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# LIEB'S PERMANENTAL DOMINANCE CONJECTURE FOR ALL SUBGROUPS OF $S_6$

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Abstract. We prove Lieb's General Subgroup Permanental Dominance Conjecture (1966) at order $n = 6$: for every subgroup $H \leq S_6$ (all 1,455 subgroups in 56 conjugacy classes), every irreducible complex character $\chi \in \mathrm{Irr}(H)$ (317 irreducible characters summed across the 56 conjugacy classes: 207 intransitive and 110 transitive), and every $6 \times 6$ complex Hermitian positive semidefinite matrix $A \succeq 0$, the normalized generalized matrix function $d_\chi^H(A)$ is real and satisfies $d_\chi^H(A) \leq \mathrm{per}(A)$. This advances the general-subgroup frontier beyond $n \leq 3$ (Tabata, 2015), $n = 4$ (Zeng, 2026), and $n = 5$ (McBee, 2026). Orders $n \geq 7$ remain open for general subgroups.

In sharp contrast to order 5, where every transitive subgroup character lies inside the homogeneous degree-5 Walled-Brauer sum-of-squares (SOS) cone, exact semidefinite classification of $S_6$ reveals a clean two-layer obstruction structure. First, all 40 intransitive conjugacy classes (207 characters) reduce via Lieb's block permanent inequalities, indicator identities, and the $3 \times 3$ Chollet inequality — including all 11 non-direct subdirect-product classes (8 in $S_4 \times S_2$ and 3 in $S_3 \times S_3$). Second, among the 110 transitive characters across 16 conjugacy classes, 100 are certified inside the degree-6 Walled-Brauer SOS cone and elementary analytic layer via 23 exact zero-floating-point master certificates (over $\mathbb{Q}$, $\mathbb{Q}(i)$, and $\mathbb{Q}(\sqrt{5})$) and 5 elementary analytic masters, while exactly 7 character orbits (10 degree-1 characters) lie strictly outside the homogeneous degree-6 Walled-Brauer SOS cone and collapse to four core targets:

(1) **Core #1** ($C_3 \wr C_2$, $\chi_{1,+1}$ over $\mathbb{Q}(\omega)$): settled by a new elementary proof combining a $2 \times 2$ Hermitian PSD Gram matrix $Q_\omega(A) \succeq 0$, a 2-term rational SOS, and zero-elevation Bernstein bounds over $\mathbb{Q}$.

(2) **Core #2** ($C_2 \wr S_3$, trivial character, i.e. Marcus's block-permanent inequality for $m = 3$ blocks of size $k = 2$, together with $C_2 \wr A_3$ at $\omega \boxtimes \mathbf{1}$): *prior-proved and Lean-verified by Daniel Tubbenhauer* (`arXiv:2611.22058`, Nov 2026) for all $m \leq 4$ and arbitrary $k \geq 1$. Full priority for Core #2 belongs to Tubbenhauer; we additionally supply a self-contained 51-term sparse rational vector-norm certificate inside our Schur-complement cone so our formal proof requires no external axioms.

(3) **Cores #3 and #4** (PGL$(2, 5)$ trivial and sign characters, plus PSL$(2, 5)$ trivial): settled via a new *Multilinear Mixed-Degree Walled-Brauer Schur-Complement Cone* on the 1,546 matchings of $K_{5,5}$, yielding 23-block exact rational certificates certified strictly positive definite over $\mathbb{Z}$ by exact integer Gershgorin congruence ($> 93\%$ diagonal dominance margin) and exact 1,546/1,546 matching equality over $\mathbb{Q}$.

The entire theorem — quantifying unconditionally over every subgroup $H \leq S_6$, every finite-dimensional irreducible complex representation of $H$, and every $6 \times 6$ complex Hermitian PSD matrix — is formalized end-to-end in Lean 4.33.1 + Mathlib (36 modules, 35,344 lines of code, zero `sorry`, zero `native_decide`, and strictly standard axioms `[propext, Classical.choice, Quot.sound]`).

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## 1. Introduction and Main Results

For a subgroup $H \leq S_n$ of the symmetric group on $\{0, 1, \ldots, n-1\}$, an irreducible complex character $\chi \in \mathrm{Irr}(H)$, and an $n \times n$ complex Hermitian positive semidefinite (PSD) matrix $A \succeq 0$, the *normalized generalized matrix function* (or subgroup immanant) is

$$d_\chi^H(A) \;:=\; \frac{1}{\chi(1)} \sum_{\sigma \in H} \chi(\sigma) \prod_{i=0}^{n-1} A_{i,\sigma(i)}.$$

Since $A^* = A$ and $\chi(\sigma^{-1}) = \overline{\chi(\sigma)}$, pairing $\sigma$ with $\sigma^{-1}$ shows $d_\chi^H(A) \in \mathbb{R}$. Schur [10] proved $d_\chi^H(A) \geq 0$ for all $A \succeq 0$. At the top of the ordering sits the permanent $\mathrm{per}(A) := d_{\mathbf{1}}^{S_n}(A)$. In 1966, Lieb [5] proved the block permanent inequality $\mathrm{per}(A) \geq \mathrm{per}(B)\,\mathrm{per}(D)$ for partitioned PSD matrices $A = \left(\begin{smallmatrix} B & C \\ C^* & D \end{smallmatrix}\right) \succeq 0$ and conjectured the *General Subgroup Permanental Dominance Conjecture*:

$$d_\chi^H(A) \;\leq\; \mathrm{per}(A) \qquad \text{for all } n \geq 1,\ H \leq S_n,\ \chi \in \mathrm{Irr}(H),\ A \succeq 0. \tag{1}$$

Our main theorem completely resolves (1) at order $n = 6$.

**Theorem 1.1** (Order-6 General Subgroup Permanental Dominance Conjecture)**.** *For every subgroup $H \leq S_6$ (all* 1,455 *subgroups in* 56 *conjugacy classes), every irreducible complex character $\chi \in \mathrm{Irr}(H)$ (*317 *irreducible characters across the* 56 *conjugacy classes:* 207 *intransitive and* 110 *transitive), and every $6 \times 6$ complex Hermitian positive semidefinite matrix $A \succeq 0$,*

$$d_\chi^H(A) \in \mathbb{R} \qquad \textit{and} \qquad d_\chi^H(A) \;\leq\; \mathrm{per}(A).$$

### 1.1. Literature Context and Prior-Priority Attribution.

Progress on (1) has advanced along two complementary axes:

- **Ordinary $S_n$ immanants and block permanents.** Following Merris–Watkins [8], Heyfron [2], Soules [11], and Barrett–Hall–Loewy [1], Pate [9] established $d_\lambda^{S_n}(A) \leq \mathrm{per}(A)$ for all partitions of $n \leq 13$, and Y. Li [4] extended ordinary $S_n$ immanants through $n \leq 15$ (see also Wanless [14] and Lavi [3]). For $S_6$, this covers the 11 characters of $S_6$ and the 5 characters of $A_6$ that are averages of $S_6$ characters, leaving proper transitive subgroups open. Crucially, in November 2026, **Daniel Tubbenhauer** [13] proved and Lean-verified Marcus's block-permanent conjecture $\mathrm{per}(A) \geq \mathrm{per}(G)$ for all $m \leq 4$ blocks of arbitrary positive size $k$ (as well as $m = 5, k \geq 3$ and asymptotic $k \geq 2\log_2 m$) via positivity in symmetric webs. At $n = 6$, Tubbenhauer's theorem **prior-settles Core #2** ($C_2 \wr S_3$ trivial, $m = 3, k = 2$, Class #51 `ch0`) and also covers $S_3 \wr C_2$ trivial ($m = 2, k = 3$, Class #53 `ch0`). We claim no novelty for Core #2; full priority belongs to Tubbenhauer [13].
- **General subgroups $H \leq S_n$.** General subgroup dominance was established for $n \leq 3$ by Tabata [12], for $n = 4$ (all 30 subgroups / 11 classes / 37 characters, with the sole non-Walled-Brauer case $A_4(\omega, \bar{\omega})$ settled by a Schur-complement Gram certificate) by Zeng [15], and for $n = 5$ (all 156 subgroups / 19 classes / 78 characters, completely inside the degree-5 Walled-Brauer SOS cone on transitive subgroups) in our prior work [7]. Order $n = 6$ is therefore the first open general-subgroup order, and also the first order exhibiting genuinely non-Walled-Brauer transitive obstructions across multiple distinct group families ($C_3 \wr C_2$, $C_2 \wr S_3$, and $\mathrm{PGL}(2, 5)$).

1.2. **Why Order 6 Is Structurally Deeper Than Order 5.** At $n = 5$, every transitive subgroup character lies inside the homogeneous degree-5 Walled-Brauer SOS cone [7]. At $n = 6$, semidefinite screening across all 317 irreducible characters exposes a sharp partition:

(1) All 40 intransitive classes (1,176 subgroups, 207 characters) reduce cleanly to orders $\leq 5$ and $3 \times 3$ Chollet, including 11 non-direct subdirect-product classes (Section 3).
(2) Of the 110 transitive characters across 16 conjugacy classes (279 subgroups), exactly 100 are resolved by 23 exact degree-6 Walled-Brauer vector-norm SOS masters, 5 elementary analytic masters, and exact character-ring induction/conjugation (Sections 4–5).
(3) Exactly 7 character orbits (10 degree-1 characters) lie *strictly outside* the homogeneous degree-6 Walled-Brauer SOS cone:
   - `#45 ch2` ($C_3 \wr C_2$, $\chi_{1,+1}$, $\mathbb{Q}(\omega)$, $\times 2$) — **Core #1** (new elementary proof, Section 6);
   - `#45 ch3` ($C_3 \wr C_2$, $\chi_{1,-1}$, $\mathbb{Q}(\omega)$, $\times 2$) — 2-line reduction (Proposition 4.2(3));
   - `#51 ch0` ($C_2 \wr S_3$, $\mathbf{1}$, $\mathbb{Q}$, $\times 1$) — **Core #2** (prior-proved by Tubbenhauer [13]; independent 51-term WB-Schur cert in Section 7) and `#48 ch2` ($C_2 \wr A_3$, $\omega \boxtimes \mathbf{1}$, $\mathbb{Q}(\omega)$, $\times 2$), reduced to Core #2 via the $3 \times 3$ permanent compound $P(A) \succeq 0$ (Proposition 7.1);
   - `#54 ch0` (PGL$(2,5)$, $\mathbf{1}$, $\mathbb{Q}$, $\times 1$) and `#54 ch1` (PGL$(2,5)$, sgn, $\mathbb{Q}$, $\times 1$) — **Cores #3 and #4** (new 23-block exact rational Multilinear Mixed-Degree WB-Schur certificates on $K_{5,5}$, Section 8), which immediately imply `#52 ch0` (PSL$(2,5)$, $\mathbf{1}$, $\mathbb{Q}$, $\times 1$) by averaging.

## 2. Preliminaries: Normalization, Slice Positivity, and $3 \times 3$ Bounds

Whenever any diagonal entry satisfies $A_{ii} = 0$, $A \succeq 0$ forces row $i$ and column $i$ to vanish, so $d_\chi^H(A) = 0 = \mathrm{per}(A)$. Otherwise $A_{ii} > 0$ for all $i$; conjugating by $D^{-1} = \mathrm{diag}(A_{00}^{-1/2}, \ldots, A_{55}^{-1/2})$ scales both $d_\chi^H(A)$ and $\mathrm{per}(A)$ by $\prod_{i=0}^5 A_{ii}^{-1} > 0$ while preserving $A \succeq 0$. Thus, whenever convenient, we normalize to unit diagonals $A_{ii} = 1$ (so $|A_{ij}| \leq 1$).

**Lemma 2.1** (Self-Contained Lieb Block Inequality and Crossing-Slice Positivity)**.** *Let* $A = \left(\begin{smallmatrix} B & C \\ C^* & D \end{smallmatrix}\right) \in \mathbb{C}^{n \times n}$ *be Hermitian PSD with* $B \in \mathbb{C}^{n_1 \times n_1}$, $D \in \mathbb{C}^{n_2 \times n_2}$ $(n_1 + n_2 = n,\ n_1 \leq n_2)$*. For* $s \in \{0, \ldots, n_1\}$*, let* $I_1 = \{0, \ldots, n_1 - 1\}$, $I_2 = \{n_1, \ldots, n - 1\}$*, and define the* $s$*-crossing slice*

$$L_s(A) \;:=\; \sum_{\substack{\sigma \in S_n \\ |\sigma(I_1) \cap I_2| = s}} \prod_{i=0}^{n-1} A_{i,\sigma(i)}.$$

*Then* $\mathrm{per}(A) = \sum_{s=0}^{n_1} L_s(A)$ *and* $L_s(A) \geq 0$ *for every* $s \in \{0, \ldots, n_1\}$*. In particular:*

(1) $\mathrm{per}(A) \geq L_0(A) = \mathrm{per}(B)\,\mathrm{per}(D)$ *(yielding* $(1,5)$, $(2,4)$, $(3,3)$*, and iterated* $(2,2,2)$ *block bounds);*
(2) *For* $n_1 = n_2 = 3$, $\mathrm{per}(A) \geq L_0(A) + L_3(A) = \mathrm{per}(B)\,\mathrm{per}(D) + |\,\mathrm{per}(C)|^2$.

*Proof.* Each $\sigma \in S_n$ with $|\sigma(I_1) \cap I_2| = s$ is uniquely specified by $s$-subsets $S, S' \subseteq I_1$, $T, T' \subseteq I_2$ ($S = \sigma^{-1}(I_2) \cap I_1$, $T = \sigma(S)$, $T' = \sigma^{-1}(I_1) \cap I_2$, $S' = \sigma(T')$) together with four bijections $I_1 \setminus S \to I_1 \setminus S'$, $S \to T$, $T' \to S'$, and $I_2 \setminus T' \to I_2 \setminus T$. Summing over these four bijections and using $\mathrm{per}(C^*_{T',S'}) = \overline{\mathrm{per}(C_{S',T'})}$ gives

$$L_s(A) = \sum_{\substack{|S| = |S'| = s \\ |T| = |T'| = s}} \mathrm{per}(B_{I_1 \setminus S,\, I_1 \setminus S'})\ \mathrm{per}(D_{I_2 \setminus T',\, I_2 \setminus T})\ \mathrm{per}(C_{S,T})\ \overline{\mathrm{per}(C_{S',T'})}.$$

For any PSD $M = V^*V$, the $k$-th permanent compound $(P_k(M))_{X,Y} := \mathrm{per}(M_{X,Y})$ is the Gram matrix of the symmetrized tensors $\frac{1}{\sqrt{k!}}\sum_{\pi\in S_k} v_{x_{\pi(1)}}\otimes\cdots\otimes v_{x_{\pi(k)}} \in \mathrm{Sym}^k(\mathbb{C}^r)$, hence $P_k(M) \succeq 0$. Relabeling $S \mapsto I_1 \setminus S$, $T \mapsto I_2 \setminus T$, the kernel $M^{(s)}_{(S,T),(S',T')} := \mathrm{per}(B_{I_1\setminus S,\, I_1\setminus S'})\,\mathrm{per}(D_{I_2\setminus T',\, I_2\setminus T})$ is the Kronecker product $P_{n_1-s}(B) \otimes (P_{n_2-s}(D))^T \succeq 0$. With $v_{(S,T)} := \overline{\mathrm{per}(C_{S,T})}$, we obtain $L_s(A) = v^*M^{(s)}v \geq 0$. Evaluating $s = 0$ and $(n_1, n_2, s) = (3, 3, 3)$ gives (1) and (2). □

**Lemma 2.2** (Elementary $3\times 3$ Chollet Inequality)**.** *For any $3\times 3$ Hermitian PSD matrices $A_1, A_2$ with unit diagonals,* $\mathrm{per}(A_1 \circ A_2) \leq \mathrm{per}(A_1)\,\mathrm{per}(A_2)$.

*Proof.* Let $x_1 = |A_{1,01}|^2$, $x_2 = |A_{1,12}|^2$, $x_3 = |A_{1,20}|^2 \in [0,1]$, $s_1 = x_1 + x_2 + x_3$, $z_1 = A_{1,01}A_{1,12}A_{1,20}$, and similarly $y_r, s_2, z_2$ for $A_2$. From $\det(A_1) = 1 - s_1 + 2\,\mathrm{Re}(z_1) \geq 0$ and $|z_1| \leq (s_1/3)^{3/2}$, we obtain $\mathrm{per}(A_1) \geq 1 + \frac{2}{3}s_1$ (if $s_1 \leq \frac{3}{4}$, $2(s_1/3)^{3/2} \leq \frac{1}{3}s_1$; if $s_1 \geq \frac{3}{4}$, $\mathrm{per}(A_1) \geq 2s_1 \geq 1 + \frac{2}{3}s_1$). Hence $\mathrm{per}(A_1)\,\mathrm{per}(A_2) \geq 1 + \frac{2}{3}(s_1 + s_2) + \frac{4}{9}s_1s_2$. Meanwhile $x_ry_r \leq \frac{1}{2}(x_r + y_r)$ and $2|z_1||z_2| \leq 2(s_1s_2/9)^{3/2} \leq \frac{2}{9}s_1s_2$ (as $s_1s_2 \leq 9$), giving $\mathrm{per}(A_1 \circ A_2) \leq 1 + \frac{1}{2}(s_1 + s_2) + \frac{2}{9}s_1s_2 \leq \mathrm{per}(A_1)\,\mathrm{per}(A_2)$. □

**Lemma 2.3** (Sharp $3\times 3$ Cycle-Modulus Bound)**.** *For every $3\times 3$ Hermitian PSD matrix $B$ with unit diagonals and cycle product $z = B_{01}B_{12}B_{20}$,* $\mathrm{per}(B) \geq 1 + 4|z|$. *Consequently, for unit-diagonal $6\times 6$ $A \succeq 0$ and any $(3,3)$-partition $T_1 \sqcup T_2$ with cycle products $z_1, z_2$,*

$$\mathrm{per}(A) \;\geq\; \mathrm{per}(A_{T_1})\,\mathrm{per}(A_{T_2}) \;\geq\; (1+4|z_1|)(1+4|z_2|) \;\geq\; 1 + 24|z_1||z_2|.$$

*Proof.* Set $t := |z|^{1/3} \in [0,1]$, so $s := |B_{01}|^2 + |B_{12}|^2 + |B_{20}|^2 \geq 3t^2$. If $t \leq \frac{1}{2}$, $\mathrm{per}(B) \geq 1 + 3t^2 - 2t^3 = (1 + 4t^3) + 3t^2(1 - 2t) \geq 1 + 4t^3$. If $t \in [\frac{1}{2}, 1]$, $\det(B) \geq 0$ gives $\mathrm{per}(B) \geq 2s \geq 6t^2 = 1 + 4t^3 + (2t - 1)(1 + 2t(1 - t)) \geq 1 + 4t^3$. Finally, $(1 + 4|z_1|)(1 + 4|z_2|) \geq 1 + 8\sqrt{|z_1||z_2|} + 16|z_1||z_2| \geq 1 + 24|z_1||z_2|$ since $|z_1|, |z_2| \leq 1$. □

## 3. Complete Resolution of All 40 Intransitive Classes (207 Characters)

Direct enumeration of $S_6$ (verified from scratch in `verify_lieb6_hard.py`) yields 1,455 subgroups in 56 conjugacy classes carrying 317 irreducible characters: 40 intransitive classes (1,176 subgroups, 207 characters) and 16 transitive classes (279 subgroups, 110 characters). The 40 intransitive classes partition into:

- 19 fixed-point classes in $S_5 \times S_1$ (78 characters): immediate from Lemma 2.1 ($(1,5)$) and order-5 dominance [7];
- 7 direct-product classes in $S_4\times S_2$ (56 characters) and 3 direct-product classes in $S_3\times S_3$ (27 characters): immediate from Lemma 2.1 ($(2,4)$ and $(3,3)$), Schur's theorem $d^K_\psi \geq 0$, and order-$\leq 4$ dominance [12, 15];
- 8 non-direct subdirect classes in $S_4 \times S_2$ (34 characters) and 3 non-direct subdirect classes in $S_3 \times S_3$ (12 characters), settled by the next two propositions.

**Proposition 3.1** (All 8 Non-Direct Subdirect Classes in $S_4 \times S_2$)**.** *Let $H \leq S_4 \times S_2$ be any of the 8 non-direct subdirect-product classes of $S_6$ acting on $O_1 \sqcup O_2$ ($|O_1| = 4$, $|O_2| = 2$). Then $d^H_\chi(A) \leq \mathrm{per}(A)$ for every $\chi \in \mathrm{Irr}(H)$ and every $A \succeq 0$.*

*Proof.* By Goursat's lemma (and direct verification in `verify_lieb6_hard.py`), each of the 8 classes satisfies $(e, \tau) \notin H$ (where $\tau$ is the transposition on $O_2$), has surjective projection $\pi_2 : H \to S_2$, and projects isomorphically $\pi_1 : H \xrightarrow{\sim} H_1 \leq S_4$ (with $H_1 \in \{C_2, V_4, C_4, D_4, S_4\}$,

never $A_4$ since $A_4$ has no index-2 subgroup). Letting $\epsilon : H_1 \to \{\pm 1\}$ be the homomorphism with $H = \{(\sigma, r) \in H_1 \times S_2 : \operatorname{sgn}(r) = \epsilon(\sigma)\}$, the indicator identity $\mathbf{1}_H(\sigma, r) = \frac{1}{2}(1+\epsilon(\sigma)\operatorname{sgn}(r))$ on $H_1 \times S_2$ yields, for $\tilde{\chi} := \chi \circ \pi_1^{-1} \in \operatorname{Irr}(H_1)$ and $A_j := A_{O_j}$,

$$\begin{aligned} d_\chi^H(A) &= \frac{1}{2} d_{\tilde{\chi}}^{H_1}(A_1)\operatorname{per}(A_2) + \frac{1}{2} d_{\tilde{\chi}\cdot\epsilon}^{H_1}(A_1)\det(A_2) \\ &\le \operatorname{per}(A_1)\frac{\operatorname{per}(A_2)+\det(A_2)}{2} \;\le\; \operatorname{per}(A_1)\operatorname{per}(A_2) \;\le\; \operatorname{per}(A), \end{aligned}$$

using order-4 dominance for $H_1$, $0 \le \det(A_2) \le \operatorname{per}(A_2)$, and Lemma 2.1. □

**Proposition 3.2** (All 3 Non-Direct Subdirect Classes in $S_3 \times S_3$)**.** *Let $H \le S_3 \times S_3$ be any of the 3 non-direct subdirect classes acting on $\{0,1,2\} \sqcup \{3,4,5\}$: $C_3^{\mathrm{diag}}$ (order 3, 3 irreps), $S_3^{\mathrm{diag}}$ (order 6, 3 irreps), or $(S_3 \times S_3)^+ = \ker(\operatorname{sgn} \boxtimes \operatorname{sgn})$ (order 18, 6 irreps). Then $d_\chi^H(A) \le \operatorname{per}(A)$ for all $\chi \in \operatorname{Irr}(H)$ and all $A \succeq 0$.*

*Proof.* Normalize $A$ to unit diagonals and set $A_1 = A_{\{0,1,2\}}$, $A_2 = A_{\{3,4,5\}}$.

(1) $C_3^{\mathrm{diag}}$: for $\chi(\sigma,\sigma) = \omega^k(\sigma)$, inserting $\mathbf{1}_{C_3^{\mathrm{diag}}}(\sigma_1,\sigma_2) = \frac{1}{3}\sum_{j=0}^2 \omega^j(\sigma_1)\omega^{-j}(\sigma_2)$ on $C_3 \times C_3$ gives $d_\chi^H(A) = \frac{1}{3}\sum_{j=0}^2 d_{\omega^{k+j}}^{C_3}(A_1) d_{\omega^{-j}}^{C_3}(A_2) \le \operatorname{per}(A_1)\operatorname{per}(A_2) \le \operatorname{per}(A)$.

(2) $S_3^{\mathrm{diag}} = \{(\sigma,\sigma) : \sigma \in S_3\}$: directly $d_\chi^{S_3^{\mathrm{diag}}}(A) = d_\chi^{S_3}(A_1 \circ A_2) \le \operatorname{per}(A_1 \circ A_2) \le \operatorname{per}(A_1)\operatorname{per}(A_2) \le \operatorname{per}(A)$ by Schur's product theorem ($A_1 \circ A_2 \succeq 0$), order-3 dominance, Lemma 2.2, and Lemma 2.1.

(3) $(S_3 \times S_3)^+$: the 4 restricted irreps $\psi_1 \boxtimes \psi_2$ satisfy $d_{\psi_1\boxtimes\psi_2}^H(A) = \frac{1}{2}(d_{\psi_1}^{S_3}(A_1) d_{\psi_2}^{S_3}(A_2) + d_{\psi_1 \operatorname{sgn}}^{S_3}(A_1) d_{\psi_2 \operatorname{sgn}}^{S_3}(A_2)) \le \operatorname{per}(A_1)\operatorname{per}(A_2)$ via $\mathbf{1}_H = \frac{1}{2}(1 + \operatorname{sgn}\boxtimes\operatorname{sgn})$; the 2 split 2-dimensional irreps $\psi_\pm$ satisfy $d_{\psi_\pm}^H(A) = \frac{1}{2}(d_\omega^{C_3}(A_1) d_{\omega^{\pm 1}}^{C_3}(A_2) + d_{\bar{\omega}}^{C_3}(A_1) d_{\bar{\omega}^{\pm 1}}^{C_3}(A_2)) \le \operatorname{per}(A_1)\operatorname{per}(A_2)$. □

## 4. Transitive Classification, Elementary Families, and the $S_3 \wr C_2$ Tower

Table 1 lists all 16 transitive subgroup conjugacy classes of $S_6$ (proved exhaustive in `TransitiveClassify.lean` via imprimitivity for $5 \nmid |H|$, and Sylow-5 plus simplicity of $A_5$ and $A_6$ when $5 \mid |H|$).

For unit-diagonal $A \succeq 0$ and any 3-edge matching $M = \{e_0, e_1, e_2\}$, iterating Lemma 2.1 yields $\operatorname{per}(A) \ge (1+x_0)(1+x_1)(1+x_2)$ ($x_r = |A_{e_r}|^2 \in [0,1]$). Checking the 8 vertices of $[0,1]^3$ gives two elementary inequalities:

$$(1+x_0)(1+x_1)(1+x_2) \;\ge\; 1+7\,x_0x_1x_2, \qquad (1+x_0)(1+x_1)(1+x_2) \;\ge\; 1+\frac{7}{3}(x_0x_1+x_1x_2+x_2x_0). \tag{2}$$

**Proposition 4.1** (Elementary Transitive Classes: $S_3$, $C_6$, $A_4$, and 5/6 of $D_6$)**.** *For every unit-diagonal $6 \times 6$ $A \succeq 0$, $d_\chi^H(A) \le \operatorname{per}(A)$ holds elementarily for:*

(1) ***Transitive** $S_3$ **(#41,** 3/3**):** Consisting of e, two $(3,3)$-cycles on $T_1 \sqcup T_2$, and three matchings $m_0, m_1, m_2$, we have $|d_\chi^{S_3}(A)| \le 1 + 2|z_1||z_2| + \sum_{r=0}^2 P_{m_r}$, bounded by averaging $\frac{1}{4}$ of Lemma 2.3 and $\frac{1}{4}$ of (2) on each $m_r$.*

(2) ***Transitive** $C_6$ **(#42,** 6/6**, all** $\zeta_6^k$**):** Generated by $g = (0,1,2,3,4,5)$, every unit-modulus character satisfies $d_\chi^{C_6}(A) \le 1 + 2|z_1||z_2| + P_{M_{\mathrm{diag}}} + 2\sqrt{P_{M_1}P_{M_2}}$ where $M_{\mathrm{diag}} = (03)(14)(25)$, $M_1 = (01)(23)(45)$, $M_2 = (12)(34)(50)$. Averaging $\frac{1}{4}$ of Lemma 2.3 on*

TABLE 1. Complete classification of all 16 transitive subgroup conjugacy classes of $S_6$ (279 subgroups, 110 irreducible characters).

| # | Group $H$ | $\lvert H\rvert$ | Conj. | $\lvert\mathrm{Irr}\rvert$ | $\le A_6$? | Proof Route |
|---|---|---|---|---|---|---|
| 41 | $S_3$ (tr.) | 6 | 20 | 3 | No | Elementary (Prop. 4.1(1)) |
| 42 | $C_6$ (tr.) | 6 | 60 | 6 | No | Elementary unit-modulus chars (Prop. 4.1(2)) |
| 43 | $A_4$ (tr.) | 12 | 15 | 4 | Yes | Elementary incl. $\omega,\bar\omega$ (Prop. 4.1(3)) |
| 44 | $D_6$ (tr.) | 12 | 60 | 6 | No | 5/6 elem. (Prop. 4.1(4)) + `D6_++` WB-SOS ($K$=67) |
| 45 | $C_3 \wr C_2$ | 18 | 20 | 9 | No | 5 DAG + `ch3` (Prop. 4.2(3)) + **Core #1** (Thm. 6.3) |
| 46 | $S_4$ (`24a`) | 24 | 15 | 5 | Yes | Character-ring DAG from WB-SOS masters |
| 47 | $S_4$ (`24b`) | 24 | 15 | 5 | No | Character-ring DAG from WB-SOS masters |
| 48 | $C_2 \wr A_3$ | 24 | 15 | 8 | No | `C2wrA3_triv` ($K$=10) + DAG + `ch2` $\le$ Core #2 |
| 49 | $C_3^2 \rtimes C_4$ | 36 | 10 | 6 | Yes | `C32C4_i` ($K$=30, $\mathbb{Q}(i)$) + Prop. 4.2(2) + DAG |
| 50 | $(S_3\times S_3)^+$ | 36 | 10 | 9 | No | `S3xS3_d1` ($K$=36) + Prop. 4.2(2) + DAG |
| 51 | $C_2 \wr S_3$ | 48 | 15 | 10 | No | 2 WB-SOS + **Core #2** (Tubbenhauer [13] + §7) |
| 52 | PSL(2, 5) | 60 | 6 | 5 | Yes | `PSL25_phi+` ($K$=22, $\mathbb{Q}(\sqrt5)$) + `ch0` $\le$ Cores #3–#4 |
| 53 | $S_3 \wr C_2$ | 72 | 10 | 9 | No | 2 WB-SOS masters + Prop. 4.2(1) + DAG |
| 54 | PGL(2, 5) | 120 | 6 | 7 | No | 4 WB-SOS masters + **Cores #3–#4** (Thm. 8.4) |
| 55 | $A_6$ | 360 | 1 | 7 | Yes | `A6_phi+` ($K$=11, $\mathbb{Q}(\sqrt5)$) + $S_6$ averages |
| 56 | $S_6$ | 720 | 1 | 11 | No | 9 rational WB-SOS masters ($K \in \{1..10\}$; cf. [9, 4]) |

$\{0,2,4\}\sqcup\{1,3,5\}$ *and* $\frac14$ *of* (2) *on* $M_{\mathrm{diag}}, M_1, M_2$ *gives* $\mathrm{per}(A) \ge 1+6|z_1||z_2|+\frac74 P_{M_{\mathrm{diag}}}+\frac72\sqrt{P_{M_1}P_{M_2}} \ge d_\chi^{C_6}(A)$.

(3) ***Transitive*** $A_4$ ***(#43,*** 4/4, ***incl.*** $\omega,\bar\omega$***):*** *Acting on blocks* $\{0,1\},\{2,3\},\{4,5\}$, $A_4$ *comprises* $e$, *three double transpositions with sum* $d_0d_1+d_1d_2+d_2d_0$, *and* 4 *inverse pairs of* $(3,3)$*-cycles on* 4 *distinct* $(3,3)$*-partitions. Weight* $\frac{1}{12}$ *on each* $(3,3)$*-partition (Lemma 2.3:* $\frac{1}{12}\cdot 24 = 2$*) plus weight* $\frac37$ *on* (2) *(*$\frac37\cdot\frac73=1$*) uses total convex weight* $\frac{4}{12}+\frac37=\frac{16}{21}<1$.

(4) ***Transitive*** $D_6$ ***(#44,*** 5/6 ***elementary):*** *Both* 2*D irreps are averages of* $C_6$*; the reflection-sign char satisfies* $d \le d_{\mathbf{1}}^{C_6} \le \mathrm{per}$*; the two alternating reflection chars are bounded by convex combinations of* $(3,3)$ *and* $(2,2,2)$ *partitions with total weights* $\frac{67}{84}<1$ *and* $\frac{313}{336}<1$ *(verified in* ***`verify_lieb6_hard.py`*** *and formalized in* ***`CertsS6E.lean`****).*

**Proposition 4.2** (The $S_3 \wr C_2$ Wreath Tower and Outside `#45 ch3`)**.** *Write* $A=\left(\begin{smallmatrix} A_1 & C \\ C^* & A_2\end{smallmatrix}\right)\succeq 0$ *with* $3\times 3$ *blocks* $A_1, A_2 \succeq 0$.

(1) *For* $S_3 \wr C_2$ *(#53), the four linear characters* $(\chi_0,\epsilon)\in\{\mathbf{1},\mathrm{sgn}\}\times\{\pm1\}$ *satisfy*

$$d_{\mathbf{1},\epsilon}^{S_3\wr C_2}(A)=\mathrm{per}(A_1)\,\mathrm{per}(A_2)+\epsilon|\,\mathrm{per}(C)|^2, \qquad d_{\mathrm{sgn},\epsilon}^{S_3\wr C_2}(A)=\det(A_1)\det(A_2)+\epsilon|\det(C)|^2.$$

*For* $(\mathbf{1},\pm1)$, *Lemma 2.1(2) gives* $d \le L_0+L_3 \le \mathrm{per}(A)$. *For* $(\mathrm{sgn},-1)$, *we obtain* $d \le \det(A_1)\det(A_2) \le \mathrm{per}(A_1)\,\mathrm{per}(A_2) \le \mathrm{per}(A)$. *For* $(\mathrm{sgn},+1)$ *(unit diagonals,* $A\succ 0$ *by continuity), Hadamard and Schur complement give* $|\det(C)|^2 \le \det(A_1)\det(A_2) \le 1$, *so* $|\det(C)|^2 \le |\det(C)|^{2/3} \le \frac13\,\mathrm{tr}(C^*C)$, *while averaging the* 6 *matchings* $\{(i,3+\pi(i))\}$ *(*$\pi\in S_3$*) gives* $\mathrm{per}(A) \ge 1+\frac13\,\mathrm{tr}(C^*C) \ge \det(A_1)\det(A_2)+|\det(C)|^2$.

(2) *For #49 (*$C_3^2\rtimes C_4$, $\pm i$ *chars) and #50 (transitive* $(S_3\times S_3)^+$, *all* 4 *linear chars), writing* $\mathrm{per}(C)=U+W$ *and* $\det(C)=U-W$ *yields* $d_{\pm i}^{\#49}(A)=\frac12(\mathrm{per}(A_1)\det(A_2)+$

$\det(A_1)\,\mathrm{per}(A_2)) \mp 2\,\mathrm{Im}(U\overline{W}) \le \frac{1}{2}(d^{S_3\wr C_2}_{\mathbf{1},+1}(A) + d^{S_3\wr C_2}_{\mathrm{sgn},+1}(A)) \le \mathrm{per}(A)$, *and similarly for #50.*

(3) *For the non-WB character* `#45 ch3` *($C_3 \wr C_2$, $\chi_{1,-1}$: $\omega$ on $C_3^2$, $-1$ on swap),*

$$d^{C_3\wr C_2}_{\chi_{1,-1}}(A) = d^{C_3}_{\omega}(A_1)\, d^{C_3}_{\omega}(A_2) - \left|d^{C_3}_{\omega}(C)\right|^2 \;\le\; \mathrm{per}(A_1)\,\mathrm{per}(A_2) \;\le\; \mathrm{per}(A).$$

## 5. Degree-6 Walled-Brauer Vector-Norm SOS Layer

Write $A = U^*U$ with columns $U = [u_0, \dots, u_5] \in \mathbb{C}^{r\times 6}$ and $P_\sigma(A) := \prod_{i=0}^{5} A_{i,\sigma(i)}$. Taking partial transposes on subsets $J \subset \{0,\dots,5\}$ yields rank-independent contraction tensors $v_\alpha^{(B)}(U)$ in the degree-6 walled Brauer sectors $B^{(t)}_{6-|J|,|J|}$ ($|J| \in \{1,2,3\}$, $t \ge 1$) whose pairwise inner products are single permutation monomials $P_{\mathrm{tbl}_B(\alpha,\beta)}(A)$ with zero coefficient on the identity permutation $e$. Every positive-weight orbit average

$$\mathrm{per}(A) - d^H_\chi(A) \;\equiv\; \frac{1}{|H|}\sum_{g\in H}\sum_{t=1}^{K} w_t \left\|\sum_{\alpha} c_{t,\alpha}\, v_\alpha^{(B_t)}(U_g)\right\|^2 \;\ge\; 0 \tag{3}$$

holds unconditionally for every rank $r \ge 1$ once $w_t > 0$ and the permutation identity (3) holds identically on all 720 permutations of $S_6$.

**Theorem 5.1** (23 Exact Degree-6 WB-SOS Master Certificates)**.** *For each of the* 23 *master transitive characters listed in Table 2, identity* (3) *holds identically across all* 720 *permutations of $S_6$ with strictly positive exact weights in $\mathbb{Q}$, $\mathbb{Q}(\sqrt{5})$ (checked by exact integer inequalities), and exact coefficients in $\mathbb{Z}$ or $\mathbb{Z}[i]$, totaling* 333 *deduplicated unique rank-*1 *contraction terms across* 423,756 *index pairs. Combined with the* 5 *elementary analytic masters of Proposition 4.1(2,4) and exact character-ring induction, normal-quotient, conjugation, and complex-conjugation identities in $\mathbb{C}[S_6]$, this settles all* 100 *INSIDE transitive characters of $S_6$ without external theorems.*

Table 2. All 23 exact zero-floating-point degree-6 Walled-Brauer SOS master certificates verified inside the pure Lean 4 kernel (`CertsS6.lean`, `CertsS6_01.18.lean`).

| Master Key | Class | Deg | Field | $K$ | Master Key | Class | Deg | Field | $K$ |
|---|---|---|---|---|---|---|---|---|---|
| `S6_det` | #56 ch1 | 1 | $\mathbb{Q}$ | 10 | `PGL25_5a` | #54 ch4 | 5 | $\mathbb{Q}$ | 17 |
| `S6_51` | #56 ch2 | 5 | $\mathbb{Q}$ | 1 | `PGL25_5b` | #54 ch5 | 5 | $\mathbb{Q}$ | 18 |
| `S6_33` | #56 ch3 | 5 | $\mathbb{Q}$ | 2 | `PSL25_phi+` | #52 ch1 | 3 | $\mathbb{Q}(\sqrt{5})$ | 22 |
| `S6_222` | #56 ch4 | 5 | $\mathbb{Q}$ | 4 | `S3wrC2_triv` | #53 ch0 | 1 | $\mathbb{Q}$ | 2 |
| `S6_21111` | #56 ch5 | 5 | $\mathbb{Q}$ | 10 | `S3wrC2_perMdet` | #53 ch2 | 1 | $\mathbb{Q}$ | 25 |
| `S6_42` | #56 ch6 | 9 | $\mathbb{Q}$ | 2 | `C2wrS3_perBdet` | #51 ch2 | 1 | $\mathbb{Q}$ | 33 |
| `S6_2211` | #56 ch7 | 9 | $\mathbb{Q}$ | 8 | `C2wrS3_2a` | #51 ch4 | 2 | $\mathbb{Q}$ | 5 |
| `S6_411` | #56 ch8 | 10 | $\mathbb{Q}$ | 3 | `S3xS3_d1` | #50 ch2 | 1 | $\mathbb{Q}$ | 36 |
| `S6_3111` | #56 ch9 | 10 | $\mathbb{Q}$ | 5 | `C32C4_i` | #49 ch2 | 1 | $\mathbb{Q}(i)$ | 30 |
| `A6_phi+` | #55 ch3 | 8 | $\mathbb{Q}(\sqrt{5})$ | 11 | `C2wrA3_triv` | #48 ch0 | 1 | $\mathbb{Q}$ | 10 |
| `PGL25_4a` | #54 ch2 | 4 | $\mathbb{Q}$ | 18 | `D6_++` | #44 ch0 | 1 | $\mathbb{Q}$ | 67 |
| `PGL25_4b` | #54 ch3 | 4 | $\mathbb{Q}$ | 18 | | | | | |

## 6. Core #1 (#45 ch2: $C_3 \wr C_2, \omega, +1$) — Elementary Rational Proof

For unit-diagonal $A = \left(\begin{smallmatrix} A_1 & C \\ C^* & A_2 \end{smallmatrix}\right) = V^*V \succeq 0$ with $3 \times 3$ blocks, exact expansion in $\mathbb{Q}(\omega)[S_6]$ gives

$$d^{C_3 \wr C_2}_{\chi_{1,+1}}(A) \;=\; d^{C_3}_{\omega}(A_1)\, d^{C_3}_{\omega}(A_2) + \left|d^{C_3}_{\omega}(C)\right|^2,$$

where $d^{C_3}_{\omega}(M) = u_0(M) + \omega u_1(M) + \omega^2 u_2(M)$ and $u_r(M) := \prod_{i=0}^{2} M_{i,(i+r) \bmod 3}$. Defining $\xi_1 := v_0 \otimes v_1 \otimes v_2 + \omega\, v_1 \otimes v_2 \otimes v_0 + \omega^2\, v_2 \otimes v_0 \otimes v_1$ and $\xi_2 := v_3 \otimes v_4 \otimes v_5 + \omega\, v_4 \otimes v_5 \otimes v_3 + \omega^2\, v_5 \otimes v_3 \otimes v_4$ in $(\mathbb{C}^r)^{\otimes 3}$, the $2 \times 2$ Gram matrix

$$Q_\omega(A) \;:=\; \frac{1}{3}\begin{pmatrix} \langle \xi_1, \xi_1 \rangle & \langle \xi_1, \xi_2 \rangle \\ \langle \xi_2, \xi_1 \rangle & \langle \xi_2, \xi_2 \rangle \end{pmatrix} \;=\; \begin{pmatrix} d^{C_3}_{\omega}(A_1) & d^{C_3}_{\omega}(C) \\ \overline{d^{C_3}_{\omega}(C)} & d^{C_3}_{\omega}(A_2) \end{pmatrix} \;\succeq\; 0.$$

Setting $D_j := d^{C_3}_{\omega}(A_j) \geq 0$, $D := D_1 D_2$, $x_r := |u_r(C)| \in [0,1]$, and $S := x_0 + x_1 + x_2$, $\det Q_\omega(A) \geq 0$ and the triangle inequality yield the joint upper bound

$$d^{C_3 \wr C_2}_{\chi_{1,+1}}(A) \;\leq\; D + \min(D, S^2). \tag{4}$$

On the other hand, taking weight $\frac{5}{11}$ on the $(3,3)$ block bound and weight $\frac{6}{11}$ on the average of the three matchings $M_r = \{(i, 3 + (i+r) \bmod 3)\}$ gives

$$\operatorname{per}(A) \;\geq\; \frac{5}{11}\operatorname{per}(A_1)\operatorname{per}(A_2) + \frac{6}{11} \cdot \frac{1}{3}\sum_{r=0}^{2}\prod_{i=0}^{2}\bigl(1 + |C_{i,(i+r) \bmod 3}|^2\bigr). \tag{5}$$

**Lemma 6.1** (Matching-Average Rational SOS)**.** *For all $x \in [0,1]$, $x^{2/3} + x^{4/3} + \frac{1}{3}x^2 \geq \frac{111}{50}x$, and hence $\frac{1}{3}\sum_{r=0}^{2}\prod_{i=0}^{2}(1 + |C_{i,(i+r) \bmod 3}|^2) \geq 1 + \frac{111}{50}S$.*

*Proof.* By AM–GM on each matching product, $\prod_{i=0}^{2}(1 + a_i) \geq 1 + 3x_r^{2/3} + 3x_r^{4/3} + x_r^2$. Setting $x = u^3$ $(u \geq 0)$, $(u^2 + u^4 + \frac{1}{3}u^6) - \frac{111}{50}u^3 = \frac{u^2}{150}\bigl(50(u^2 - \frac{5}{8})^2 + \frac{425}{2}(u - \frac{333}{425})^2 + \frac{151}{13600}\bigr) \geq 0$. □

**Lemma 6.2** (Sharp $3\times 3$ Bound Relating $\operatorname{per}(B)$ and $d^{C_3}_{\omega}(B)$)**.** *For unit-diagonal $3\times 3$ $B \succeq 0$, set $u := d^{C_3}_{\omega}(B) - 1$. Then: (1) $u \leq \frac{1}{3}$; and (2) whenever $u \in [0, \frac{1}{3}]$, $\operatorname{per}(B) \geq 1 + \frac{11}{5}u + \frac{1}{5}u^2$.*

*Proof.* Let $z = B_{01}B_{12}B_{20} = x + iy$ and $s = |B_{01}|^2 + |B_{12}|^2 + |B_{20}|^2$. Then $\operatorname{per}(B) = 1 + s + 2x$, $\det(B) = 1 - s + 2x \geq 0$ (so $x \geq -\frac{1}{2}$), and $u = -x - \sqrt{3}y$, whence $|z|^2 = \frac{1}{3}(4x^2 + 2ux + u^2)$ and AM–GM gives $(1+2x)^3 \geq s^3 \geq 9(4x^2 + 2ux + u^2)$. For (1), if $u > \frac{1}{3}$, then for $x \geq -\frac{1}{3}$ we have $9((u+x)^2 + 3x^2) > 9(4x^2 + \frac{2}{3}x + \frac{1}{9}) = (1+2x)^3 + 8x^2(3-x) \geq (1+2x)^3$, while for $x \in [-\frac{1}{2}, -\frac{1}{3})$ we have $9(4x^2 + 2ux + u^2) \geq 27x^2 > 3 > \frac{1}{27} > (1+2x)^3$, contradiction. For (2), let $T(u) := \frac{11}{5}u + \frac{1}{5}u^2$ and $y_{\mathrm{end}}(u) := \frac{1+T(u)}{2} \leq \frac{79}{90} < \frac{9}{10}$. Assume $2x < T(u)$ (else $s \geq 0 \geq T(u) - 2x$). *Case A ($x \leq \frac{T(u)-1}{4}$):* Set $y := 1 + 2x \in [0, y_{\mathrm{end}}(u)]$. On $y \in [0, \frac{9}{10}]$, $f_A(u,y) := 9((y-1)^2 + u(y-1) + u^2) - y^3$ has $\partial_y f_A = -3(y-3)^2 + 9 + 9u \leq -\frac{123}{100} < 0$, so $f_A(u,y) \geq g_A(u) := f_A(u, y_{\mathrm{end}}(u))$. Every degree-6 Bernstein coefficient of $g_A$ (zero degree elevation) on $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{7}{24}]$, and $[\frac{7}{24}, \frac{1}{3}]$ is strictly positive (`verify_lieb6_hard.py`, `Core1.lean`), contradicting $(1+2x)^3 \geq 9(4x^2 + 2ux + u^2)$. *Case B ($x \in [\frac{T(u)-1}{4}, \frac{T(u)}{2}]$):* Set $w := T(u) - 2x \in [0, y_{\mathrm{end}}(u)]$; proving $s \geq w$ reduces to $f_B(u,w) := 9((T(u)-w)^2 + u(T(u)-w) + u^2) - w^3 \geq 0$. On $u \in [0, \frac{1}{4}]$, $w^3 \leq y_{\mathrm{end}}(u)w^2$ reduces $f_B$ to a quadratic in $w$ whose reduced negative discriminant $u^2 h_B(u)$ has all degree-4 Bernstein coefficients of $h_B$ on $[0, \frac{1}{4}]$ at least $\frac{1179}{128} > 0$. On $u \in [\frac{1}{4}, \frac{1}{3}]$, $\partial_{ww} f_B = 18 - 6w > 0$ gives $\partial_w f_B(u,w) \leq \partial_w f_B(u, y_{\mathrm{end}}(u)) =: d_B(u) < 0$ (all degree-4 Bernstein coefficients of $-d_B$ on $[\frac{1}{4}, \frac{1}{3}]$ are $\geq \frac{147}{1024} > 0$), so $f_B(u,w) \geq f_B(u, y_{\mathrm{end}}(u)) \equiv g_A(u) > 0$. □

**Theorem 6.3** (Core #1: $C_3 \wr C_2, \chi_{1,+1}$)**.** *For every $6 \times 6$ Hermitian PSD matrix $A \succeq 0$, $d^{C_3 \wr C_2}_{\chi_{1,+1}}(A) \le \mathrm{per}(A)$.*

*Proof.* Normalize to unit diagonals and set $D_j = d^{C_3}_{\omega}(A_j) \in [0, \frac{4}{3}]$, $P(u) := 1 + \frac{11}{5}u + \frac{1}{5}u^2$, and $R(u_1, u_2) := \frac{5}{11}P(u_1)P(u_2) + \frac{6}{11}$. If $D_1, D_2 \le 1$, then $D \le 1$, so (4) gives $d_{\chi_{1,+1}} \le 1 + \min(1, S^2) \le 1 + \frac{333}{275}S \le \mathrm{per}(A)$ by (5) and Lemma 6.1. Otherwise replace any $D_j < 1$ by 1 (increasing $D + \min(D, S^2)$ while preserving $\mathrm{per}(A_j) \ge 1 = P(0)$) and set $u_j = \max(D_j - 1, 0) \in [0, \frac{1}{3}]$, $D = (1 + u_1)(1 + u_2) \in [1, \frac{16}{9}]$. By (4), (5), Lemma 6.1, and Lemma 6.2, it suffices to prove $\max_{S \ge 0}(D + \min(D, S^2) - \frac{333}{275}S) = \max(D, 2D - \frac{333}{275}\sqrt{D}) \le R(u_1, u_2)$ on $[0, \frac{1}{3}]^2$. Every power-basis coefficient of $R(u_1, u_2) - (1 + u_1)(1 + u_2)$ is nonnegative, giving $D \le R$. For $2D - \frac{333}{275}\sqrt{D}$, if $D \le (\frac{333}{275})^2$ it is $\le D \le R$; otherwise $s = \sqrt{D} \in [\frac{6}{5}, \frac{4}{3}]$ satisfies $s \ge \frac{15}{38}s^2 + \frac{12}{19}$ (difference $\frac{15}{38}(\frac{4}{3} - s)(s - \frac{6}{5}) \ge 0$), and every degree-$(2,2)$ Bernstein coefficient of $R(u_1, u_2) - (2D - \frac{333}{275}(\frac{15}{38}D + \frac{12}{19}))$ on $[0, \frac{1}{3}]^2$ is $\ge \frac{119}{22275} > 0$. □

## 7. Core #2 (#51 ch0: $C_2 \wr S_3$ Trivial) and #48 ch2

Partition $\{0, \dots, 5\}$ into three $2 \times 2$ blocks $B_0 = \{0, 1\}$, $B_1 = \{2, 3\}$, $B_2 = \{4, 5\}$, and form the $3 \times 3$ permanent compound matrix $P(A)_{IJ} := \mathrm{per}(A_{B_I, B_J})$.

**Proposition 7.1** (Compound-Matrix Reduction for `#48 ch2`)**.** *$P(A)$ is $3 \times 3$ Hermitian PSD: writing $A = V^*V$ with block column pairs $V_I = [u_I, w_I] \in \mathbb{C}^{r \times 2}$ and $X_I := u_I w_I^T + w_I u_I^T \in \mathrm{Sym}_r(\mathbb{C})$, we have $P(A)_{IJ} = \frac{1}{2}\mathrm{tr}(X_I^* X_J)$, a Gram matrix. Moreover, identically on $S_6$,*

$$d^{C_2 \wr S_3}_{\mathbf{1}}(A) = \mathrm{per}(P(A)), \qquad d^{C_2 \wr A_3}_{\omega \boxtimes \mathbf{1}}(A) = d^{C_3}_{\omega}(P(A)),$$

*and hence by order-3 dominance applied to $P(A) \succeq 0$, $d^{C_2 \wr A_3}_{\omega \boxtimes \mathbf{1}}(A) \le d^{C_2 \wr S_3}_{\mathbf{1}}(A)$.*

**Theorem 7.2** (Core #2: Marcus Block-Permanent Inequality at $m = 3, k = 2$)**.** *For every $6 \times 6$ Hermitian PSD matrix $A \succeq 0$, $d^{C_2 \wr S_3}_{\mathbf{1}}(A) = \mathrm{per}(P(A)) \le \mathrm{per}(A)$, and consequently $d^{C_2 \wr A_3}_{\omega \boxtimes \mathbf{1}}(A) \le \mathrm{per}(A)$.*

*Attribution and Independent Self-Contained Verification.* Theorem 7.2 was **previously proved and Lean-verified by Daniel Tubbenhauer** [13] (for all $m \le 4$ blocks of arbitrary size $k \ge 1$). Full priority for Core #2 belongs to Tubbenhauer. To render our Lean 4 formalization 100% self-contained with zero external axioms, `Core2.lean` and `core2_exact_cert.json` (verified independently by `verify_core2_exact.py`) also supply a direct 51-term sparse rank-1 exact rational vector-norm certificate (1,046 entries) inside the Multilinear Mixed-Degree Walled-Brauer Schur-Complement Cone introduced in Section 8, matching all 1,546/1,546 matchings of $K_{5,5}$ identically over $\mathbb{Q}$ under the 16-element $(D_4 \times \{\mathrm{id}, T\})$-average. □

## 8. Cores #3 & #4 ($\mathrm{PGL}(2,5)$) and #52 ch0 ($\mathrm{PSL}(2,5)$): Multilinear Mixed-Degree WB-Schur Certificates

Fix the sharply 3-transitive action of $H = \mathrm{PGL}(2,5) \cong S_5$ (order 120) in $S_6$ with point stabilizer $F_{20} := \mathrm{Stab}_H(0) \le S_{\{1,\dots,5\}} \cong S_5$ of order 20, and let $\chi \in \{\mathbf{1}, \mathrm{sgn}\}$.

**Lemma 8.1** (Dense Schur-Complement Reduction to the 1,546 Matchings of $K_{5,5}$)**.** *The set $\mathcal{U} := \{A \in \mathcal{H}_6 : A \succ 0$ and $A_{i,0} \ne 0$ for all $i\}$ is dense in the $6 \times 6$ PSD cone. For every*

*$A \in \mathcal{U}$, setting $v_i := A_{i,0}/\sqrt{A_{0,0}} \neq 0$ and defining the rescaled $5 \times 5$ Schur complement $\widetilde{S}_{a,b} := (A_{a+1,b+1} - v_{a+1}\overline{v_{b+1}})/(v_{a+1}\overline{v_{b+1}})$ ($a, b \in \{0, \ldots, 4\}$), we have $\widetilde{S} \succ 0$ and*

$$\frac{\operatorname{per}(A) - d_\chi^H(A)}{\prod_{i=0}^5 |v_i|^2} \;=\; \operatorname{per}\big(J_6 + (0 \oplus \widetilde{S})\big) - d_\chi^H\big(J_6 + (0 \oplus \widetilde{S})\big) \;=\; \sum_{M \in \mathcal{M}(K_{5,5})} c_\chi(M) \prod_{(a,b) \in M} \widetilde{S}_{a,b},$$

*where $\mathcal{M}(K_{5,5})$ is the set of all $\sum_{m=0}^5 \binom{5}{m}^2 m! = 1{,}546$ matchings of $K_{5,5}$ and $c_\chi(M) \in \mathbb{Z}$.*

*Proof.* For any $A \succeq 0$ and $\varepsilon > 0$, $A_1 := A + \varepsilon I_6 \succ 0$ and $A_2 := A_1 + \delta J_6 \in \mathcal{U}$ for $\delta \in (0, \varepsilon) \setminus \{-(A_1)_{i,0}\}$. Conjugating the positive definite Schur complement $S \succ 0$ by $\operatorname{diag}(v_1^{-1}, \ldots, v_5^{-1})$ preserves $\widetilde{S} \succ 0$, and $A = D_v(J_6 + (0 \oplus \widetilde{S}))D_v^*$ scales every permutation monomial by $\prod_{i=0}^5 |v_i|^2$. Expanding $\prod_{i \geq 1,\, \sigma(i) \geq 1}(1 + \widetilde{S}_{i-1,\sigma(i)-1})$ produces a sum over subsets $M$ with distinct rows and distinct columns, i.e. matchings of $K_{5,5}$. □

**Lemma 8.2** (Multilinear Mixed-Degree Walled-Brauer Schur-Complement Gram Positivity)**.** *Write $\widetilde{S} = W^*W \succeq 0$ with columns $w_0, \ldots, w_4 \in \mathbb{C}^5$ ($\widetilde{S}_{a,b} = \langle w_a, w_b \rangle$, conjugate-linear in the first slot). Fix $J \subseteq \{0, \ldots, 4\}$, $J^c = \{0, \ldots, 4\} \setminus J$, and integers $0 \leq p \leq |J^c|$, $0 \leq q \leq |J|$. Let $\mathcal{B}(J, p, q)$ consist of all triples $\alpha = (\text{pairs}_\alpha, P_\alpha, Q_\alpha)$ where $P_\alpha = (c_1, \ldots, c_p)$ has distinct entries in $J^c$, $Q_\alpha = (d_1, \ldots, d_q)$ has distinct entries in $J$, and $\text{pairs}_\alpha = ((j_1, x_1), \ldots, (j_t, x_t))$ has $0 \leq t \leq \min(|J| - q, |J^c| - p)$, distinct $j_1 < \cdots < j_t \in J \setminus Q_\alpha$, and distinct $x_1, \ldots, x_t \in J^c \setminus P_\alpha$. Define*

$$\psi_\alpha(\widetilde{S}) \;:=\; \Big(\prod_{r=1}^t \widetilde{S}_{j_r,x_r}\Big)\Big(\bigotimes_{k=1}^p w_{c_k}\Big) \otimes \Big(\bigotimes_{\ell=1}^q \overline{w_{d_\ell}}\Big) \;\in\; (\mathbb{C}^5)^{\otimes p} \otimes (\overline{\mathbb{C}^5})^{\otimes q}.$$

*Then for every $\alpha, \beta \in \mathcal{B}(J, p, q)$, $\langle \psi_\alpha(\widetilde{S}), \psi_\beta(\widetilde{S}) \rangle = \prod_{(a,b) \in M(\alpha,\beta)} \widetilde{S}_{a,b}$ is a valid multilinear matching monomial $M(\alpha, \beta) \in \mathcal{M}(K_{5,5})$:*

$$M(\alpha, \beta) = \{(x_r, j_r) : (j_r, x_r) \in \text{pairs}_\alpha\} \cup \text{pairs}_\beta \cup \{(c_k, c_k')\}_{k=1}^p \cup \{(d_\ell', d_\ell)\}_{\ell=1}^q.$$

*Hence for any real symmetric $X \succeq 0$, any $g \in S_5$, and any $\tau \in \{\mathrm{id}, T\}$,*

$$\sum_{\alpha,\beta \in \mathcal{B}(J,p,q)} X_{\alpha,\beta} \prod_{(a,b) \in (g,\tau) \cdot M(\alpha,\beta)} \widetilde{S}_{a,b} \;\geq\; 0 \qquad (\forall \widetilde{S} \succeq 0).$$

*Proof.* Conjugate-linearity turns $\widetilde{S}_{j_r,x_r}$ into $\overline{\widetilde{S}_{j_r,x_r}} = \widetilde{S}_{x_r,j_r}$ and $\langle \overline{w_{d_\ell}}, \overline{w_{d_\ell'}} \rangle = \widetilde{S}_{d_\ell',d_\ell}$. Because $J \cap J^c = \emptyset$, the row indices $\{x_r\} \subseteq J^c \setminus P_\alpha$, $P_\alpha \subseteq J^c$, $\{j_r'\} \subseteq J \setminus Q_\beta$, and $Q_\beta \subseteq J$ are pairwise disjoint, and symmetrically so are the four column sets; thus $M(\alpha, \beta) \in \mathcal{M}(K_{5,5})$. Spectral decomposition of $X \succeq 0$ expresses the sum as a nonnegative combination of squared Hilbert-space norms, and $(g, \tau)$ maps $\widetilde{S} \succeq 0$ to $P_g \widetilde{S} P_g^T \succeq 0$ or $P_g \widetilde{S}^T P_g^T \succeq 0$. □

**Lemma 8.3** (Exact Integer Gershgorin Strict Positive-Definiteness Criterion)**.** *If symmetric $X \in \mathbb{Z}^{m \times m}$ and lower-triangular $U \in \mathbb{Z}^{m \times m}$ with $U_{ii} > 0$ satisfy strict diagonal dominance of $M := UXU^T$ ($M_{ii} > \sum_{j \neq i} |M_{ij}|$ for all $i$), then $X \succ 0$.*

*Proof.* $\det(U) = \prod_i U_{ii} > 0$, so for any $z \neq 0$, $y := U^{-T} z \neq 0$ satisfies $z^* X z = y^* M y = \sum_i (M_{ii} - \sum_{j \neq i} |M_{ij}|)|y_i|^2 + \sum_{i<j} |M_{ij}||y_i + \operatorname{sgn}(M_{ij}) y_j|^2 > 0$. □

**Theorem 8.4** (Cores #3 & #4 ($\mathrm{PGL}(2,5)$) and `#52 ch0` ($\mathrm{PSL}(2,5)$))**.** *For every $6 \times 6$ Hermitian PSD matrix $A \succeq 0$,*

$$d_{\mathbf{1}}^{\mathrm{PGL}(2,5)}(A) \;\leq\; \operatorname{per}(A), \qquad d_{\mathrm{sgn}}^{\mathrm{PGL}(2,5)}(A) \;\leq\; \operatorname{per}(A), \qquad d_{\mathbf{1}}^{\mathrm{PSL}(2,5)}(A) \;\leq\; \operatorname{per}(A).$$

*Proof.* For each $\chi \in \{\mathbf{1}, \mathrm{sgn}\}$, the ancillary files `core3_exact_cert.json` ($D = 1{,}000$, min diagonal-dominance margin 93.17%) and `core4_exact_cert.json` ($D = 50{,}000$, min margin 93.87%) specify 23 blocks $(J_b, p_b, q_b)$ ($|J_b| \le 2$, $p_b + q_b \le 4$), exact rational symmetric matrices $X^{(b)} = \frac{1}{\mathrm{den}_b} X_{\mathbb{Z}}^{(b)}$, and exact lower-triangular integer witnesses $U^{(b)}$. Exact integer arithmetic in `verify_lieb6_hard.py` (and inside Lean 4 in `Cores34.lean`) verifies symmetry, $U_{ii}^{(b)} > 0$, and strict diagonal dominance of $U^{(b)} X_{\mathbb{Z}}^{(b)} (U^{(b)})^T$ on all 23/23 blocks (so $X^{(b)} \succ 0$ by Lemma 8.3), and checks that the full 40-element $(F_{20} \times \{\mathrm{id}, T\})$-average matches $c_\chi(M)$ identically over $\mathbb{Q}$ on all 1,546/1,546 matchings of $K_{5,5}$. Lemmas 8.2 and 8.1 prove $d_\chi^{\mathrm{PGL}(2,5)}(A) \le \mathrm{per}(A)$ on $\mathcal{U}$ and hence on all $A \succeq 0$ by continuity. Finally, $\mathbf{1}_{\mathrm{PSL}(2,5)} = \frac{1}{2}(1 + \mathrm{sgn})$ on $\mathrm{PGL}(2,5)$ gives $d_{\mathbf{1}}^{\mathrm{PSL}(2,5)}(A) = \frac{1}{2}(d_{\mathbf{1}}^{\mathrm{PGL}(2,5)}(A) + d_{\mathrm{sgn}}^{\mathrm{PGL}(2,5)}(A)) \le \mathrm{per}(A)$. □

Combining Sections 3–8 completes the proof of Theorem 1.1.

## 9. End-to-End Lean 4.33.1 Formalization and Reproducibility

Theorem 1.1 is formalized end-to-end in Lean 4.33.1 + Mathlib in `formal/Conjectures/LiebOrder6/` (36 modules, 35,344 LOC, zero `sorry`, zero `admit`, zero `native_decide`, zero custom axioms). The top-level theorem `LiebOrder6.lieb_order6_all_subgroups` quantifies directly over every `H : Subgroup (Equiv.Perm (Fin 6))`, every finite-dimensional irreducible complex representation $\rho$ of $H$ (`Representation` over $\mathbb{C}$), and every $6 \times 6$ complex Hermitian PSD matrix `A` (`A.PosSemidef`). Key engineering features include:

- **Pure-kernel WB-SOS verification** (`FastWBSOS.lean`, `OrbitsS6.lean`, `CertsS6_01..18.lean`): Deduplicating all 23 masters into 333 unique rank-1 contraction terms (423,756 index pairs), replacing Lehmer encoding by 5-digit direct orbit dispatch, evaluating sums strictly in binary trees, and sharding across 18 LPT-balanced modules enables pure C++ kernel `decide +kernel` checking (strictly `[propext]`) in under 5 minutes on 18 cores.
- **Zero-trust axiom audit:** Running `#print axioms` on all 55 audited headline theorems of `Conjectures.LiebOrder6` confirms dependence strictly on `propext`, `Classical.choice`, and `Quot.sound`.

All standalone Python verifiers (`verify_lieb6_hard.py`, `verify_core2_exact.py`), exact certificates (`core2_exact_cert.json`, `core3_exact_cert.json`, `core4_exact_cert.json`), and the complete Lean 4 codebase are publicly archived at https://github.com/coracle-us/conjectures (release `v0.3.0`).

**Methodology Note.** Certificate search, algebraic rationalization, elementary proofs, and Lean 4 formalization were conducted with an error-corrected multi-agent AI research workflow inspired by Loader, Oppenheim, and Osborne [6], with independent adversarial referee audits and full Lean 4 kernel verification.

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