hexagon:2611.01308v1

A Proof of the Intersection Conjecture for Traceless Matrix Invariants

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Abstract

Padellaro, Ramgoolam, and Suzuki conjectured that, at finite N, the invariant polynomials in two traceless matrices are represented by the intersection of a Young-diagram truncation of the permutation centralizer algebra with the span of derangements. We prove that restriction of multi-traces induces the asserted isomorphism. The proof identifies the intersection with the orthogonal complement of the kernel of restriction and extends to any finite number of matrices. Applied to the one-loop operator of Padellaro, Ramgoolam, and Suzuki, the same kernel identity gives their proposed action on the intersection.

1 The intersection

Padellaro, Ramgoolam, and Suzuki prove the stable case $N \geq L$ in Proposition 4.5 of [1]. Their Conjecture 1 concerns the intersection when the cutoff on Young diagrams is active. The connection between derangements and tensor powers of the adjoint representation also appears in work of Benkart and Doty [2].

Let $N \geq 1$, $V = \mathbb{C}^N$, and let $\mathbf{m} = (m_1, \ldots, m_k)$ be a multidegree with $k \geq 1$ and $L = \sum_i m_i$. Let $K = S_{m_1} \times \cdots \times S_{m_k}$ permute the corresponding blocks of tensor positions. Write

$$A = \mathbb{C}[S_L]^K, \qquad A_N = \bigoplus_{\substack{R \vdash L \\ \ell(R) \leq N}} P_R A,$$

where K acts by conjugation and P_R is the central idempotent for the irreducible representation indexed by R. Let D and F be the subspaces of A supported on permutations with no fixed points and with at least one fixed point, respectively. Give A the inner product induced by the orthonormal permutation basis of $\mathbb{C}[S_L]$. Both supports are K-invariant, so $D = F^\perp$ inside A. Set $D_N = A_N \cap D$.

Write $H_N = \mathbb{C}[\mathfrak{gl}_N^k]_{\mathbf{m}}^{U(N)}$ and $H'_N = \mathbb{C}[\mathfrak{sl}_N^k]_{\mathbf{m}}^{SU(N)}$ for the indicated multihomogeneous spaces, with the groups acting by simultaneous conjugation. For the permutation representation ρ on $V^{\otimes L}$, define

$$E(\alpha)(X_1, \ldots, X_k) = \mathrm{Tr}_{V^{\otimes L}}\left(\rho(\alpha)(X_1^{\otimes m_1} \otimes \cdots \otimes X_k^{\otimes m_k})\right).$$

Let $T = \mathrm{res} \circ E$, where res restricts all matrices to their traceless parts.

Theorem 1. *The map T is surjective and $D_N = (\ker T)^\perp$ in A. In particular, $T|_{D_N} : D_N \to H'_N$ is an isomorphism. For $k = 2$, this is Conjecture 1 of [1].*

Proof. The first fundamental theorem for simultaneous conjugation generates the invariant polynomials by traces of words (see [3]). The $U(N)$-invariants agree with the $GL_N(\mathbb{C})$-invariants because $U(N)$ is Zariski dense. Products of traces of the prescribed multidegree are the evaluations of permutations in S_L. Averaging a permutation over conjugation by K preserves its evaluation, so $E : A \to H_N$ is surjective.

Polarization shows that the tensors $X_1^{\otimes m_1} \otimes \cdots \otimes X_k^{\otimes m_k}$ span $\mathrm{End}(V^{\otimes L})^K$ under the identification $\mathrm{End}(V^{\otimes L}) \cong \mathrm{End}(V)^{\otimes L}$. For $\alpha \in A$, both $\rho(\alpha)$ and $\rho(\alpha)^*$ are K-invariant. If $E(\alpha) = 0$, the trace pairing of $\rho(\alpha)$ with every tensor in that span vanishes. In particular, $\mathrm{Tr}(\rho(\alpha)\rho(\alpha)^*) = 0$, hence $\rho(\alpha) = 0$. The reverse implication follows from the definition of E. By Schur–Weyl duality only the summands indexed by $\ell(R) \le N$ act on $V^{\otimes L}$. Thus

$$\ker E = \bigoplus_{\ell(R)>N} P_R A = A_N^{\perp},$$

where the last equality uses the self-adjointness of the P_R.

Put $t_i = \mathrm{Tr}(X_i)$ and $X_i^0 = X_i - (t_i/N)I_N$. The central scalar subgroup of $U(N)$ acts trivially by conjugation, so $SU(N)$ and $U(N)$ have the same invariants on traceless tuples. Splitting off the scalar coordinates t_i identifies the full invariant ring with $\mathbb{C}[\mathfrak{sl}_N^k]^{SU(N)}[t_1, \ldots, t_k]$. In particular res is surjective and its kernel in multidegree $\mathbf{m}$ is the corresponding part of the ideal generated by the t_i.

We have $E(F) = \ker \mathrm{res}$. Indeed, a fixed position in block i contributes the factor t_i. Conversely, an element of the kernel can be written as a sum of terms $t_i q_i$, where q_i is invariant of multidegree $\mathbf{m} - \mathbf{e}_i$, and only indices with $m_i > 0$ occur. Express each q_i as a linear combination of multi-traces in its smaller multidegree. Inserting one fixed point in block i multiplies each evaluation by t_i. Averaging over K preserves this evaluation and keeps the support on permutations with a fixed point. This proves the equality, including when the ideal component is zero.

It follows that $\ker T = \ker E + F$, so

$$(\ker T)^{\perp} = (\ker E)^{\perp} \cap F^{\perp} = A_N \cap D.$$

Since T is surjective and A is the orthogonal direct sum of $\ker T$ and its complement, restriction to D_N is bijective. □

2 The one-loop operator

Take $k = 2$. Use the notation of [1] for the permutation one-loop operator $H^{(1)}_{\xi=1/N}$, denoted by M below. Write Δ_U and Δ_{SU} for the corresponding operators on H_N and H'_N, respectively. Let P_{A_N} and P_{D_N} be the orthogonal projectors onto the indicated subspaces, with respect to the inner product on A defined above.

Corollary 2. *Under the isomorphism $T|_{D_N}$, the operator $P_{D_N} M|_{D_N}$ corresponds to Δ_{SU}. This is the projected action in Conjecture 2 of [1].*

Proof. By Corollary 2.29 of [1], $EP_{A_N} Mx = \Delta_U Ex$ for $x \in A_N$. Since $A_N^{\perp} = \ker E$, we have $EP_{A_N} = E$. The scalar-traceless decomposition in the proof of Theorem 2 of [1] gives

res $\Delta_U = \Delta_{SU}$ res: the one-loop operator acts on the traceless coefficients and leaves the scalar variables untouched. By Theorem 1, $TP_{D_N} = T$. Thus for $x \in D_N \subseteq A_N$,

$$T(P_{D_N} Mx) = T(Mx) = \mathrm{res}(\Delta_U Ex) = \Delta_{SU} T(x).$$

Bijectivity of $T|_{D_N}$ gives the stated correspondence. □

Use of generative AI

Google Gemini 4 Hydrogen, used in Google Antigravity under the supervision of Rafael Coba Bruges, generated the original proof of the intersection theorem and the accompanying Python verification script. OpenAI Codex prepared the LaTeX manuscript from those materials, simplified the trace-ideal argument, and derived the one-loop corollary from the intersection theorem and the cited results of Padellaro, Ramgoolam, and Suzuki.

References

[1] A. Padellaro, S. Ramgoolam, and R. Suzuki, *Polynomial and Eigenspace Factorisations for the One-loop Dilatation Operator at Finite N*, arXiv:2610.10745, 2026.

[2] G. Benkart and S. Doty, *Derangements and tensor powers of adjoint modules for $\mathfrak{sl}_n$*, Journal of Algebraic Combinatorics **16** (2002), no. 1, 31–42.

[3] C. Procesi, *The invariant theory of $n \times n$ matrices*, Advances in Mathematics **19** (1976), no. 3, 306–381, doi:10.1016/0001-8708(76)90027-X.