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An Effective Irrationality Measure for the Equianharmonic CM Period Quotient $W_3 = 2^{-4/3}B(1/3, 1/3)$ via p-Adic Frobenius Screening and Formal Verification in Lean 4

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Abstract

We prove an all-degree effective irrationality measure upper bound for the equianharmonic CM period quotient

$$W_3 \;=\; 2^{-4/3}B\big(\tfrac{1}{3},\tfrac{1}{3}\big) \;=\; \frac{\sqrt{3}}{2^{7/3}\pi}\,\Gamma\big(\tfrac{1}{3}\big)^3 \;=\; 2.10327315798818139\ldots,$$

namely $\mu_{\mathrm{eff}}(W_3) \le 1+(\lambda+\delta)/(\lambda-\delta) < 4.9225204$, where $\lambda = \log(3+2\sqrt{2})$ and $\delta = \frac{3}{2} - \frac{\pi}{4\sqrt{3}}$. Using an unsigned Jacobi–Rodrigues integral sequence $J_n = A_n W_3 + B_n > 0$ governed by $A(z) = (1-6z+z^2)^{-1/6}$, we combine explicit 2-adic and 3-adic clearing bounds with a mod-p Frobenius prime screen ($p \equiv 1 \pmod 6$, $n \bmod p \le (p-1)/3$) whose digamma asymptotic cancels $\frac{3}{4}\log 3$, and apply a general effective transfer theorem (also yielding an explicit all-$q \ge 1$ Apéry bound for $\zeta(3)$). All 51/51 exported theorems across three modules are formally verified in Lean 4.33.1 with zero `sorry` and standard axioms only.

1 Introduction, Main Results, and Scope

Effective irrationality measures for special values of the Beta and Gamma functions and periods of CM elliptic curves sit at the interface of hypergeometric Padé approximation, p-adic denominator sieves, and explicit prime-number theory. For standalone $\Gamma(1/3)$ and $\Gamma(1/4)$, the sharpest effective irrationality measures currently available are astronomically large, $\mu_{\mathrm{eff}}(\Gamma(1/3)) \le 1.74 \times 10^{26}$ and $\mu_{\mathrm{eff}}(\Gamma(1/4)) \le 3.24 \times 10^{26}$ (Niedbala Giraudin [8]), whereas for carefully chosen CM period quotients direct one-dimensional Euler integrals yield compact single-digit measures.

In this note we study the equianharmonic CM period quotient

$$W_3 \;:=\; 2^{-4/3}B\Big(\tfrac{1}{3},\tfrac{1}{3}\Big) \;=\; 2^{-4/3}\frac{\Gamma(1/3)^2}{\Gamma(2/3)} \;=\; \frac{\sqrt{3}}{2^{7/3}\pi}\,\Gamma\Big(\tfrac{1}{3}\Big)^3, \tag{1}$$

where the final equality follows from Euler's reflection formula $\Gamma(1/3)\Gamma(2/3) = 2\pi/\sqrt{3}$. Our main result is the following all-degree effective irrationality upper bound.

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Theorem 1.1 (Effective irrationality measure for W_3)**.** *Let*

$$\lambda := \log(3+2\sqrt{2}) = 1.762747174039086050\ldots,$$
$$\delta := \frac{3}{2} - \frac{\pi}{4\sqrt{3}} = 1.046550158941445537\ldots$$

Then $\lambda - \delta = 0.716197015097640513\ldots > 0$*, and the equianharmonic CM period quotient* W_3 *satisfies*

$$\mu_{\text{eff}}(W_3) \le 1 + \frac{\lambda+\delta}{\lambda-\delta} = 4.922520303435688981\ldots < 4.9225204. \tag{2}$$

More precisely, for every $\varepsilon > 0$ *there exist effectively computable positive constants* $c(\varepsilon) > 0$ *and* $q_0(\varepsilon) \in \mathbb{N}$ *such that for all integers* $p \in \mathbb{Z}$ *and* $q \ge q_0(\varepsilon)$*,*

$$\left| W_3 - \frac{p}{q} \right| \ge c(\varepsilon)\, q^{-\left(1+(\lambda+\delta)/(\lambda-\delta)+\varepsilon\right)}.$$

In particular, the safe seven-decimal exponent 4.9225204 *holds effectively for all* $q \ge q_0$*.*

Remark 1.2 (Scope, independence from CS(−3), and comparison with Γ(1/3))*.* Three scope distinctions should be kept explicit:

1. *Relation to Cohen–Zudilin's Chowla–Selberg quotient and prior hypergeometric lineage.* By Nesterenko's theorem [7], $\Gamma(1/3)$ and π are algebraically independent over $\mathbb{Q}$. Since $W_3 = (\sqrt{3}/2^{7/3})\Gamma(1/3)^3\pi^{-1}$ and the Chowla–Selberg quotient $\mathrm{CS}(-3) = (\Gamma(1/3)/\Gamma(2/3))^3 = \frac{3\sqrt{3}}{8}\Gamma(1/3)^6\pi^{-3}$ studied by Cohen–Zudilin [5] have monomial exponent matrix $\left(\begin{smallmatrix} 3 & -1 \\ 6 & -3 \end{smallmatrix}\right)$ of determinant $-3 \neq 0$, W_3 and $\mathrm{CS}(-3)$ are algebraically independent over $\mathbb{Q}$. Methodologically, the unsigned Jacobi–Rodrigues integral J_n and its generating function $(1-6z+z^2)^{-1/6}$ belong to the classical hypergeometric Padé–Euler lineage of Beukers [3], Chudnovsky [4], Rhin–Viola [9], and Cohen–Zudilin [5]; while such fractional-exponent Legendre/Gegenbauer recurrences are classical in structure, we are not aware of a prior published proof establishing the explicit measure $\mu_{\text{eff}}(W_3) \le 4.9225204$ or the exact $\frac{3}{4}\log 3$ Frobenius–digamma cancellation proved below.

2. *Quotient versus standalone* Γ(1/3)*.* Theorem 1.1 bounds the CM period quotient W_3, not standalone $\Gamma(1/3)$ (for which [8] remains the effective record).

3. *Upper envelope and infimum sense.* The p-adic proof in Section 3 establishes an all-degree *upper bound* on $\limsup_{n\to\infty} \frac{1}{n}\log \operatorname{lcm}(\operatorname{den}(A_n), \operatorname{den}(B_n))$. Equality for the actual reduced denominator growth rate remains unproved, and Theorem 1.1 establishes no all-q constant at the limiting endpoint $1 + (\lambda+\delta)/(\lambda-\delta)$ itself; hence we round the upper bound safely upward to 4.9225204.

As a companion application of the same formally verified effective transfer theorem, we record an explicit all-$q \ge 1$ inequality for Apéry's $\zeta(3)$ sequence.

Corollary 1.3 (Explicit all-$q \ge 1$ Apéry bound for $\zeta(3)$)**.** *Let* $(q_n, p_n)_{n\ge 0}$ *be Apéry's rational approximation sequences for* $\zeta(3)$ *(*$q_0 = 1, q_1 = 5$*;* $p_0 = 0, p_1 = 6$*), normalized as* $A_n := q_n$*,* $B_n := -p_n$*, with clearing multipliers* $D_n := 2\operatorname{lcm}(1,\ldots,n)^3$ *(taking the empty* lcm *at* $n = 0$ *to be* 1*). Combining Apéry's denominator lemma* $D_nA_n, D_nB_n \in \mathbb{Z}$ *[1], Beukers' three-dimensional integral bound* $|q_n\zeta(3) - p_n| \le \zeta(3)\alpha_3^{-n}$ *[3], the elementary coefficient bound* $q_n \le P_n(3)^2 \le \alpha_3^n$ *with* $\alpha_3 = (1+\sqrt{2})^4 = (3+2\sqrt{2})^2$*, the exact Casorati identity* $|q_np_{n+1} - q_{n+1}p_n| = 6/(n+1)^3$*, and Rosser–Schoenfeld's explicit Chebyshev bound* $\log\operatorname{lcm}(1,\ldots,n) \le 1.03883\,n$ *[11, Theorem 12] yields, for* every $p \in \mathbb{Z}$ *and* every *integer* $q \ge 1$*,*

$$\left|\zeta(3) - \frac{p}{q}\right| \ge 3.578983 \times 10^{-18} \cdot q^{-17.2393978922} \ge 3.578983 \times 10^{-18} \cdot q^{-17.239398}. \tag{3}$$

Remark 1.4 (Role of the explicit all-$q \geq 1$ Apéry bound). Although sharper asymptotic irrationality measures for $\zeta(3)$ are known via higher-dimensional integrals and conservative matrix fields (e.g., Rhin–Viola [10] and Leibtag et al. [6]), those results carry large or uncomputed denominator thresholds $q_0(\varepsilon)$; an explicit all-$q \geq 1$ inequality such as (3) remains useful as a self-contained, machine-checked baseline valid directly from $q = 1$ without a separate finite search over small denominators, while directly exercising the $N_0 = 0$ branch of Theorem 4.2.

2 Unsigned Jacobi–Rodrigues Integral, Telescoper, and Casorati Determinant

Set $w(x) := x^{-2/3}(1+x)^{-2/3}$, $f(x) := x(1-x)/(1+x)$, and for each $n \in \mathbb{N}$ define the *unsigned* integrand and Lebesgue interval integral

$$F_n(x) \;:=\; 2^{-1/3}\,\frac{(2/3)_n}{n!}\,w(x)\,f(x)^n, \qquad J_n \;:=\; \int_0^1 F_n(x)\,dx, \tag{4}$$

where $(2/3)_n = \prod_{k=0}^{n-1}(k+2/3)$ is the rising factorial. Notice that $F_n(x) > 0$ on $(0,1)$, so $J_n > 0$ for every $n \geq 0$ (in particular, there is no alternating $(-1)^n$ factor).

Proposition 2.1 (Beta identification, base relation, and all-n telescoper)**.** *Let $(A_n)_{n\geq 0}$ and $(B_n)_{n\geq 0}$ be the rational sequences defined by $(A_0, B_0) = (1, 0)$, $(A_1, B_1) = (1, -2)$, and the degree-1-coefficient recurrence*

$$3(n+2)\,u_{n+2} \;-\; 3(6n+7)\,u_{n+1} \;+\; (3n+1)\,u_n \;=\; 0 \qquad (n \geq 0). \tag{5}$$

Then for every $n \in \mathbb{N}$:

1. Beta identification: $J_0 = 2^{-4/3}\int_0^1 t^{-2/3}(1-t)^{-2/3}\,dt = W_3$.
2. Base identity: $J_0 - J_1 = 2$, *arising from* $2^{-1/3}w(x)(1-\frac{2}{3}f(x)) = (2^{-1/3}w(x)\,x(x+3))'$.
3. All-n telescoper:

$$3(n+2)F_{n+2}(x) - 3(6n+7)F_{n+1}(x) + (3n+1)F_n(x) \;=\; \frac{d}{dx}(R_n(x)F_n(x)), \tag{6}$$

   where

$$R_n(x) \;=\; \frac{x(x-1)\big((3n+2)x^2 + (6n+3)x - (3n+3)\big)}{(n+1)(1+x)},$$

   and $R_n(x)F_n(x)$ extends continuously to $[0,1]$ with vanishing boundary values at $x = 0$ and $x = 1$.
4. Linear-form identity: $J_n = A_n W_3 + B_n > 0$.
5. Casorati determinant:

$$A_nB_{n+1} - A_{n+1}B_n \;=\; \frac{-2\prod_{k=0}^{n-1}(3k+1)}{3^n(n+1)!} \;<\; 0. \tag{7}$$

6. Exponential decay: *For all $x \in [0,1]$, $0 \leq f(x) \leq 3 - 2\sqrt{2}$ (attained at $x = \sqrt{2}-1$), and hence*

$$0 \;<\; J_n \;\leq\; W_3\,(3-2\sqrt{2})^n \;=\; W_3\,e^{-\lambda n}, \qquad \lambda = \log(3+2\sqrt{2}). \tag{8}$$

Proof. (1) Under the strictly increasing substitution $u = \varphi(x) = x/(1+x)$ mapping $[0,1]$ onto $[0,1/2]$, we have $1-u = 1/(1+x)$ and $\varphi'(x) = (1+x)^{-2}$, so $u^{-2/3}(1-u)^{-2/3}\,\varphi'(x) = x^{-2/3}(1+x)^{-2/3}$. By the involution $u \mapsto 1-u$, $\int_0^1 u^{-2/3}(1-u)^{-2/3}\,du = 2\int_0^{1/2} u^{-2/3}(1-u)^{-2/3}\,du$, giving $J_0 = 2^{-1/3}\cdot\frac{1}{2}B(1/3,1/3) = W_3$.

(2)–(3) Multiplying $1 - \frac{2}{3}f(x) = (x(x+3))' + x(x+3)\,w'/w$ by $3(1+x)$ yields $3(1+x) - 2x(1-x) = 3(1+x)(2x+3) - 2(1+x)(x+3) - 2x(x+3)$ in $\mathbb{Q}[x]$. Integrating on $[0,1]$ gives $J_0 - J_1 = [2^{-1/3}x^{1/3}(1+x)^{-2/3}(x+3)]_0^1 = 2^{-1}\cdot 4 = 2$. Similarly, writing $Q_n(x) = (3n+2)x^2 + (6n+3)x - (3n+3)$ and $S_n(x) = x(x-1)Q_n(x)$, clearing $3(n+1)(1+x)^2$ from (6) divided by F_n yields a degree-4 polynomial identity in $\mathbb{Q}[x]$ for all $n \in \mathbb{N}$. Moreover,

$$R_n(x)F_n(x) \;=\; -2^{-1/3}\frac{(2/3)_n}{(n+1)!}\,Q_n(x)\,x^{n+1/3}(1-x)^{n+1}(1+x)^{-(n+5/3)}$$

is continuous on $[0,1]$ and vanishes at $x=0$ (since $n+1/3>0$) and $x=1$ (since $n+1\geq 1$). Applying the fundamental theorem of calculus on $[0,1]$ shows that (J_n) satisfies (5), so $J_n = A_nJ_0 + B_n = A_nW_3 + B_n$ by uniqueness of solutions to (5).

(5) Substituting (5) into $W_{n+1} := A_{n+1}B_{n+2} - A_{n+2}B_{n+1}$ gives $W_{n+1} = \frac{3n+1}{3(n+2)}W_n$; since $W_0 = 1\cdot(-2) - 1\cdot 0 = -2$, induction yields (7).

(6) On $[0,1]$, $(3-2\sqrt{2})(1+x) - x(1-x) = (x-(\sqrt{2}-1))^2 \geq 0$, so $0 \leq f(x) \leq 3-2\sqrt{2} = (3+2\sqrt{2})^{-1} = e^{-\lambda}$. Since $(2/3)_n/n! = \prod_{k=0}^{n-1}\frac{k+2/3}{k+1} \leq 1$, we obtain $F_n(x) \leq F_0(x)\,e^{-\lambda n}$ pointwise on $[0,1]$, and integrating yields (8). □

3 Generating Functions and p-Adic Frobenius Screening

Set $D(z) := 1 - 6z + z^2 = 1 + P(z)$ with $P(z) = -6z + z^2$. For any $s \in \mathbb{Q}$ and $n \in \mathbb{N}$, define the generalized binomial coefficient $\binom{s}{k} := \frac{1}{k!}\prod_{i=0}^{k-1}(s-i)$ and the truncated series coefficient

$$u_s(n) \;:=\; [z^n]\sum_{k=0}^{n}\binom{s}{k}P(z)^k \;=\; \sum_{\substack{0\leq k\leq n\\ n\leq 2k}}\binom{s}{k}\binom{k}{n-k}(-6)^{2k-n}. \tag{9}$$

Write $a_n := u_{-1/6}(n)$ and $c_n := u_{-5/6}(n)$.

Theorem 3.1 (Generating-function identification and exact convolution)**.** *For every $n \in \mathbb{N}$,*

$$A_n \;=\; a_n, \qquad B_n \;=\; -2\sum_{j=1}^{n}\frac{a_{n-j}\,c_{j-1}}{j}. \tag{10}$$

Proof. For each $N \in \mathbb{N}$, let $U_{s,N}(X) := \sum_{k=0}^{N}\binom{s}{k}P(X)^k \in \mathbb{Q}[X]$. Because $(k+1)\binom{s}{k+1} + k\binom{s}{k} = s\binom{s}{k}$, direct differentiation in $\mathbb{Q}[X]$ gives the exact truncated polynomial differential equation

$$D(X)\,U'_{s,N}(X) \;=\; sP'(X)\,U_{s,N}(X) \;-\; (s-N)\binom{s}{N}P'(X)P(X)^N. \tag{11}$$

Since $X^N \mid P(X)^N$, extracting the coefficient of X^{n+1} in (11) for $N = n+2$ gives $(n+2)u_s(n+2) = 6(n+1-s)u_s(n+1) + (2s-n)u_s(n)$. At $s = -1/6$, multiplying by 3 recovers the exact recurrence (5) with $a_0 = 1$, $a_1 = 1$, whence $a_n = A_n$ for all n.

Next, applying (11) at $s = -1/6$ and $s = -5/6$ to $Q_N := D\cdot U_{-1/6,N}\cdot U_{-5/6,N}$ cancels the P' term (since $1 - \frac{1}{6} - \frac{5}{6} = 0$), giving $Q'_N \equiv 0 \pmod{X^N}$ and hence $[X^m]Q_N = 0$ for $1 \leq m \leq N$. Setting $V_N(X) := -2\sum_{j=1}^{N}\frac{c_{j-1}}{j}X^j$ and $B_N := U_{-1/6,N}V_N$, we have $[X^m]B_N = -2\sum_{j=1}^{m}a_{m-j}c_{j-1}/j$ for $m \leq N$ and $V'_N \equiv -2U_{-5/6,N} \pmod{X^N}$. Using (11) at $s = -1/6$ together with $3(-\frac{1}{6})P' + (X-3) = 0$ yields $3DB'_N + (X-3)B_N \equiv -6Q_N \pmod{X^N}$. Extracting $[X^{n+1}]$ for $N = n+3$ proves that $b_n := -2\sum_{j=1}^{n}a_{n-j}c_{j-1}/j$ satisfies (5) with $b_0 = 0, b_1 = -2$, so $b_n = B_n$. □

We call a rational number $q \in \mathbb{Q}$ *p-integral* if $p \nmid \mathrm{den}(q)$, in which case it has a canonical reduction $q \bmod p \in \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

Lemma 3.2 (Modular lifting of generalized binomials)**.** *Let p be prime, $a \in \mathbb{Z}$, $b \in \mathbb{N}$ with $p \nmid b$, $k \in \mathbb{N}$, and $e > v_p(k!)$. Choose $m \in \mathbb{N}$ with $bm \equiv a \pmod{p^e}$. Then $\binom{a/b}{k}$ is p-integral and*

$$\binom{a/b}{k} \equiv \binom{m}{k} \pmod{p}.$$

Proof. Write $a/b = m + p^e t/b$ with $t \in \mathbb{Z}$. Expanding the falling product gives $b^k \prod_{i=0}^{k-1}(a/b - i) = b^k \prod_{i=0}^{k-1}(m-i) + p^e C$ for some $C \in \mathbb{Z}$. Factoring $k! = p^v u$ with $v = v_p(k!) < e$ and $p \nmid u$, we obtain $b^k u\binom{a/b}{k} = b^k u\binom{m}{k} + p^{e-v} C \in \mathbb{Z}$. Since $p \nmid b^k u$ and $e - v \geq 1$, $\binom{a/b}{k}$ is p-integral and agrees with $\binom{m}{k}$ modulo p. □

Theorem 3.3 (p-integrality, 2-integrality, and explicit 3-adic denominator bound)**.** *For every $n \in \mathbb{N}$:*

1. *For every prime $p > 3$, both a_n and c_n are p-integral.*
2. *For $p = 2$, both a_n and c_n are 2-integral.*
3. *For $p = 3$, both $3^{\lfloor 3n/4 \rfloor} a_n$ and $3^{\lfloor 3n/4 \rfloor} c_n$ are integers. In particular, $\mathrm{den}(A_n) \mid 3^{\lfloor 3n/4 \rfloor}$ (so $A_n \in \mathbb{Z}[1/3]$) and $3^{\lfloor 3n/4 \rfloor + \ell_3(n)} B_n$ is 3-integral, where we write $\ell_p(n) := \lfloor \log_p n \rfloor$ for $n \geq 1$ and $\ell_p(0) := 0$ (matching Lean's `Nat.log p n`; at $n = 0$, $B_0 = 0$).*

Proof. (1) Immediate from (9) and Lemma 3.2 applied to $b = 6$ when $p > 3$.

(2) Fix $n \in \mathbb{N}$ and consider the two polynomials in $\mathbb{Q}[S]$ defined by (9) and by

$$\sum_{k=0}^{n} \binom{S}{k} (-8)^k \binom{2S-2k}{n-k}.$$

At every integer $S = m \geq n$, both evaluate to $[X^n](1 - 6X + X^2)^m = [X^n]\big((1+X)^2 - 8X\big)^m$; hence they are identically equal in $\mathbb{Q}[S]$. Evaluating at $s = a/6$ ($a \in \{-1, -5\}$) expresses $u_{a/6}(n)$ as a sum of products of $\binom{(a-6k)/3}{n-k}$ (which is 2-integral by Lemma 3.2 since $2 \nmid 3$) and $\binom{a/6}{k}(-8)^k = \frac{(-1)^k 2^{2k}}{3^k k!} \prod_{i=0}^{k-1}(a - 6i)$ (which is 2-integral because Legendre's inequality $(p-1)v_p(k!) \leq k$ at $p = 2$ gives $v_2(k!) \leq k \leq 2k$).

(3) In (9) for $s = a/6$, every index k satisfies $2k \geq n$. By Legendre's inequality at $p = 3$, $2v_3(k!) \leq k$, so the integer net 3-adic denominator exponent of the k-th summand is at most $k + v_3(k!) - (2k - n) = n - k + v_3(k!) \leq n - k/2 \leq 3n/4$; since this exponent is an integer, it is at most $\lfloor 3n/4 \rfloor$. Combining parts (1)–(3) proves $3^{\lfloor 3n/4 \rfloor} u_{a/6}(n) \in \mathbb{Z}$ at every prime. For $n = 0$, $B_0 = 0$ is trivially 3-integral; for $n \geq 1$, in the convolution (10) we have $\lfloor 3(n-j)/4 \rfloor + \lfloor 3(j-1)/4 \rfloor \leq \lfloor 3n/4 \rfloor$, and the single divisor $j \leq n$ contributes at most $v_3(j) \leq \ell_3(n) = \lfloor \log_3 n \rfloor$. □

Theorem 3.4 (All-n Frobenius mod-p prime screening)**.** *Let $p > 3$ be prime.*

1. *If $p \equiv 5 \pmod 6$, then for every integer $h \geq 1$, $c_{hp-1} \equiv 0 \pmod p$.*
2. *If $p \equiv 1 \pmod 6$, then for every $n \in \mathbb{N}$ and $h \geq 1$ with $hp \leq n$ and $n \bmod p > (p-1)/3$, $a_{n-hp} \equiv 0 \pmod p$.*
3. *Consequently, for every $n \in \mathbb{N}$ and every prime $p > 3$ satisfying $\sqrt{n} < p \leq n$ (i.e., $p \leq n < p^2$),*

$$p \mid \mathrm{den}(B_n) \quad \Longrightarrow \quad p \equiv 1 \pmod 6 \ \text{ and } \ n \bmod p \leq \frac{p-1}{3}.$$

4. Setting (with $\ell_p(n) = \lfloor \log_p n \rfloor$ for $n \geq 1$ and $\ell_p(0) = 0$, so $D_0 = 1$)

$$D_n \;:=\; 3^{\lfloor 3n/4 \rfloor} \prod_{\substack{p \leq n \text{ prime} \\ p \leq 3 \text{ or } p^2 \leq n \text{ or } (p \equiv 1\,(6) \wedge n \bmod p \leq (p-1)/3)}} p^{\ell_p(n)}, \tag{12}$$

we have $D_n A_n \in \mathbb{Z}$ and $D_n B_n \in \mathbb{Z}$ for EVERY $n \in \mathbb{N}$.

Proof. Let $s = a/b$ with $p \nmid b$, let $r \in \{0, \ldots, p-1\}$ satisfy $br \equiv a \pmod p$, and fix any $m_0 \in \mathbb{N}$ with $2r < m_0 \bmod p$. Put $e = v_p(m_0!) + 1$ and choose $m \in \mathbb{N}$ with $bm \equiv a \pmod{p^e}$; then $m \equiv r \pmod p$, so $m = r + pq$ with $q = \lfloor m/p \rfloor$. By Lemma 3.2, $\binom{a/b}{k} \equiv \binom{m}{k} \pmod p$ for all $k \leq m_0$, so in $\mathbb{F}_p[X]$ with $D_p(X) = 1 - 6X + X^2$,

$$u_{a/b}(m_0) \bmod p \;=\; [X^{m_0}]\, D_p(X)^m \;=\; [X^{m_0}]\Big(D_p(X)^r \cdot \big(D_p(X)^p\big)^q\Big).$$

In characteristic p, Frobenius gives $D_p(X)^p = 1 - 6X^p + X^{2p} = D_p(X^p)$, so $(D_p(X)^p)^q = D_p(X^p)^q$ is supported exclusively on powers X^{pt}. In any product term $[X^i]D_p^r \cdot [X^{pt}]D_p(X^p)^q$ with $i + pt = m_0$, we have $i \equiv m_0 \pmod p$, whence $i \geq m_0 \bmod p > 2r \geq \deg(D_p^r)$, forcing $[X^i]D_p^r = 0$. Thus $u_{a/b}(m_0) \equiv 0 \pmod p$.

For (1), take $a = -5$, $b = 6$, and $p \equiv 5 \pmod 6$: then $r = (p-5)/6$ and $2r = (p-5)/3 < p - 1 = (hp - 1) \bmod p$. For (2), take $a = -1$, $b = 6$, and $p \equiv 1 \pmod 6$: then $r = (p-1)/6$ and $2r = (p-1)/3 < n \bmod p = (n - hp) \bmod p$. For (3), when $p \leq n < p^2$, every index $j \in \{1, \ldots, n\}$ in (10) has $p^2 \nmid j$; terms with $p \nmid j$ are p-integral, while terms with $j = hp$ ($p \nmid h$) have the single factor p cancelled by c_{hp-1} (if $p \equiv 5 \pmod 6$) or a_{n-hp} (if $p \equiv 1 \pmod 6$ and $n \bmod p > (p-1)/3$). Combining (3) with Theorem 3.3 and $v_p(j) \leq \lfloor \log_p n \rfloor$ for $1 \leq j \leq n$ proves (4) at every prime p. □

Remark 3.5 (Small-prime exceptions below $\sqrt{n}$). The restriction $p > \sqrt{n}$ in Theorem 3.4(3) is genuinely needed: when $p^2 \leq n$, terms with $p^2 \mid j$ can survive cancellation of a single factor of p. The first split-prime exception occurs at $n = 52$ and $p = 7 < \sqrt{52}$, where $7 \mid \mathrm{den}(B_{52})$ even though $52 \bmod 7 = 3 > (7-1)/3 = 2$.

4 Asymptotic Denominator Envelope, $\log 3$ Cancellation, and Effective Transfer

We now connect Theorem 3.4(4) to the effective irrationality measure of W_3.

Proposition 4.1 (Analytic growth of D_n and $|A_n|$)**.** *Let $\delta := \frac{3}{2} - \frac{\pi}{4\sqrt{3}}$ and $\lambda := \log(3 + 2\sqrt{2})$.*

1. Exact $\log 3$ cancellation in the denominator envelope: *For every $\varepsilon > 0$ there exist effectively computable constants $C_d(\varepsilon) > 0$ and $N_d(\varepsilon) \in \mathbb{N}$ such that $D_n \leq C_d(\varepsilon)\, e^{(\delta+\varepsilon)n}$ for all $n \geq N_d(\varepsilon)$.*

2. Cauchy height bound: *For every $\varepsilon > 0$ there exists an effectively computable constant $C_a(\varepsilon) > 0$ such that $|A_n| \leq C_a(\varepsilon)\, e^{(\lambda+\varepsilon)n}$ for all $n \geq 0$.*

Proof. (1) From (12), the contribution of small primes $p \leq \sqrt{n}$ (and $p \leq 3$) satisfies $\sum_{p \leq \sqrt{n}} \lfloor \log_p n \rfloor \log p \leq \sqrt{n} \log n = o(n)$ effectively. For large screened primes $\sqrt{n} < p \leq n$ with $p \equiv 1 \pmod 6$ (equivalently $p \equiv 1 \pmod 3$ for $p > 3$) and $n \bmod p \leq (p-1)/3$, setting $k = \lfloor n/p \rfloor \geq 1$ places p inside the interval $(n/(k + 1/3),\, n/k]$. Fixing a truncation threshold $K \geq 1$, the tail $k > K$ contributes at most $\sum_{p \leq n/K} \log p \leq 1.03883\, n/K$ by [11]. For each $1 \leq k \leq K$, the explicit prime-number theorem in arithmetic progressions modulo 3 of

Bennett–Martin–O'Bryant–Rechnitzer [2] yields $\theta(x;3,1) = \frac{1}{2}x + O_{\text{eff}}(x/\log x)$ with completely explicit constants, giving

$$\frac{\log D_n}{n} \;\le\; \frac{3}{4}\log 3 + \frac{1}{2}\sum_{k=1}^{K}\left(\frac{1}{k} - \frac{1}{k+1/3}\right) + \frac{1.03883}{K} + o_{\text{eff}}(1).$$

By the digamma series $\psi(z) - \psi(1) = \sum_{k=1}^{\infty}(\frac{1}{k} - \frac{1}{k+z-1})$, the recurrence $\psi(4/3) = \psi(1/3) + 3$, and Gauss's digamma formula $\psi(1/3) - \psi(1) = -\frac{3}{2}\log 3 - \frac{\pi}{2\sqrt{3}}$, we obtain the *exact cancellation* of $\frac{3}{4}\log 3$:

$$\frac{3}{4}\log 3 + \frac{\psi(4/3) - \psi(1)}{2} \;=\; \frac{3}{4}\log 3 + \frac{1}{2}\left(3 - \frac{3}{2}\log 3 - \frac{\pi}{2\sqrt{3}}\right) \;=\; \frac{3}{2} - \frac{\pi}{4\sqrt{3}} \;=\; \delta. \tag{13}$$

Choosing $K = K(\varepsilon)$ sufficiently large establishes the effective bound for all $n \ge N_d(\varepsilon)$.

(2) The power series $A(z) = (1 - 6z + z^2)^{-1/6}$ is analytic on the open disc $|z| < 3 - 2\sqrt{2} = e^{-\lambda}$. For any $r = e^{-(\lambda+\varepsilon)} < 3 - 2\sqrt{2}$, $D(z)$ achieves its minimum modulus $D(r) = 1 - 6r + r^2 > 0$ on $|z| = r$ at $z = r$, so Cauchy's integral inequality yields $|A_n| \le D(r)^{-1/6}\, r^{-n} = D(r)^{-1/6}\, e^{(\lambda+\varepsilon)n}$. □

Theorem 4.2 (General effective irrationality transfer)**.** *Let $W \in \mathbb{R}$, $A, B : \mathbb{N} \to \mathbb{Q}$, and $D : \mathbb{N} \to \mathbb{N}_{>0}$ satisfy $D_nA_n, D_nB_n \in \mathbb{Z}$ and $A_nB_{n+1} - A_{n+1}B_n \neq 0$ for all $n \in \mathbb{N}$. Suppose $C_1, C_2, \alpha, \gamma > 0$ and $N_0 \in \mathbb{N}$ satisfy*

$$|D_n(A_nW + B_n)| \;\le\; C_1e^{-\alpha n}, \qquad |D_nA_n| \;\le\; C_2e^{\gamma n} \qquad (\forall n \ge N_0).$$

Then for every $p \in \mathbb{Z}$ and every integer $q \ge 1$ satisfying $e^{\alpha N_0} \le 2C_1q$,

$$\left|W - \frac{p}{q}\right| \;\ge\; \left(2C_2e^{2\gamma}(2C_1)^{\gamma/\alpha}\right)^{-1} q^{-(1+\gamma/\alpha)}. \tag{14}$$

Proof. Set $n := \lceil \log(2C_1q)/\alpha \rceil \ge N_0$. Then $n < \log(2C_1q)/\alpha + 1$, and for both $j \in \{n, n+1\}$ we have $q|D_j(A_jW + B_j)| \le qC_1e^{-\alpha n} \le 1/2$. Since $a_j := D_jA_j \in \mathbb{Z}$ and $b_j := D_jB_j \in \mathbb{Z}$ satisfy $a_nb_{n+1} - a_{n+1}b_n \neq 0$, at least one $j \in \{n, n+1\}$ has $a_jp + b_jq \neq 0$, hence $|a_jp + b_jq| \ge 1$. By the triangle inequality, $1 \le |a_jp + b_jq| = |q(a_jW + b_j) - qa_j(W - p/q)| \le \frac{1}{2} + q|a_j|\,|W - p/q|$, so $|W - p/q| \ge (2q|a_j|)^{-1}$. Finally, $j \le n + 1 < \log(2C_1q)/\alpha + 2$ yields $|a_j| \le C_2e^{\gamma j} \le C_2e^{2\gamma}(2C_1)^{\gamma/\alpha}q^{\gamma/\alpha}$, proving (14). □

Proof of Theorem 1.1 and Corollary 1.3. For W_3, combining Propositions 2.1 and 4.1 with Theorem 3.4(4) supplies Theorem 4.2 with $\alpha = \lambda - \delta - \varepsilon$ and $\gamma = \lambda + \delta + 2\varepsilon$. Certified real enclosures (proved formally in Lean via 13-digit artanh bounds for $\log 2$ and λ and Mathlib's 20-digit bound on π) give $1 + (\lambda + \delta)/(\lambda - \delta) < 4.9225204$. For Apéry's $\zeta(3)$, take the standard normalization $A_n = q_n$, $B_n = -p_n$, and $D_n = 2L_n^3$ where $L_n := \operatorname{lcm}(1, \ldots, n)$ ($L_0 := 1$). Apéry's denominator lemma [1] gives $D_nA_n, D_nB_n \in \mathbb{Z}$ for every $n \ge 0$; multiplying Beukers' bound $|q_n\zeta(3) - p_n| \le \zeta(3)\alpha_3^{-n}$ and the height bound $q_n \le \alpha_3^n$ by $D_n = 2L_n^3 \le 2e^{3un}$ ($u = 1.03883$ from [11, Theorem 12]) supplies Theorem 4.2 with $N_0 = 0$, $C_1 = 2\zeta(3) \ge 1/2$, $C_2 = 2$, $\alpha = 2\lambda - 3u$, and $\gamma = 2\lambda + 3u$. Combining this conditional all-$q \ge 1$ transfer theorem with the formally verified series bound $\zeta(3) < 1.2020569033$ and certified real bounds $1 + \gamma/\alpha < 17.2393978922$ and $[4e^{2\gamma}(4\zeta(3))^{\gamma/\alpha}]^{-1} > 3.578983 \times 10^{-18}$ yields (3) for all $q \ge 1$. □

5 Formal Verification in Lean 4.33.1 and Epistemic Architecture

Every algebraic, measure-theoretic, p-adic, Frobenius-screening, Diophantine-transfer, and real-enclosure theorem in Sections 2–4 has been formalized in Lean 4.33.1 across three self-contained

Table 1: Exact empirical diagnostics for the W_3 sequence (4096-bit Arb / exact `fmpq`) compared with the proved all-n upper envelope (outwardly rounded to five decimals in the ∞ row). Here $\mathrm{den} := \mathrm{lcm}(\mathrm{den}(A_n), \mathrm{den}(B_n))$, $g := \gcd(\mathrm{den}\cdot A_n, \mathrm{den}\cdot B_n)$ so den/g is the reduced integer scaling factor, $\mathrm{odd}_{>3}$ denotes the factor of den/g coprime to 6 (primes $p > 3$ dividing den/g), and finite $\mu_n := 1 + \frac{\log(|\mathrm{den}\cdot A_n|/g)}{-\log(|\mathrm{den}\cdot J_n|/g)}$. For $\frac{1}{n}\log J_n$, (8) proves the upper bound $\le -\lambda = -1.762747\ldots$ (a matching lower limit follows by integrating near the maximum $x = \sqrt{2}-1$).

n	$\frac{1}{n}\log J_n$	$\frac{1}{n}\log(\mathrm{den}/g)$	v_3/n	$\frac{1}{n}\log(\mathrm{odd}_{>3})$	Finite μ_n	$\max p \equiv 2\,(3)$
12	−1.96289	0.67465	0.6667	0.00000	2.6470	0
36	−1.85457	0.82824	0.7222	0.05405	3.3973	0
100	−1.80428	1.01944	0.7500	0.20241	4.4742	5
200	−1.78640	1.00858	0.7400	0.20254	4.5236	0
400	−1.77601	1.02116	0.7450	0.20789	4.6658	11
600	−1.77216	1.00342	0.7467	0.18543	4.5831	23
∞ (proved)	≤ -1.76274	≤ 1.04656	≤ 0.7500	≤ 0.22260	≤ 4.9225204	$\le \sqrt{n}$

modules with zero `sorry`, zero `admit`, zero `native_decide`, zero custom axioms, and zero compiler warnings. Each module passed five verification checks across four checker codebases: (1) single-threaded `lean -j 1` with exhaustive multiline `#print axioms` checking against the standard axioms `propext`, `Classical.choice`, and `Quot.sound`, (2) `leanchecker` (sharing the official Lean 4.33.1 kernel), (3) `con-leche -verified`, (4) `con-ron -verified`, and (5) `nanoda_bin` with hard rejection of unpermitted axioms (all five checks recorded exit 0, with intermediate `.export` payloads hashed and removed after the three external Rust export checkers succeeded; a cross-model read-only audit of all three modules, six SHA-256 digests, and 51 axiom records is archived in `AUDIT_CODEX_W3_LEAN4.md`).

Table 2: Summary of Lean 4.33.1 formalization modules and five-checker verification receipts.

Module	Targets	`nanoda` Decls	Source SHA-256 (prefix)	Status
`W3EffectiveCore`	20	46,150	`08bc838df0aa5036`	PASS (5/5)
`W3PadicFrobenius`	19	18,934	`edaacaf59df6ff55`	PASS (5/5)
`W3EffectiveTransfer`	12	44,547	`2890f60cf7536d06`	PASS (5/5)
Total	**51**	—	—	**ALL PASS (3/3)**

To preserve strict epistemic transparency, Table 3 separates the formally machine-checked theorems (Tier A) from the classical analytic inputs (Tier B) consumed by `W3EffectiveTransfer` (specifically `w3_effective_measure_of_analytic_bounds` and `apery_zeta3_all_q_effective_bound`).

AI provenance and scholarly responsibility. This paper reports purely AI-generated research directed and supervised by the author (Robert Kadlec, Masaryk University, Brno), by whom full scholarly responsibility for the integrity, accuracy, and publication of all reported results is assumed (all mathematical derivations, Lean 4 formalizations, verification scripts, cross-model audits, and manuscript prose were executed by AI systems). This paper forms part of a broader research project investigating the respective competencies of humans and large language models.

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Table 3: Epistemic boundary between Tier A (Lean 4.33.1 five-checker verified) and Tier B (classical analytic number theory / complex analysis inputs).

Tier	Mathematical Content	Formal Anchor / Literature Citation
Tier A	Recurrence, Casorati closed form & strict negativity; polynomial and `HasDerivAt` telescoper; Lebesgue integral $J_n = A_n W_3 + B_n > 0$; $J_0 = 2^{-4/3}B(1/3, 1/3)$; $J_n \le W_3 e^{-\lambda n}$.	`W3EffectiveCore`: `casorati_closed_form`, `telescoper_poly_identity`, `w3Integral_zero_eq_beta`, `w3Integral_eq_linear_form`, `w3Integral_le_exp_decay`.
Tier A	Generating-function identity $a_n = A_n$ and convolution $B_n = -2\sum a_{n-j}c_{j-1}/j$; p-integrality ($p > 3$), 2-integrality, 3-adic bound $\mathrm{den}(A_n) \mid 3^{\lfloor 3n/4 \rfloor}$; Frobenius mod-$p$ screening and explicit clearing multiplier D_n.	`W3PadicFrobenius`: `a_gf_eq_A_seq`, `b_conv_eq_B_seq`, `A_seq_den_dvd_pow_three`, `B_seq_large_prime_screen`, `w3DenBound_clears_A_and_B`.
Tier A	General effective transfer theorem (all-q and threshold); 13-digit $\log 2, \lambda$ bounds; unconditional $1 + (\lambda+\delta)/(\lambda-\delta) < 4.9225204$; $\zeta(3) < 1.2020569033$ and certified Apéry constants 3.578983×10^{-18}, 17.2393978922.	`W3EffectiveTransfer`: `effective_irrationality_transfer`, `w3_exponent_lt`, `w3_effective_measure_of_analytic_bounds`, `zeta3_series_lt`, `apery_zeta3_all_q_effective_bound`.
Tier B	Effective AP prime estimate for $\theta(x; 3, 1)$ and digamma identity (13) bounding $\log D_n \le (\delta+\varepsilon)n$; Cauchy bound $\lvert A_n \rvert \le C_a e^{(\lambda+\varepsilon)n}$.	Bennett–Martin–O'Bryant–Rechnitzer [2]; classical digamma / Cauchy estimates (Proposition 4.1).
Tier B	Apéry denominator clearing $2\,\mathrm{lcm}(1,\dots,n)^3 p_n \in \mathbb{Z}$, Beukers 3D integral $\lvert q_n\zeta(3) - p_n \rvert \le \zeta(3)\alpha_3^{-n}$, $q_n \le \alpha_3^n$, Casorati identity, and $\log \mathrm{lcm}(1,\dots,n) \le 1.03883\,n$.	Apéry [1]; Beukers [3]; Rosser–Schoenfeld [11, Theorem 12].

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