formal series does.

The local extensions glue along the lift. In the ordered Borel incidence of Proposition 5.7, the peripheral element has Cartan projection $t = \tau(u)$ [Boa11, Theorems A and B]. In a frame of the given Borel the same root-height elimination, with invertible coefficients $1 - \alpha(t)$, conjugates the element to t. The remaining ordered frame changes lie in the torus. In this frame attach the model $d + \Xi(u)\,dw/w$ to the punctured flat bundle using the horizontal solution $w^{-\Xi(u)}$ on the universal cover of an annulus. Its monodromy is t. In each torus character the chosen exponent specifies the regular-singular lattice at the smooth marking divisor by Proposition 5.6 [Del70, Chapter II, Proposition 5.4, Remark 5.5(ii) and Corollary 5.6]. The handle data lie in the kernel G of the abelianization, so the abelianized flat line on the complement has its given horizontal section. Its residue in the model is -1. The section is therefore w times a local frame at the pole and gives the prescribed $(\mathcal{O}_C(x), d_x)$. Off the marking graph, flat frames on simply connected opens are glued by the handle words evaluated on the representation. These matrices are holomorphic in the representation parameters, as is the annular gluing in the chosen logarithm chart. Conversely, parallel transport of a connection depends holomorphically on its coefficients, and the normalizing gauge above recovers precisely this extension and its flag.

On an overlap in the lift the exponents agree. In toral frames a flat isomorphism of the ordered punctured models has the form $w^{-\Xi(u)} g_0 w^{\Xi(u)}$, with g_0 constant in w. Single-valuedness and preservation of the flag put g_0 in the centralizer of t in the Borel, which is the torus. Thus the isomorphism extends as g_0 across the pole. Replacing w by w times a holomorphic unit changes the horizontal solution by a holomorphic torus gauge, using a local logarithm of the unit. Thus on overlaps in $\operatorname{Lie} T$ the identity of the punctured data has a unique extension to the lattices. Uniqueness gives the cocycle identity on triple overlaps.

If the handle monodromies lay in a proper parabolic, their commutator product t would lie

in its derived group. Any parabolic containing the regular element t contains its centralizing torus, since it contains a maximal torus through t. This would put t on a deleted Levi wall. An invariant logarithmic reduction restricts to such a flat reduction on the complement, since the extra factor of H is central. Stability is therefore automatic on both sides. Fixing the peripheral element to t with its ordered flag leaves the G-tuple modulo T, giving $D_t//T$ at a fixed mark. Conjugation equivariance gives inverse analytic chart maps on the weighted coarse spaces.

Take a finite good cover of S by surface marking charts. The coordinate at the pole is relative to the moving graph and the paths for holonomy move by isotopy, so the fixed-mark comparisons vary continuously with the mark. A marking change substitutes words in the handles and conjugates the peripheral element and its Borel by the same word. The ordered constant model conjugates with them. A peripheral power fixes this model and its flag, and the central corrections from the moving root multiply the lifted handles by locally constant factors that cancel in every commutator. Thus the extension commutes with these locally constant algebraic marking changes. Writing Y_t^{mov} for the ordered Betti fiber with S retained, we obtain identifications $\beta_u : Y_{1,u} \to Y^{\mathrm{mov}}_{\tau(u)}$ over the entire lift. Its two ends project to t_*. By Theorem 7.2 they satisfy $\beta_{v-\ell} \circ \Phi_1 = \beta_v$.

On the marking cover form the finite derived limit of the ordinary mixed Hodge module direct images, with pullback by the chart changes as transition maps. Proposition 5.16 identifies its rational realization with the corresponding limit of cochain complexes [Tub25, §2.1 and Proposition 4.2.6]. Each chart intersection is contractible, so cochain descent identifies the result with ordinary cohomology of the total fiber over S.

Here the rational hypotheses suffice for the variation in Proposition 8.1. On V the fixed-mark fiber is Y_t and Theorem 3.5 gives ordinary lissity and base change. Near z these hold for the flagged incidence by Theorem 5.24. Finite sums and cones preserve local constancy and commute with restriction to a point. Thus the limit has locally constant cohomology with ordinary base change on both regions. Along the lift, the restrictions of any class from $\mathcal{Y}$ form a flat section of this pulled-back local system. Thus $\rho_{1,v-\ell}\rho_{1,v}^{-1}$ is Betti parallel transport P_γ in the initial and final logarithmic charts. Closing the lift by the endpoint identification gives

$$\begin{aligned}\rho_{1,v}\mathsf{T}\rho_{1,v}^{-1} &= \Phi_1^*\rho_{1,v-\ell}\rho_{1,v}^{-1}\\ &= \Phi_1^*\beta_{v-\ell}^*\,\mathsf{P}_\gamma\,(\beta_v^*)^{-1} = \beta_v^*\,\mathsf{P}_\gamma\,(\beta_v^*)^{-1}.\end{aligned}$$

At v the chosen exponent agrees with the logarithm window about z in Theorem 5.24. Uniqueness identifies the lattices on this overlap, so restriction along $u = av$ transports the same marked cohomology between t_* and z.

The logarithm lowers weights on the moving family. The fiber base-change maps are isomorphisms in mixed Hodge modules since they are so on rational realization. Thus each fixed-mark term carries its algebraic cohomology weights. After the smooth-base dimension shift, the cohomology Hodge modules of the limit give graded-polarizable variations by Proposition 5.13 [Sai90, Theorem 3.27 and its algebraic remark]. Write W^{mov} for their flat weight filtration. The spectral sequence of the finite cover has sums of fixed-mark cohomology groups in its first page. They are Hodge–Tate on V by Theorem 3.5, and subquotients and extensions preserve this property. The Hodge numbers of the pure graded variations are locally constant, so the same conclusion holds along the radial segment to z.

The single operator T preserves the full lattice transported from $H^*(Y_{0,v},\mathbb{Z})/\mathrm{torsion}$, since T_ℓ is an automorphism of that finite-type coarse space. Its intersections with the rational weight subspaces induce full invariant lattices on their successive quotients. On each nonzero graded

piece the flat polarization is definite up to sign, since there is just one Hodge type (j, j). The isometries of a definite lattice form a finite group, as there are only finitely many choices for the image of each basis vector of prescribed norm. Choose one $M > 0$ killing the graded monodromy of this loop in every degree. The odd gradeds vanish, so $\mathsf{T}^M - 1$ sends W_j^{mov} into W_{j-2}^{mov} and is nilpotent. Put

$$N = M^{-1} \log(\mathsf{T}^M), \qquad N_{\lambda,u} = \rho_{\lambda,u} N \rho_{\lambda,u}^{-1}.$$

The logarithm is a finite rational sum. It is a derivation on ordinary cohomology, since the polynomial $\exp(tN)$ preserves products for $t = Mq$ at all positive integers q, and differentiation at zero gives the Leibniz rule. By flat transport $N_{1,0}$ lowers the weights of the marked Betti fiber by at least two.

Passage to the native scalar weights. Let pr denote projection from S times the scalar space to the scalar space on either side. Use the unit on S for pr^* and integration against its complex orientation for pr_*. Define

$$f' = \mathrm{pr}_*\, L_0 N_{0,0}^2 J_0\, \mathrm{pr}^*\,. \tag{8.1}$$

The endpoint identification of Theorem 5.24 carries this to $\mathrm{pr}_*\, L_1 N_{1,0}^2 J_1\, \mathrm{pr}^*$ under Ψ. In the constant scalar diagram the limit is

$$C^*(\text{nerve of the cover}, \mathbb{Q}) \otimes R\Gamma(B_z, I_{B_z}).$$

The nerve cochains consist of copies of $\mathbb{Q}(0)$, so every degree of $H^*(S, \mathbb{Q})$ has diagram weight zero and the second factor has its native intersection-cohomology weights. Both the unit and the orientation trace preserve these weights. The semismall maps of Theorem 5.24 commute with the marking changes as Hodge-module maps. The inclusion J_1 raises weight by $2b$ and L_1 lowers it by $2b$, from their opposite Tate twists. In the harmonic identification this gives $f' W_j \subset W_{j-4}$. The two ordinary shifts of the flag maps also cancel, leaving the degree -2 of the curve trace.

Specialization over the full logarithmic coefficient base. Let $\mathcal{E} \to S$ be the bundle with fiber at $\xi \in S$ given by

$$\mathcal{E}_\xi = \bigoplus_{i=1}^{r} H^0(C, K_C(x(\xi))^{\otimes d_i}),$$

where d_i are the degrees of the basic invariants of $\mathrm{Lie}\, G$. The first cohomology of these line bundles vanishes by degree. The holomorphic coefficients form the closed subbundle $S \times \mathcal{A}_G \subset \mathcal{E}$, and the abelianized field is fixed. On the Higgs line $\lambda = 0$, $u = sv$, write $q : \mathcal{Y}_{\mathrm{line}} \to \mathcal{E} \times \mathbb{A}^1_s$ for the coefficient map together with s.

We check properness of q with the moving pole. Locally on S, embed the moving twist $K_C(x(\xi))$ in a fixed sufficiently positive line bundle on C by multiplication by a section nonzero on each curve fiber. Such sections exist by relative generation after increasing the degree. Divisibility of a field by this section is vanishing on its relative Cartier zero divisor, a closed condition. Saturation on the curve gives the same invariant parabolic reductions before and after multiplication. This closed substack therefore gives a closed subspace on the good quotient by Proposition 5.15 [Alp13, Lemma 4.14]. The coefficients embed by the corresponding powers of the section. Properness of the fixed-twist map in Proposition 5.3 [CFH+25, Proposition 3.28(i)] now gives properness for the moving unweighted map, also after fixing the abelianization. Over the unweighted quotient the flag incidence with its small polarization is projective. Indeed on the quotient charts of Proposition 5.2 [AHR20, Theorem 4.12] it is the relative Proj of the invariant section algebra of the polarized flag incidence. With tiny weights a nonzero integral stability degree keeps its sign,

and the flag polarization tests the reductions of degree zero. Fixing the ordered residue to sv imposes closed equations. This proves properness over $\mathcal{E} \times \mathbb{A}^1$.

Scaling the Higgs field identifies the punctured line-family with $Y_{0,v} \times \mathbb{G}_m$. Conjugating T_ℓ by scaling gives an automorphism over $\mathcal{E} \times \mathbb{G}_m$, since T_ℓ preserves the coefficients and scaling multiplies the ith coefficient by s^{d_i}. Let $q_0 : Y_{0,0} \to \mathcal{E}$ be the special fiber map and put $\mathcal{K} = Rq_{0*}\mathbb{Q}_{Y_{0,0}}$. Lemma 8.2 applies in the orbifold charts of Theorem 5.24. In detail, smoothness gives the specialization isomorphism $\mathbb{Q}_{Y_{0,0}} \simeq \psi\mathbb{Q}_{\mathcal{Y}_{\mathrm{line}}}$ for unshifted nearby cycles on a smooth chart. Finite proper pushforward and the rational invariant projector give the same isomorphism on the coarse chart. Proper commutation with q then specializes the punctured pullback to an automorphism U of $\mathcal{K}$ over $\mathcal{E}$.

For its action on ordinary cohomology, pull the punctured product to the universal cover of $\mathbb{G}_m$. Restriction to a lift of 1 identifies its cohomology with $H^*(Y_{0,v})$. Restriction to the zero slice in the definition of nearby cycles gives a map from the same cohomology to $H^*(\mathcal{E}, \psi Rq_*\mathbb{Q}) \simeq H^*(Y_{0,0})$. Precomposing this map with pullback from $\mathcal{Y}_{\mathrm{line}}$ gives ordinary restriction to $Y_{0,0}$ by naturality of the specialization unit. Both endpoint restrictions from $\mathcal{Y}_{\mathrm{line}}$ are isomorphisms by Theorem 5.24. The comparison intertwines the two actions since the punctured automorphism lies over $\mathcal{E}$. Thus U acts by $\rho_{0,0}\mathsf{T}\rho_{0,0}^{-1}$.

Bounded constructible complexes on $\mathcal{E}$ have finite-dimensional morphism spaces, by constructibility of derived Hom and finite triangulation. Take $\mathcal{N} = \log U_{\mathrm{u}}$, a rational polynomial in U as in Lemma 8.3. Under hypercohomology the commuting semisimple and unipotent factors map to the corresponding factors of the cohomology action. Its Mth power is unipotent, so $\mathcal{N}$ induces precisely $N_{0,0}$. Push J_0, L_0 to $\mathcal{E}$ and compose them with $\mathcal{N}^2$ as in (8.1). Their shifts cancel, giving an endomorphism of the closed pushforward of $R(\mathrm{id}_S \times h)_*(\mathbb{Q}_S \boxtimes I_{X_z})$. Closed pushforward is fully faithful. We thus have an endomorphism on $S \times \mathcal{A}_G$. Pushing to $\mathcal{A}_G$ and composing with the unit $\mathbb{Q} \to R\Gamma(S, \mathbb{Q})$ and the trace $R\Gamma(S, \mathbb{Q}) \to \mathbb{Q}[-2]$ gives a morphism

$$\mathcal{F}' : Rh_*I_{X_z} \longrightarrow (Rh_*I_{X_z})[-2]$$

inducing f'. Shift both sides by $\dim X_z$ to use perverse normalization. Truncating at $k - d_h$ on the source gives truncation at $k - d_h - 2$ before the shift $[-2]$ on the target. Taking images in intersection cohomology gives $f'P_k \subset P_{k-2}$.

The adjoint cup descends on the unweighted central family. Let $\mathcal{V}$ be the universal adjoint vector bundle with fiber $\operatorname{Lie} G$ on the curve times the moduli stack. Finite stabilizers on the flagged stack identify its rational cohomology with coarse cohomology. In particular $A = \int_C \operatorname{ch}_2(\mathcal{V})$ and the flag class y_ℓ of Theorem 7.2 are ordinary classes on $\mathcal{Y}$. To compare A with θ along the scalar support we descend the line $\det R\Gamma_C(\mathcal{V} \otimes F)$, for the line F of degree $g - 1$ defining θ.

At an unweighted semistable Higgs point, $\mathcal{V}$ is semistable of degree zero by Proposition 5.1 [HZ25, Corollary 2.33]. Here the central summand of $\operatorname{Lie} H$ is trivial and splitting it off preserves semistability of slope zero. A semisimple stabilizer element has constant eigenvalues on C and splits $\mathcal{V}$ into invariant eigenbundles. Each has nonpositive degree and the sum of their degrees is zero, so each has degree zero. This degree assertion also holds on the central $u = 0$ line at $\lambda \neq 0$. The ordered residue is central modulo the nilradical, so the adjoint residue is nilpotent. After division by λ, the connection on each eigenbundle has residue of trace zero. Its determinant therefore has a holomorphic connection and degree zero.

Choose a rational section to write a line bundle L on C as the line bundle of a divisor. Adding a point to the divisor increases the Euler characteristic by one, by the length-one quotient. Starting with $\chi(\mathcal{O}_C) = 1 - g$ gives $\chi(L) = 1 - g + \deg L$ for every line bundle L. A line in the

generic fiber of a bundle extends by saturation to a line subbundle with locally free quotient on the smooth curve. Induction thus filters each eigenbundle $\mathcal{B}_\mu$ of eigenvalue μ by subbundles with line quotients. Additivity gives

$$\chi(\mathcal{B}_\mu \otimes F) = \deg \mathcal{B}_\mu + \mathrm{rk}(\mathcal{B}_\mu)(1 - g + \deg F) = 0.$$

The semisimple element acts on the determinant by the product of $\mu^{\chi(\mathcal{B}_\mu \otimes F)}$ over its eigenvalues, hence by one. Unipotent elements act trivially on a line, so the character of every closed stabilizer is trivial. Proposition 5.15 gives the descended line, including at strictly semistable points [Alp13, Theorem 10.3].

At $\lambda = u = 0$ the flagged stack with moving mark is a smooth complex algebraic orbifold of finite type by Construction 5.18 and Theorem 5.24. The universal $\mathcal{V}$ is a vector bundle on its product with the smooth projective connected complex curve C. Its determinant character is trivial on the connected semisimple adjoint group, so $c_1(\mathcal{V}) = 0$. With F pulled back from C, Lemma 6.15 identifies the first Chern class of the pulled-back determinant with $\int_C \mathrm{ch}_2(\mathcal{V})$. Under the rational coarse-space identification this is $A|_{Y_{0,0}}$. On $S \times X_z$ the central tensor correcting the moving mark acts trivially on $\mathcal{V}$. By uniqueness of descent the line there is the pullback of the scalar determinant, with class $\mathrm{pr}^*\,\theta$. Consequently both J_0 and L_0 intertwine m_A and $m_{\mathrm{pr}^*\,\theta}$ by the projection formula on the full unweighted target.

The double commutator on the scalar intersection complex. Put $\eta = B_{\mathrm{ad}}(\ell, \ell) x^* c_1(T_C)$, pulled back from S. On $Y_{0,v}$ the translation formula of Theorem 7.2 gives $(T_\ell^k)^* A = A + k y_\ell + k^2 \eta/2$. The underlying bundle and flag are unchanged by scaling the field, so these classes pull back from $Y_{0,v}$ in the punctured product. The formula is thus an identity of ordinary cup morphisms on that product. All three classes extend over $\mathcal{Y}_{\mathrm{line}}$. Naturality of the specialization unit and proper pushforward give, in $\mathrm{Hom}(\mathcal{K}, \mathcal{K}[2])$,

$$U^k[2]\, m_A\, U^{-k} = m_A + k\, m_{y_\ell} + \tfrac{1}{2} k^2 m_\eta.$$

Lemma 8.3 gives $[\mathcal{N}, m_A] = m_{y_\ell}$, with shifts understood in compositions. Since the two cup products commute,

$$[[\mathcal{N}^2, m_A], m_A] = [\mathcal{N} m_{y_\ell} + m_{y_\ell} \mathcal{N}, m_A] = 2 m_{y_\ell^2}.$$

This last cup product can be pushed forward along the flag-forgetful map before taking Hitchin coefficients. Over its unweighted target the composition $2 L_0 m_{y_\ell^2} J_0$ has degree four on the closed pushforward of $\mathbb{Q}_S \boxtimes I_{X_z}$. Pull it through this closed embedding and apply the unit and trace along S. This gives $D_X : I_{X_z} \to I_{X_z}[2]$. The intertwining of both flag maps with the descended cup, and the projection formula for pr, identify the derived commutator on $\mathcal{A}_G$ as

$$[[\mathcal{F}', Rh_* m_\theta], Rh_* m_\theta] = Rh_* D_X.$$

Let $X^{\mathrm{rs}} \subset X_z$ be the regularly stable open. Over $S \times X^{\mathrm{rs}}$ the scalar flags form the full flag bundle π of the adjoint bundle evaluated at x. Here J_0 is flag pullback followed by Gysin for the inclusion of the flags, and L_0 is restriction followed by flag integration. The scalar flags are cut out by the vanishing of the nilradical residue. At a regularly stable scalar point the obstruction to prescribing the framed first-order residue lies in degree two of the adjoint Higgs deformation complex, dual to the infinitesimal adjoint stabilizer, which vanishes. Thus the residue cuts out the flags regularly, with normal the nilradical bundle. Residue trivializes $K_C(x)|_x$, and the adjoint

pairing identifies the nilradical bundle with $T_\pi^\vee$, so self-intersection contributes $(-1)^b c_b(T_\pi)$. Thus $D_X|_{X^{\mathrm{rs}}}$ is cup by

$$2(-1)^b \operatorname{pr}_* \pi_*(c_b(T_\pi) y_\ell^2) = \frac{4(-1)^b |\mathcal{W}|}{r} B_{\mathrm{ad}}(\ell,\ell)\ \operatorname{pr}_* \operatorname{ch}_2(\mathcal{V}|_x).$$

The equality is the flag integral in Theorem 7.2. The same vanishing of the obstruction space gives a smooth stack on the regularly stable locus. Its central stabilizers act trivially on deformations, so X^{rs} is a smooth finite-type variety. Both the central inertia on the curve and these stabilizers act trivially on the adjoint. It descends to a vector bundle of trivial determinant on $C \times X^{\mathrm{rs}}$, and its evaluation at the mark is its pullback by $x \times \mathrm{id}_{X^{\mathrm{rs}}}$. Writing $\delta = \deg(x : S \to C)$, we have $x_*[S] = \delta[C]$. Applying Lemma 6.15 on this smooth base with the same line F on C, we obtain

$$\operatorname{pr}_* \operatorname{ch}_2(\mathcal{V}|_x) = \delta \int_C \operatorname{ch}_2(\mathcal{V}|_{C \times X^{\mathrm{rs}}}) = \delta\, \theta|_{X^{\mathrm{rs}}}.$$

The restriction injectivity of Theorem 6.16 now gives the identity

$$D_X = \frac{4(-1)^b \delta |\mathcal{W}|}{r} B_{\mathrm{ad}}(\ell,\ell)\, m_\theta \quad \text{in } \operatorname{Hom}(I_{X_z}, I_{X_z}[2]).$$

Normalization and equality of subspaces. The roots span the real character space, so $B_{\mathrm{ad}}(\ell,\ell) = \sum_\alpha \alpha(\ell)^2 > 0$. Multiply $\mathcal{F}'$ by

$$\frac{(-1)^{b+1} r}{2\delta |\mathcal{W}| B_{\mathrm{ad}}(\ell,\ell)}$$

and average its conjugates under $H^1(C, Z(G))$. Central twists act over $\mathcal{A}_G$, preserve the adjoint determinant, and act algebraically on the matched Betti space. The averaged morphism therefore induces a rational equivariant f with the same perverse and weight bounds. The identity for D_X gives $[[f,e],e] = -2e$.

On $\bigoplus_m \mathrm{IH}^m(X_z)$ take the two filtrations with steps P_k and $\Psi^* W_{2k}$. Both are finite, exhaustive and separated. Theorem 6.16 gives e shift at most two for both filtrations and hard Lefschetz on both gradeds with center R. Here the scalar weights are Hodge–Tate, so the kth graded for the second filtration is $\Psi^* \operatorname{Gr}^W_{2k}$. The bound by four on native weight gives f shift at most -2 for this filtration. Lemma 8.4 thus gives $P_k = \Psi^* W_{2k}$ as subspaces of the common rational cohomology. Intersecting with $\mathrm{IH}^m(X_z)$ gives the degreewise identity, and $\operatorname{Gr}^W_{2k+1} = 0$ gives the adjacent odd step. After extending scalars to $\mathbb{C}$, applying each character projector of $H^1(C, Z(G))$ gives the asserted equalities on its summand. □

9 Components and central isogenies

We retain the curve, the full Hitchin base and the harmonic homeomorphism of Setup 2.1. We work on the reduced coarse spaces. In genus one a finite cover by torus moduli computes the defect of the Hitchin map in each flat type. In every positive genus a central isogeny expresses the two filtrations in a given type in terms of the scalar factors.

Proposition 9.1. *For a proper morphism of schemes and a finite abelian étale coefficient sheaf, the higher direct images commute with base change and their geometric stalks are the corresponding cohomology groups of the fibers.*

For the projection of a constant family of curves, this identifies the sheaves of lifting obstructions in degree two and of twists in degree one [Art73, Exposé XII, Théorème 5.1(iii) and Corollaire 5.2(iii)].

On an elliptic curve, fix oriented generators of the fundamental group. Lift the two holonomies of a semisimple commuting pair to $Z(G)^\circ \times G_{\mathrm{sc}}$, where G_{sc} is the simply connected cover of $[G,G]$. Their commutator is $(1,c)$ in the kernel of the isogeny to G. Central changes of lifts preserve c. In the central extension used to parametrize representations, the central $\mathbb{R}$ acts by $s \mapsto \exp(su)$ for $u \in \mathrm{Lie}(Z(G)^\circ)$. The quotient by this $\mathbb{R}$ is $\mathbb{Z}^2$, so the ordinary representations are precisely those of types $(0,c)$, including nontrivial c [FGN19, §§2.2–2.3, Remark 2.3].

Definition 9.2. For a semisimple commuting pair in G of lifted commutator $(1,c)$, choose a maximal torus S_c of its common centralizer and put $L_c = Z_G(S_c)$. The *residual torus* and the *relative Weyl group* are

$$T_c = L_c/[L_c,L_c] \simeq S_c/(S_c \cap [L_c,L_c]), \qquad \mathcal{W}_c = N_G(S_c)/L_c.$$

Indeed $S_c = Z(L_c)^\circ$, since the connected center of L_c contains S_c and centralizes the pair. The conjugacy class of S_c depends only on the sector [LNY24, §6.2].

Proposition 9.3. *For a connected complex reductive group on an elliptic curve, in each flat type the Higgs and central-representation coarse spaces admit bijective normalizations by residual Weyl quotients. On the Higgs cover the Hitchin map is induced by projection to the Lie algebra of the residual torus followed by the invariant-polynomial map.*

The cocharacter lattice in both covers is $X_*(T_c)$. The isogeny $S_c \to T_c$ identifies their Lie algebras [FGN19, Theorem 3.14, Proposition 4.1, Theorem 5.3 and Corollary 5.4]. For the closed-point parametrization one fixes a commuting pair in $[L_c,L_c]$ of lifted commutator $(1,c)$.

Proposition 9.4. *In a fixed lifted-commutator sector, the conjugacy classes of semisimple commuting pairs are parametrized by pairs in the residual torus modulo its relative Weyl group. The fixed pair in the derived Levi has finite centralizer in that derived group.*

Before passing to T_c, the parameters multiply the two entries of this fixed pair by elements of S_c. This describes the map on closed points [LNY24, §6.2, Proposition 6.2.1].

Proposition 9.5. *For a smooth projective family of curves and connected complex reductive structure group the ordinary relative Hitchin morphism of semiharmonic Higgs moduli is projective.*

We use projectivity over the full base after restricting to a fixed curve and an open and closed flat type [Cat21, Proposition 2.2.2].

Proposition 9.6. *On a complex algebraic pseudomanifold the rational middle-perversity intersection complex with constant generic coefficients is determined intrinsically by the topology with its ordinary normalization.*

In particular the harmonic homeomorphism identifies $I_{M_{\mathrm{B}}}$ with $I_{M_{\mathrm{Dol}}}$ by pullback, and hence identifies ordinary intersection cohomology in the same degree [GM83, §§4.1–4.3]. On a pure-dimensional union we take the sum of the intersection complexes of its irreducible components.

Lemma 9.7. *Let $q : Y \to X$ be a finite surjection of reduced complex varieties, with X irreducible and Y pure of dimension $\dim X$. The inclusion of constants on the generic fiber induces a split inclusion $\mathrm{IC}_X \hookrightarrow q_*\mathrm{IC}_Y$, also for the intersection Hodge modules. If a finite group Γ acts on Y over X and acts transitively on the geometric generic fiber, this inclusion identifies IC_X with $(q_*\mathrm{IC}_Y)^\Gamma$.*

Proof. Let $n = \dim X$. Choose a smooth dense open $U \subset X$ over which q is finite étale with smooth source. Finite direct image is perverse exact. Indeed, finite fibers preserve the stalk support inequalities for the ordinary cohomology sheaves, and proper duality gives the cosupport inequalities. The strict inequalities for middle extension across the complement of U are preserved for the same reason. Thus $q_*\mathrm{IC}_Y$ is the middle extension of the permutation local system $(q|_{q^{-1}U})_*\mathbb{Q}[n]$. The diagonal inclusion of constants in this system is split by the trace divided by the degree. Both maps extend by middle extension. For a transitive action the averaging projector on the generic fiber has precisely this diagonal as image.

The permutation system is a variation of type $(0,0)$ before shifting. Diagonal and trace are maps of variations. Projective direct image and uniqueness of the pure extension with strict support give the same splitting in Hodge modules, by Propositions 5.12 and 5.13 [Sai88, Théorème 5.3.1] [Sai90, Théorème 3.21]. Its ordinary direct image to a point therefore gives the intrinsic weight filtration on the downstairs summand. □

Lemma 9.8. *Let $Z_1, \dots, Z_s$ be irreducible complex quasi-projective varieties, with $s \geq 1$, and put $Z = \prod_i Z_i$. The identity on constants over the product of the smooth loci extends to an isomorphism $\boxtimes_i I_{Z_i} \simeq I_Z$. For every integer m, the resulting cross product is an isomorphism of mixed Hodge structures*

$$\bigoplus_{m_1+\cdots+m_s=m} \bigotimes_{i=1}^{s} \mathrm{IH}^{m_i}(Z_i) \longrightarrow \mathrm{IH}^m(Z).$$

Proof. We identify the intersection Hodge module on Z and use the adjunction for the maps to a point to construct the cross product.

The coefficient on the product. Write $n_i = \dim Z_i$, $n = \sum_i n_i$, and let U be the product of the smooth loci. On U the rational complex $\boxtimes_i \mathrm{IC}_{Z_i}$ is $\mathbb{Q}_U[n]$. The support and cosupport bounds add on product strata. On each stratum outside U a boundary factor makes each of the bounds strict. This characterizes IC_Z by middle extension, and shifting by $-n$ gives the asserted identification of the I_{Z_i} on the product.

Let M_i be the intersection Hodge module on Z_i. The exterior product takes the rational parts to $\boxtimes_i \mathrm{IC}_{Z_i}$, just identified with IC_Z. Since rational realization is faithful and exact on the perverse hearts, this product is concentrated in perverse degree zero and has no subobject or quotient supported outside U. Its restriction to U is the constant variation shifted by n, of Hodge-module weight n. Thus its weight step W_{n-1} and its quotient by W_n are supported outside U and both vanish. The product equals its polarizable graded piece of weight n. Proposition 5.13 identifies this extension of the constant variation with the intersection Hodge module on Z.

The cross product on cohomology. Let $a_i : Z_i \to \mathrm{pt}$ and $a : Z \to \mathrm{pt}$ be the structure maps, and put $E_i = M_i[-n_i]$. Taking the exterior product of the counits $a_i^* Ra_{i*}E_i \to E_i$ gives a map $a^*(\bigotimes_i Ra_{i*}E_i) \to \boxtimes_i E_i$. Its adjoint is a morphism in the derived category of mixed Hodge structures

$$\bigotimes_i Ra_{i*}E_i \longrightarrow Ra_*(\boxtimes_i E_i).$$

The structure maps are of finite type and the complexes are bounded and constructible. Proposition 5.16 therefore identifies the realized adjunctions with those for rational sheaves [Tub25, §2.1 and Proposition 4.2.6]. The underlying map is the ordinary cross product for the I_{Z_i}, a quasi-isomorphism by rational Künneth. Forgetting the mixed Hodge structures detects isomorphisms. Since their tensor product is exact, tensoring cocycles identifies the degree-m cohomology of the

source with the displayed sum over multidegrees. The target in that degree is $\mathrm{IH}^m(Z)$ with its intrinsic mixed Hodge structure by the coefficient identification above. □

For a complex torus S write $X_*(S) = \mathrm{Hom}(\mathbb{G}_m, S)$ and put

$$J(S) = \mathrm{Pic}^0(C) \otimes_{\mathbb{Z}} X_*(S), \qquad V(S) = H^0(C, K_C) \otimes_{\mathbb{C}} \mathrm{Lie}\, S.$$

In a basis of cocharacters the degree-zero Higgs space of S parametrizes $\dim S$ degree-zero line bundles with their Higgs fields. Its Hitchin map is the projection $h_S : J(S) \times V(S) \to V(S)$.

Lemma 9.9. *Let C be a smooth projective connected complex curve of genus g, let S be a complex torus, and let $\beta : V(S) \to A$ be a finite morphism of complex varieties. Then βh_S has defect $g \dim S$. For every $m, k \in \mathbb{Z}$, its perverse filtration in degree m is zero for $k < m$ and full for $k \geq m$. The group $H^m(S^{2g})$ is pure of weight $2m$ for every m.*

Proof. Put $a = g \dim S$, the dimension of both $J(S)$ and $V(S)$. The image of β has dimension a, and each nonempty reduced fiber of βh_S is a finite disjoint union of copies of $J(S)$. The relative square thus has dimension $3a$ and the source has dimension $2a$. Writing $V = V(S)$, the product projection gives

$$Rh_{S*}\mathbb{Q}_{J(S)\times V}[2a] \simeq \bigoplus_{j=0}^{2a} H^j(J(S)) \otimes \mathbb{Q}_V[a][a-j].$$

Since $\mathbb{Q}_V[a]$ is perverse, the jth term has perverse degree $j - a$. Finite direct image by β preserves perverse truncations and global hypercohomology. At cutoff $k - a$ the surviving terms have $j \leq k$, and contractibility of V leaves only $j = m$ in degree $m - 2a$. Finally, $S^{2g} \simeq (\mathbb{C}^\times)^{2a}$. The Gysin sequence for $\{0, \infty\} \subset \mathbb{P}^1$ gives $H^1(\mathbb{C}^\times) = \mathbb{Q}(-1)$ and vanishing in higher degrees. For $a > 0$, each nonzero degree-m term in Lemma 9.8 is therefore $\mathbb{Q}(-m)$. For $a = 0$ the space S^{2g} is a point. □

Proposition 9.10. *For a connected complex reductive G and a smooth projective connected complex curve of genus zero or one, on each flat component the filtrations P_k, $\Psi^* W_{2k}$ and $\Psi^* W_{2k+1}$ on IH^m are zero for $k < m$ and the whole space for $k \geq m$.*

Proof. We compute the genus-zero filtration directly. In genus one we realize each flat type by finite maps from torus moduli and compute the two filtrations on their intersection-complex summands.

The genus-zero component. Here $\pi_1 C$ is trivial, so M_{B} and its harmonic partner are points. The defect is zero. The unique nonzero cohomology group is $\mathbb{Q}$ in degree zero, with perversity and weight both zero.

Finite covers in an elliptic sector. Fix c and the groups of Definition 9.2. Choose a nonzero differential on C. Proposition 9.3 gives the normalizations $(J(T_c) \times \mathrm{Lie}\, S_c)/\mathcal{W}_c$ and $T_c^2/\mathcal{W}_c$. To describe the covers before the isogeny $S_c \to T_c$, choose the commuting pair (x_c, y_c) in $[L_c, L_c]$ of Proposition 9.4. The Betti morphism

$$q_{\mathrm{B},c} : S_c^2 \longrightarrow M_{\mathrm{B}}, \qquad (s_1, s_2) \longmapsto [(s_1 x_c, s_2 y_c)]$$

is surjective and has finite fibers by the closed-point parametrization. It is finite as a morphism. Indeed, record the two individual conjugacy classes in $(G/\!/G)^2$. Each of x_c, y_c belongs to a maximal torus of L_c containing S_c. In its respective maximal torus the translate $S_c x_c$ or $S_c y_c$ is closed, and the map to $G/\!/G$ is the finite Weyl quotient. The composite from S_c^2 to $(G/\!/G)^2$ is

therefore finite. Its base change to M_{B} contains the graph of $q_{\mathrm{B},c}$ as a closed subvariety, proving finiteness.

The subgroup generated by x_c, y_c lies in their finite common centralizer in $[L_c, L_c]$. Its flat bundle admits a parallel compact reduction, so its Higgs partner has zero Higgs field. Multiplying this fixed bundle by the torus Higgs family given by the Poincaré bundles and extending the structure group from $S_c \times [L_c, L_c]$ to G gives an algebraic map $q_{\mathrm{D},c} : J(S_c) \times V(S_c) \to M_{\mathrm{Dol}}$. For a torus Higgs bundle with harmonic holonomies (s_1, s_2), its image has harmonic holonomies $(s_1 x_c, s_2 y_c)$ by tensor compatibility in Proposition 5.10 [Sim11, Theorems 6.10, 8.8 and 9.7]. Thus $q_{\mathrm{D},c}$ is surjective with finite fibers. Its composite with h is $\beta_c h_{S_c}$, where, in the chosen trivialization of K_C,

$$\beta_c : V(S_c) = \operatorname{Lie} S_c \;\hookrightarrow\; \operatorname{Lie} T \longrightarrow (\operatorname{Lie} T)/\mathcal{W} = \mathcal{A}_G.$$

Here T is a maximal torus of G containing S_c and $\mathcal{W}$ its Weyl group. Invariant polynomials evaluate on the toral Higgs field, so β_c is a closed linear inclusion followed by a finite map. Now $\beta_c h_{S_c}$ is proper and M_{Dol} is separated over $\mathcal{A}_G$. The graph construction makes $q_{\mathrm{D},c}$ proper, hence finite.

The defect and the two jumps. Put $e_c = \dim S_c$. Finiteness gives $\dim M_{\mathrm{Dol}} = 2e_c$. Each nonempty Hitchin fiber is the finite image of a finite union of copies of $J(S_c)$, and the image of h has dimension e_c. Consequently

$$\dim(M_{\mathrm{Dol}} \times_{\mathcal{A}_G} M_{\mathrm{Dol}}) = 3e_c, \qquad d_h = e_c.$$

Lemma 9.7 realizes $\mathrm{IC}_{M_{\mathrm{Dol}}}$ as a summand of $q_{\mathrm{D},c*}\mathbb{Q}[2e_c]$. Truncation, hypercohomology and the image defining P_k respect this splitting. The defect just computed is the one in Lemma 9.9 for $\beta_c h_{S_c}$, so its degree-m jump at m passes to $\mathrm{IH}^m(M_{\mathrm{Dol}})$. The finite surjection $q_{\mathrm{B},c} : S_c^2 \to M_{\mathrm{B}}$ has irreducible target of the same dimension as its source. Applying Lemma 9.7 to this map identifies $\mathrm{IH}^m(M_{\mathrm{B}})$ as a Hodge summand of $H^m(S_c^2)$. By Lemma 9.9 with $g = 1$ and $S = S_c$, this ambient group is pure of weight $2m$. The two filtrations therefore have the stated zero and full subspaces in each degree. Proposition 9.6 identifies these subspaces under the given Ψ^*. □

Example 9.11. For elliptic PGL_2 the neutral sector has residual rank one. In the other sector the lifts $A, B \in \mathrm{SL}_2$ anticommute. Since A is conjugate to $-A$, its eigenvalues are $i, -i$. In this eigenbasis B exchanges the two lines, and diagonal conjugation puts its off-diagonal entries equal to $1, -1$. This sector is a point, with $S_c = 1$ and $[L_c, L_c] = \mathrm{PGL}_2$. Thus the two components have dimensions $2, 0$ and defects $1, 0$. A uniform truncation at $k - a$ would place their degree-zero units first at $a - 1$ and a, respectively. The componentwise cutoff places each unit at level zero, its half-weight.

On a disjoint union of types we take the direct sum of the resulting filtered spaces.

Proposition 9.12. *Suppose $g \geq 1$ and $T_0 \times \prod_\nu G_\nu \to G$ is a central isogeny with finite kernel F_0, with T_0 a torus and the G_ν simply connected simple. Fix a lifting obstruction $(1, (z_\nu)) \in F_0$. In the corresponding type the two filtered intersection cohomologies for G are the $H^1(C, F_0)$-invariant subspaces of the tensor products of the torus and the scalar factors (X_{z_ν}, B_{z_ν}). Each filtration on the tensor product is its convolution filtration. The harmonic comparison is induced by the tensor product of the factor comparisons. In degree m the torus comparison identifies the single perverse jump m with the single weight jump $2m$.*

Proof. We construct the two finite quotients in the prescribed type, identify their harmonic maps, and compute the defect and the filtrations on the invariant summands.

The lifting obstruction and the Betti quotient. Write $\rho : \widetilde{G} = T_0 \times \prod_\nu G_\nu \to G$ for the isogeny, $f = (1, (z_\nu))$, and $\Gamma = H^1(C, F_0)$. Choose oriented surface generators with boundary word $\delta = \prod_{j=1}^{g} [\alpha_j, \beta_j]$. Over the representation scheme, lifting the $2g$ holonomies along ρ gives the restriction of the finite étale F_0^{2g}-torsor $\widetilde{G}^{2g} \to G^{2g}$. Evaluated on the lifts, δ takes values in F_0 with torus coordinate one. Central twists and conjugation preserve its value. Its values give open and closed loci on the character variety.

These values record the entire flat topological type. Indeed, $\pi_1(G)$ is the kernel of the covering $\operatorname{Lie} T_0 \times \prod_\nu G_\nu \to G$. An element of this kernel is torsion exactly when its vector coordinate is zero. Projection identifies these elements with $F_0 \cap (\{1\} \times \prod_\nu G_\nu)$. The obstruction to extending a trivialization across the two-cell of C is the lifted boundary word, whose value in this kernel is $(0, (z_\nu))$. Every such value occurs. In a compact form of each G_ν take a maximal torus of rank r_ν containing z_ν and a representative $\dot{w}$ of a Coxeter element $w = s_1 \cdots s_{r_\nu}$. On the real Lie algebra of the torus,

$$1 - w = \sum_{j=1}^{r_\nu} s_1 \cdots s_{j-1}(1 - s_j).$$

The image lines of the summands are spanned by the transformed simple coroots, unitriangular in the simple coroot basis. A vector fixed by w is thus fixed by all s_j and is zero. The torus map $v \mapsto v\, w(v)^{-1}$ is surjective, so $[v, \dot{w}] = z_\nu$ for some v. Use these compact pairs on one handle and identities on the others.

Over f the space of lifted tuples is the product of T_0^{2g} and the scalar word fibers. Taking conjugation invariants factor by factor commutes with taking invariants under $\Gamma = F_0^{2g}$. In the opposite order one first recovers the G-tuples in the prescribed type and then takes their conjugation quotient. Consequently the map

$$q_{\mathrm{B}} : Y_{\mathrm{B}} = T_0^{2g} \times \prod_\nu B_{z_\nu} \longrightarrow M_{\mathrm{B}}$$

is the finite orbit quotient. The action multiplies each handle lift using all the projections of its entry of F_0.

Higgs lifts and the finite quotient in families. Put $N = \operatorname{ord}(f)$ and let $\mathcal{G}_N$ be the gerbe of Nth roots of the line bundle $\mathcal{O}_C(p)$. In the circle bundle of this degree-one line bundle the lifted surface word is the fiber generator, with the chosen orientation. The root gerbe takes the quotient of that fiber group by $N\mathbb{Z}$. Thus the lifted word on $\mathcal{G}_N$ is $\delta = \tau$, with τ the marked central generator of order N. Consider semiharmonic $\widetilde{G}$-Higgs bundles with inertia $\tau \mapsto f$. The torus factor has trivial inertia and its rational degrees vanish, so it descends to C with degree zero. For $n_\nu = \operatorname{ord}(z_\nu)$ the map of root gerbes $\mathcal{G}_N \to \mathcal{G}_{n_\nu}$ sends a root line M to M^{N/n_ν} and preserves the marking of the surface word. Its relative inertia μ_{N/n_ν} acts trivially in the νth factor. Descent on neutralizing charts gives precisely the bundles on $\mathcal{G}_{n_\nu}$ with the marked inertia z_ν. The Higgs fields descend too, since the canonical bundle on each gerbe is pulled back from C. This includes $n_\nu = 1$ and $N = 1$.

After extension by ρ the entire inertia acts trivially. Descent to C therefore gives an algebraic morphism

$$q_{\mathrm{D}} : Y_{\mathrm{D}} = (J(T_0) \times V(T_0)) \times \prod_\nu X_{z_\nu} \longrightarrow M_{\mathrm{Dol}}.$$

Semistability is preserved in both directions under a finite central isogeny. Parabolic reductions correspond via the isomorphisms of flag varieties, and a positive power of each testing character

kills the kernel, so the degree inequalities agree. Rational characteristic classes correspond since BF_0 has trivial positive rational cohomology. On the gerbe these tests can be made on a finite étale curve cover with a root line. Indeed a cyclic cover of C of degree N exists for $g \geq 1$, and the pulled-back degree-N line has an Nth root by divisibility of the degree-zero Picard variety. A destabilizing reduction pulls back along a finite étale cover. Conversely, if a pullback is unstable, its Harder–Narasimhan reduction on a Galois refinement is invariant by uniqueness and descends. Trace gives injectivity of pullback on rational characteristic classes.

For a family of G-Higgs bundles of this type over $C \times U$, let $\mathcal{L}$ be its F_0-gerbe of local lifts of the underlying bundle. The differential $d\rho$ lifts the Higgs field uniquely. A lift on $\mathcal{G}_N \times U$ with the prescribed inertia is an equivalence of $\mathcal{L}$ with the pushout of that root gerbe along $\tau \mapsto f$. On local objects such an equivalence assigns a lift with its descent isomorphisms. On each geometric fiber the class of $\mathcal{L}$ in $H^2_{\text{ét}}(C, F_0) \simeq F_0$ is the image of the clutching class, namely f. The pushed-out gerbe has the same class because $\mathcal{O}_C(p)$ has degree one and its marked inertia maps to f. For $\pi_U : C \times U \to U$, the sheaf associated to the presheaf $U' \mapsto H^j_{\text{ét}}(C \times U', F_0)$ is $R^j\pi_{U*}F_0$. Proposition 9.1 identifies it with the constant sheaf of fiber cohomology. In degree two this gives equality of the two gerbe classes étale locally on U, hence the desired lifts. Two lifts with identified image differ by an F_0-torsor. Their inertia actions agree, so this torsor descends to $C \times U$. In degree one the same argument identifies that torsor, locally on U, with a fixed member of Γ.

It follows by étale descent that a Γ-invariant map from the upstairs moduli to an algebraic space descends uniquely to the downstairs moduli. Indeed it has the same value on any two local lifts, and their automorphisms act trivially on maps to an algebraic space. By the universal property of the coarse spaces this identifies $M_{\text{Dol}} = Y_{\text{D}}/\Gamma$. Thus q_{D} is finite, including at points with a twist stabilizer.

The harmonic identification. On $\mathcal{G}_N$ the local harmonic operators for a product of groups are the products of the factor operators. They pull back on neutralizing charts and commute with extension by ρ (Proposition 5.10). A trivial-inertia local system descends to C, where its holonomies are the ρ-images of the lifted generators. A twist by an F_0-torsor has finite unitary holonomy and zero Higgs field. The product harmonic homeomorphism is consequently Γ-equivariant and satisfies

$$\Psi \, q_{\text{D}} = q_{\text{B}}\big(\Psi_{T_0} \times \prod_{\nu} \Psi_\nu\big),$$

where $\Psi_\nu : X_{z_\nu} \to B_{z_\nu}$ and Ψ_{T_0} are the factor comparisons. The identification of the invariant intersection complex in Lemma 9.7 is normalized by pulling back constants. By Proposition 9.6 the square transports the rational intersection complexes and their averaging projectors. The two smooth covering loci meet after transport in a dense open in the classical topology. There both inclusions send 1 to 1 on every sheet. On each irreducible support the middle extension of the rank-one constant system has scalar endomorphisms, so the inclusions agree on the invariant intersection cohomologies.

The defect of the product Hitchin map. The invariant polynomials for $\widetilde{G}$ and G agree under $d\rho$. Inertia is central on each factor and acts trivially on its adjoint bundle. Thus the Higgs invariants take values in the ordinary powers of K_C, and the identification of $\mathcal{A}_G$ with the product of factor bases takes hq_{D} to the product of the factor Hitchin maps h_i. The group Γ acts over this base. Proposition 9.5 gives projectivity of h. Its composite with q_{D} is projective, and fixing a point in every other factor gives projectivity of each h_i.

The map

$$Y_{\text{D}} \times_{\mathcal{A}_G} Y_{\text{D}} \longrightarrow M_{\text{Dol}} \times_{\mathcal{A}_G} M_{\text{Dol}}$$

is finite surjective, since lifts of two points over the same base point still have the same Higgs invariants. Dimensions are preserved both here and on the sources of the Hitchin maps. The upstairs relative square is the product of the factor relative squares, giving $d_h = \sum_i d_{h_i}$. In particular, for $g \geq 2$ the same finite quotient for $G_\nu \to G_\nu/Z(G_\nu)$ takes X_{z_ν} onto the ordinary adjoint component over the identical base. With $R_\nu = (g-1)\dim G_\nu$, Proposition 5.3 for K_C gives total dimension $2R_\nu$ and fibers of dimension R_ν there [CFH+25, Propositions 3.27(ii) and 3.28(i)]. Its relative square has dimension $3R_\nu$, so $d_{h_\nu} = R_\nu$. The torus contributes $g \dim T_0$ by Lemma 9.9.

Convolution on the invariant summands. Write $Z_i^{\mathrm{D}}, Z_i^{\mathrm{B}}$ for the Higgs and Betti spaces of the factors, including the torus. Lemma 9.8 identifies the intersection complex on each product of irreducible components with the exterior product of their intersection complexes.

Set $K_i = (Rh_{i*}\mathrm{IC}_{Z_i^{\mathrm{D}}})[-d_{h_i}]$, so that truncation at k computes the normalized filtration on that factor. Proper Künneth and additivity of the defects identify the corresponding product complex with $\boxtimes_i K_i$. By Proposition 5.12 each K_i splits into its perverse cohomology sheaves shifted to their degrees. External products of perverse sheaves are perverse by the product support and cosupport inequalities. A tensor term from degrees j_i is therefore retained by truncation at k exactly when $\sum_i j_i \leq k$. Under Künneth this gives the convolution of the factor image filtrations, independently of the chosen splittings.

For each finite quotient Lemma 9.7 identifies the downstairs intersection complex with the averaging summand of the upstairs direct image, taking the sum over the components above each irreducible component. Perverse truncation and the image on hypercohomology commute with this projector. On the Betti side it is a Hodge projector. By Lemma 9.8 the Betti cross product identifies each multidegree summand with its tensor product mixed Hodge structure, whose weight filtration is the convolution of the factor weights. Thus in degree m the two subspaces identify with

$$P_k\mathrm{IH}^m(M_{\mathrm{Dol}}) \simeq \left(\bigoplus_{\sum_i m_i = m} \sum_{\sum_i k_i \leq k} \bigotimes_i P_{k_i}\mathrm{IH}^{m_i}(Z_i^{\mathrm{D}}) \right)^\Gamma,$$

$$W_\ell\mathrm{IH}^m(M_{\mathrm{B}}) \simeq \left(\bigoplus_{\sum_i m_i = m} \sum_{\sum_i \ell_i \leq \ell} \bigotimes_i W_{\ell_i}\mathrm{IH}^{m_i}(Z_i^{\mathrm{B}}) \right)^\Gamma.$$

Naturality of external products in the harmonic square makes Ψ^* the restriction of the tensor product of the factor comparisons. Lemma 9.9 with $\beta = \mathrm{id}$ gives the torus jumps at m and $2m$, respectively. □

In particular, if W_{2a} and W_{2a+1} both pull back to P_a in every factor, each tensor term can use the even weight indices $2k_i$. The cutoffs $2k$ and $2k+1$ then both retain exactly the terms with $\sum_i k_i \leq k$, also after taking Γ-invariants.

10 The intersection-cohomology comparison

We work in a fixed flat type, with P normalized by the Hitchin defect d_h of that component. In genus at least two the comparison descends from the torus and simple factors of a central cover. Its description on a projective compactification uses the following weight formula of Saito [Sai90, Theorems 2.14 and 3.21, formulas (4.2.11)–(4.2.12) and (4.5.1)–(4.5.2)].

Proposition 10.1. *Let $j : B \hookrightarrow \overline{B}$ be an affine open immersion of complex varieties with $\overline{B}$ projective, and let E be a mixed Hodge module on B. Then j_*E is a mixed Hodge module. For all $a, t \in \mathbb{Z}$, the image of $H^t(\overline{B}, W_a j_*E)$ in $H^t(B, E)$ is $W_{a+t}H^t(B, E)$.*

Here the cutoff W_a is taken on the extension j_*E in $\mathrm{MHM}(\overline{B})$. For fixed a, t, its image in cohomology depends only on B and E.

Theorem 10.2. *Let C be a smooth projective connected complex curve and let G be a connected complex reductive group. In each flat topological type, for every $m, k \in \mathbb{Z}$,*

$$P_k\mathrm{IH}^m(M_{\mathrm{Dol}}) = \Psi^* W_{2k}\mathrm{IH}^m(M_{\mathrm{B}}) = \Psi^* W_{2k+1}\mathrm{IH}^m(M_{\mathrm{B}}).$$

Proof. In genera zero and one, Proposition 9.10 gives zero for each of the three subspaces when $k < m$ and all of $\mathrm{IH}^m(M_{\mathrm{Dol}})$ when $k \geq m$. Suppose $g \geq 2$. We compare the filtrations on the factors of a central isogeny and then pass to invariants in their tensor product.

Comparison on the factors. Let $T_0 = Z(G)^\circ$ and write $\prod_{\nu=1}^s G_\nu$ for the simply connected cover of $[G, G]$, decomposed into simple factors. Multiplication gives a central isogeny $T_0 \times \prod_{\nu=1}^s G_\nu \to G$ with finite kernel F_0. For a representation in our component, lift the holonomies of the surface generators through this isogeny. The product of their commutators lies in F_0 and has torus coordinate 1. Since F_0 is central, this product is independent of the lifts. Write $(1, (z_\nu))$ for the resulting obstruction. On the full intersection cohomology of each scalar pair (X_{z_ν}, B_{z_ν}), Theorem 8.5 identifies P_a with both $\Psi^* W_{2a}$ and $\Psi^* W_{2a+1}$ for every a. On the torus factor Proposition 9.12 places degree m_0 at perversity m_0 and weight $2m_0$, giving the same two equalities.

Descent to the chosen component. Fix a multidegree $m_0 + \cdots + m_s = m$, numbering the torus factor by 0. Let V_ν be the degree-m_ν intersection cohomology of factor ν, identifying the two sides by harmonic pullback. On this space $W_j V_\nu = P_{\lfloor j/2 \rfloor} V_\nu$. If $\sum_\nu j_\nu \leq 2k + 1$, then $\sum_\nu \lfloor j_\nu/2 \rfloor \leq k$. Conversely, if $\sum_\nu a_\nu \leq k$, the indices $j_\nu = 2a_\nu$ have sum at most $2k$. Thus the convolution filtrations satisfy

$$W_{2k+1}\left(\bigotimes_{\nu=0}^{s} V_\nu\right) \subseteq \sum_{a_0 + \cdots + a_s \leq k} \bigotimes_{\nu=0}^{s} P_{a_\nu} V_\nu \subseteq W_{2k}\left(\bigotimes_{\nu=0}^{s} V_\nu\right).$$

The middle term is the perverse step P_k, and both inclusions are equalities since W is increasing. By Proposition 9.12, summing over the multidegrees and taking $H^1(C, F_0)$-invariants gives the filtration steps on the chosen component. The tensor product of the factor pullbacks induces Ψ^* there. □

The odd weight gradeds of $\mathrm{IH}^m(M_{\mathrm{B}})$ therefore vanish. This accounts for the two consecutive cutoffs in the compactified description.

Corollary 10.3. *Let $g \geq 2$ and fix a flat component with $\dim M_{\mathrm{B}} = 2d$ and $d_h = d$. Choose an affine open immersion $j : M_{\mathrm{B}} \hookrightarrow \overline{B}$ with $\overline{B}$ projective, and put $E = \mathrm{IC}_{M_{\mathrm{B}}}(d)$ in $\mathrm{MHM}(M_{\mathrm{B}})$. For every m, k, the harmonic image of $P_k\mathrm{IH}^m(M_{\mathrm{Dol}})(d)$ in $\mathrm{IH}^m(M_{\mathrm{B}})(d)$ equals the image of*

$$H^{m-2d}(\overline{B}, W_a(j_*E)) \longrightarrow \mathrm{IH}^m(M_{\mathrm{B}})(d)$$

both for $a = 2k - m$ and for $a = 2k + 1 - m$.

Proof. Since $d_h = d$, the truncation defining P_k on $Rh_*\mathrm{IC}_{M_{\mathrm{Dol}}}$ is ${}^{\mathrm{p}}\tau_{\le k-d}$. On the Betti side put $t = m - 2d$. The normalization $I_{M_{\mathrm{B}}} = \mathrm{IC}_{M_{\mathrm{B}}}[-2d]$ gives the identifications

$$H^t(\overline{B}, j_*E) = H^t(M_{\mathrm{B}}, E) = \mathrm{IH}^m(M_{\mathrm{B}})(d)$$

of mixed Hodge structures. By Proposition 10.1 the image in the corollary is $W_{a+t}(\mathrm{IH}^m(M_{\mathrm{B}})(d))$. The Tate factor $\mathbb{Q}(d)$ has weight $-2d$, which takes the weight $2d$ of $\mathrm{IC}_{M_{\mathrm{B}}}$ to zero. On cohomology this twist gives

$$W_{a+t}(\mathrm{IH}^m(M_{\mathrm{B}})(d)) = (W_{a+t+2d}\mathrm{IH}^m(M_{\mathrm{B}}))(d) = (W_{a+m}\mathrm{IH}^m(M_{\mathrm{B}}))(d).$$

At $a = 2k - m$ and $a = 2k + 1 - m$ the untwisted bounds are $2k$ and $2k + 1$, respectively. By Theorem 10.2 the pullback $\Psi^* \otimes \mathrm{id}_{\mathbb{Q}(d)}$ carries both images onto $(P_k\mathrm{IH}^m(M_{\mathrm{Dol}}))(d)$. □

A Details of the use of generative AI

This manuscript was produced by the Pharos project `piwi-reductive`. Pharos is built on Rethlas [Ju+26]. The project recorded separately the mathematical target, the expert's guidance, the accepted proof dependencies and the review of the assembled manuscript.

A.1 How the system was run

The project was initialized on 22 November 2026 at 07:01:10 UTC. The first research run began at 07:06:38 UTC. The main agent used `gpt-6.2-astra` at the `ultra` reasoning setting. Seven proof workers used the same model, four at `max` and three at `xhigh`. Separate model sessions provided exploratory reading and proof audits. A verifier using `gpt-6.2-astra` at `max` evaluated proof submissions before they entered the project's graph of accepted results.

The verifier accepted the exact target at 12:18:56 UTC, approximately five hours and twelve minutes after launch. Its dependency graph contains 45 results, with eight results on a longest chain. The same run produced a separate proof of the scalar lowering theorem. The eight technical manuscript sections then underwent individual review. A whole-paper review required explicit marked-interval gluing, earlier placement of the fixed-curve determinant identity and a family comparison along the full lifted coroot loop. The revised whole-paper review passed on 22 November 2026. These reviews were automated. No human referee or proof-assistant certification is recorded for this manuscript.

A.2 The problem as posed

The following is the wording of the project question. The two references to the supplied conversations are printed by their link captions.

```
# PI=WI for connected complex reductive groups

Prove the intersection-cohomology \(PI=WI\) conjecture for the untwisted coarse moduli spaces of
a smooth projective complex curve \(C\) and a connected complex reductive group \(G\).

Let (M_B=\operatorname{Hom}(\pi_1\(C\),G)/\!/G) be the character variety and
\(M_{\mathrm{Dol}}\) the corresponding coarse moduli space of polystable \(G\)-Higgs bundles in
the topological types matched by nonabelian Hodge theory. Let \(\Psi:M_{\mathrm{Dol}}\to M_B\)
be the nonabelian Hodge homeomorphism and \(h:M_{\mathrm{Dol}}\to\mathcal A_G\) the Hitchin map.
With rational coefficients, establish in each degree \(m\) and at every filtration index \(k\)
```

```
$$
P_k IH^m(M_{\mathrm{Dol}};\mathbf Q)
=\Psi^* W_{2k} IH^m(M_B;\mathbf Q)
=\Psi^* W_{2k+1} IH^m(M_B;\mathbf Q).
$$
Here \(P\) is the normalized perverse Leray filtration for (h) with intersection-complex
coefficients, and \(W\) is the mixed Hodge weight filtration of the Betti intersection
cohomology.

The formulation comes from the first shared conversation. The even and odd weight indexing and
the proposed filtered comparison are discussed in the second shared conversation. The
conversations provide research context for this goal.
```

A.3 Guidance given during the run

The launch record of 21 November 2026 includes the following instruction.

> First run is 10 hours.

The proof workers were stopped after the target was accepted, within that limit.

The launch record also transcribes the following question from the second supplied discussion.

> For the PI=WI conjecture, what approach can be used to deduce the final comparison besides cameral cover and cocharacter induction? I am considering a structure like mixed hodge module containing multiple filtrations where non-abelian Hodge will interchange two of them.

The investigation used this question to formulate comparisons in terms of actual filtered images. Corollary 10.3 describes the Hitchin step on coarse intersection cohomology as an image from the weight filtration of a compactified Betti mixed Hodge module.

References

[Ju+26] H. Ju, G. Gao, J. Jiang, B. Wu, Z. Sun, S. Liu, L. Chen, Y. Wang, Y. Wang, Z. Wang, W. He, P. Wu, L. Xiao, R. Liu, B. Dai, and B. Dong, *Automated Conjecture Resolution with Formal Verification*, arXiv:2604.03789.

[AMM98] Anton Alekseev, Anton Malkin and Eckhard Meinrenken, *Lie group valued moment maps*, 1998.

[Alp13] Jarod Alper, *Good moduli spaces for Artin stacks*, 2013.

[AHR20] Jarod Alper, Jack Hall and David Rydh, *A Luna étale slice theorem for algebraic stacks*, 2020.

[Art73] Michael Artin, *Théorème de changement de base pour un morphisme propre*, 1973.

[BKV19] David Baraglia, Masoud Kamgarpour and Rohith Varma, *Complete integrability of the parahoric Hitchin system*, 2019.

[BL95] Arnaud Beauville and Yves Laszlo, *Un lemme de descente*, 1995.

[BBD82] A. A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux pervers*, 1982.

[Boa11] Philip Boalch, *Riemann–Hilbert for tame complex parahoric connections*, 2011.

[Boa14] P. P. Boalch, *Geometry and braiding of Stokes data; Fission and wild character varieties*, 2014.

[BFG+26] B. Bolognese, C. Felisetti, A. Gross, A. Küronya, G. Nam and M. Ulirsch, *$P = W$ conjectures for varieties with nilpotent fundamental group*, 2026.

[Bra21] Lukas Braun, *The local fundamental group of a Kawamata log terminal singularity is finite*, 2021.

[DHM25] Ben Davison, Lucien Hennecart and Sebastian Schlegel Mejia, *BPS Lie algebras for totally negative 2-Calabi-Yau categories and nonabelian Hodge theory for stacks*, 2025.

[dCHM10] M. A. de Cataldo, T. Hausel and L. Migliorini, *Topology of Hitchin systems and Hodge theory of character varieties: the case A_1*, 2010.

[Cat21] Mark Andrea A. de Cataldo, *Projective Compactification of Dolbeault Moduli Spaces*, 2021.

[CFH+25] Mark Andrea de Cataldo, Roberto Fringuelli, Andres Fernandez Herrero and Mirko Mauri, *Hitchin fibrations are Ngô fibrations*, 2025.

[Del70] Pierre Deligne, *Équations différentielles à points singuliers réguliers*, 1970.

[Del71] Pierre Deligne, *Théorie de Hodge : II*, 1971.

[Del74] Pierre Deligne, *Théorie de Hodge, III*, 1974.

[Dem70] Michel Demazure, *Groupes réductifs : déploiements, sous-groupes, groupes-quotients*, 1970.

[FM22] Camilla Felisetti and Mirko Mauri, *P=W conjectures for character varieties with symplectic resolution*, 2022.

[HZ25] Andres Fernandez Herrero and Siqing Zhang, *Meromorphic Hodge moduli spaces for reductive groups in arbitrary characteristic*, 2025.

[FGN19] Emilio Franco, Oscar García-Prada and P. E. Newstead, *Higgs bundles over elliptic curves for complex reductive groups*, 2019.

[GM83] Mark Goresky and Robert MacPherson, *Intersection homology II*, 1983.

[Gro61] Alexander Grothendieck, *Éléments de géométrie algébrique III. Étude cohomologique des faisceaux cohérents, première partie*, 1961.

[GHJ+97] K. Guruprasad, J. Huebschmann, L. Jeffrey and A. Weinstein, *Group systems, groupoids, and moduli spaces of parabolic bundles*, 1997.

[Har75] Robin Hartshorne, *On the de Rham cohomology of algebraic varieties*, 1975.

[HR15] Tamás Hausel and Fernando Rodriguez-Villegas, *Cohomology of large semiprojective hyperkähler varieties*, 2015.

[HMMS22] T. Hausel, A. Mellit, A. Minets and O. Schiffmann, $P = W$ *via* $\mathcal{H}_2$, 2022.

[Kal06] D. Kaledin, *Symplectic singularities from the Poisson point of view*, 2006.

[LNY24] Penghui Li, David Nadler and Zhiwei Yun, *Functions on the commuting stack via Langlands duality*, 2024.

[MS24] Davesh Maulik and Junliang Shen, *The P=W conjecture for GL_n*, 2024.

[MMS18] M. Mauri, E. Mazzon and M. Stevenson, *On the geometric* $P = W$ *conjecture*, 2018.

[MM22] M. Mauri and L. Migliorini, *Hodge-to-singular correspondence for reduced curves*, 2022.

[Mel25] Anton Mellit, *Toric stratifications of character varieties*, 2025.

[Ope26] OpenAI, $P = W$ *in composite rank for fixed determinant*, 2026.

[Sai88] Morihiko Saito, *Modules de Hodge polarisables*, 1988.

[Sai90] Morihiko Saito, *Mixed Hodge Modules*, 1990.

[Sai00] Morihiko Saito, *Mixed Hodge complexes on algebraic varieties*, 2000.

[She17] Vivek Shende, *The weights of the tautological classes of character varieties*, 2017.

[Shu25] Cheng Shu, *Singularities of character varieties*, 2025.

[Sim94] Carlos T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety II*, 1994.

[Sim94b] Carlos T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety I*, 1994.

[Sim97] Carlos Simpson, *The Hodge filtration on nonabelian cohomology*, 1997.

[Sim11] Carlos T. Simpson, *Local systems on proper algebraic V-manifolds*, 2011.

[Tub25] Swann Tubach, *Mixed Hodge modules on stacks*, 2025.